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*Research article*

# Threshold behavior in a stochastic SIRS epidemic model with a logarithmic Ornstein-Uhlenbeck process

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**Abstract:** This study investigates the threshold behavior of a population-varying stochastic susceptible-infectious-recovered-susceptible (SIRS) model driven by a logarithmic Ornstein-Uhlenbeck process. Introducing the logarithmic Ornstein-Uhlenbeck process to account for random environmental fluctuations enhances the biological significance of the model. By applying the Itô stochastic integral, we construct a suitable Lyapunov function, proving the existence and uniqueness of the global positive solution of the model, thereby ensuring biological feasibility. Then, a critical threshold parameter $R_0^s$ is derived: If $R_0^s > 1$, the system admits a unique invariant probability measure; if $R_0^s < 1$, the infection dies out almost surely around the disease-free equilibrium. Furthermore, near the quasi-equilibrium point, the invariant probability density admits a local Gaussian approximation and converges weakly to a normal distribution as the environmental noise tends to zero. Numerical simulations in MATLAB illustrate the theoretical results, further reveal the sensitivity of the stochastic threshold to the key parameters, and confirm that population variation influences the threshold structure and long-term infection level.

**Keywords:** threshold; SIRS model; Markov semigroup; stationary distribution; Ornstein-Uhlenbeck process

## 1. Introduction

In recent years, infectious diseases such as COVID-19, avian influenza, and monkeypox have frequently emerged around the world, severely impacting public health and socio-economic systems. Mathematical modeling is of great significance for understanding disease transmission dynamics. The classic SIRS model established by Kermack and McKendrick [1] describes the dynamic changes in the population of susceptible, infected, and recovered compartments, with the assumption that the population is constant. However, the classical model has limitations when dealing with demographic changes and high-mortality diseases such as HIV and AIDS.

Busenberg and Driessche [2] proposed a variable-population SIRS epidemic model, and the model is given by

$$\begin{cases} \dot{S}(t) = \theta N - \mu S - \dfrac{\bar{\beta} S I}{N} + \epsilon R, \\ \dot{I}(t) = \dfrac{\bar{\beta} S I}{N} - (\mu + \nu + \gamma) I, \\ \dot{R}(t) = \gamma I - (\mu + \epsilon) R. \end{cases} \tag{1.1}$$

The biological significance of each variable is illustrated in [2], and no further explanation will be provided here. In Table 1, the meanings of the parameters are given.

**Table 1.** Significance of parameters.

| Parameter | Interpretation |
|---|---|
| $\theta$ | Birth rate |
| $\mu$ | Disease-free death rate |
| $\gamma$ | Recovery rate of infectives |
| $\epsilon$ | Immunity loss rate in recovered individuals |
| $\nu$ | Additional death rate among infectives |
| $\bar{\beta}$ | Rate of infection |

To facilitate mathematical processing, Busenberg and Driessche [2] performed a normalizing transformation of the model, and defined $s = S/N$, $i = I/N$, $r = R/N$. After normalizing the original model (1.1), the new system of equations becomes

$$\begin{cases} \dot{s}(t) = \theta - \theta s - \bar{\beta} s i + \epsilon r + \nu s i, \\ \dot{i}(t) = \bar{\beta} s i - (\theta + \nu + \gamma) i + \nu i^2, \\ \dot{r}(t) = \gamma i - (\theta + \epsilon) r + \nu i r. \end{cases} \tag{1.2}$$

One of the central objectives in mathematical epidemiology is to determine critical thresholds that delineate the persistence or extinction of infectious diseases. Next, we introduce the classical epidemiological threshold $R_0 = \dfrac{\bar{\beta}}{\theta + \nu + \gamma}$, and according to [2], we have the following:

(i) When $R_0 < 1$, the basic reproduction number lies below the epidemic threshold. In this case, there exists a unique disease-free equilibrium $E^0 = (s^0, i^0, r^0)$, and it is globally asymptotically stable within the invariant set, which indicating that the infection will die out and eradication is biologically feasible.

(ii) When $R_0 > 1$, the threshold is exceeded, the disease-free equilibrium $E_0$ becomes unstable, and there is a globally asymptotically stable endemic equilibrium point $E^* = (s^*, i^*, r^*)$, signifying sustained infection and the necessity of control measures to reduce transmission.

However, empirical evidence shows that real-world epidemic dynamics are governed by random perturbations stemming from environmental changes and population movements. These stochastic influences can undermine deterministic predictions; indeed, even when deterministic theory predicts persistence, random perturbations may drive the infection to extinction [3, 4]. Consequently, it is necessary to introduce a stochastic version of the basic reproduction number, denoted $R_0^s$, and a precise definition and understanding of $R_0^s$ is crucial for characterizing threshold behavior under uncertainty.

In deterministic epidemic models, parameter perturbations are typically introduced either by incorporating white noise proportional to the key parameters [5, 6] or by modeling the parameters as following an Ornstein-Uhlenbeck (OU) process [7–9]. Research has shown that the OU process is more realistic than the traditional white noise approach. For example, Wang et al. [10] have proven that the OU process exhibits continuity, an asymptotic distribution, and practical applicability. The study by Mamis and Farazmand [11] indicates that white noise models usually fail to capture the key characteristics of major epidemic outbreaks, while using an OU process to characterize uncertainty in disease transmission can predict the epidemic more accurately. According to Zhou et al. [12], when an OU process is used to constrain the parameter for the stochastic process, its variance becomes very small over short time intervals, enabling the OU process to avoid certain biological shortcomings. Therefore, stochastic models described by an OU process are more practical than those based on Gaussian white noise.

In practical modeling, directly using the OU process may lead to negative parameter values, which are unreasonable in biology. To address this problem, researchers have introduced the logarithmic Ornstein-Uhlenbeck (Log-OU) process, which ensures that parameters remain positive by taking the logarithm of the OU process. Shi and Jiang [13] studied a stochastic susceptible-infectious-susceptible (SIS) model with Log-OU process, obtaining threshold conditions for extinction. Han et al. [14] applied the OU process to develop a stochastic COVID-19 model and analyzed the behavioral characteristics of the model solutions in detail. These studies indicate that the Log-OU process has biological significance, which is more meaningful than previous stochastic modeling methods. The Log-OU process addresses a key limitation of traditional noise modeling: It ensures biologically plausible positive transmission rates while allowing us to rigorously define a stochastic threshold $R_0^s$ that generalizes $R_0$ to noisy environments. Thus, we consider

$$d\ln\beta(t) = \rho\left(\ln\bar{\beta} - \ln\beta(t)\right) dt + \sigma\, dB(t),$$

where $\rho$ represents reversion speed, $\bar{\beta}$ denotes the long-term average transmission rate, and $\sigma$ denotes the noise intensity. Reference [15] shows that $\beta(t)$ converges to a log-normal distribution characterized by a mean $\bar{\beta}e^{\frac{\sigma^2}{4\rho}}$ and a specific variance. The variance gradually approaches zero as the time interval shortens. Based on the above discussion, we define

$$G(t) = \ln\beta(t) - \ln\bar{\beta}.$$

It can be derived that $G(t)$ satisfies the following OU process:

$$dG(t) = -\rho\, G(t)\, dt + \sigma\, dB(t).$$

Thus, the stochastic SIRS model with a logarithmic OU process is given by

$$\begin{cases} dS(t) = \left[\theta N - \mu S - \dfrac{\bar{\beta}e^{G(t)}SI}{N} + \epsilon R\right] dt, \\ dI(t) = \left[\dfrac{\bar{\beta}e^{G(t)}SI}{N} - (\mu + \nu + \gamma)I\right] dt, \\ dR(t) = \left[\gamma I - (\mu + \epsilon)R\right] dt, \\ dG(t) = -\rho G(t)dt + \sigma dB(t). \end{cases} \tag{1.3}$$

The innovation of the stochastic SIRS model lies in the introduction of the Log-OU process as the stochastic perturbation for the transmission rate and in focusing on the threshold parameter $R_0^s$ under environmental noise. This approach not only preserves biological realism but also enables the explicit characterization of how environmental noise reshapes the epidemic threshold. This allows the model to simulate the effects of external environmental changes on disease transmission more realistically. By analyzing this model, we aim to explore its dynamic properties, such as the stationary distribution, extinction, and persistence, to better reflect the uncertainty and complexity inherent in disease transmission.

In particular, we investigate the threshold behavior that separates disease eradication from persistence, providing biologically meaningful insights into how environmental fluctuations affect epidemic outcomes. The main results of this work confirm that $R_0^s$ is the critical threshold parameter governing the long-term dynamics of the stochastic SIRS system. Biologically, $R_0^s$ reflects the combined influence of disease transmission and environmental perturbations: Even when a deterministic model predicts persistence, if $R_0^s < 1$, random fluctuations can still drive the infection to extinction; conversely, $R_0^s > 1$ suggests that the transmission potential is sufficient to overcome the effects of noise and thus maintain the outbreak.

Same as above, by normalizing (1.3), we develop the system that limits the system variables to [0, 1], simplifying and standardizing the analysis:

$$\begin{cases} di(t) = \left[\bar{\beta}e^{G(t)}(1-i-r)i - (\theta+\nu+\gamma)i + \nu i^2\right]dt, \\ dr(t) = \left[\gamma i - (\theta+\epsilon)r + \nu i r\right]dt, \\ dG(t) = -\rho G(t)\,dt + \sigma\,dB(t). \end{cases} \tag{1.4}$$

This normalization simplifies model analysis by lowering the dimensions of the system based on the interdependencies between variables.

To make the effect of population variability explicit, we compare (1.4) with the corresponding stochastic SIRS model with constant total population and the same Log-OU perturbation:

$$\begin{cases} di_c(t) = [\bar{\beta}e^{G(t)}(1-i_c-r_c)i_c - (\theta+\gamma)i_c]dt, \\ dr_c(t) = [\gamma i_c - (\theta+\epsilon)r_c]dt, \\ dG(t) = -\rho G(t)\,dt + \sigma\,dB(t). \end{cases}$$

For this constant-population counterpart, the associated stochastic threshold is

$$R_{0,c}^s = \frac{\bar{\beta}e^{\sigma^2/(4\rho)}}{\theta+\gamma},$$

whereas for the varying-population model (1.4), the threshold is

$$R_0^s = \frac{\bar{\beta}e^{\sigma^2/(4\rho)}}{\theta+\nu+\gamma}.$$

Therefore, although the total population does not appear explicitly in the normalized system, the influence of population variability is retained through the disease-induced mortality parameter $\nu$. More precisely, it enters both the stochastic threshold and the $\nu$-dependent nonlinear drift terms in (1.4), namely $\nu i^2$ and $\nu ir$. This shows that the normalized system still preserves the dynamical effect of varying population size.

The diffusion terms in stochastic systems can generally be divided into degenerate and non-degenerate diffusion terms. For stochastic models with non-degenerate diffusion terms, the stationary distribution is typically analyzed using Khasminskii's theory [16] and Lyapunov exponent theory. However, for models with degenerate diffusion terms, the analysis becomes more complex, and the conventional methods used in [16, 17] cannot be applied. Therefore, it is necessary to employ Markov semigroup theory [18].

In stochastic epidemic models, especially in stochastic systems with degenerate perturbations, invariant probability densities provide useful information for describing the long-term behavior of the system. The Fokker-Planck equation gives a formal description of the evolution of probability densities. However, in degenerate settings, it is usually difficult to obtain explicit density information directly from the corresponding partial differential equation. For this reason, researchers often combine probabilistic techniques with approximation methods. Zhou et al. proposed a normal approximation method [19] and developed local approximations of invariant probability densities for epidemic models [5]. More recently, Zhou et al. [20] established a mathematical framework for local and global approximations of invariant probability densities in stochastic Kolmogorov systems with small diffusion. Motivated by these developments, in the present paper we establish the threshold and invariant measure results, mainly through probabilistic arguments and Markov semigroup techniques, while the density-based discussion is used primarily to interpret the local probabilistic structure near the quasi-equilibrium point.

Nevertheless, stochastic SIRS models with variable populations, Log-OU perturbations, and degenerate diffusion terms have received relatively limited attention. This motivates us to investigate the threshold dynamics, invariant measure, and extinction behavior of system (1.4). In Section 2, we first establish the existence and uniqueness of its positive solution. In Section 3, under the condition $R_0^s > 1$, we focus on the existence and uniqueness of a stationary distribution for model (1.4). In Section 4, we present a local approximation of the invariant probability density near the quasi-equilibrium point. In Section 5, assuming $R_0^s < 1$, we prove the extinction property of (1.4). Finally, Section 6 presents numerical simulations and sensitivity analysis.

## 2. Existence and uniqueness of solutions to the system (1.4)

In this section, we establish the existence and uniqueness of the global positive solution to system (1.4). Moreover, we define the feasible region as

$$\Gamma = \{(i, r, G) \in \mathbb{R}_+^2 \times \mathbb{R} : i + r < 1\}, \tag{2.1}$$

which is positively invariant for model (1.4) almost surely (a.s.).

**Theorem 2.1.** *For any initial condition* $(i(0), r(0), G(0)) \in \Gamma$*, system* (1.4) *has a unique solution* $(i(t), r(t), G(t))$ *for all* $t \geq 0$*. Moreover, this solution remains in* $\Gamma$ *with probability one.*

*Proof.* Since the coefficients in system (1.4) satisfy the local Lipschitz condition, the theory in [21] guarantees the existence of a unique local solution for any $(i(0), r(0), G(0)) \in \Gamma$. To extend this solution globally, it suffices to show that the explosion time $\tau_e = \infty$ almost surely. The general approach involving a Lyapunov function and appropriate stopping times is well established (see Theorem 2.1 of [22]). Define

$$V(i, r, G) = (i - 1 - \ln i) + (r - 1 - \ln r) + [(1 - i - r) - 1 - \ln(1 - i - r)] + (e^G - G - 1). \tag{2.2}$$

Then,

$$dV(i, r, G) = LV(i, r, G)dt + \left(e^G - 1\right)\sigma dB(t), \tag{2.3}$$

where

$$\begin{aligned} LV(i, r, G) = {} & 2\theta + \nu + \gamma + \epsilon - 3\nu i - \frac{\gamma i}{r} - \frac{\theta i}{1 - i - r} - \frac{(\theta + \epsilon)r}{1 - i - r} \\ & + \bar{\beta}e^G[i - (1 - i - r)] - \rho G(e^G - 1) + \frac{\sigma^2}{2}e^G. \end{aligned} \tag{2.4}$$

Let $f(G) = -\rho G(e^G - 1) + \left(\bar{\beta} + \frac{\sigma^2}{2}\right)e^G$, and observe that $f(G) \to -\infty$ as $G \to -\infty$ or $G \to +\infty$. Therefore, we can find a $k_0$ satisfying $\sup_{G\in\mathbb{R}} f(G) < k_0$. Then, we get

$$LV(i, r, G) \leq 2\theta + \nu + \gamma + \epsilon + k_0 := K. \tag{2.5}$$

Then,

$$dV(i, r, G) \leq Kdt + \left(e^G - 1\right)\sigma dB(t). \tag{2.6}$$

The following proof is essentially identical to Theorem 2.1 in [22] and is therefore omitted here. □

## 3. Existence and uniqueness of a stationary distribution

### 3.1. Markov semigroup

We define a diffusion process $(i, r, G)^T$ with a transition probability density function $P(t, i, r, G, A_1)$, where $A_1 \in \Sigma$, such that

$$P(t, i, r, G, A_1) = \text{Prob}\left\{(i(t), r(t), G(t))^T \in A_1\right\},$$

where $A_1$ is a measurable set in the state space $\Gamma$ defined by the $\sigma$-algebra $\Sigma$. Let $m$ be a $\sigma$-finite measure. Then, in the $\sigma$-finite measure space $(\Gamma, \Sigma, m)$, the density space is defined as $\mathbb{D} = \{g \in L^1(\Gamma) : g \geq 0, \|g\|_1 = 1\}$. For $\{P(t)\}_{t\geq 0}$, its long-term behavior is fundamentally classified as follows.

