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Research article

Traveling wavefront solutions in a nonlocal diffusion model with nonlocal delay effect

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Abstract: This paper is devoted to investigating the existence of traveling wavefront solutions with a sufficiently large wave speed for a nonlocal diffusion model with nonlocal delay effect using a perturbation method. Our proof is based on an abstract formulation of the wave profile as a solution to an operator equation in a specific Banach space, combined with the Fredholm theory and the Banach contraction mapping principle. Several numerical simulations are presented to illustrate our main results.

Keywords: nonlocal diffusion; nonlocal delay; traveling wavefront solution; perturbation method

1. Introduction

Diffusion is a prevalent feature of ecological systems, thereby describing how the irregular microscopic movements of biological individuals give rise to certain macroscopic or overall regular motions of the group. Classical diffusion adheres to Fick's second law, which gives rise to Laplacian diffusion, also referred to as local diffusion. Local diffusion is confined to dilute systems, where concentrations or densities are low. However, in many biological systems such as embryological development, the densities of the involved cells are not low, and a local or short-range diffusive flux proportional to the gradient is insufficiently accurate [1]. In such cases, it is reasonable to incorporate long-range effects (nonlocal diffusion), which can be modeled through the formulation of an integral equation [2–7].

Another deficiency of simple reaction-diffusion models is that the birth function is assumed to act instantaneously. In other words, the future state of the growth function is solely determined by the present state and is independent of past history. However, as is well known, a time delay exists to account for the time required to reach maturity, the finite gestation period, and other relevant factors. In such situations, the principle of causality is often only a first approximation of the actual scenario, and a more realistic model should incorporate some of the past states of the growth function. Furthermore, the effects of diffusion and time delays are not independent of each other; individuals that were at a specific location at a previous time may no longer be at the same spatial point at the present time. From both theoretical and practical application perspectives, it is more reasonable to adopt a nonlocal per capita growth rate to incorporate the potential dispersal of individuals during the maturation period [8–16].

Based on the aforementioned considerations, in this paper, we investigate the following nonlocal diffusion model with nonlocal delay effect:

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]\mathrm{d}y + u(x,t)f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)u(x+y,t-\tau)\right), \tag{1.1}$$

where $x, y \in \mathbb{R}$ are the spatial variables, and $t \geq 0$ is the time. The function $u(x,t)$ denotes the population density of biological individuals at position x and time t. The integral $\int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]\mathrm{d}y$ denotes the nonlocal diffusion of biological individual, where the function J is radially symmetric, smooth, nonnegative, and satisfies the following:

$$\int_{\mathbb{R}} J(y)\mathrm{d}y = 1 \quad \text{and} \quad \overline{J} \triangleq \int_{\mathbb{R}} J(y)|y|\mathrm{d}y < \infty,$$

where $\triangleq$ means "is defined as". $J(x-y)$ denotes the probability distribution of a biological individual arriving at location x from location y, hence, $\int_{\mathbb{R}} J(x-y)u(y,t)\mathrm{d}y$ is the rate at which individuals arrive at location x from all other places, and $-u(x,t) = -\int_{\mathbb{R}} J(x-y)u(x,t)\mathrm{d}y$ is the rate at which they leave location x for all other places. Moreover, the reaction term $u(x,t)f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)u(x+y,t-\tau)\right)$ denotes the proliferation and the reduction due to the death of the individuals, and $f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)u(x+ y,t-\tau)\right)$ may account for all the competitive pressure when an individual born at location y can survive the immature period $[0,\tau]$ and has moved to location x when becoming mature (τ time units after birth); μ is a bounded measure on $\mathbb{R}^N$, which satisfies the following:

$$\int_{\mathbb{R}} \mathrm{d}\mu(y) = 1 \quad \text{and} \quad \overline{\mu} \triangleq \int_{\mathbb{R}} |y|\mathrm{d}|\mu(y)| < \infty.$$

Given their significant role in governing the long-time behavior of dynamical systems with a diffusion process, traveling wave solutions have long been a topic of enduring interest. For nonlocal diffusion models with and without delay effects, the existence, uniqueness, stability, and regularity of traveling wavefront solutions have also been extensively investigated in the literature (see, for example, [11, 17–26]). However, in most existing studies, the existence of traveling wavefront solutions is established via the upper-lower solution method, which requires the system to satisfy a comparison principle. More precisely, the system must be either monotonic or quasi-monotonic. Furthermore, these studies only provide theoretical proofs for the existence of traveling wavefront solutions, with a lack of corresponding numerical simulations to visually verify the theoretical results. To overcome the limitations of the aforementioned method and provide numerical illustrations that facilitate the visualization of wavefront behavior, in this paper, we will establish the existence of traveling wavefront solutions for System (1.1) by applying a perturbation method and conduct numerical simulations using an irregular finite difference scheme.

Our approach is based on an abstract formulation of the wave profile as a solution to an operator equation in a specific Banach space. To apply the Banach contraction mapping principle, we analyze the associated Fredholm operator and perform careful estimations of the nonlinear perturbation. This idea has been explored in the literature (e.g., [27–30]). However, this perturbation method has only been applied to the study of the existence of traveling wavefront solutions for local diffusion models. In this paper, we expand its scope of application and apply it to the study of the existence of traveling wavefront solutions for nonlocal diffusion models. Compared with [27–30], the major distinction of the present paper lies in the fact that replacing local diffusion with nonlocal diffusion fundamentally alters the corresponding operator equation. More specifically, as the diffusion process evolves from local diffusion to the more complex nonlocal diffusion, the inhomogeneous linear differential equation of the wave profile (see Eq (3.3)) is actually simplified from second-order to first-order. Nevertheless, the inhomogeneous term becomes more complex, which, in turn, complicates the estimation of the nonlinear operator. Furthermore, our approach reflects a natural connection between the existence of traveling wavefront solutions for System (1.1) and the existence of heteroclinic solutions for the corresponding reaction equation:

$$\dot{u}(t) = u(t)f\left(u(t-\tau)\right). \tag{1.2}$$

Now, we state our main result.

Theorem 1.1. *Assume the following:*

(H1). *f is a $C^k(k \geq 2)$-smooth function, $f(0) > 0$, and there exists a unique non-zero constant $u_0^* > 0$ such that $f(u_0^*) = 0$ and $\dot{f}(u_0^*) < 0$;*

(H2). *f is nonincreasing with $mf(u_0^*(m+1)) < 0$ for $m \neq 0 \in \mathbb{R}$;*

(H3). *$|f(u_0^*(m+1))| \leq |u_0^*\dot{f}(u_0^*)||m|$ for $m \in \mathbb{R}$;*

(H4). *$0 \leq \tau \leq -\frac{3}{2u_0^*\dot{f}(u_0^*)}$ ($\frac{3}{2}$-type condition).*

Then, there is a sufficiently large $c^* > 0$ such that for $c > c^*$, System (1.1) has a traveling wavefront solution $u(\xi) = u(x,t)$ that connects zero equilibrium 0 to nonzero equilibrium u_0^*, where $\xi = x + ct$, that is,

$$u(-\infty) \triangleq \lim_{\xi \to -\infty} u(\xi) = 0 \quad \text{and} \quad u(+\infty) \triangleq \lim_{\xi \to +\infty} u(\xi) = u_0^*.$$