**Definition 3.1.** *The semigroup is said to exhibit:*

*(i) Asymptotic stability if there exists a unique invariant density $g^* \in \mathbb{D}$ such that $\lim_{t\to\infty} \|P(t)g - g^*\|_1 = 0$ for all $g \in \mathbb{D}$.*

*(ii) Sweeping property on a set $A_1 \in \Sigma$ if $\lim_{t\to\infty} \int_{A_1} P(t)g\, dm = 0$ for all $g \in \mathbb{D}$.*

*(iii) Foguel Alternative holds for an integral Markov semigroup $\{P(t)\}$ with transition kernel $k(t, \mathbf{x}, \mathbf{y})$ satisfying $\int_\Gamma k(t, \mathbf{x}, \mathbf{y})\, dm(\mathbf{s}) = 1$ for all $\mathbf{y} \in \Gamma$ and $t > 0$, and $\int_0^\infty P(t)g(\mathbf{x})\, dt > 0$ almost everywhere for all $g \in \mathbb{D}$, in which case the semigroup is either asymptotically stable or sweeping.*

### 3.2. Formal Fokker-Planck description

The diffusion process $(i, r, G)^\top$ admits a transition density $\Phi(t, i, r, G)$. Formally, the evolution of $\Phi$ is governed by the Fokker-Planck equation:

$$\frac{\partial \Phi}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2(G^2\Phi)}{\partial G^2} - \frac{\partial(f_1(i, r, G)\Phi)}{\partial i} - \frac{\partial(f_2(i, r, G)\Phi)}{\partial r} - \frac{\partial(f_3(i, r, G)\Phi)}{\partial G}, \tag{3.1}$$

where $f_k(i, r, G)$ $(k = 1, 2, 3)$ represents the drift terms as described in system (1.4).

*3.3. Proof of stationary distribution*

**Lemma 3.1.** *For every $(i_0, r_0, G_0)^T \in \Gamma$, $P(t, i_0, r_0, G_0, A_1)$ is absolutely continuous with respect to the Lebesgue measure, and thus can be described by $k(t, i, r, G; i_0, r_0, G_0)$, which is continuous in all its arguments.*

*Proof.* Define $\mathbf{m}(\zeta, \vartheta, v) \in \Gamma$ and $\mathbf{n}(\zeta, \vartheta, v) \in \Gamma$; the Lie bracket $[\mathbf{m}, \mathbf{n}]$ can be given by

$$[\mathbf{m}, \mathbf{n}]_j(x) = \sum_{k=1}^{d} \left( m_k \frac{\partial n_j}{\partial x_k} - n_k \frac{\partial m_j}{\partial x_k} \right), \quad j = 1, 2, 3.$$

Then, we get

$$\mathbf{m}(\zeta, \vartheta, v) = \left( \bar{\beta} e^v (1 - \zeta - \vartheta)\zeta - (\theta + \nu + \gamma)\zeta + \nu\zeta^2, \ \gamma\zeta - (\theta + \epsilon)\vartheta + \nu\zeta\vartheta, \ -\rho v \right)^T,$$

$$\mathbf{n}(\zeta, \vartheta, v) = (0, 0, \sigma)^T, \quad (\zeta, \vartheta, v) \in \Gamma.$$

Let $\mathbf{c} = [\mathbf{m}, \mathbf{n}]$. Then,

$$\mathbf{c} = \left( -\bar{\beta}\sigma e^v (1 - \zeta - \vartheta)\zeta, \ 0, \ \sigma\rho \right)^T.$$

Define $\mathbf{d} = [\mathbf{m}, \mathbf{c}]$. Then,

$$\begin{aligned} \mathbf{d}_1 = [\mathbf{m}, \mathbf{c}]_1 = \Bigg\{ & - \Big[ \beta e^v (1 - \zeta - \vartheta)\vartheta - (\theta + \nu + \gamma)\zeta + \nu\zeta^2 \Big] \cdot \sigma(1 - 2\zeta - \vartheta) \\ & + \sigma\zeta \Big[ \gamma\zeta - (\theta + \epsilon)\vartheta + \nu\zeta\vartheta \Big] + (1 - \zeta - \vartheta)\zeta \Big[ \rho v \sigma + \sigma\bar{\beta} e^v (1 - 2\zeta - \vartheta) \\ & - \sigma(\theta + \nu + \gamma) + 2\sigma\nu\zeta - \sigma\rho \Big] \Bigg\} \bar{\beta} e^v, \end{aligned}$$

$$\mathbf{d}_2 = [\mathbf{m}, \mathbf{c}]_2 = \bar{\beta}\sigma e^v (1 - \zeta - \vartheta)\zeta(\gamma + \nu\vartheta) \neq 0,$$

$$\mathbf{d}_3 = [\mathbf{m}, \mathbf{c}]_3 = \sigma\rho^2 \neq 0.$$

Then,

$$[\mathbf{n}, \mathbf{c}, \mathbf{d}] = \begin{pmatrix} 0 & -\bar{\beta}\sigma e^v (1 - \zeta - \vartheta)\zeta & d_1 \\ 0 & 0 & \bar{\beta}\sigma e^v (1 - \zeta - \vartheta)\zeta(\gamma + \nu\vartheta) \\ \sigma & \sigma\rho & \sigma\rho^2 \end{pmatrix},$$

$$|\mathbf{n}, \mathbf{c}, \mathbf{d}| = -\bar{\beta}^2 \sigma^3 e^{2v} (1 - \zeta - \vartheta)^2 \zeta^2 (\gamma + \nu\vartheta) < 0.$$

Consequently, $\mathbf{n}, \mathbf{c}, \mathbf{d}$ are linearly independent on $\Gamma$ . For any $(\zeta, \vartheta, v) \in \Gamma$, we can obtain that $\mathbf{n}, \mathbf{c}, \mathbf{d}$ span the space $\Gamma$. According to Hörmander theorem [23], $P(t, i_0, r_0, G_0, A_1)$ has a continuous density $k(t, i, r, G; i_0, r_0, G_0)$, $k \in C^\infty((0, \infty) \times \Gamma \times \Gamma)$. □

We define $X = (i, r, G)^T$ and $X_0 = (i_0, r_0, G_0)^T$ as usual. Consider the Fréchet derivative $D_{X_0;\varphi}$ of $h \to X_{\varphi+h}(T)$ from $L^2([0, T]; \mathbb{R})$ to $X$. If $D_{X_0,\varphi}$ is full rank, for $X = X_\varphi(T)$, we have $k(T, i, r, G; i_0, r_0, G_0) > 0$. Define

$$\Psi(t) = \mathbf{f}'(X_\varphi(t)) + \mathbf{g}'(X_\varphi(t))\varphi,$$

where

$$\mathbf{f} = \begin{pmatrix} \overline{\beta} e^{G_\varphi}(1 - i_\varphi - r_\varphi)i_\varphi - (\theta + \nu + \gamma)i_\varphi + \nu i^2 \\ \gamma i_\varphi - (\theta + \epsilon)r_\varphi + \nu i_\varphi r_\varphi \\ -\rho G_\varphi \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}.$$

Then, the Jacobian matrix $\mathbf{f}'$ is

$$\mathbf{f}' = \begin{pmatrix} \overline{\beta} e^{G_\varphi}(1 - 2i_\varphi - r_\varphi) - (\theta + \nu + \gamma) + 2\nu i_\varphi & -\overline{\beta} e^{G_\varphi} i_\varphi & \overline{\beta} e^{G_\varphi}(1 - i_\varphi - r_\varphi)i_\varphi \\ \gamma + \nu r_\varphi & \nu i_\varphi - (\theta + \epsilon) & 0 \\ 0 & 0 & -\rho \end{pmatrix}.$$

For $0 \leq t_0 \leq t \leq T$, let $W(t_0, t_0) = \mathbf{I}d$ and $W(t, t_0)$ represent the matrix solution satisfying

$$\frac{\partial W(t, t_0)}{\partial t} = \Psi(t)W(t, t_0),$$

where $\mathbf{I}d$ is the identity matrix. Thus, $D_{X_0;\varphi}$ can be represented as

$$D_{X_0;\varphi} = \int_0^T W(T, s)\mathbf{g}(s)h(s)\, ds.$$

**Lemma 3.2.** *If for any $t \in [0, T]$, $x(0) = x_0 > 0$, $x'(t) + ax(t) > 0$ ($a > 0$), then $x(t) > 0$, $\forall t \in [0, T]$.*

*Proof.* According to the method of variation of constants for first-order nonhomogeneous linear differential equations, then $x(t) = e^{-at}x_0 + \int_0^t e^{-a(t-s)}h(s)\, ds > 0$. □

**Lemma 3.3.** *Consider an initial state vector $X_0 = (i_0, r_0, G_0)^\top \in \Gamma$ and $(i, r, G)^\top \in \Gamma$; then, there exists a positive constant $T$ satisfying $k(T, i, r, G; i_0, r_0, G_0) > 0$, where the constraint space $\Gamma$ follows definition Eq* (2.1).

*Proof.* Step 1: Full rank of the Fréchet derivative

Firstly, prove that derivative $D_{X_0;\varphi}$ has rank 3. Consider an arbitrary $\varepsilon$ within the interval $(0, T)$, and define

$$h(t) = \frac{\mathbf{1}_{[T-\varepsilon,T]}(t)}{G_\varphi(t)}, \quad t \in [0, T].$$

Here, $\mathbf{1}_{[T-\varepsilon,T]}$ is the characteristic function on the terminal temporal segment $[T - \varepsilon, T]$.

Through asymptotic expansion analysis of the equation $W(T, s) = I + \Psi(T)(T - s) + o(T - s)$, then

$$D_{X_0;\varphi}h = \varepsilon\mathbf{v} + \frac{\varepsilon^2\Psi(T)\mathbf{v}}{2} + o(\varepsilon^2).$$

For the function $\mathbf{v} = (0, 0, \sigma)^T$,

$$\Psi(t)\mathbf{v} = \begin{pmatrix} \overline{\beta}\sigma e^G(1 - i - r)i \\ 0 \\ -\rho\sigma \end{pmatrix}.$$

We define the matrix $B \triangleq \Psi^2(t)$ as

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}.$$

We compute

$$\Psi^2(t)\mathbf{v} = \begin{pmatrix} \sigma B_{13} \\ \sigma B_{23} \\ \sigma\rho^2 \end{pmatrix}.$$

Then, the determinant of the matrix is

$$\left|\mathbf{v}, \Psi(t)\mathbf{v}, \Psi^2(t)\mathbf{v}\right| = \sigma^3\bar{\beta}^2 e^{2G}(1-i-r)^2 i^2(\gamma+\nu r) \neq 0.$$

$\mathbf{v}, \Psi(t)\mathbf{v}$ and $\Psi^2(t)\mathbf{v}$ are linearly independent because the determinant is non-zero. Therefore, $D_{X_0;\varphi}$ has **rank 3**.

Step 2: Existence of the control function

Next, we demonstrate that for any given points $(i_0, r_0, G_0)^\top \in \Gamma$ and $(i_1, r_1, G_1)^\top \in \Gamma$, there exists $\varphi$ and $T > 0$ satisfying

$$(i_\varphi(0), r_\varphi(0), G_\varphi(0))^\top = (i_0, r_0, G_0)^\top, \quad (i_\varphi(T), r_\varphi(T), G_\varphi(T))^\top = (i_1, r_1, G_1)^\top.$$

On $[0, T]$, system (1.4) is equivalently governed by

$$\begin{cases} i'_\varphi = \bar{\beta}e^{G_\varphi}(1-i_\varphi-r_\varphi)i_\varphi - (\theta+\nu+\gamma)i_\varphi + \nu i_\varphi^2, \\ r'_\varphi = \gamma i_\varphi - (\theta+\epsilon)r_\varphi + \nu i_\varphi r_\varphi, \\ G'_\varphi = -\rho G_\varphi + \sigma\varphi. \end{cases} \tag{3.2}$$

Hence, we obtain that the variable $r_\varphi \in C^2([0, T]; \mathbb{R}_+)$ satisfies the inequalities given below:

$$\begin{cases} r_\varphi > 0, \\ i_\varphi = \dfrac{r'_\varphi + (\theta+\epsilon)r_\varphi}{\gamma+\nu r_\varphi} > 0, \\ i_\varphi + r_\varphi = \dfrac{r'_\varphi + (\theta+\epsilon)r_\varphi}{\gamma+\nu r_\varphi} + r_\varphi < 1, \\ \bar{\beta}e^{G_\varphi}(1-i_\varphi-r_\varphi)i_\varphi = i'_\varphi + (\theta+\nu+\gamma)i_\varphi - \nu i_\varphi^2 > 0, \end{cases} \tag{3.3}$$

which is mathematically equivalent to

$$\begin{cases} r_\varphi > 0, \\ r'_\varphi + (\theta+\epsilon)r_\varphi > 0, \\ r'_\varphi + \nu r_\varphi^2 + (\theta+\nu+\gamma+\epsilon)r_\varphi - \gamma < 0, \\ (\gamma+\nu r_\varphi)r''_\varphi - 2\nu r'^2_\varphi + \gamma(2\theta+\epsilon+\nu+\gamma)r'_\varphi + \left[\nu(\theta+\nu+\gamma) - 2\nu(\theta+\epsilon)\right] r_\varphi r'_\varphi \\ \quad + \nu(\theta+\epsilon)(\nu+\gamma-\epsilon)r_\varphi^2 + \gamma(\theta+\epsilon)(\theta+\nu+\gamma)r_\varphi := F(r_\varphi, r'_\varphi, r''_\varphi) > 0. \end{cases} \tag{3.4}$$