Remark 1.1. *Note that hypothesis (H1) implies that the reaction equation (1.2) only admits an unstable node 0 and a nonzero equilibrium u_0^*, and there exists a positive number τ^* such that for each $0 \leq \tau < \tau^*$, u_0^* is locally asymptotically. Hypotheses (H2), (H3), and (H4) are desired hypotheses in order to ensure u_0^* is globally attractive in the admissible initial conditions. In addition, the traditional standard growth function $f(u) = k(1-u)$ with $k > 0$ obviously satisifies the above assumptions about f. In this case, we have $u_0^* = 1 > 0$, $u_0^*\dot{f}(u_0^*) = -k < 0$, and $f(0) = k > 0$; hence, $mf(u_0^*(m+1)) = -km^2 < 0$ and $|f(u_0^*(m+1))| = k|m| = |u_0^*\dot{f}(u_0^*)||m|$.*

For convenience, we introduce some notations. Let $C = C(\mathbb{R}, \mathbb{R})$ be the Banach space of continuous and bounded functions equipped with the supremum norm, $C^1 = C^1(\mathbb{R}, \mathbb{R}) = \{\psi \in C : \dot{\psi} \in C\}$ be the Banach space equipped with the standard norm $\|\psi\|_{C^1} = \|\psi\|_C + \|\dot{\psi}\|_C$, and $C_0 = \{\psi \in C : \lim_{t\to\pm\infty} \psi(t) = 0\}$ equipped with the same norm as C and $C_0^1 = \{\psi \in C_0 : \dot{\psi} \in C\}$ equipped with the same norm as C^1. For a continuous function $\Psi : [a-\tau, b] \to \mathbb{R}$, as usual we let $\Psi_t \in \mathcal{C} \triangleq C([-\tau, 0], \mathbb{R})$, $t \in [a, b]$ be defined by $\Psi_t(s) = \Psi(t + s)$ for $s \in [-\tau, 0]$, and for $\Psi_t \in \mathcal{C}$, we denote the norm by $\|\Psi_t\|_{\mathcal{C}} = \sup_{s\in[-\tau,0]} \|\Psi_t(s)\|_{\mathbb{R}}$.

2. Heteroclinic solutions of the reaction equation

In this section, we establish the existence of heteroclinic solutions that connects the zero equilibrium 0 to the nonzero equilibrium u_0^* for System (1.2), i.e., we have the following result.

Lemma 2.1. *Assume that (H1)–(H4) hold; then, System (1.2) has a family of heteroclinic solutions $u^\star(t)$ from 0 to u_0^*. Namely, System (1.2) has a family of solutions $u^\star(t)$ defined for all $t \in \mathbb{R}$ such that*

$$u^\star(-\infty) \triangleq \lim_{t\to-\infty} u^\star(t) = 0 \quad \text{and} \quad u^\star(+\infty) \triangleq \lim_{t\to+\infty} u^\star(t) = u_0^*.$$

Proof. Under the above assumptions (H1), System (1.2) only has two equilibria: 0 and u_0^*. The characteristic equations of System (1.2) that correspond to the equilibria 0 and u_0^* are

$$\lambda - f(0) = 0, \tag{2.1}$$

and

$$\lambda - u_0^* \dot{f}(u_0^*) e^{-\lambda\tau} = 0, \tag{2.2}$$

respectively. From Eq (2.1), it can be seen that the zero equilibrium 0 is an unstable node. Next, we consider the root of Eq (2.2). Obviously, when $\tau = 0$, $\lambda = u_0^* \dot{f}(u_0^*) < 0$. When τ has a tiny increase from 0, we still have $\mathrm{Re}\{\lambda\} < 0$. When the value of τ further increases such that $\mathrm{Re}\{\lambda\} \geq 0$, we assume that τ^* is the minimum value τ such that $\mathrm{Re}\{\lambda\} \geq 0$; then, λ crosses through the imaginary axis. Now, because of $\dot{f}(u_0^*) \neq 0$, we have $\lambda \neq 0$. When $\tau = \tau^*$, there exists $\omega > 0$ such that $\mathrm{i}\omega - u_0^* \dot{f}(u_0^*) e^{-\mathrm{i}\omega\tau^*} = 0$, which is equivalent to the following:

$$-u_0^* \dot{f}(u_0^*) \cos(\omega\tau^*) + \mathrm{i}\{\omega + u_0^* \dot{f}(u_0^*) \sin(\omega\tau^*)\} = 0.$$

Hence, we have

$$\cos(\omega\tau^*) = 0 \quad \text{and} \quad \sin(\omega\tau^*) = \frac{\omega}{-u_0^* \dot{f}(u_0^*)} > 0,$$

and

$$\omega = -u_0^* \dot{f}(u_0^*) \quad \text{and} \quad \tau^* = \frac{\frac{\pi}{2} + 2n\pi}{-u_0^* \dot{f}(u_0^*)}, \quad n = 0, 1, 2, \cdots.$$

Suppose that when $\tau = \tau^*$, the root of $\lambda - u_0^* \dot{f}(u_0^*) e^{-\lambda\tau} = 0$ first crosses through the imaginary axis; then, we have $n = 0$ (i.e., $\tau^* = -\frac{\pi}{2u_0^* \dot{f}(u_0^*)}$). Thus, if $0 < \tau < -\frac{\pi}{2u_0^* \dot{f}(u_0^*)}$, then $\mathrm{Re}\{\lambda\} < 0$. Since $0 \leq \tau \leq -\frac{3}{2u_0^* \dot{f}(u_0^*)} < -\frac{\pi}{2u_0^* \dot{f}(u_0^*)}$, it follows that all eigenvalues that correspond to the equilibrium u_0^* have negative real parts and no limit cycle branches out from the equilibrium u_0^*.

Let $g(t) = \frac{u(t)}{u_0^*} - 1$; then, we transform Eq (1.2) into the following:

$$\dot{g}(t) = (1 + g(t)) f\left(u_0^*[g(t-\tau) + 1]\right). \tag{2.3}$$

From Corollary 3.8 in [31], we know that the solutions $g(t)$ of Eq (2.3) with the initial conditions $g(s) \geq -1$ for $s \in [-\tau, 0)$, $g(0) > -1$ satisfy $g(t) \to 0$ as $t \to \infty$, i.e., the solutions $u(t)$ of System (1.2) with the initial conditions $u(s) \geq 0$ for $s \in [-\tau, 0)$, $u(0) > 0$ satisfy $u(t) \to u_0^*$ as $t \to \infty$.