Moreover, from (3.2), we derive the system for $t \in [0, T]$:

$$\begin{cases} r_\varphi(t) = r_t, \\ r'_\varphi(t) = \gamma i_t - (\theta + \epsilon - \nu i_t) r_t, \\ r''_\varphi(t) = (\gamma + \nu r_t)\bar{\beta} e^{G_t}(1 - i_t - r_t) i_t - \dfrac{1}{\gamma + \nu r_t} \Big\{ -2\nu \left(r'_t\right)^2 \\ \qquad\qquad +\gamma\left(2\theta + \epsilon + \nu + \gamma\right)\right) r'_t + \left[\nu(\theta + \nu + \gamma) - 2\nu(\theta + \epsilon)\right] r_t r'_t \\ \qquad\qquad + \gamma(\theta + \epsilon)(\theta + \nu + \gamma) r_t + \nu(\theta + \epsilon)(\nu + \gamma - \epsilon) r_t^2 \Big\}. \end{cases} \tag{3.5}$$

Now, we aim to find an appropriate $T$ and establish a differentiable function $r_\varphi \in C^2([0, T]; \mathbb{R}_+)$ satisfying

$$\begin{cases} r_\varphi(0) \triangleq r_0, \\ r'_\varphi(0) = \gamma i_0 - (\theta + \epsilon - \nu i_0) r_0 \triangleq r'_0, \\ r''_\varphi(0) = (\gamma + \nu r_0)\bar{\beta} e^{G_0}(1 - i_0 - r_0) i_0 - \dfrac{1}{\gamma + \nu r_0} \Big\{ -2\nu \left(r'_0\right)^2 \\ \qquad\qquad +\gamma\left(2\theta + \epsilon + \nu + \gamma\right)\right) r'_0 + \left[\nu(\theta + \nu + \gamma) - 2\nu(\theta + \epsilon)\right] r_0 r'_0 \\ \qquad\qquad + \gamma(\theta + \epsilon)(\theta + \nu + \gamma) r_0 + \nu(\theta + \epsilon)(\nu + \gamma - \epsilon) r_0^2 \Big\} \triangleq r''_0, \end{cases} \tag{3.6}$$

and

$$\begin{cases} r_\varphi(T) \triangleq r_1, \\ r'_\varphi(T) = \gamma i_1 - (\theta + \epsilon - \nu i_1) r_1 \triangleq r'_1, \\ r''_\varphi(T) = (\gamma + \nu r_1)\bar{\beta} e^{G_1}(1 - i_1 - r_1) i_1 - \dfrac{1}{\gamma + \nu r_1} \Big\{ -2\nu \left(r'_1\right)^2 \\ \qquad\qquad +\gamma\left(\theta + \epsilon + (\theta + \nu + \gamma)\right) r'_1 + \left[\nu(\theta + \nu + \gamma) - 2\nu(\theta + \epsilon)\right] r_1 r'_1 \\ \qquad\qquad + \gamma(\theta + \epsilon)(\theta + \nu + \gamma) r_1 + \nu(\theta + \epsilon)(\nu + \gamma - \epsilon) r_1^2 \Big\} \triangleq r''_1. \end{cases} \tag{3.7}$$

Next, we shall prove that when $r_\varphi \in C^2([0, T]; \mathbb{R}_+)$ makes the last inequality of (3.3) hold, then the remaining three inequalities also hold. In fact, when the last inequality in (3.3) holds, we can get that

$$i'_\varphi + (\theta + \nu + \gamma) i_\varphi > 0, \quad \forall t \in [0, T]. \tag{3.8}$$

Since $i_\varphi(0) = i_0 > 0$, then from Lemma 3.2, there is $i_\varphi(t) > 0$ for any $t \in [0, T]$. Furthermore, assuming $i_\varphi(t) > 0$ holds, we can also deduce that

$$r'_\varphi + (\theta + \epsilon) r_\varphi > 0, \quad \forall t \in [0, T], \tag{3.9}$$

and given $r_\varphi(0) = r_0 > 0$, by Lemma 3.2,

$$r_\varphi(t) > 0, \quad \forall t \in [0, T]. \tag{3.10}$$

Similarly, from the last inequality in (3.3) and $\bar{\beta} e^{G_\varphi} > 0$, then

$$i_\varphi(t) + r_\varphi(t) < 1, \quad \forall t \in [0, T]. \tag{3.11}$$

The function $r_\varphi$ is constructed over three distinct temporal domains: $[0,\tau]$, $[\tau, T-\tau']$, and $[T-\tau', T]$, where $\tau$ and $\tau'$ will be determined later and $\tau+\tau' < T$. On $[0,\tau]$, $r_\varphi$ is constructed as follows:

$$r_\varphi(t) = \frac{At^3}{6} + \frac{r_0''}{2}t^2 + r_0't + r_0, \quad t \in [0,\tau], \tag{3.12}$$

where $\tau$ is a sufficiently small number which will be chosen later, the coefficients $r_0, r_0', r_0''$ are given in (3.6), A satisfies $i_\varphi'(\tau) = 0$, and

$$i_\varphi'(\tau) = \frac{(\gamma+\nu r_\varphi(\tau))r_\varphi''(\tau) + \gamma(\theta+\epsilon)r_\varphi'(\tau) - \nu(r_\varphi'(\tau))^2}{(\gamma+\nu r_\varphi(\tau))^2} = 0. \tag{3.13}$$

That is,

$$\begin{aligned} &-\frac{\nu\tau^4}{12}A^2 + \left[\tau\left(\frac{\nu r_0''\tau^2}{2} + \nu r_0'\tau + \nu r_0 + \gamma\right) + \frac{\nu\tau^3}{6}r_0'' + (\theta+\epsilon)\gamma\frac{\tau^2}{2} - \nu\tau^2(r_0''\tau + r_0')\right]A \\ &+ r_0''\left(\gamma + \nu r_0 + \frac{\nu r_0''\tau^2}{2} + \nu r_0'\tau\right) + (\theta+\epsilon)\gamma r_0''\tau + r_0'(\theta+\epsilon)\gamma - \nu r_0''^2\tau^2 - 2\nu r_0' r_0''\tau \\ &- \nu r_0'^2 = 0. \end{aligned} \tag{3.14}$$

Simplifying (3.14), we have

$$\begin{aligned} &-\frac{\nu\tau^2}{12}(A\tau)^2 + \left[-\frac{1}{3}\nu r_0''\tau^2 + \frac{1}{2}(\theta+\epsilon)\gamma\tau + (\gamma+\nu r_0)\right]\cdot A\tau + (\gamma+\nu r_0)r_0'' \\ &+ (\theta+\epsilon)\gamma r_0' - \nu(r_0')^2 + (\theta+\epsilon)\gamma r_0''\tau - \frac{\nu}{2}r_0''^2\tau^2 - \nu r_0' r_0''\tau = 0. \end{aligned}$$

Thus,

$$-\frac{\nu\tau^2}{12}(A\tau)^2 + B(\tau)\cdot(A\tau) + C(\tau) = 0, \tag{3.15}$$

where

$$B(\tau) = (\gamma+\nu r_0) + o_1(\tau),$$

$$C(\tau) = (\gamma+\nu r_0)r_0'' + (\theta+\epsilon)\gamma r_0' - \nu(r_0')^2 + o_2(\tau) = (\gamma+\nu r_0)^2 i_0' + o_2(\tau),$$

and $\lim_{\tau\to 0^+} o_j(\tau) = 0 (j = 1,2,...)$,

$$o_1(\tau) = \frac{1}{2}(\theta+\epsilon)\gamma\tau - \frac{1}{3}\nu r_0''\tau^2,$$

$$o_2(\tau) = (\theta+\epsilon)\gamma r_0''\tau - \frac{\nu}{2}r_0''^2\tau^2 - \nu r_0' r_0''\tau.$$

As $\tau$ is a sufficiently small number, let

$$A\tau = \frac{B(\tau) - \sqrt{B^2(\tau) + \frac{\nu}{3}\tau^2\cdot C(\tau)}}{\nu\tau^2/6} = -\frac{2C(\tau)}{B(\tau) + \sqrt{B^2(\tau) + \frac{\nu}{3}\tau^2\cdot C(\tau)}}, \tag{3.16}$$

and then, from (3.15) and (3.16), we have

$$A\tau = -(\gamma + \nu r_0)i_0'(0) + o_3(\tau). \tag{3.17}$$

On $[0, \tau]$, from (3.12), we can obtain

$$r_\varphi(0) = r_0, \quad r_\varphi'(0) = r_0', \quad r_\varphi''(0) = r_0''. \tag{3.18}$$

According to (3.12) and (3.17), we get

$$\begin{aligned}
&r_\varphi(t) = r_0 + o_4(\tau), \quad r_\varphi'(t) = r_0' + o_5(\tau), \\
&r_\varphi''(t) = At + r_0'' = -\frac{t}{\tau}\left[(\gamma + \nu r_0)i_0' + o_3(\tau)\right] + r_0'' = -\frac{t}{\tau}(\gamma + \nu r_0)i_0' + r_0'' + o_6(\tau).
\end{aligned} \tag{3.19}$$

In addition, we obtain

$$i_\varphi(t) = i_0 + o_7(\tau), \quad i_\varphi'(t) = \frac{(\gamma + \nu r_0)r_0'' + (\theta + \epsilon)\gamma r_0' - \nu(r_0')^2}{(\gamma + \nu r_0)^2} + o_8(\tau).$$

Therefore,

$$\begin{aligned}
&i_\varphi'(t) + (\theta + \nu + \gamma)i_\varphi(t) - \nu i_\varphi^2(t) \\
&= \left(1 - \frac{t}{\tau}\right)i_0' + (\theta + \nu + \gamma)i_0 - \nu i_0^2 + o_9(\tau), \quad t \in [0, \tau],
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
F(r_\varphi, r_\varphi', r_\varphi'') &= (\gamma + \nu r_\varphi)^2\left[i_\varphi'(t) + (\theta + \nu + \gamma)i_\varphi(t) - \nu i_\varphi^2(t)\right] \\
&= (\gamma + \nu r_0)^2\left[\left(1 - \frac{t}{\tau}\right)i_0' + (\theta + \nu + \gamma)i_0 - \nu i_0^2\right] + o_{10}(\tau).
\end{aligned} \tag{3.21}$$

Next, we shall prove (3.4) holds. Since

$$\begin{cases}
r_\varphi(0) > 0, \\
r_\varphi'(0) + (\theta + \epsilon)r_\varphi(0) > 0, \\
r_\varphi'(0) + \nu r_\varphi(0)^2 + (\theta + \epsilon + \gamma + \nu)r_\varphi(0) - \gamma < 0, \\
(\gamma + \nu r_\varphi(0))r_\varphi''(0) - 2\nu r_\varphi'(0)^2 + \gamma(2\theta + \epsilon + \nu + \gamma))r_\varphi'(0) \\
\quad + \left[\nu(\theta + \nu + \gamma) - 2\nu(\theta + \epsilon)\right]r_\varphi(0)r_\varphi'(0) + \gamma(\theta + \epsilon)(\theta + \nu + \gamma)r_\varphi(0) \\
\quad + \nu(\theta + \epsilon)(\nu + \gamma - \epsilon)r_\varphi^2(0) > 0.
\end{cases} \tag{3.22}$$

Then, there exists a sufficiently small $\tau$ such that for $t \in (0, \tau)$,

$$\begin{cases}
r_\varphi = r_0 + o_4(\tau) > 0, \\
r_\varphi' + (\theta + \epsilon)r_\varphi = r_0' + (\theta + \epsilon)r_0 + o_{11}(\tau) > 0, \\
r_\varphi' + \nu r_\varphi^2 + (\theta + \nu + \gamma + \epsilon)r_\varphi - \gamma = r_0' + \nu r_0^2 + (\theta + \nu + \gamma + \epsilon)r_0 - \gamma + o_{12}(\tau) < 0, \\
F(r_\varphi, r_\varphi', r_\varphi'') = (\gamma + \nu r_0)r_0'' - 2\nu r_0'^2 + \gamma(2\theta + \epsilon + \nu + \gamma)r_0' \\
\qquad + \left[\nu(\theta + \nu + \gamma) - 2\nu(\theta + \epsilon)\right]r_0 r_0' + \gamma(\theta + \epsilon)(\theta + \nu + \gamma)r_0 \\
\qquad + \nu(\theta + \epsilon)(\nu + \gamma - \epsilon)r_0^2 + o_{13}(\tau) > 0.
\end{cases} \tag{3.23}$$

From (3.23), it follows that the first three inequalities in (3.4) are satisfied. As for the fourth inequality, we can analyze this from two different perspectives.

**Case 1.** If $i_0' \geq 0$, then by virtue of (3.21), we obtain

$$\begin{aligned} F(r_\varphi, r_\varphi', r_\varphi'') &= (\gamma + \nu r_0)^2 \left[\left(1 - \frac{t}{\tau}\right) i_0' + (\theta + \nu + \gamma) i_0 - \nu i_0^2\right] + o_{10}(\tau) \\ &\geq (\gamma + \nu r_0)^2 \left[(\theta + \nu + \gamma) i_0 - \nu i_0^2\right] \\ &\geq (\gamma + \nu r_0)^2 \left[(\theta + \gamma) i_0 + \nu i_0 (1 - i_0)\right] > 0. \end{aligned} \tag{3.24}$$

**Case 2.** If $i_0' < 0$, then by virtue of (3.21), we obtain

$$\begin{aligned} F(r_\varphi, r_\varphi', r_\varphi'') &= (\gamma + \nu r_0)^2 \left[\left(1 - \frac{t}{\tau}\right) i_0' + (\theta + \nu + \gamma) i_0 - \nu i_0^2\right] + o_{10}(\tau) \\ &\geq (\gamma + \nu r_0)^2 \left[i_0' + (\theta + \nu + \gamma) i_0 - \nu i_0^2\right] > 0. \end{aligned} \tag{3.25}$$

Hence, for $t \in [0, \tau]$, $r_\varphi$ satisfies (3.4), (3.6), and (3.7). Likewise, a $C^2$-function $r_\varphi : [T - \tau', T]$ can be constructed to ensure

$$r_\varphi(T) = r_1, \quad r_\varphi'(T) = r_1', \quad r_\varphi''(T) = r_1'', \quad i_\varphi'(T - \tau') = 0,$$

and $r_\varphi$ satisfies the inequality (3.5) for $t \in [T - \tau', T]$.