From Eq (2.1), we know that the unstable subspace E_u of the trivial solution in the usual phase space C is spanned by $\chi(s) = e^{f(0)s}$, $s \in [-\tau, 0]$. Let E_s be the subspace in C so that $C = E_s \oplus E_u$; then, there exists $\eta_0 > 0$ and a C^1-map $\hbar : E_u \to E_s$ with $\hbar(0) = 0$ and $D\hbar(0) = 0$ such that a local unstable manifold of equilibrium 0 is give by $\eta\chi + \hbar(\eta\chi)$ for $\eta \in (-\eta_0, \eta_0)$. Choose $\eta_0 > 0$ sufficiently small so that the operator norm $\|D\hbar(\eta\chi)\|_C < e^{-f(0)\tau}$ for $\eta \in (0, \eta_0)$. Then, pick up $\eta \in (0, \eta_0)$ and consider $\nu = \eta\chi + \hbar(\eta\chi)$; we have the following:

$$\nu(s) = \eta e^{f(0)s} + \hbar(\eta\chi)(s) > \eta e^{f(0)s} - e^{-f(0)\tau}\eta\|\chi\|_C \geq \eta\left(e^{f(0)s} - e^{-f(0)\tau}\right) \geq 0.$$

Therefore, the solution starting from the point ν on the local unstable manifold of equilibrium 0 is positive and tends to equilibrium u_0^* due to the global attractivity of equilibrium u_0^*. This completes the proof. □

Remark 2.1. *We need to normalize the function f before using the Corollary 3.8 in [31]. Moreover, due to the biological interpretation, only positive solutions are to be considered and therefore admissible. Hence, we only select admissible initial conditions $u(s) \geq 0$ for $s \in [-\tau, 0)$, $u(0) > 0$.*

3. Operator equation

In this section, we convert System (1.1) into an operator equation that involves a linear operator and a nonlinear perturbation. By substituting $U(\xi) = u(x, t)$ into System (1.1), we obtain the wave profile equation of System (1.1) as follows:

$$\left[\int_{\mathbb{R}} J(y)U(\xi - y)\mathrm{d}y - U(\xi)\right] - c\dot{U}(\xi) + U(\xi) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)U(\xi + y - c\tau)\right) = 0. \tag{3.1}$$

Let $\kappa(\xi) = U(c\xi)$ and $\varepsilon = \frac{1}{c}$; we rewrite Eq (3.1) as follows:

$$\left[\int_{\mathbb{R}} J(y)\kappa(\xi - \varepsilon y)\mathrm{d}y - \kappa(\xi)\right] - \dot{\kappa}(\xi) + \kappa(\xi) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)\kappa(\xi + \varepsilon y - \tau)\right) = 0. \tag{3.2}$$

It is obvious that when c is large, ε is small. If $\varepsilon = 0$, then Eq (3.2) is reduced to the reaction equation (1.2).

We further transform Eq (3.2) by introducing the variable $\vartheta(\xi) = \kappa(\xi) - u^\star(\xi)$, where $u^\star$ is the heteroclinic orbit of System (1.2) in Lemma 2.1. Hence, we obtain the equation for ϑ as follows:

$$-\dot{\vartheta}(\xi) - \vartheta(\xi) + [I\vartheta](\xi) + H(\varepsilon, \xi, \vartheta) = 0, \tag{3.3}$$

where the linear operator I and the nonlinear operator H are defined as

$$[I\vartheta](\xi) = [1 + f(u^\star(\xi - \tau))]\vartheta(\xi) + u^\star(\xi)\dot{f}(u^\star(\xi - \tau))\vartheta(\xi - \tau),$$

and

$$H(\varepsilon, \xi, \vartheta) = \left[\int_{\mathbb{R}} J(y)[\vartheta + u^\star](\xi - \varepsilon y) \mathrm{d}y - [\vartheta + u^\star](\xi) \right] - u^\star(\xi) f\left(u^\star(\xi - \tau)\right)$$
$$+ [\vartheta + u^\star](\xi) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)[\vartheta + u^\star](\xi + \varepsilon y - \tau) \right) - [I\vartheta](\xi) + \vartheta(\xi),$$

respectively. It follows that $\vartheta(\xi) \in C_0$ is a bounded solution of Eq (3.3) if and only if $\vartheta(\xi)$ satisfies the following:

$$[L\vartheta](\xi) = \mathcal{H}(\varepsilon, \xi, \vartheta), \quad \xi \in \mathbb{R}, \tag{3.4}$$

where the linear operator L and the nonlinear operator $\mathcal{H}$ are defined as

$$[L\vartheta](\xi) = \vartheta(\xi) - \int_{-\infty}^{\xi} e^{-(\xi - t)}[I\vartheta](t) \mathrm{d}t \quad \text{and} \quad \mathcal{H}(\varepsilon, \xi, \vartheta) = \int_{-\infty}^{\xi} e^{-(\xi - t)} H(\varepsilon, t, \vartheta) \mathrm{d}t,$$

respectively. By this means, we show that System (1.1) has a traveling wavefront solution $u(\xi)$ that connects 0 to u_0^* if and only if Eq (3.4) has a solution $\vartheta \in C_0$. Therefore, the remainder of this paper is dedicated to proving that the operator equation (3.4) has a solution $\vartheta(\xi) \in C_0$ using the Banach fixed point theorem. For this purpose, we need further detailed properties of the linear operator L and the nonlinear operator $\mathcal{H}$.

4. Properties of L and $\mathcal{H}$

The aim of this section is to analyze properties of the linear operator L and the nonlinear operator $\mathcal{H}$ that allow us to solve (3.4) by applying the Banach fixed point theorem. First, we investigate the properties of the linear operator L. Let $\mathcal{N}(L)$ denote the null space of the linear operator L. From the definition of L, it follows that the necessary and sufficient condition for $L\vartheta = 0$ is as follows:

$$\dot{\vartheta}(\xi) = f(u^\star(\xi - \tau))\vartheta(\xi) + u^\star(\xi) \dot{f}\left(u^\star(\xi - \tau)\right) \vartheta(\xi - \tau).$$

Define the linear operator $T : C^1 \to C$ by the following:

$$(T\vartheta)(\xi) = \dot{\vartheta}(\xi) - f(u^\star(\xi - \tau))\vartheta(\xi) - u^\star(\xi) \dot{f}\left(u^\star(\xi - \tau)\right) \vartheta(\xi - \tau).$$

Therefore, $L\vartheta = 0$ if and only if $T\vartheta = 0$(i.e., we have the following result).

Lemma 4.1. $\dim \mathcal{N}(L) = \dim \mathcal{N}(T)$, *where $\mathcal{N}(T)$ is the null space of the operator T.*

We define the formal adjoint equation of $T\vartheta = 0$ as follows:

$$\dot{\varphi}(\xi) + f(u^\star(\xi))\varphi(\xi) + u^\star(\xi + \tau) \dot{f}\left(u^\star(\xi)\right) \varphi(\xi + \tau) = 0. \tag{4.1}$$

For the solutions of System (4.1), we have the following result.

Lemma 4.2. *If $\varphi \in C^1$ is a solution to System* (4.1)*, then $\varphi = 0$.*

Proof. Let φ be a bounded solution of Equation (4.1) and $h(\xi) = \varphi(-\xi)$ for $\xi \in \mathbb{R}$. Then, we have the following:

$$\dot{h}(\xi) = Q(\xi)h(\xi) \triangleq f(u^\star(-\xi))h_\xi(0) + u^\star(-\xi + \tau) \dot{f}\left(u^\star(-\xi)\right) h_\xi(-\tau). \tag{4.2}$$

The limiting equation of Eq (4.2) as $\xi \to -\infty$ is as follows:

$$\dot{h}(\xi) = Q(-\infty)h(\xi) \triangleq u_0^* \dot{f}\left(u_0^*\right) h(\xi - \tau), \tag{4.3}$$

since the linear delay differential equation (4.3) and the linearized equation of System (1.2) at u_0^* have the same eigenvalues. We conclude that all eigenvalues associated with Eq (4.3) have negative real parts by the process of proof in Lemma 2.1. Let $\{\mathcal{J}(\xi)\}_{\xi \geq 0}$ be the semigroup generated by the solutions of Eq (4.3), that is, $\mathcal{J}(\xi) : C \to C$ and $h(\xi)$ is the solution of Eq (4.3) with initial condition $\varphi_0(s)$ for $s \in [-\tau, 0]$, where $h(\xi)$ is defined by $h(\xi + s) = h_\xi(s) = (J(\xi)\varphi_0)(s)$ for $\xi \geq 0$ and $s \in [-\tau, 0]$. Moreover, let $\mathcal{X}(s) : [0, \infty) \to \mathbb{R}$ be the solution of Eq (4.3) with the initial condition

$$\mathcal{X}(s) = \begin{cases} 1, & \text{for } s = 0, \\ 0, & \text{for } s \in [-\tau, 0). \end{cases}$$

Then, there are positive constants $\gamma > 0$ and $\beta > 0$ such that for $\xi \geq 0$, we have the following:

$$\|\mathcal{J}(\xi)\varphi_0\|_C \leq \gamma e^{-\beta\xi}\|\varphi_0\|_C, \;\; \|\mathcal{X}(s)\|_{\mathbb{R}} \leq \gamma e^{-\beta\xi}, \;\; \varphi_0 \in C. \tag{4.4}$$

Let $\varrho > 0$ be such that $\varrho\gamma e^{\beta\tau} < \beta$. Since $Q(\xi) \to Q(-\infty)$ as $\xi \to -\infty$, there is a ξ^* such that

$$|Q(\xi) - Q(-\infty)| \leq \varrho, \;\; \xi \leq \xi^*. \tag{4.5}$$

We rewrite Eq (4.2) as follows:

$$\dot{h}(\xi) = Q(-\infty)h(\xi) + [Q(\xi) - Q(-\infty)]h(\xi). \tag{4.6}$$

By the variation of constants formula, solutions of Eq (4.6) can be expressed as follows:

$$h_\xi(s) = [\mathcal{J}(\xi - \zeta)h_\zeta](s) + \int_\zeta^{\xi+s} \mathcal{X}(\xi + s - z)[Q(z) - Q(-\infty)]h_z \mathrm{d}z, \tag{4.7}$$

for $\zeta \leq \xi$ and $s \in [-\tau, 0]$. In view of inequalities (4.4) and (4.5) and Eq (4.7), we have the following:

$$\begin{aligned} \|h_\xi\|_C &\leq \gamma e^{-\beta(\xi-\zeta)}\|h_\zeta\|_C + \varrho\gamma \int_\zeta^{\xi+s} e^{-\beta(\xi+s-z)}\|h_z\|_C \mathrm{d}z \\ &\leq \gamma e^{-\beta(\xi-\zeta)}\|h_\zeta\|_C + \varrho\gamma e^{\beta\tau} \int_\zeta^{\xi+s} e^{-\beta(\xi-z)}\|h_z\|_C \mathrm{d}z \\ &\leq \gamma e^{-\beta(\xi-\zeta)}\|h_\zeta\|_C + \varrho\gamma e^{\beta\tau} \int_\zeta^{\xi} e^{-\beta(\xi-z)}\|h_z\|_C \mathrm{d}z, \end{aligned}$$

for $\zeta \leq \xi \leq \xi^*$. Or equivalently,

$$e^{\beta\xi}\|h_\xi\|_C \leq \gamma e^{\beta\zeta}\|h_\zeta\|_C + \varrho\gamma e^{\beta\tau} \int_\zeta^{\xi} e^{\beta z}\|h_z\|_C \mathrm{d}z. \tag{4.8}$$

Applying the Gronwall inequality to inequality (4.8) yields the following:

$$e^{\beta\xi}\|h_\xi\|_C \leq \gamma e^{\beta\zeta}\|h_\zeta\|_C e^{\varrho\gamma e^{\beta\tau}(\xi-\zeta)}.$$

From the above inequality, we have the following:

$$\|h_\xi\|_C \le \gamma e^{\beta(\zeta-\xi)}\|h_\zeta\|_C e^{\varrho\gamma e^{\beta\tau}(\xi-\zeta)} = \gamma e^{-(\beta-\varrho\gamma e^{\beta\tau})(\xi-\zeta)}\|h_\zeta\|_C, \quad \text{for } \zeta \le \xi \le \xi^*. \tag{4.9}$$

Note that h_ζ is bounded. By letting $\zeta \to -\infty$ in inequality (4.9), we immediately have the following:

$$\|h_\xi\|_C = 0, \quad \text{for } \xi \le \xi^*. \tag{4.10}$$

The uniqueness of the solution of Eq (4.6) implies that $h_\xi = 0$ for all $\xi \in \mathbb{R}$, and hence $\varphi = 0$. The proof is completed. □

Let $\mathcal{R}(T)$ denote the range of the operator T. We have the following result.

Lemma 4.3. $\mathcal{R}(T) = C$.

Proof. It follows from the process of proof in Lemmma 2.1 that the operator T is Fredholm (see p7 in [32]). Moreover, we have the following:

$$\mathcal{R}(T) = \left\{ w \in C : \int_{-\infty}^{\infty} \varphi(\xi)w(\xi)\mathrm{d}\xi = 0 \text{ for an arbitrary bounded solution } \varphi \text{ of System (4.1)} \right\}.$$

In view of Lemma 4.2, we have $\mathcal{R}(T) = C$. The proof is completed. □

Next, we prove that $\mathcal{R}(L) = C_0$, where $\mathcal{R}(L)$ denotes the range of L, that is, for any $p \in C_0$, we need to show that there exists $\vartheta \in C_0$ such that $L\vartheta = p$, or equivalently,

$$\vartheta(\xi) - \int_{-\infty}^{\xi} e^{-(\xi-t)}[I\vartheta](t)\mathrm{d}t = p(\xi), \tag{4.11}$$

has a solution in C_0. To this end, we let $q(\xi) = \vartheta(\xi) - p(\xi)$. Upon a substitution, we obtain the following equation for q:

$$q(\xi) = \int_{-\infty}^{\xi} e^{-(\xi-t)}\,[Iq]\,(t) + [Ip]\,(t)\mathrm{d}t.$$

Differentiating both sides of the above equation with respect to ξ yields the following:

$$\dot{q}(\xi) = [Iq]\,(\xi) + [Ip]\,(\xi) - q(\xi).$$

Combining with the definition of the operator T yields the following:

$$(Tq)(\xi) = Y(\xi) \triangleq [Ip](\xi), \tag{4.12}$$

From the expression of I and $p \in C_0$, we obtain $Y \in C_0$. For Eq (4.12), we have the following result.

Lemma 4.4. *Let $Y \in C_0$ be a given function. If q is a bounded solution to the equation $Tq = Y$, then $q \in C_0^1$.*

Proof. From Lemma 4.3, we know $\mathcal{R}(T) = C$. Thus, for any $Y \in C_0$, there exists $q \in C^1$ that satisfies $Tq = Y$. By the definition of the operator T, we have the following:

$$\dot{q}(\xi) - f(u^\star(\xi-\tau))q(\xi) - u^\star(\xi)\dot{f}\left(u^\star(\xi-\tau)\right)q(\xi-\tau) = Y(\xi). \tag{4.13}$$

When $\xi \to -\infty$, Eq (4.13) asymptotically tends to the following limiting system:

$$\dot{q}(\xi) = f(0)q(\xi). \tag{4.14}$$

Any bounded solution to System (4.14) is the zero solution. Similarly, as $\xi \to +\infty$, we obtain the following:

$$\dot{q}(\xi) = u_0^* \dot{f}(u_0^*) q(\xi - \tau). \tag{4.15}$$

We know that the ω-limit set of every bounded solution to Eq (4.15) is just the zero solution $q = 0$. Therefore, from Theorem 1.8 in [33], we conclude that every bounded solution component of System (4.13) satisfies $\lim_{|\xi|\to\infty} q(\xi) = 0$ (i.e., $q \in C_0^1$). The proof is completed. □

In view of Lemma 4.4, System (4.12) has a solution $q \in C_0^1$. Consequently, $\vartheta = q + p \in C_0$ is a solution to Eq (4.11). Therefore, we obtain the following conclusion.