Finally, on $[\tau, T - \tau']$, in order to satisfy the fourth inequality in (3.3), it is necessary to ensure that $i_\varphi'(t) + (\theta + \nu + \gamma) i_\varphi(t) - \nu i_\varphi^2(t) > 0$ holds while meeting the constraints $i_\varphi'(\tau) = 0$, $i_\varphi'(T - \tau') = 0$. Accordingly, introducing $w_\varphi(t) = \ln(i_\varphi(t))$ simplifies the requirement to verifying that $w_\varphi'(t) > -(\theta + \gamma)$, $w_\varphi'(\tau) = 0$ and $w_\varphi'(T - \tau') = 0$. In this case, (3.23) holds. By choosing $T$ sufficiently large, $r_\varphi : [0, \tau] \cup [T - \tau', T] \to \mathbb{R}$ can be expanded to a $C^2$-function $r_\varphi$ defined on $[0, T]$, ensuring that (3.4), (3.6), and (3.7) hold.

Then by virtue of the second, third, and fourth equations of (3.3), we can find a $C^1$-function $i_\varphi$ and a continuous function $G_\varphi$ that satisfy (3.4), (3.6), and (3.7). Thus, the proof is completed. □

**Lemma 3.4.** *[18] The Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets.*

*Proof.* From Lemma 3.1, $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup with a density $k(t, i, r, G)$. By Lemma 3.3,

$$\int_0^\infty P(t) g dt > 0 \quad \text{a.s. on } \Gamma, \ \forall g \in \mathcal{D},$$

from $k(t, i, r, G) > 0$, $P(t)g = \int_\Gamma k(t, \mathbf{x}) g(\mathbf{x}) m(d\mathbf{x})$, and $\mathbf{x} = (i, r, G)^\top$. The conclusion then follows from Lemma 3.3. □

We consider the stochastic differential equation (SDE)

$$dX(t) = f(X(t))dt + g(X(t))dB(t), \tag{3.26}$$

where $X(0) \in \mathbb{R}^d$ is the initial value. The functions $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are assumed to be Borel measurable.

**Lemma 3.5.** *[24] Let $\mathbb{A} \subset \mathbb{R}^d$ be a bounded closed set with a boundary $\Gamma$. If, for any $X(0) \in \mathbb{R}^d$, the condition*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}(t, X(0), \mathbb{A})dt > 0 \quad a.s.$$

*holds, where $\mathbb{P}(t, X(0), \mathbb{A})$ is the transition probability of $X(t)$, so the solution of system* (3.26) *has the Feller property. Moreover, there exists at least one invariant probability measure $\Gamma(\cdot)$ for system* (3.26) *in $\mathbb{R}^d$.*

**Theorem 3.1.** *Let*

$$R_0^s = \frac{\bar{\beta} e^{\frac{\sigma^2}{4\rho}}}{\theta + \nu + \gamma} > 1,$$

*and let $\Psi(t, i, r, G)$ denote the probability density function of the distribution of $(i, r, G)^\top$, $\forall t > 0$. Then, a unique invariant probability measure $\varkappa$ exists on $\Gamma$ with density $\Psi^*(i, r, G)$ such that*

$$\lim_{t\to\infty} \int_\Gamma |\Psi(t, i, r, G) - \Psi^*(i, r, G)|\, di\, dr\, dG = 0,$$

*which implies that $\{P(t)\}_{t\geq 0}$ is asymptotically stable.*

*Proof.* Applying Lemma 3.3, $\{P(t)\}_{t\geq 0}$ satisfies the Foguel alternative. Next, we need to rule out the possibility of sweeping.

**Step 1: Construction of a nonnegative function**

Define the function $V_1$:

$$V_1 = -\ln(1 - i - r).$$

According to Itô's formula and (1.4),

$$\begin{aligned} \mathcal{L}^* V_1 &= -\frac{\theta + \epsilon r}{1 - i - r} + \theta + (\bar{\beta} e^G - \nu) i \\ &\leq -\frac{\theta + \epsilon r}{1 - i - r} + \theta + \bar{\beta} e^{\frac{\sigma^2}{4\rho}} + \bar{\beta}\left(e^G - e^{\frac{\sigma^2}{4\rho}}\right) \\ &:= -\frac{\theta + \epsilon r}{1 - i - r} + \theta + \bar{\beta} e^{\frac{\sigma^2}{4\rho}} + f_1(G), \end{aligned} \tag{3.27}$$

where $f_1(G) = \bar{\beta}\left(e^G - e^{\frac{\sigma^2}{4\rho}}\right)$. Then, we compute that

$$\begin{aligned} \mathcal{L}^*(-\ln i) &= -\bar{\beta} e^G (1 - i - r) + (\theta + \nu + \gamma) - \nu i \\ &= -(\theta + \nu + \gamma)(R_0^s - 1) + \bar{\beta} e^G (i + r) - \nu i - \bar{\beta}\left(e^G - e^{\frac{\sigma^2}{4\rho}}\right). \end{aligned} \tag{3.28}$$

By the arithmetic-geometric mean inequality, it follows that

$$e^G \leq k e^{2G} + \frac{1}{4k}, \tag{3.29}$$

where $k$ is a positive constant to be determined later. By substituting (3.29) into (3.28), it follows that

$$\begin{aligned}\mathcal{L}^*(-\ln i) &\le -(\theta+\nu+\gamma)(R_0^s-1)+\bar\beta\left(ke^{2G}+\frac{1}{4k}\right)(i+r)-\bar\beta\left(e^G-e^{\frac{\sigma^2}{4\rho}}\right)\\ &\le -(\theta+\nu+\gamma)(R_0^s-1)+\frac{\bar\beta}{4k}(i+r)+\bar\beta k e^{\frac{\sigma^2}{\rho}}+f_2(G)+\bar\beta k\left(e^{2G}-e^{\frac{\sigma^2}{\rho}}\right)\\ &= -(\theta+\nu+\gamma)(R_0^s-1)+\frac{\bar\beta}{4k}(i+r)+\bar\beta k e^{\frac{\sigma^2}{\rho}}+f_2(G)+f_3(G),\end{aligned}\tag{3.30}$$

where $f_2(G)=-\bar\beta\left(e^G-e^{\frac{\sigma^2}{4\rho}}\right)$ and $f_3(G)=\bar\beta k\left(e^{2G}-e^{\frac{\sigma^2}{\rho}}\right)$. Choose

$$k=\frac{(\theta+\nu+\gamma)(R_0^s-1)}{2\bar\beta e^{\frac{\sigma^2}{\rho}}},$$

such that $\bar\beta k e^{\frac{\sigma^2}{\rho}}=\frac{1}{2}(\theta+\nu+\gamma)(R_0^s-1)$. Then, (3.30) becomes

$$\mathcal{L}^*(-\ln i)\le -\frac{1}{2}(\theta+\nu+\gamma)(R_0^s-1)+\frac{\bar\beta}{4k}(i+r)+f_2(G)+f_3(G).\tag{3.31}$$

Then,

$$\mathcal{L}^*\left(\frac{r}{\theta+\epsilon}\right)=-r+\frac{\gamma+\nu r}{\theta+\epsilon}i.\tag{3.32}$$

Define

$$V_2=-\ln i+\frac{\bar\beta r}{4k(\theta+\epsilon)}.$$

Then, combining (3.31) and (3.32), we have

$$\begin{aligned}\mathcal{L}^*(V_2) &\le -\frac{1}{2}(\theta+\nu+\gamma)(R_0^s-1)+\frac{\bar\beta}{4k}i+\frac{\bar\beta(\gamma+\nu r)}{4k(\theta+\epsilon)}i+f_2(G)+f_3(G)\\ &\le -\frac{1}{2}(\theta+\nu+\gamma)(R_0^s-1)+\frac{\bar\beta}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right)i+f_2(G)+f_3(G).\end{aligned}\tag{3.33}$$

Let $V_3=-\ln r$ and $V_4=e^G-G-1$; then we have

$$\mathcal{L}^*V_3=-\gamma\frac{i}{r}+(\theta+\epsilon)-\nu i\le -\gamma\frac{i}{r}+(\theta+\epsilon),\tag{3.34}$$

and

$$\mathcal{L}^*V_4=-\rho G(e^G-1)+\frac{\sigma^2e^G}{2}.\tag{3.35}$$

Let

$$V'=V_1+MV_2+V_3+V_4,$$

where $M$ is a sufficiently large constant chosen to satisfy

$$-\frac{M}{2}(\theta+\nu+\gamma)(R_0^s-1)+K_1\le -2,\tag{3.36}$$

that is

$$M \geq \frac{4 + 2K_1}{(\theta + \nu + \gamma)(R_0^s - 1)}, \tag{3.37}$$

and

$$K_1 := 2\theta + \epsilon + \bar{\beta} e^{\frac{\sigma^2}{4\rho}} + \sup_{G \in \Gamma} \left\{ -\rho G(e^G - 1) + \frac{\sigma^2 e^G}{2} \right\} < \infty. \tag{3.38}$$

We note that $V'(i, r, G)$ remains continuous and increases to infinity as $(i, r, G)^\top$ approaches the boundary of $\Gamma$. Consequently, $V'(i, r, G)$ is bounded from below, achieving its minimum at some point $(i_m, r_m, G_m)^\top$ within the interior of $\Gamma$. Define a $C^2$-function $V : \Gamma \to \overline{\mathbb{R}}_+$ as

$$V(i, r, G) = V'(i, r, G) - V'(i_m, r_m, G_m).$$

From (3.27), (3.33)–(3.35), we have

$$\begin{aligned} \mathcal{L}^* V \leq & -\frac{M}{2}(\theta + \nu + \gamma)(R_0^s - 1) - \frac{\theta + \epsilon r}{1 - i - r} + M\frac{\bar{\beta}}{4k}\left(1 + \frac{\gamma + \nu}{\theta + \epsilon}\right) i - \gamma \frac{i}{r} \\ & + 2\theta + \epsilon + \bar{\beta} e^{\frac{\sigma^2}{4\rho}} - \rho G(e^G - 1) + \frac{\sigma^2 e^G}{2} + f_1(G) + M f_2(G) + M f_3(G) \\ := & F(i, r, G) + f_1(G) + M f_2(G) + M f_3(G), \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} F(i, r, G) = & -\frac{M}{2}(\theta + \nu + \gamma)(R_0^s - 1) - \frac{\theta + \epsilon r}{1 - i - r} + M\frac{\bar{\beta}}{4k}\left(1 + \frac{\gamma + \nu}{\theta + \epsilon}\right) i - \gamma \frac{i}{r} \\ & + 2\theta + \epsilon + \bar{\beta} e^{\frac{\sigma^2}{4\rho}} - \rho G(e^G - 1) + \frac{\sigma^2 e^G}{2}. \end{aligned} \tag{3.40}$$

**Step 2: Construction of a compact set**

Let

$$U_\varepsilon = \left\{ (i, r, G) \in \Gamma : |G| \leq \frac{1}{\varepsilon},\ i \geq \varepsilon,\ r \geq \varepsilon,\ i + r \leq 1 - \varepsilon^2 \right\}, \tag{3.41}$$

where $\varepsilon > 0$ is chosen to be sufficiently small so that the given conditions hold:

$$\frac{\bar{\beta} M}{4k}\left(1 + \frac{\gamma + \nu}{\theta + \epsilon}\right)\varepsilon \leq 1, \tag{3.42}$$

$$-\frac{\gamma}{\varepsilon} + \frac{\bar{\beta} M}{4k}\left(1 + \frac{\gamma + \nu}{\theta + \epsilon}\right) \leq 1, \tag{3.43}$$

$$-\frac{\epsilon}{\varepsilon} + \frac{\bar{\beta} M}{4k}\left(1 + \frac{\gamma + \nu}{\theta + \epsilon}\right) \leq 1. \tag{3.44}$$

$M$ was mentioned in Eq (3.37), and $K_1$ was mentioned in Eq (3.38). Next, the set $\Gamma \setminus U_\varepsilon$ is divided into the following five subsets $D_\varepsilon^j$, $j = 1, 2, .., 5$, where

$$\begin{aligned} & D_\varepsilon^1 = \left\{ (i, r, G) \in \Gamma : G > \frac{1}{\varepsilon} \right\}, \quad D_\varepsilon^2 = \left\{ (i, r, G) \in \Gamma : G < -\frac{1}{\varepsilon} \right\}, \\ & D_\varepsilon^3 = \{ (i, r, G) \in \Gamma : i < \varepsilon \}, \quad D_\varepsilon^4 = \left\{ (i, r, G) \in \Gamma : r < \varepsilon^2,\ i \geq \varepsilon \right\}, \\ & D_\varepsilon^5 = \left\{ (i, r, G) \in \Gamma : i + r > 1 - \varepsilon^2,\ r \geq \varepsilon \right\}. \end{aligned}$$

Thus, $\Gamma \setminus U_\varepsilon = D_\varepsilon^1 \cup D_\varepsilon^2 \cup D_\varepsilon^3 \cup D_\varepsilon^4 \cup D_\varepsilon^5$. We will prove that, for each $j$ and every $(i,r,G)^\top \in D_\varepsilon^i$, the inequality $F(i,r,G) \leq -1$ is satisfied.

**Case 1.** If $(i,r,G)^\top \in D_\varepsilon^1$, we have

$$\lim_{G\to\infty}\left\{-\rho G(e^G-1)+\frac{\sigma^2 e^G}{2}\right\} = -\infty,$$

$$\begin{aligned} F(i,r,G) &= -\frac{M}{2}(\theta+\nu+\gamma)(R_0^s-1) + M\frac{\bar{\beta}}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right)i - \gamma\frac{i}{r} \\ &\quad + 2\theta+\epsilon+\bar{\beta}e^{\frac{\sigma^2}{4\rho}} - \rho G(e^G-1) + \frac{\sigma^2 e^G}{2} - \frac{\theta+\epsilon r}{1-i-r} \\ &\leq -1. \end{aligned} \tag{3.45}$$

**Case 2.** If $(i,r,G)^\top \in D_\varepsilon^2$,

$$\lim_{G\to-\infty}\left\{-\rho G(e^G-1)+\frac{\sigma^2 e^G}{2}\right\} = -\infty,$$

$$\begin{aligned} F(i,r,G) &= -\frac{M}{2}(\theta+\nu+\gamma)(R_0^s-1) + M\frac{\bar{\beta}}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right)i - \gamma\frac{i}{r} \\ &\quad + 2\theta+\epsilon+\bar{\beta}e^{\frac{\sigma^2}{4\rho}} - \rho G(e^G-1) + \frac{\sigma^2 e^G}{2} - \frac{\theta+\epsilon r}{1-i-r} \\ &\leq -1. \end{aligned} \tag{3.46}$$

**Case 3.** If $(i,r,G)^\top \in D_\varepsilon^3$, according to (3.36) and (3.42), we can obtain

$$\begin{aligned} F(i,r,G) &\leq -\frac{M}{2}(\theta+\nu+\gamma)(R_0^s-1) + M\frac{\bar{\beta}}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right)i + 2\theta+\epsilon+\bar{\beta}e^{\frac{\sigma^2}{4\rho}} \\ &\quad - \rho G(e^G-1) + \frac{\sigma^2 e^G}{2} \\ &\leq -\frac{M}{2}(\theta+\nu+\gamma)(R_0^s-1) + K_1 + \frac{\bar{\beta}M}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right)\varepsilon \\ &\leq -2 + \frac{\bar{\beta}M}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right)\varepsilon \\ &\leq -1. \end{aligned} \tag{3.47}$$

**Case 4.** If $(i,r,G)^\top \in D_\varepsilon^4$, according to (3.36) and (3.43), it can be derived that

$$F(i,r,G) \leq -\frac{\gamma}{\varepsilon} + \frac{\bar{\beta}M}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right) - 2 \leq -1. \tag{3.48}$$

**Case 5.** If $(i,r,G)^\top \in D_\varepsilon^5$, according to (3.36) and (3.44), we have

$$\begin{aligned} F(i,r,G) &\leq -\frac{\epsilon r}{1-i-r} + \frac{\bar{\beta}M}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right) - 2 \\ &\leq -\frac{\epsilon}{\varepsilon} + \frac{\bar{\beta}M}{4k}\left(1+\frac{\gamma+\nu}{\theta+\epsilon}\right) - 2 \\ &\leq -1. \end{aligned} \tag{3.49}$$

From the previous discussion, for the set $\Gamma \setminus U_\varepsilon = D_\varepsilon^1 \cup D_\varepsilon^2 \cup D_\varepsilon^3 \cup D_\varepsilon^4 \cup D_\varepsilon^5$, a sufficiently small $\varepsilon$ exists for which

$$F(i, r, G) \le -1, \quad \text{for any } (i, r, G)^\top \in \Gamma \setminus U_\varepsilon. \tag{3.50}$$

Let

$$K_2 := \sup_{(i,r,G)^\top \in \Gamma} F(i, r, G),$$

and then we get

$$F(i, r, G) \le K_2 < \infty, \quad \text{for any } (i, r, G)^\top \in \Gamma. \tag{3.51}$$

**Step 3: Existence of the invariant measure**

By Lemmas 3.1, 3.3–3.5, and based on the Lyapunov function $V(i, r, G)$ constructed in Steps 1 and 2, then

$$\mathcal{L}^* V(i, r, G) \le F(i, r, G) + f_1(G) + M f_2(G) + M f_3(G),$$

where the function $F(i, r, G)$ is strictly negative outside a compact set $U_\varepsilon$, and bounded above inside it. The process $G(t)$ evolves according to an Ornstein–Uhlenbeck process, which ensures ergodicity. In particular, it holds almost surely that

$$\frac{1}{t}\int_0^t e^{G(s)}\, ds \to e^{\frac{\sigma^2}{4\rho}}, \ \ \frac{1}{t}\int_0^t e^{2G(s)}\, ds \to e^{\frac{\sigma^2}{\rho}}, \quad \text{as } t \to \infty, \tag{3.52}$$

as established in [13] and supported by classical ergodic theorems [25]. As a result, the time averages of all $G$-dependent perturbation terms vanish asymptotically. Combining these facts, we conclude that the long-time average of the generator satisfies

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}[F(i(s), r(s), G(s))]\, ds \ge 0. \tag{3.53}$$

Given the structure of $F$, which is less than or equal to $-1$ outside $U_\varepsilon$ and bounded above by a constant $K_2$ within $U_\varepsilon$, we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{\{(i(s),r(s),G(s))\in U_\varepsilon\}}\, ds \ge \frac{1}{K_2 + 1} > 0 \quad \text{a.s.} \tag{3.54}$$

Then, by Fatou's lemma, it follows that the time-averaged measure of the process in $U_\varepsilon$ is strictly positive, almost surely:

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t P(s, (i(s), r(s), G(s)), U_\varepsilon)\, ds \ge \frac{1}{K_2 + 1} > 0 \quad \text{a.s.} \tag{3.55}$$

Hence, the Markov semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable with respect to the compact set $U_\varepsilon$, and sweeping does not occur. This implies the existence of a unique invariant probability measure $\varkappa$ with density $\Psi^*(i, r, G)$. We conclude that Theorem 3.1 is fully established.

□

## 4. Probability density function

This section is devoted to a local Gaussian approximation of the invariant probability density near the quasi-endemic equilibrium of system (1.4). The quasi-endemic equilibrium $\bar{P} = (\bar{i}, \bar{r})^\top$ is defined, which can be determined by

$$\begin{cases} \bar{\beta}(1 - \bar{i} - \bar{r})\bar{i} - (\theta + \nu + \gamma)\bar{i} + \nu\bar{i}^2 = 0, \\ \gamma\bar{i} - (\theta + \epsilon)\bar{r} + \nu\bar{i}\bar{r} = 0. \end{cases} \tag{4.1}$$

As mentioned in the introduction, one can verify that $\bar{P}$ exists under the condition $R_0 = \dfrac{\bar{\beta}}{\theta + \nu + \gamma} > 1$. By solving Eq (4.1), it follows that $\bar{i}$ is equal to $i^*$, and $i^*$ is strictly greater than zero. Similarly, $\bar{r}$ is equal to $r^*$, and $r^*$ is strictly positive. $i^*, r^*$ are given in the first section. Let $(u_1, u_2, u_3)^\top = (i - \bar{i}, r - \bar{r}, G)^\top$. The linearized version of system (1.4) around $\bar{P}$ is given by

$$\begin{cases} du_1 = (-c_{11}u_1 - c_{12}u_2 - c_{13}u_3)\, dt, \\ du_2 = (c_{21}u_1 - c_{22}u_2)\, dt, \\ du_3 = -\rho u_3\, dt + \sigma dB(t), \end{cases} \tag{4.2}$$

where

$$c_{11} = [R_0(\theta + \nu + \gamma) - \nu]\bar{i},\ \ c_{12} = \bar{\beta}\bar{i},\ \ c_{13} = -\bar{\beta}(1 - \bar{i} - \bar{r})\bar{i},\ \ c_{21} = \nu\bar{r} + \gamma,\ \ c_{22} = \frac{\gamma\bar{i}}{\bar{r}}.$$

Let $u_1 = i - \bar{i}, \quad u_2 = r - \bar{r}, \quad u_3 = G$, and

$$\mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

With this notation, the dynamics can be concisely expressed as

$$d\mathbf{U}(t) = C\mathbf{U}(t)dt + Hd\mathbf{B}(t), \tag{4.3}$$

where

$$C = \begin{pmatrix} -c_{11} & -c_{12} & -c_{13} \\ c_{21} & -c_{22} & 0 \\ 0 & 0 & -\rho \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}.$$

**Theorem 4.1.** *If $R_0 > 1$, then the linearized system* (4.3) *around the quasi-endemic equilibrium $\bar{P}$ admits a unique Gaussian invariant measure with distribution $N_3(0, \Sigma)$. Consequently, near $\bar{P}$, the invariant probability density of system* (1.4) *is locally approximated by*

$$\Psi^*(u_1, u_2, u_3) = (2\pi)^{-\frac{3}{2}} \cdot |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{U}^\top\Sigma^{-1}\mathbf{U}\right).$$

*In this expression, $\Sigma$ is a positive definite matrix whose explicit form is presented below:*

$$\Sigma = \omega^2 J^{-1}\Sigma_0(J^{-1})^\top,$$

*with*

$$\omega = c_{13}c_{21}\sigma, \quad \Sigma_0 = \begin{pmatrix} \frac{c_2}{2(c_1c_2-c_3)} & 0 & -\frac{1}{2(c_1c_2-c_3)} \\ 0 & \frac{1}{2(c_1c_2-c_3)} & 0 \\ -\frac{1}{2(c_1c_2-c_3)} & 0 & \frac{c_1}{2c_3(c_1c_2-c_3)} \end{pmatrix},$$

$$J = \begin{pmatrix} -c_{21}(c_{11}+c_{21}) & c_{22}^2 - c_{12}c_{21} & -c_{13}c_{21} \\ c_{21} & -c_{22} & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

*where*

$$c_1 = c_{11} + c_{22} + \rho, \quad c_2 = c_{11}c_{22} + c_{12}c_{21} + \rho(c_{11} + c_{22}), \quad c_3 = \rho\,(c_{11}c_{22} + c_{12}c_{21})\,.$$

*Proof.* Note that (4.3) constitutes a first-order linear stochastic differential system. According to Øksendal [26], given any $U(0) = (u_1(0), u_2(0), u_3(0))^T$, one may derive an equivalent representation of system (4.3) as follows:

$$U(t) = e^{Ct}U(0) + \int_0^t e^{C(t-s)}HdB(s).$$

Note that $H$ is a constant matrix, and by a standard theory [21], the expression $\int_0^t e^{C(t-s)}HdB(s)$ is normally distributed at time $t$ with zero mean and covariance matrix $\bar{\Sigma}(t)$; that is, it follows the distribution $\mathbb{N}_3(\mathbf{0}_3, \bar{\Sigma}(t))$, where

$$\bar{\Sigma}(t) = \int_0^t e^{C(t-s)HH^\top C^\top(t-s)}ds.$$

Thus, the state vector of system (4.3) at time $t$ exhibits a normal distribution given by $\mathbb{N}_3(e^{Ct}U(0), \bar{\Sigma}(t))$.

Let $I$ be the identity matrix, and then the characteristic polynomial of the matrix $C$ can be expressed by

$$\phi_C(\lambda) = \det(C - \lambda I) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3,$$

where

$$c_1 = c_{11} + c_{22} + \rho, \quad c_2 = c_{11}c_{22} + c_{12}c_{21} + \rho(c_{11} + c_{22}), \quad c_3 = \rho\,(c_{11}c_{22} + c_{12}c_{21})\,.$$

From $c_{jk} > 0$, $(j, k = 1, 2)$, we can get $c_{k'} > 0 (k' = 1, 2, 3)$, and

$$c_1c_2 - c_3 = c_{11}c_{22}(c_{11} + c_{22}) + (c_{11} + c_{22})c_{12}c_{21} + \rho(c_{11} + c_{22})^2 + \rho^2(c_{11} + c_{22}) > 0.$$

It can be concluded that all eigenvalues of $C$ possess negative real parts, meaning the quasi-steady state equilibrium point is locally asymptotically stable. From the stability theory for the zero solution of a general linear equation [27], we have $\lim\limits_{t\to\infty} e^{Ct} = 0$, $\lim\limits_{t\to\infty} e^{Ct}U(0) = 0$, and the limiting covariance matrix is defined by

$$\Sigma := \lim_{t\to\infty}\bar{\Sigma}(t) = \lim_{t\to\infty}\int_0^t e^{C(t-s)HH^\top C^\top(t-s)}ds = \int_0^\infty e^{Ct}HH^\top e^{C^\top t}dt.$$

This guarantees that $\{U(t)\}_{t\geq 0}$ admits a unique invariant probability measure $\mathbb{N}_3(\mathbf{0}_3, \Sigma)$.

$H^2$ is positive semi-definite, and $\Sigma$ inherits this property. As time tends to infinity, the transient distribution of $U(t)$ converges to the normal distribution $\mathbb{N}_3(\mathbf{0}_3, \Sigma)$, implying that the stationary solution $(u_1(t), u_2(t), u_3(t))^\top$ ultimately conforms to a unique Gaussian density $\Psi^*(u_1, u_2, u_3)$.

Moreover, the integral $\int_0^\infty e^{Ct}HH^\top e^{C^\top t}dt$ cannot be easily computed in a closed form. We therefore approach the study of $\Sigma$ by considering its associated matrix equation, which leads to

(i) $\int_0^\infty \frac{d}{dt}(e^{C(t)}HH^\top e^{C^\top(t)})ds = C\Sigma + \Sigma C^T$,

(ii) $\int_0^\infty \frac{d}{dt}(e^{Ct}HH^\top e^{C^\top t})dt = -H^2$,

and we can deduce that

$$H^2 + C\Sigma + \Sigma C^T = 0. \tag{4.4}$$

Having established the necessary groundwork, the accurate form of $\Sigma$ can now be presented, and its positive definiteness verified. Define $C_1 = J_1 C J_1^T$. The matrix $J_1$ takes the form

$$J_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and we compute that

$$C_1 = \begin{pmatrix} -\rho & 0 & 0 \\ -c_{13} & -c_{11} & -c_{12} \\ 0 & c_{21} & -c_{22} \end{pmatrix}.$$

Let $C_2 = J_2 C_1 J_2^T$, where

$$J_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{21} & -c_{22} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$C_2 = \begin{pmatrix} -\rho & 0 & 0 \\ -c_{13}c_{21} & -c_{11} - c_{21} & -c_{11}c_{22} - c_{12}c_{21} \\ 0 & 1 & 0 \end{pmatrix}.$$

Finally, let $C_3 = J_3 C_2 J_3^T$, where the matrix $J_3$ takes the form

$$J_3 = \begin{pmatrix} -c_{13}c_{21} & -c_{11} - c_{22} & -c_{11}c_{22} - c_{12}c_{21} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we get

$$C_3 = \begin{pmatrix} -c_1 & -c_2 & -c_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

When the condition $\bar{\beta} > 2\nu$ is satisfied, then we have $c_k > 0\,(k = 1, 2, 3)$, $c_1c_2 - c_3 > 0$. Let

$$J = J_3 J_2 J_1 = \begin{pmatrix} -c_{21}(c_{11} + c_{22}) & c_{22}^2 - c_{12}c_{21} & -c_{13}c_{21} \\ c_{21} & -c_{22} & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and we can equivalently recast (4.4) as follows:

$$JH^2J^\top + C_3 J\Sigma J^\top + J\Sigma J^\top C_3{}^\top = 0,$$

i.e.,

$$H_0^2 + C_3\Sigma_0 + \Sigma_0 C_3^\top = 0, \tag{4.5}$$

where

$$H_0 = \mathrm{diag}(1,0,0), \quad \omega = c_{13}c_{21}\sigma, \quad \Sigma_0 = \frac{1}{\omega^2}J\Sigma J^\top.$$

By solving Eq (4.5), we have

$$\Sigma_0 = \begin{pmatrix} \frac{c_2}{2(c_1c_2-c_3)} & 0 & -\frac{1}{2(c_1c_2-c_3)} \\ 0 & \frac{1}{2(c_1c_2-c_3)} & 0 \\ -\frac{1}{2(c_1c_2-c_3)} & 0 & \frac{c_1}{2c_3(c_1c_2-c_3)} \end{pmatrix}. \tag{4.6}$$

Hence, by virtue of Lemma 3 in [19], $\Sigma_0$ is established to be positive definite, and thus the matrix $\Sigma = \omega^2 J^{-1}\Sigma_0(J^{-1})^T$ inherits this property.