Lemma 4.5. $\mathcal{R}(L) = C_0$.

Next, we study the property of the nonlinear operator $\mathcal{H}$. For the nonlinear operator $\mathcal{H}(\varepsilon, \xi, \vartheta)$, we have the following result.

Lemma 4.6. *There exists $\delta > 0$ and $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$ and $\vartheta \in B(\delta) \subset C_0$, we have the following:*

$$|\mathcal{H}(\varepsilon, \xi, \vartheta)| = O(\varepsilon) + O(\varepsilon\|\vartheta\|_C) + O(\|\vartheta\|_C^2) \text{ as } \varepsilon \to 0^+,$$

where $B(\delta)$ be the open ball in C_0 with radius δ and centered at the origin.

Proof. We divide our proof into five steps.
Step 1. We rewrite $\mathcal{H}(\varepsilon, \xi, \vartheta)$ as follows:

$$\mathcal{H}(\varepsilon, \xi, \vartheta) = \sum_{i=1}^{4} \int_{-\infty}^{\xi} e^{-(\xi - t)} N_i(\varepsilon, t, \vartheta) \mathrm{d}t,$$

where

$$\begin{aligned}
N_1(\varepsilon, t, \vartheta) =& \vartheta(t) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)[\vartheta + u^\star](t + \varepsilon y - \tau)\right) - \vartheta(t) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t + \varepsilon y - \tau)\right) \\
&+ u^\star(t) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)[\vartheta + u^\star](t + \varepsilon y - \tau)\right) - u^\star(t) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t + \varepsilon y - \tau)\right) \\
&- u^\star(t) \dot{f}\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t + \varepsilon y - \tau)\right)\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) \vartheta(t + \varepsilon y - \tau)\right), \\
N_2(\varepsilon, t, \vartheta) =& \vartheta(t) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t + \varepsilon y - \tau)\right) - f(u^\star(t - \tau))\vartheta(t) - u^\star(t)\dot{f}(u^\star(t - \tau))\vartheta(t - \tau) \\
&+ u^\star(t) \dot{f}\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t + \varepsilon y - \tau)\right)\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) \vartheta(t + \varepsilon y - \tau)\right), \\
N_3(\varepsilon, t, \vartheta) =& u^\star f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t + \varepsilon y - \tau)\right) - u^\star(t) f\left(u^\star(t - \tau)\right),
\end{aligned}$$

and

$$N_4(\varepsilon,t,\vartheta)=\left[\int_{\mathbb{R}} J(y)[\vartheta+u^\star](t-\varepsilon y)\mathrm{d}y-[\vartheta+u^\star](t)\right].$$

Step 2. We claim that for each $\vartheta\in B(\delta)\subset C_0$ and $\varepsilon\in(0,\varepsilon^*]$, we have the following:

$$\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}N_1(\varepsilon,t,\vartheta)\mathrm{d}t\right| = O(\|\vartheta\|_C^2) \text{ as } \varepsilon\to 0^+.$$

In fact, since $N_1(\varepsilon,t,\vartheta)$ is C^k-smooth ($k\geq 2$), we see that $\frac{\partial N_1(\varepsilon,t,\vartheta)}{\partial\vartheta}$ and $\frac{\partial^2 N_1(\varepsilon,t,\vartheta)}{\partial\vartheta^2}$ are continuous in $\vartheta\in B(\delta)\subset C_0$. Moreover, $N_1(\varepsilon,t,0)=\frac{\partial N_1(\varepsilon,t,0)}{\partial\vartheta}\equiv 0$. Hence, there exists $\rho>0$ such that $|N_1(\varepsilon,t,\vartheta)|\leq \rho\|\vartheta\|_C^2$ uniformly for $\varepsilon\in(0,\varepsilon^*]$. In addition, there exists $K_1>0$ such that $\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}\mathrm{d}t\right|\leq K_1$. Thus, for each $\vartheta\in B(\delta)\subset C_0$ and $\varepsilon\in(0,\varepsilon^*]$, we have the following:

$$\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}N_1(\varepsilon,t,\vartheta)\mathrm{d}t\right| \leq \left|\int_{-\infty}^{\xi} e^{-(\xi-t)}\mathrm{d}t\right|\rho\|\vartheta\|_C^2 = O(\|\vartheta\|_C^2) \text{ as } \varepsilon\to 0^+.$$

Step 3. Next, we estimate $\int_{-\infty}^{\xi} e^{-(\xi-t)}N_2(\varepsilon,t,\vartheta)\mathrm{d}t$. Noticing that $u^\star$ is C^1-smooth, we obtain

$$\left|u^\star(t+\varepsilon y-\tau)-u^\star(t-\tau)\right|\leq \varepsilon|y|\left\|\dot{u^\star}\right\|_C,$$

and hence

$$\left|\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[u^\star(t+\varepsilon y-\tau)-u^\star(t-\tau)\right]\right|\leq \varepsilon\bar{\mu}\left\|\dot{u^\star}\right\|_C. \tag{4.16}$$

Since f is C^k-smooth for some $k\geq 3$, there exist two constants $K_2>0$ and $K_3>0$ such that

$$\left|\int_0^1 \dot{f}\left(\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[(1-\sigma)u^\star(t+\varepsilon y-\tau)+\sigma u^\star(t-\tau)\right]\right)\mathrm{d}\sigma\right|\leq K_2, \tag{4.17}$$

and

$$\left|\int_0^1 \ddot{f}\left(\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[(1-\sigma)u^\star(t+\varepsilon y-\tau)+\sigma u^\star(t-\tau)\right]\right)\mathrm{d}\sigma\right|\leq K_3. \tag{4.18}$$

From Inequalities (4.16) and (4.17), we obtain the following:

$$\begin{aligned}
&\left|f\left(\int_{\mathbb{R}}\mathrm{d}\mu(y)u^\star(t+\varepsilon y-\tau)\right)-f\left(u^\star(t-\tau)\right)\right|\\
&\leq\left|\int_0^1 \dot{f}\left(\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[(1-\sigma)u^\star(t+\varepsilon y-\tau)+\sigma u^\star(t-\tau)\right]\right)\mathrm{d}\sigma\right|\\
&\quad\times\left|\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[u^\star(t+\varepsilon y-\tau)-u^\star(t-\tau)\right]\right|\\
&\leq \varepsilon K_2\bar{\mu}\left\|\dot{u^\star}\right\|_C.
\end{aligned} \tag{4.19}$$

In addition, using Inequalities (4.16) and (4.18), we get the following:

$$\begin{aligned}
&\left|\dot f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)u^\star(t+\varepsilon y-\tau)\right)-\dot f\left(u^\star(t-\tau)\right)\right| \\
\le &\left|\int_0^1 \ddot f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y)\left[(1-\sigma)u^\star(t+\varepsilon y-\tau)+\sigma u^\star(t-\tau)\right]\right)\mathrm{d}\sigma\right| \\
&\times\left|\int_{\mathbb{R}} \mathrm{d}\mu(y)\left[u^\star(t+\varepsilon y-\tau)-u^\star(t-\tau)\right]\right| \\
\le &\varepsilon K_3\bar\mu\left\|\dot{u^\star}\right\|_C .
\end{aligned} \tag{4.20}$$