The invariant measure of system (1.4) near the equilibrium point $\varkappa$ can be approximated by $\mathbb{N}_3(\bar{E}, \Sigma)$. The density function of the invariant measure is given as

$$\Psi^*(u_1, u_2, u_3) = (2\pi)^{-\frac{3}{2}} \cdot |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(u_1, u_2, u_3)\Sigma^{-1}(u_1, u_2, u_3)^\top\right).$$

Thus, Theorem 4.1 is proved. □

**Lemma 4.1.** *If $R_0 > 1$ and $\nu < \sqrt{\dfrac{4\gamma^2 i^*(\bar{\beta}-\nu)}{\bar{\beta}r^*}}$, then for any arbitrarily small $\sigma$, we have*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |\mathbf{Y}(s) - \mathbf{Y}^*|^2\, ds \le \overline{\kappa}(\sigma).$$

*Here, $\mathbf{Y} = (i, r, G)^\top$, $\mathbf{Y}^* = (i^*, r^*, G^*)^\top$ denotes the positive equilibrium of system* (1.4)*, $G^* = 0$, and*

$$\overline{\kappa}(\sigma) = \max\left\{\frac{\bar{\beta}^2}{2pa}\left(e^{\frac{\sigma^2}{\rho}} - 2e^{\frac{\sigma^2}{4\rho}} + 1\right), \frac{\sigma^2}{\rho}\right\},$$

*where $a = \min\left\{\bar{\beta} - \nu - \dfrac{w}{2}, \dfrac{\bar{\beta}i^*}{r^*} - \dfrac{\nu^2\bar{\beta}^2}{2w\gamma^2}\right\}$, $\dfrac{\nu^2\bar{\beta}r^*}{2\gamma^2 i^*} < w < 2(\bar{\beta} - \nu)$.*

*Proof.* Similar to the first section, we obtain that $(i^*, r^*, G^*)^\top$ exists provided that $R_0 > 1$, and this equilibrium point of (1.4) satisfies

$$\begin{cases} \bar{\beta}e^{G^*}(1 - i^* - r^*)i^* - (\theta + \nu + \gamma)i^* + \nu(i^*)^2 = 0, \\ \gamma i^* - (\theta + \epsilon)r^* + \nu i^* r^* = 0, \end{cases} \tag{4.7}$$

where $i^*$ and $r^*$ are given in the first section. Define

$$V(i, r) = V_1(i) + \frac{\bar{\beta}}{\gamma}V_2(r),$$

where

$$V_1(i) = i - i^* - i^*\ln\frac{i}{i^*}, \quad V_2(r) = \frac{1}{2}(r - r^*)^2.$$

Then, we get

$$\begin{aligned}\mathcal{L}V_1 &= \left[\bar{\beta}e^{G}(1-i-r)-(\theta+\nu+\gamma)+\nu i\right](i-i^*)\\ &= \left[\bar{\beta}e^{G}(1-i-r)-\bar{\beta}(1-i^*-r^*)-\nu i^*+\nu i\right](i-i^*)\\ &= -(\bar{\beta}-\nu)(i-i^*)^2-\bar{\beta}(i-i^*)(r-r^*)+(\bar{\beta}e^{G}-\bar{\beta})(1-i-r)\cdot(i-i^*),\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathcal{L}V_2 &= (r-r^*)\left\{\gamma i-(\theta+\epsilon)r+\nu ir-\left[\gamma i^*-(\theta+\epsilon)r^*+\nu i^*r^*\right]\right\}\\ &= (r-r^*)\left\{(\gamma+\nu r)(i-i^*)+\left[\nu i^*-(\theta+\epsilon)\right](r-r^*)\right\}\\ &= (\gamma+\nu r)(i-i^*)(r-r^*)-\frac{\gamma i^*}{r^*}(r-r^*)^2.\end{aligned} \tag{4.9}$$

Then, we have

$$\begin{aligned}\mathcal{L}V &= -(\bar{\beta}-\nu)(i-i^*)^2+\frac{\nu\bar{\beta}}{\gamma}r(i-i^*)(r-r^*)-\frac{\bar{\beta}i^*}{r^*}(r-r^*)^2\\ &\quad+(\bar{\beta}e^{G}-\bar{\beta})(1-i-r)\cdot(i-i^*).\end{aligned}$$

Using Young's inequality, we get

$$\mathcal{L}V \leq -(\bar{\beta}-\nu)(i-i^*)^2+\frac{\nu\bar{\beta}}{\gamma}\cdot|i-i^*|\cdot|r-r^*|-\frac{\bar{\beta}i^*}{r^*}(r-r^*)^2+|\bar{\beta}e^{G}-\bar{\beta}|.$$

Choose $w>0$ to satisfy the following inequality:

$$\frac{\nu^2\bar{\beta}r^*}{2\gamma^2 i^*}<w<2(\bar{\beta}-\nu).$$

Additionally, if $\nu$ satisfies

$$\nu<\sqrt{\frac{4\gamma^2 i^*(\bar{\beta}-\nu)}{\bar{\beta}r^*}}, \tag{4.10}$$

it follows that

$$\mathcal{L}V \leq -\left(\bar{\beta}-\nu-\frac{w}{2}\right)(i-i^*)^2-\left(\frac{\bar{\beta}i^*}{r^*}-\frac{\nu^2\bar{\beta}^2}{2w\gamma^2}\right)(r-r^*)^2+|\bar{\beta}e^{G}-\bar{\beta}|. \tag{4.11}$$

Let

$$a:=\min\left\{\bar{\beta}-\nu-\frac{w}{2},\frac{\bar{\beta}i^*}{r^*}-\frac{\nu^2\bar{\beta}^2}{2w\gamma^2}\right\}.$$

Integrating (4.11) over the interval $[0,t]$ and taking the expectation yields

$$\begin{aligned}\frac{V(i(t),r(t))-V(i(0),r(0))}{t} \leq &-\frac{a}{t}\int_0^t\left[(i(s)-i^*)^2+(r(s)-r^*)^2\right]ds\\ &+\int_0^t|\bar{\beta}e^{G(s)}-\bar{\beta}|\,ds,\end{aligned} \tag{4.12}$$

and we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t|\bar{\beta}e^{G(s)}-\bar{\beta}|\,ds\le\left(\frac{1}{t}\int_0^t(\bar{\beta}e^{G(s)}-\bar{\beta})^2\,ds\right)^{\frac{1}{2}}.$$

Considering the strong law of large numbers [25] and Lemma 1.1 in [13], it follows that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t(\bar{\beta}e^{G(s)}-\bar{\beta})^2\,ds&=\lim_{t\to\infty}\frac{1}{t}\int_0^t\bar{\beta}^2e^{2G(s)}\,ds-2\bar{\beta}\lim_{t\to\infty}\frac{1}{t}\int_0^t\bar{\beta}e^{G(s)}\,ds+\bar{\beta}^2,\\&=\bar{\beta}^2e^{\frac{\sigma^2}{\rho}}-2\bar{\beta}^2e^{\frac{\sigma^2}{4\rho}}+\bar{\beta}^2\quad\text{a.s.}\end{aligned}\tag{4.13}$$

Taking the limit superior in (4.12) and using the positivity of $V(t)$, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left[(i(s)-i^*)^2+(r(s)-r^*)^2\right]ds\le\frac{\bar{\beta}}{a}\left(e^{\frac{\sigma^2}{\rho}}-2e^{\frac{\sigma^2}{4\rho}}+1\right)^{\frac{1}{2}}:=\kappa(\sigma).$$

Also, from Lemma 1.1 in [13], one can obtain that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^tG^2(s)\,ds\le\frac{\sigma^2}{\rho}.\tag{4.14}$$

Then,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left[(i(s)-i^*)^2+(r(s)-r^*)^2+G^2(s)\right]ds\le\max\left\{\kappa(\sigma),\frac{\sigma^2}{\rho}\right\}:=\bar{\kappa}(\sigma),\tag{4.15}$$

where $\lim_{\sigma\to0}\bar{\kappa}(\sigma)=0$. The proof is completed. □

The following theorem follows from Lemma 4.1 and [20].

**Theorem 4.2.** *If $R_0>1$ and $\nu<\sqrt{\dfrac{4\gamma^2i^*(\bar{\beta}-\nu)}{\bar{\beta}r^*}}$, for every sufficiently small $\sigma$, then*

*(i) The invariant joint probability distribution $\tilde{\varkappa}$ (respectively, invariant joint probability density function $\Psi(\cdot)$ ) is globally approximated by $\mathbb{N}^3(\mathbf{Y}^*,\Sigma^{(3)})$ limited on $\Gamma$ and*

$$\Psi^*(\mathbf{Y}(t))=(2\pi)^{-\frac{3}{2}}|\Sigma^{(3)}|^{-\frac{1}{2}}e^{-\frac{1}{2}(\mathbf{Y}-\mathbf{Y}^*)^\top(\Sigma^{(3)})^{-1}(\mathbf{Y}-\mathbf{Y}^*)},\tag{4.16}$$

*where the set $\Gamma$ is given in the second section.*

*(ii) For any $k\in(0,2]$, we have*

$$\lim_{\sigma\to0^+}\int_{\Gamma'}|\mathbf{i}-\mathbf{Y}^*|^k|\Psi(\mathbf{i})-\Psi^*(\mathbf{i})|di=0.\tag{4.17}$$

*Proof.* The conclusion follows from the general approximation framework in [20] once the estimate in Lemma 4.1 is verified for system (1.4). For completeness, we briefly indicate the argument in the present setting.

First, from Lemma 4.1, we obtain

$$\frac{V(\mathbf{Y}(t)) - V(\mathbf{Y}(0))}{t} \leq -\frac{a}{t}\int_0^t |\mathbf{Y}(s) - \mathbf{Y}^*|^2\, ds + a\kappa(\sigma).$$

Taking expectations and letting $t \to \infty$, we deduce that

$$\limsup_{t\to\infty} \mathbb{E}\left(\frac{1}{t}\int_0^t |\mathbf{Y}(s) - \mathbf{Y}^*|^2\, ds\right) \leq \bar{\kappa}(\sigma),$$

where

$$\bar{\kappa}(\sigma) = \max\left\{\frac{\bar{\beta}}{a}\left(e^{\frac{\sigma^2}{\rho}} - 2e^{\frac{\sigma^2}{4\rho}} + 1\right)^{\frac{1}{2}},\ \frac{\sigma^2}{\rho}\right\}.$$

This result shows that, for small $\sigma$, the invariant measure concentrates near $\mathbf{Y}^*$.

Moreover, by a change of variable, we find that $\{\mathbf{U}(t) + \mathbf{Y}^*\}_{t\geq 0}$ possesses a unique invariant measure $\mathbb{N}_3(\mathbf{Y}^*, \Sigma^{(3)})$ with density (4.16).

Standard arguments then yield (4.17) for any $k \in (0, 2]$. The proof is completed. □

## 5. Extinction

Since we are concerned with disease transmission dynamics, we investigate the extinction property of the following system:

$$\begin{cases} \dot{i}(t) = \bar{\beta}e^{G(t)}(1 - i - r)i - (\theta + \nu + \gamma)i + \nu i^2, \\ \dot{r}(t) = \gamma i - (\theta + \epsilon)r + \nu ir. \end{cases} \tag{5.1}$$

**Theorem 5.1.** *If* $R_0^s = \dfrac{\bar{\beta}e^{\frac{\sigma^2}{4\rho}}}{\theta + \nu + \gamma} < 1$, *system* (5.1) *possesses a disease-free equilibrium given by* $E_s^0 = (i^0, r^0) = (0, 0)$*, which is locally asymptotically stable (LAS).*

*Proof.* First, define the variables $\hat{i} = i - i^0$, $\hat{r} = r - r^0$. By linearizing system (5.1), we can obtain the following linearized equations:

$$\begin{cases} \dfrac{d\hat{i}(t)}{dt} = [\bar{\beta}e^{G(t)} - (\theta + \nu + \gamma)]\hat{i}, \\ \dfrac{d\hat{r}(t)}{dt} = \gamma\hat{i} - (\theta + \epsilon)\hat{r}. \end{cases} \tag{5.2}$$

Then,

$$\begin{aligned} \frac{d(\ln \hat{i})}{dt} &= \bar{\beta}e^{G} - (\theta + \nu + \gamma) \\ &= \bar{\beta}e^{\frac{\sigma^2}{4\rho}} - (\theta + \nu + \gamma) + \bar{\beta}\left(e^{G} - e^{\frac{\sigma^2}{4\rho}}\right) \\ &= (R_0^s - 1)(\theta + \nu + \gamma) + \bar{\beta}\left(e^{G} - e^{\frac{\sigma^2}{4\rho}}\right). \end{aligned} \tag{5.3}$$

By integrating $\ln \hat{I}(t)$ over the interval $[0, t]$ and subsequently dividing the resulting expression by $t$, we deduce that

$$\frac{\ln \hat{i}(t) - \ln \hat{i}(0)}{t} = (R_0^s - 1)(\theta + \nu + \gamma) + \bar{\beta}\left(\frac{1}{t}\int_0^t e^{G(s)}\, ds - e^{\frac{\sigma^2}{4\rho}}\right) \quad \text{a.s.} \tag{5.4}$$

According to Lemma 1.1 in [13],

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{G(s)}\, ds = e^{\frac{\sigma^2}{4\rho}} \quad \text{a.s.} \tag{5.5}$$

Taking the limit superior in (5.4) and incorporating the result from (5.5), it follows that when $R_0^s < 1$,

$$\limsup_{t\to\infty} \frac{\ln \hat{i}(t)}{t} = (R_0^s - 1)(\theta + \nu + \gamma) < 0 \quad \text{a.s.} \tag{5.6}$$

This means

$$\lim_{t\to\infty} \hat{i}(t) = 0 \quad \text{a.s.} \tag{5.7}$$

Next, we get the solution

$$\hat{r}(t) = \gamma e^{-(\theta+\epsilon)t} \int_0^t e^{(\theta+\epsilon)v} \hat{i}(v) dv. \tag{5.8}$$