Considering the following: $\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}u^\star(t)\dot f\left(u^\star(t-\tau)\right)\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[\vartheta(t+\varepsilon y-\tau)-\vartheta(t-\tau)\right]\mathrm{d}t\right|$. By exchanging the order of integration and integration by parts, we have the following:

$$\begin{aligned}
&\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}u^\star(t)\dot f\left(u^\star(t-\tau)\right)\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[\vartheta(t+\varepsilon y-\tau)-\vartheta(t-\tau)\right]\mathrm{d}t\right| \\
=&\left|\int_0^1\left[\int_{-\infty}^{\xi}\varepsilon e^{-(\xi-t)}u^\star(t)\dot f\left(u^\star(t-\tau)\right)\int_{\mathbb{R}}\mathrm{d}\mu(y)y\dot\vartheta(t+\sigma\varepsilon y-\tau)\mathrm{d}t\right]\mathrm{d}\sigma\right| \\
\le&\left|\varepsilon\int_0^1\left[u^\star(\xi)\dot f\left(u^\star(\xi-\tau)\right)\int_{\mathbb{R}}\mathrm{d}\mu(y)y\vartheta(\xi+\sigma\varepsilon y-\tau)\right]\mathrm{d}\sigma\right| \\
&+\left|\int_0^1\left[\int_{-\infty}^{\xi}\varepsilon e^{-(\xi-t)}u^\star(t)\dot f\left(u^\star(t-\tau)\right)\int_{\mathbb{R}}\mathrm{d}\mu(y)y\vartheta(t+\sigma\varepsilon y-\tau)\mathrm{d}t\right]\mathrm{d}\sigma\right| \\
&+\left|\int_0^1\left[\int_{-\infty}^{\xi}\varepsilon e^{-(\xi-t)}\dot{u^\star}(t)\dot f\left(u^\star(t-\tau)\right)\int_{\mathbb{R}}\mathrm{d}\mu(y)y\vartheta(t+\sigma\varepsilon y-\tau)\mathrm{d}t\right]\mathrm{d}\sigma\right| \\
&+\left|\int_0^1\left[\int_{-\infty}^{\xi}\varepsilon e^{-(\xi-t)}u^\star(t)\ddot f\left(u^\star(t-\tau)\right)\dot u^\star(t)\int_{\mathbb{R}}\mathrm{d}\mu(y)y\vartheta(t+\sigma\varepsilon y-\tau)\mathrm{d}t\right]\mathrm{d}\sigma\right| \\
\le&\varepsilon\left(1+K_1\right)\bar\mu\left\|u^\star\right\|_C\left\|\dot f\right\|_C\|\vartheta\|_C+\varepsilon K_1\bar\mu\left\|\dot{u^\star}\right\|_C\left\|\dot f\right\|_C\|\vartheta\|_C+\varepsilon K_1\bar\mu\left\|u^\star\right\|_C\left\|\dot{u^\star}\right\|_C\left\|\ddot f\right\|_C\|\vartheta\|_C.
\end{aligned} \tag{4.21}$$

It follows from Inequalities (4.19), (4.20), and (4.21) that

$$\begin{aligned}
\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}N_2(\varepsilon,t,\vartheta)\mathrm{d}t\right|\le&\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}\left[f\left(\int_{\mathbb{R}}\mathrm{d}\mu(y)u^\star(t+\varepsilon y-\tau)\right)-f(u^\star(t-\tau))\right]\vartheta(t)\mathrm{d}t\right| \\
&+\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}u^\star(t)\left[\dot f\left(\int_{\mathbb{R}}\mathrm{d}\mu(y)u^\star(t+\varepsilon y-\tau)\right)-\dot f(u^\star(t-\tau))\right]\right. \\
&\left.\times\int_{\mathbb{R}}\mathrm{d}\mu(y)\vartheta(t+\varepsilon y-\tau)\mathrm{d}t\right| \\
&+\left|\int_{-\infty}^{\xi} e^{-(\xi-t)}u^\star(t)\dot f(u^\star(t-\tau))\int_{\mathbb{R}}\mathrm{d}\mu(y)\left[\vartheta(t+\varepsilon y-\tau)-\vartheta(t-\tau)\right]\mathrm{d}t\right| \\
\le&\varepsilon K_1K_2\bar\mu\left\|\dot{u^\star}\right\|_C\|\vartheta\|_C+\varepsilon K_1K_3\bar\mu\left\|u^\star\right\|_C\left\|\dot{u^\star}\right\|_C\|\vartheta\|_C+\varepsilon K_1\bar\mu\left\|\dot{u^\star}\right\|_C\left\|\dot f\right\|_C\|\vartheta\|_C \\
&+\varepsilon\left(1+K_1\right)\bar\mu\left\|u^\star\right\|_C\left\|\dot f\right\|_C\|\vartheta\|_C+\varepsilon K_1\bar\mu\left\|u^\star\right\|_C\left\|\dot{u^\star}\right\|_C\left\|\ddot f\right\|_C\|\vartheta\|_C \\
=&O\left(\varepsilon\|\vartheta\|_C\right),\quad \varepsilon\to0^+.
\end{aligned}$$

Step 4. For $\int_{-\infty}^{\xi} e^{-(\xi-t)} N_3(\varepsilon, t, \vartheta)\mathrm{d}t$, we have the following:

$$\begin{aligned}\left|\int_{-\infty}^{\xi} e^{-(\xi-t)} N_3(\varepsilon, t, \vartheta)\mathrm{d}t\right| =& \left|\int_{-\infty}^{\xi} e^{-(\xi-t)} \left[u^\star(t) f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t+\varepsilon y-\tau)\right) - u^\star(t) f\left(u^\star(t-\tau)\right)\right]\mathrm{d}t\right| \\ \leq& \left|\int_{-\infty}^{\xi} e^{-(\xi-t)}\mathrm{d}t\right| \left|f\left(\int_{\mathbb{R}} \mathrm{d}\mu(y) u^\star(t+\varepsilon y-\tau)\right) - f\left(u^\star(t-\tau)\right)\right| \left\|u^\star\right\|_C \\ \leq& \varepsilon K_1 K_2 \bar{\mu} \left\|u^\star\right\|_C \left\|\dot{u^\star}\right\|_C \\ =& O(\varepsilon) \ \text{ as } \varepsilon \to 0^+.\end{aligned}$$

Step 5. In order to estimate $\int_{-\infty}^{\xi} e^{-(\xi-t)} N_4(\varepsilon, t, \vartheta)\mathrm{d}t$, we first rewrite $N_4(\varepsilon, t, \vartheta)$ as follows:

$$N_4(\varepsilon, t, \vartheta) = \int_{\mathbb{R}} J(y)[\vartheta(t-\varepsilon y) - \vartheta(t)]\mathrm{d}y + \int_{\mathbb{R}} J(y)[u^\star(t-\varepsilon y) - u^\star(t)]\mathrm{d}y.$$

Similar to the previous discussion, we obtain

$$|u^\star(t-\varepsilon y) - u^\star(t)| \leq \varepsilon |y| \left\|\dot{u^\star}\right\|_C,$$

and

$$\left|\int_{\mathbb{R}} J(y)[u^\star(t-\varepsilon y) - u^\star(t)]\mathrm{d}y\right| \leq \varepsilon \overline{J} \left\|\dot{u^\star}\right\|_C.$$

Hence, we have the following:

$$\left|\int_{-\infty}^{\xi} e^{-(\xi-t)} \int_{\mathbb{R}} J(y)[u^\star(t-\varepsilon y) - u^\star(t)]\mathrm{d}y\mathrm{d}t\right| = O(\varepsilon) \ \text{ as } \varepsilon \to 0^+.$$

Next, we estimate $\left|\int_{-\infty}^{\xi} e^{-(\xi-t)} \int_{\mathbb{R}} J(y)[\vartheta(t-\varepsilon y) - \vartheta(t)]\mathrm{d}y\mathrm{d}t\right|$. By exchanging the order of integration and integration by parts, we have the following:

$$\begin{aligned}\left|\int_{-\infty}^{\xi} e^{-(\xi-t)} \int_{\mathbb{R}} J(y)[\vartheta(t-\varepsilon y) - \vartheta(t)]\mathrm{d}y\mathrm{d}t\right| =& \left|\int_0^1 \left[\int_{-\infty}^{\xi} \varepsilon e^{-(\xi-t)} \int_{\mathbb{R}} J(y) y \dot{\vartheta}(t-\sigma\varepsilon y)\mathrm{d}y\mathrm{d}t\right]\mathrm{d}\sigma\right| \\ \leq& \left|\int_0^1 \varepsilon \left[\int_{\mathbb{R}} J(y) y \vartheta(t-\sigma\varepsilon y)\mathrm{d}y\right]\mathrm{d}\sigma\right| \\ &+ \left|\int_0^1 \left[\int_{-\infty}^{\xi} \varepsilon e^{-(\xi-t)} \int_{\mathbb{R}} J(y) y \vartheta(t-\sigma\varepsilon y)\mathrm{d}y\mathrm{d}t\right]\mathrm{d}\sigma\right| \\ \leq& \varepsilon(1+K_1)\overline{J}\|\vartheta\|_C.\end{aligned}$$

Hence, we get the following:

$$\left|\int_{-\infty}^{\xi} e^{-(\xi-t)} N_4(\varepsilon, t, \vartheta)\mathrm{d}t\right| = O(\varepsilon) + O(\varepsilon\|\vartheta\|_C) \ \text{ as } \varepsilon \to 0^+.$$

Then, we have the following:

$$|\mathcal{H}(\varepsilon, \xi, \vartheta)| = O(\varepsilon) + O(\varepsilon\|\vartheta\|_C) + O(\|\vartheta\|_C^2) \ \text{ as } \varepsilon \to 0^+.$$

This completes the proof. □

5. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1 using the Banach fixed point theorem. Let X be the subspace in C_0 such that

$$C_0 = X \oplus \mathcal{N}(L).$$

It is well known that $X \subseteq C_0$ is a Banach space. Let $T = L|_X$ be the restriction of L to X. From Lemma 4.5, we know that $T : X \to C_0$ is one-to-one and onto. Based on the Banach inverse operator theorem, $T^{-1} : C_0 \to X$ is a bounded linear operator. Let $\|T^{-1}\| = \|T^{-1}\|_{\mathcal{L}(C_0,X)}$. For each $\vartheta \in C_0$, there are unique $\psi \in X$ and $\phi \in \mathcal{N}(L)$ such that $\vartheta = \psi + \phi$. It follows that ϑ is a solution of Eq (3.4) if and only if

$$L\psi = \mathcal{H}(\varepsilon, \cdot, \psi + \phi),$$

or, equivalently, if and only if ψ is a solution of the equation

$$\psi = T^{-1}\mathcal{H}(\varepsilon, \cdot, \psi + \phi). \tag{5.1}$$

From Lemma 4.6, we know that there are $\delta > 0$, $\varepsilon^* > 0$, and $0 < \varpi < 1$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and $\vartheta, \psi, \phi \in B(\delta) \subset C_0$,

$$\|\mathcal{H}(\varepsilon, \cdot, \vartheta)\|_C \leq \frac{1}{3\|T^{-1}\|}(\|\vartheta\|_C + \delta). \tag{5.2}$$

and

$$\|\mathcal{H}(\varepsilon, \cdot, \phi) - \mathcal{H}(\varepsilon, \cdot, \psi)\|_C \leq \frac{\varpi}{\|T^{-1}\|}\|\phi - \psi\|_C, \tag{5.3}$$

For each fixed $\phi \in \overline{B(\delta)} \cap \mathcal{N}(L)$, Inequality (5.2) implies that

$$\left\|T^{-1}\mathcal{H}(\varepsilon, \cdot, \psi + \phi)\right\|_C \leq \frac{1}{3}(\|\psi + \phi\|_C + \delta) \leq \delta, \quad \varepsilon \in (0, \varepsilon^*], \quad \psi \in X \cap \overline{B(\delta)}.$$

Hence, together with Inequality (5.3), we know that the mapping $\mathcal{S} : \left(X \cap \overline{B(\delta)}\right) \times (\mathcal{N}(L) \cap \overline{B(\delta)}\big) \times (0, \varepsilon^*) \to X \cap \overline{B(\delta)}$ given by

$$\mathcal{S}(\psi, \phi, \varepsilon) = T^{-1}\mathcal{H}(\varepsilon, \cdot, \psi + \phi),$$

is a uniform contraction mapping of $\psi \in X \cap \overline{B(\delta)}$. Therefore, for each $(\phi, \varepsilon) \in \left(\mathcal{N}(L) \cap \overline{B(\delta)}\right) \times (0, \varepsilon^*)$, there is a unique fixed point $\psi_{(\phi,\varepsilon)}$ of the mapping $\mathcal{S}(\cdot, \phi, \varepsilon)$. In other words, $\psi_{(\phi,\varepsilon)}$ is the unique solution in $X \cap \overline{B(\delta)}$ of Eq (5.1). Moreover, for fixed $\varepsilon \in (0, \varepsilon^*)$, $\vartheta_{(\phi,\varepsilon)} = \psi_{(\phi,\varepsilon)} + \phi$ is a solution of Eq (3.4). Hence, for any $c > c^* = \frac{1}{\varepsilon^*}$, System (1.1) has a traveling wavefront solution $u = u^\star + \vartheta$ that connects 0 to u_0^*. This completes the proof of Theorem 1.1.

Remark 5.1. *In fact, for the proof of Theorem 1, namely, the problem of solving operator equation (3.4), we can also use the Lyapunov-Schmidt reduction and the implicit function theorem to prove it. However, the discussion regarding the linear operator L and the nonlinear operator H requires additional properties. The detailed process of using the implicit function theorem to solve the operator equation can be referred to in [34].*

6. Numerical simulations

In order to verify our theoretical results, we perform numerical simulations by using an irregular finite difference scheme [35]. Consider the following one-dimensional Logistic growth model with nonlocal diffusion and a nonlocal delay effect:

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]\mathrm{d}y + ru(x,t)\left[1-\frac{1}{K}\int_{\mathbb{R}} B(y)u(x+y,t-\tau)\mathrm{d}y\right], \tag{6.1}$$

where r denotes the intrinsic growth rate, and K is the carrying capacity. Additionally, the corresponding reaction equation is as follows:

$$\dot{u}(\xi) = ru(\xi)\left[1-\frac{u(\xi-\tau)}{K}\right]. \tag{6.2}$$

To facilitate the numerical simulation, we choose the normal distribution kernel function for both nonlocal diffusion and nonlocal delays, i.e.,

$$G(x) \triangleq J(x) = B(x) = \frac{1}{\sqrt{2\pi}\alpha}e^{-\frac{x^2}{2\alpha^2}},$$

where α denotes the standard deviations of the population. It is obvious that System (6.1) admits two constant equilibria: $u_0 = 0$ and $u_* = K$. Then, by introducing the traveling wave variable $\xi = x + ct$, the traveling wavefront solution of System (6.1) satisfies the following boundary value problem:

$$\begin{cases} c\phi' - (G*\phi - \phi) = r\phi[1 - \frac{1}{K}(G*\phi_\tau)], \\ \phi(-\infty) = 0, \quad \phi(+\infty) = K, \end{cases} \tag{6.3}$$

where $G*\phi = \int_{\mathbb{R}} G(y)\phi(\xi-y)dy$, and $G*\phi_\tau = \int_{\mathbb{R}} G(y)\phi(\xi+y-c\tau)dy$.