From (5.6), one deduces the existence of a null set $\mathbb{N}$ with $P(\mathbb{N}) = 0$. Then for every $\omega \notin \mathbb{N}$ and any $\hat{\varepsilon} > 0$, there exists a time $T = T(\omega)$ such that for all $t \geq T$, it follows that

$$\hat{i}(t, \omega) \leq e^{(R_0^s - 1)(\theta+\nu+\gamma)+\hat{\varepsilon}} := e^{m+\hat{\varepsilon}},$$

where $m = (R_0^s - 1)(\theta + \nu + \gamma) < 0$. For each $\omega \in \Psi^*$ and $t > T(\omega)$,

$$\begin{aligned} |\hat{r}(t, \omega)| &= \gamma e^{-(\theta+\epsilon)t} \int_0^T e^{(\theta+\epsilon)v} \hat{i}(v, \omega) dv + \gamma e^{-(\theta+\epsilon)t} \int_T^t e^{(\theta+\epsilon)v} \hat{i}(v, \omega) dv \\ &\leq \gamma e^{-(\theta+\epsilon)t} \int_0^T e^{(\theta+\epsilon)v} e^{m+\hat{\varepsilon}} dv + \gamma e^{-(\theta+\epsilon)t} \int_T^t e^{(\theta+\epsilon)v} e^{m+\hat{\varepsilon}} dv. \end{aligned}$$

Then,

$$\limsup_{t\to\infty} \frac{1}{t} \ln |\hat{r}(t, \omega)| \leq -(\theta + \epsilon) \vee (m + \hat{\varepsilon}).$$

Letting $\hat{\varepsilon} \to 0$ leads to

$$\limsup_{t\to\infty} \frac{1}{t} \ln |\hat{r}(t, \omega)| \leq -(\theta + \epsilon) \vee m.$$

Then, $\lim_{t\to\infty} \hat{r}(t) = 0$ almost surely. Thus, the disease $i(t)$ converges to $i^0$ and $\hat{r}(t)$ converges to $r^0$ exponentially with probability one. This completes the proof. □

**Remark 5.1.** *When $\sigma = 0$, the random threshold $R_0^s$ is the same as the deterministic threshold $R_0$. Based on Theorems 3.1 and 5.1, we conclude that $R_0^s$ serves as the threshold for system* (1.4). *This result demonstrates that the stochastic system dynamics governed by this threshold preserve all essential characteristics of its deterministic counterpart.*

## 6. Numerical simulations

Next, we design experiments of numerical simulation to verify aforementioned results of stochastic system (1.4). The initial parameters of the system are set as $(i(0), r(0), G(0))^T = (0.15, 0.1, 0)^T$. Meanwhile, the Milstein high-order discretization method, proposed in [28], is applied in the system, and therefore we obtain the following equation:

$$\begin{cases} i^{j+1} = i^j + [\bar{\beta}e^{G^j}(1 - i^j - r^j)i^j - (\theta + \nu + \gamma)i^j + \nu(i^j)^2]\Delta t, \\ r^{j+1} = r^j + [\gamma i^j - (\theta + \epsilon)r^j + \nu i^j r^j]\Delta t, \\ G^{j+1} = G^j + (-\rho G^j)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_j, \end{cases} \tag{6.1}$$

where $(i^j, r^j, G^j)^T$ represents the corresponding value of system (6.1) at the $j$th iteration, $\Delta t$ is the time step, and the random term $\varepsilon_j$ $(j = 1, 2, \ldots, n)$ follows $\mathbb{N}(0, 1)$. In order to verify the theoretical results, we will carry out a numerical simulation informed by empirical parameter data of the pandemic influenza. We choose real parameter values from published references, as presented in Table 2. Examples 6.1–6.3 are devoted to illustrating the existence of a stationary distribution, the local density approximation near the quasi-endemic equilibrium, and the extinction behavior, respectively. In Example 6.4, we further investigate the sensitivity of the stochastic threshold to the key parameters $\bar{\beta}$, $\sigma$, and $\rho$. Example 6.5 further compares the varying-population model with its constant-population counterpart to illustrate the effect of population variation.

**Table 2.** Parameter values of system (1.4).

| Parameters | Value | References |
|---|---|---|
| $\theta$ | 0.20548 | [29] |
| $\gamma$ | 0.07140 | [29] |
| $\epsilon$ | 0.00274 | [29] |
| $\nu$ | 0.00010 | [30] |
| $\bar{\beta}$ | 0.40000 | Estimated |
| $\rho$ | 0.10000 | Estimated |

**Example 6.1.** *We set parameter* $\sigma = 0.1$ *and others as shown in Table 2 to numerically illustrate the existence of a stationary distribution. Then we compute that* $R_0^s = 1.4812 > 1$. *As shown in Figure 1, system* (1.4) *possesses a solution* $(i(t), r(t), G(t))^T$ *which follows a stationary distribution* $\Psi(\cdot)$ *by Theorem 3.1, implying that the disease will persist.*

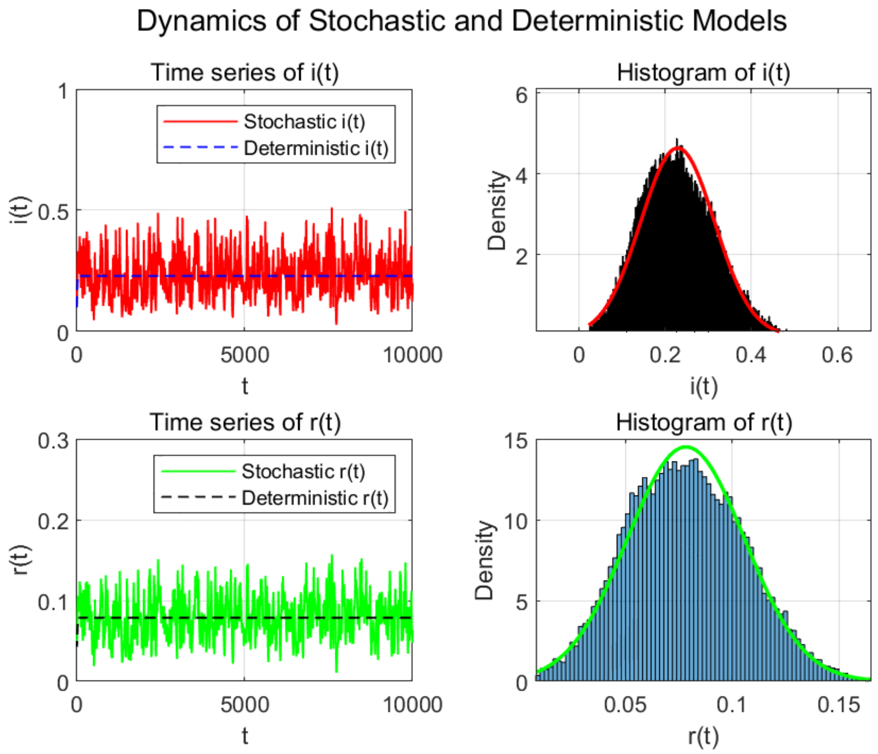


**Figure 1.** The graphs on the left-hand side illustrate trajectories of $i(t)$ and $r(t)$ in both stochastic and deterministic models over $t \in [0, 10000]$. The graphs on the right-hand side present the marginal density functions along with corresponding frequency histograms for $i(t)$ and $r(t)$.

**Example 6.2.** *The same parameter values as Example 6.1 are used to prove there is a probability density around the quasi-local popular equilibrium point* $\mathbf{Y}^*$*. Direct computation yields that* $\mathbf{Y}^* = (i^*, r^*, G^*)^T = (0.2292, 0.0786, 0)^T$ *and* $R_0 = 1.4447 > 1$*. Thus, in view of Theorem 4.1, the state vector* $(i(t), r(t), G(t))^T$ *for system* (1.4) *follows a multivariate normal distribution, namely,* $\Psi^*(i, r, G) \sim \mathbb{N}(\mathbf{Y}^*, \Sigma)$*, with the covariance matrix given by*

$$\Sigma = \begin{pmatrix} 0.0079 & 0.0024 & 0.0071 \\ 0.0024 & 0.0008 & 0.0015 \\ 0.0071 & 0.0015 & 0.0222 \end{pmatrix}.$$

*Moreover, the marginal distributions corresponding to* $\Psi(i, r, G)$ *are*

$$\phi_i(i) = 4.4885 \times \exp\left(-\frac{(i - \bar{i})^2}{0.0158}\right), \quad \phi_r(r) = 14.1047 \times \exp\left(-\frac{(r - \bar{r})^2}{0.0016}\right).$$

Figure 2 illustrates the empirical density $\Psi(T_0, i, r)$ of (6.1) in a 2D setting at $T_0 = 1 \times 10^4, 2 \times 10^4$, and $3 \times 10^4$, and the function $\Psi^*(\cdot)$ in a 2D setting exhibits very similar density to what is shown in Figure 2.

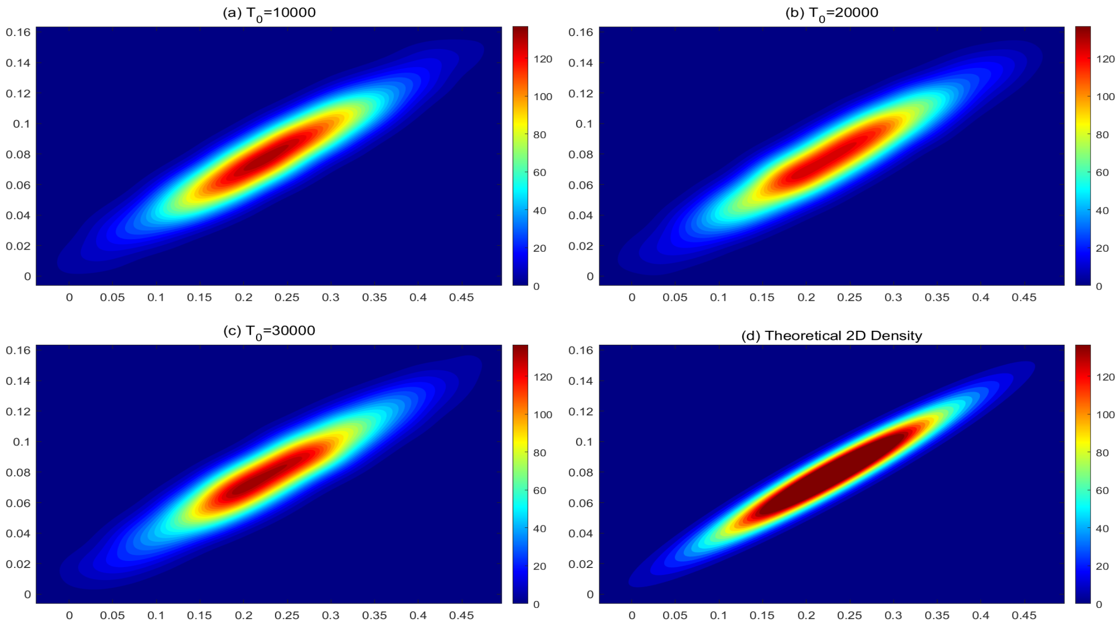


**Figure 2.** (a)–(c) The empirical density $\Psi(T_0, i, r)$ of (6.1) in a 2D setting at iteration time $T_0$ equals $1 \times 10^4, 2 \times 10^4, 3 \times 10^4$, respectively. (d) The function $\Psi^*(\cdot)$ in a 2D setting. We use the same set of parameters as in Example 6.1. All contour plots use the 'jet' colormap, where blue denotes lower density, and red is higher.

To further support this similarity, Figure 3 depicts the empirical marginal densities $\Psi_i(i)$ and $\Psi_r(r)$ (i.e., the density function curves fit well to $i$ and $r$) when $T_0 = 1 \times 10^4, 2 \times 10^4$, and $3 \times 10^4$, each in a different color. In this figure, $\Psi_i^*(i)$ and $\Psi_r^*(r)$ exhibit a high degree of overlap with those specific curves.

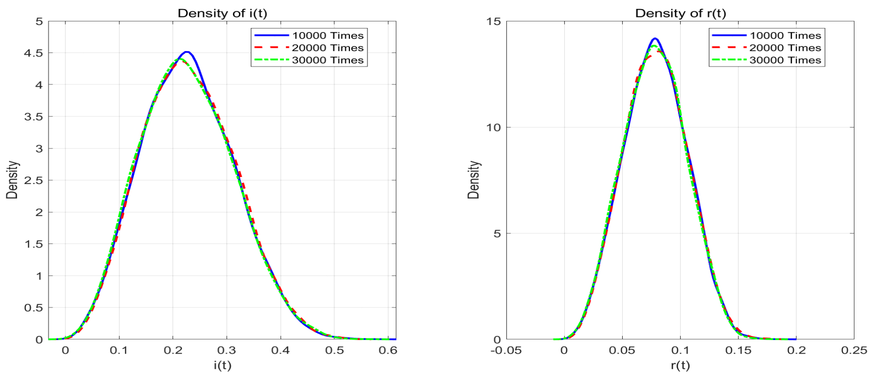


**Figure 3.** Iteration times $T_0 = 1 \times 10^4$ (blue), $2 \times 10^4$ (red), $3 \times 10^4$ (green) indicate the empirical marginal densities $\Psi_i(i)$ and $\Psi_r(r)$ of (6.1). The black lines indicate the marginal densities of $\Psi^*(i, r, G)$ (i.e., $\Psi_i^*(i)$ and $\Psi_r^*(r)$). We set parameters the same as in Example 6.1.

**Example 6.3.** *We set* $\sigma = 0.01$, $\bar{\beta} = 0.25$, *and the other parameter values are the same as those in Table 2 to validate that disease will become extinct under small noise. Then we have* $R_0^s = 0.9031 < 1$. *According to Theorem 5.1, the disease becomes extinct exponentially in the long term; see Figure 4.*

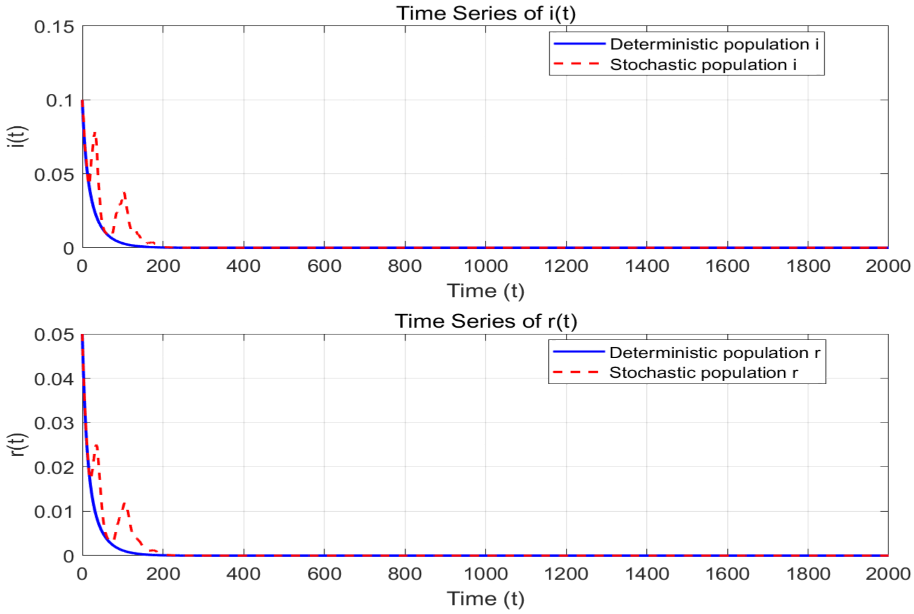


**Figure 4.** Time series of proportions of infected (i) and removed (r) populations in deterministic and stochastic systems. The noise intensity of the random system (1.4) is $\sigma = 0.01$.

**Example 6.4.** *We further investigate how the key parameters $\bar{\beta}$, $\sigma$, and $\rho$ affect the stochastic threshold and the corresponding dynamical behaviors of system* (1.4)*. In this example, the remaining parameters are fixed as in Table 2. According to the theoretical results established in Sections 3 and 5, when $R_0^s > 1$, system* (1.4) *admits a stationary distribution, whereas when $R_0^s < 1$, the disease becomes extinct.*

Figure 5 shows that $R_0^s$ increases monotonically with $\bar{\beta}$ and $\sigma$, while it decreases as $\rho$ increases. The dashed horizontal line $R_0^s = 1$ separates the extinction region from the stationary-distribution region, and the vertical dotted line marks the corresponding critical parameter value. For the baseline setting used in this example, the critical values are $\beta_c = 0.270141$, $\sigma_c = 0.101041$, and $\rho_c = 0.097950$.

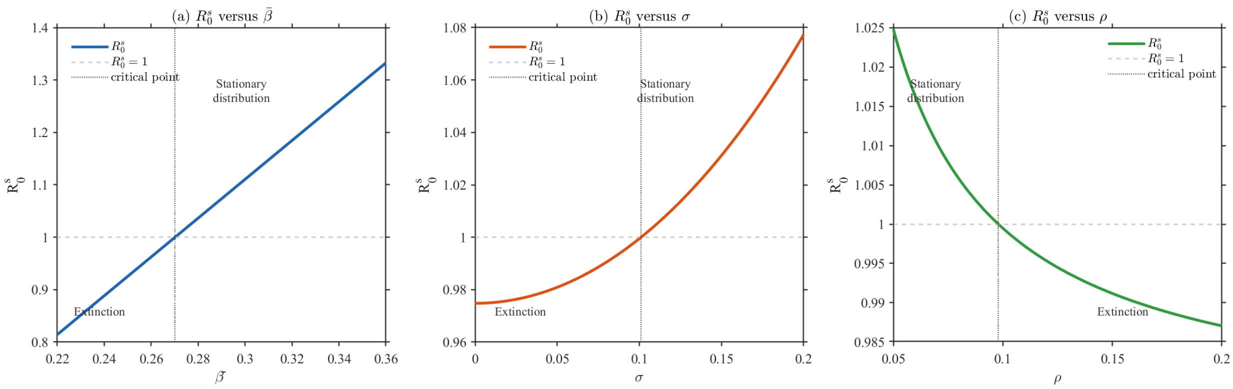


**Figure 5.** Sensitivity of the stochastic threshold $R_0^s$ with respect to $\bar{\beta}$, $\sigma$, and $\rho$. Panels (a)–(c) show the dependence of $R_0^s$ on the average transmission rate, the environmental noise intensity, and the mean-reversion speed, respectively. The horizontal dashed line denotes $R_0^s = 1$, and the vertical dotted line indicates the corresponding critical value.

To further illustrate the effect of parameter variation on the sample-path behavior, Figure 6 presents the trajectories of $i(t)$ and $r(t)$ under three sensitivity scenarios associated with $\bar{\beta}$, $\sigma$, and $\rho$. Panels (a) and (b) correspond to $\bar{\beta} = 0.2400,\ 0.2820$, and $0.3100$; panels (c) and (d) correspond to $\sigma = 0.0000,\ 0.1400$, and $0.2000$; and panels (e) and (f) correspond to $\rho = 0.2000,\ 0.0800$, and $0.0500$. For each group, the three parameter values represent cases below the threshold, slightly above the threshold, and well above the threshold, respectively. The corresponding stochastic thresholds are approximately $0.8884,\ 1.0439$, and $1.1475$ for the $\bar{\beta}$ cases; $0.9748,\ 1.0238$, and $1.0773$ for the $\sigma$ cases; and $0.9871,\ 1.0057$, and $1.0248$ for the $\rho$ cases. It can be observed that when $R_0^s < 1$, the infected component tends to decay to a very low level, whereas when $R_0^s > 1$, the trajectories fluctuate around positive levels. These observations are consistent with the theoretical threshold results.

The numerical results indicate that increasing $\bar{\beta}$ or $\sigma$ raises the stochastic threshold, whereas increasing $\rho$ reduces it. These results further illustrate the roles of the average transmission rate, the environmental noise intensity, and the mean-reversion speed in shaping the threshold behavior of system (1.4).

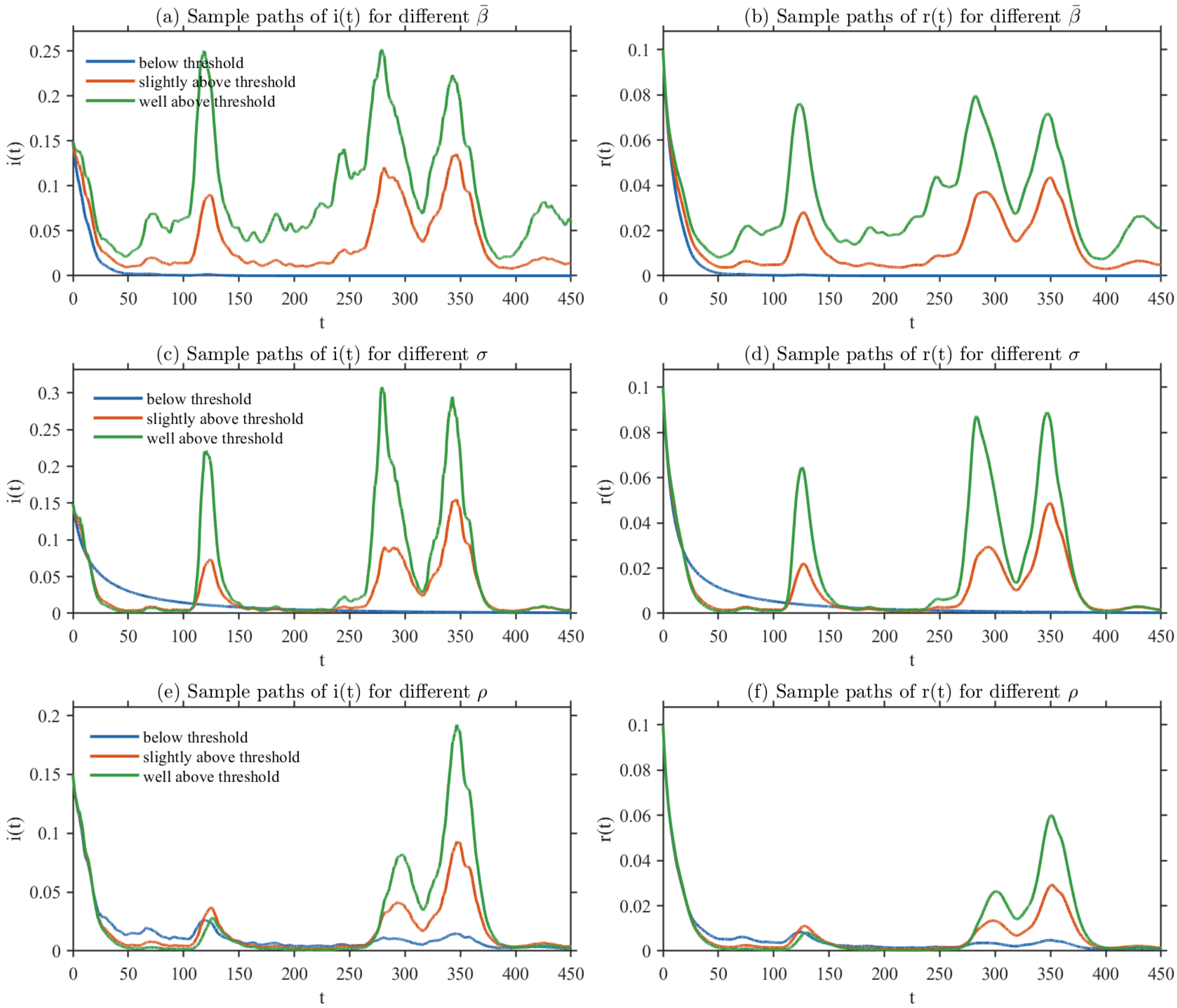


**Figure 6.** Representative sample paths of $i(t)$ and $r(t)$ under different values of $\bar{\beta}$, $\sigma$, and $\rho$.

**Example 6.5.** *We set the parameters as in Example 6.1. The case $\nu = 0$ corresponds to the constant-population model, while $\nu > 0$ gives the varying-population model. To examine the effect of $\nu$, we take $\nu = 0,\ 0.02,\ 0.06,\ 0.10,\ 0.12,$ and $0.14$. For each value of $\nu$, the same realization of the Brownian motion is used. We use $M = 200$ sample paths and compute the long-time average of $i(t)$ over $[300, 500]$. For these parameter values, solving $R_0^s(\nu) = 1$ yields the critical value $\nu_c \approx 0.133246$. Hence, increasing $\nu$ decreases the stochastic threshold.*

Figure 7 shows that the infected level decreases as $\nu$ increases. The long-time mean of $i(t)$ decreases from about 0.2308 for $\nu = 0$ to about 0.0097 for $\nu = 0.14$. This agrees with the threshold formula: Since $R_0^s(\nu)$ is decreasing in $\nu$, larger $\nu$ makes disease persistence less likely. In particular, when $\nu = 0.14$, $R_0^s(\nu) \approx 0.9838 < 1$, and the infected level becomes very small.

These observations also have a clear biological interpretation. The parameter $\nu$ represents the disease-induced mortality rate, and its increase reduces both the stochastic threshold and the long-time infected level. This suggests that population variation, although no longer explicitly appearing after normalization, still affects the epidemic outcome through the modified threshold and the $\nu$-dependent drift terms. From an applied perspective, the comparison between the varying-population model and its constant-population counterpart indicates that ignoring population variation may overestimate the persistence level of an infection in a fluctuating environment.

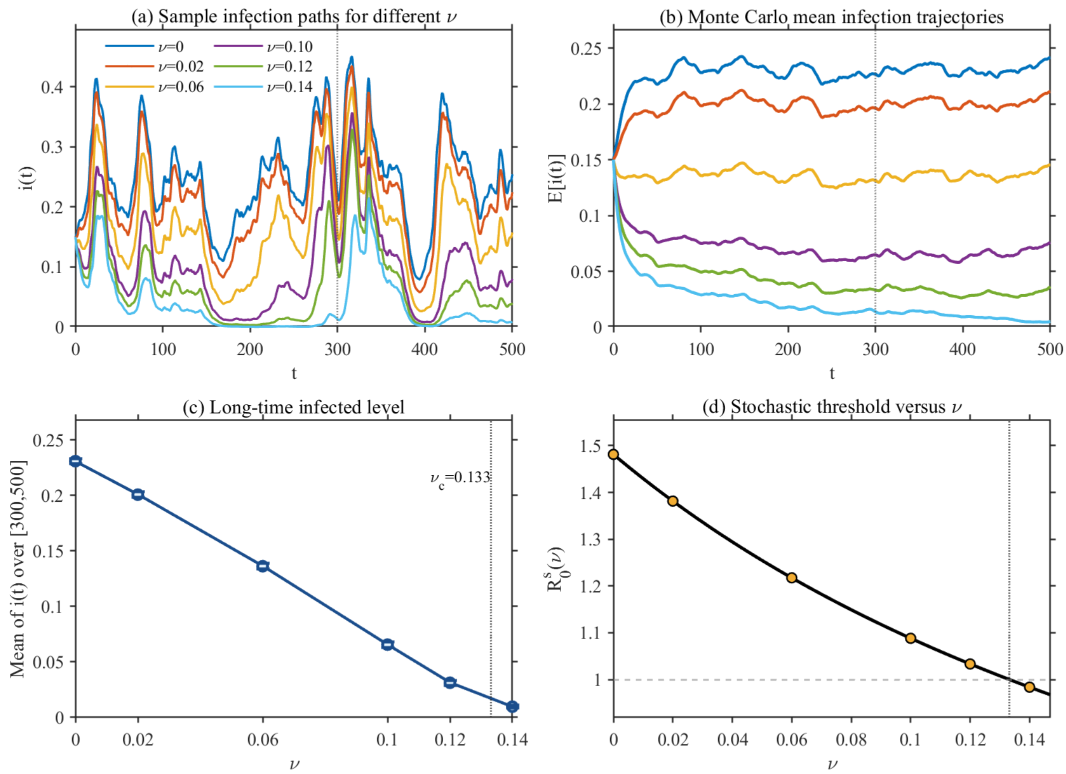


**Figure 7.** Comparison between the constant-population model and the varying-population model for different values of $\nu$ under the same Log-OU perturbation. Panels (a) and (b) show representative sample paths and Monte Carlo mean trajectories of $i(t)$, respectively. Panel (c) shows the long-time mean of $i(t)$ over $[300, 500]$. Panel (d) shows $R_0^s(\nu)$ as a function of $\nu$.

## 7. Conclusions

In this paper, we studied a stochastic SIRS epidemic model with varying population size, where the transmission rate is driven by a Log-OU process. We established the existence and uniqueness of the global positive solution, identified the stochastic threshold $R_0^s$, and showed that the model exhibits extinction when $R_0^s < 1$ and admits a unique invariant probability measure when $R_0^s > 1$. These results show that $R_0^s$ governs the long-term behavior of the system under environmental fluctuations.

We further showed that the linearized system near the quasi-endemic equilibrium admits a Gaussian invariant measure, yielding a local Gaussian approximation of the invariant probability density. Moreover, under suitable parameter conditions and sufficiently small noise intensity, the invariant joint probability distribution can be globally approximated by a normal distribution.

In addition, the comparison with the constant-population counterpart indicates that, although the total population no longer appears explicitly after normalization, the effect of population variation is retained through the disease-induced mortality parameter $\nu$, which influences both the threshold structure and the long-term infected level. Numerical results further indicate that the average transmission rate, the noise intensity, and the mean-reversion speed jointly influence the threshold and the corresponding epidemic outcome. Future work may consider multidimensional OU perturbations, higher-dimensional stochastic epidemic systems, and data-driven calibration of the proposed model.

## Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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