For numerical simulations, a finite computational domain $(-L, L)$ for ξ (here we set $L = 40$) and a finite time interval $(0, T)$ are required, with L and T chosen sufficiently large. The numerical calculations are based on the discretized regularization equation, solved by the fixed-point iteration method, which globally converges to the unique fixed point via the Banach contraction mapping principle. The iteration is terminated when the norm $\|\phi^{n+1} - \phi^n\|_{C^1([-L,L])}$ is less than 10^{-6}. Moreover, we consider the following second-order differential problem with artificial viscosity terms $-\theta\phi''$ on both sides of the equation to ensure the stability of numerical simulation:

$$\begin{cases} c\phi' - \theta\phi'' + \phi = G*\tilde{\phi} + r\tilde{\phi}[1 - \frac{1}{K}(G*\tilde{\phi}_\tau)] - \theta\tilde{\phi}'', \\ \phi(-L) = 0, \quad \phi(L) = K, \end{cases} \tag{6.4}$$

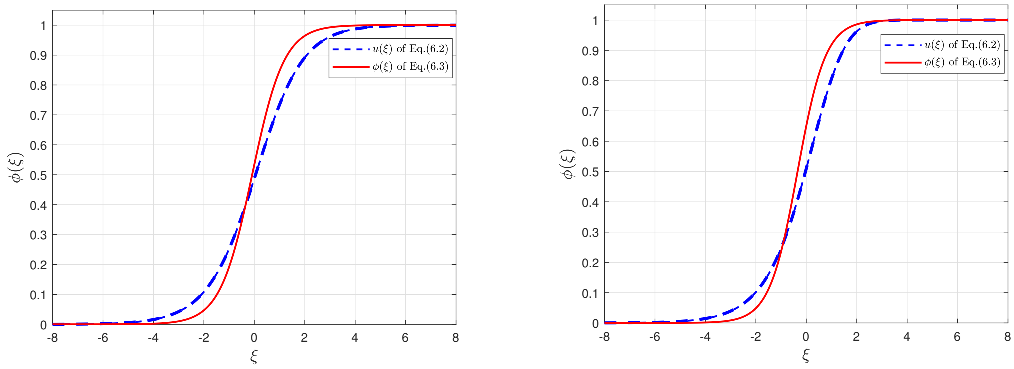
where $\tilde{\phi} \in C^2([-L, L])$, with $\tilde{\phi}(-L) = 0$, $\tilde{\phi}(L) = K$, and $\theta > 0$ is the regularization parameter.

The fixed-point iteration method is employed to solve Eq (6.4). More precisely, we start from a given initial solution $\tilde{\phi} = \frac{u_* e^{\lambda_* \xi}}{1+e^{\lambda_* \xi}}$, where λ_* is the critical eigenvalue that satisfies the following system with the critical speed c^*:

$$\begin{cases} c^*\lambda_* - \left(e^{\frac{\alpha^2\lambda_*{}^2}{2}} - 1\right) - r = 0, \\ c^* - \alpha^2\lambda_* e^{\frac{\alpha^2\lambda_*{}^2}{2}} = 0. \end{cases}$$

Substitute the initial solution $\tilde{\phi}$ into the right side of Eq (6.4) to obtain the next iterative solution ϕ. Moreover this procedure is repeated until $\tilde{\phi}$ and ϕ sufficiently converge, at which point the final ϕ is considered the approximate solution of Eq (6.3).

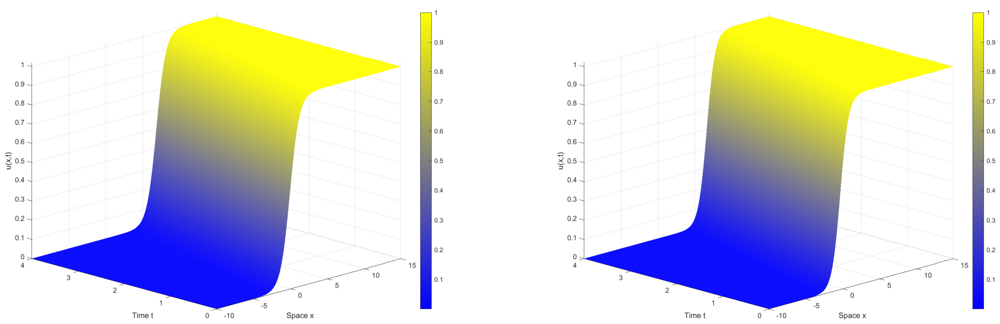
Based on typical biological backgrounds, we select the following parameters that satisfy the theoretical assumptions in Theorem 1.1. By letting $K = 1$, $r = 1$, $\theta = 0.05$, and $\alpha = 1$, we have the critical speed $c^* = 1.648721$ and the critical eigenvalue $\lambda_* = 1$. Choosing the wave speed $c = 1.731157 > c^*$ with different time delays $\tau = 0.1$ and $\tau = 0.4$, the numerical solutions of System (6.3) are shown by the red solid line in Figure 1 (a),(b) . Additionally, the blue dashed lines in Figure 1(a),(b) represent the heteroclinic solutions of Eq (6.2) with different time delays $\tau = 0.1$ and $\tau = 0.4$ that connect 0 to u_*.



(a) (b)

Figure 1. Solutions of Systems (6.2) and (6.3) with different delays: (a) $\tau = 0.1$ and (b) $\tau = 0.4$.

By translating back to the original variable, the three-dimensional numerical traveling wavefront solutions of System (6.1) with different delays $\tau = 0.1$ and $\tau = 0.4$ are shown in Figure 2.



(a) (b)

Figure 2. Traveling wavefronts of System (6.1) with different delays: (a) $\tau = 0.1$ and (b) $\tau = 0.4$.

7. Conclusions

In this paper, we studied the existence of traveling wavefront solutions with a sufficiently large wave speed for a nonlocal diffusion model with nonlocal delay effect by applying a perturbation method. The proof was completed by transforming the abstract formulation of the wave profile into an operator equation in a Banach space, combined with the Fredholm theory and the Banach contraction mapping principle. In addition, we also proved the existence of heteroclinic solutions of the reaction equation (1.2). Numerical simulations were provided to validate our results.

By observing the results of numerical simulations, it seems that traveling wavefront solutions are monotonic and unique; thus, it remains to be seen whether we can theoretically prove the monotonicity and uniqueness of traveling wavefront solutions. In addition, the stability of traveling wavefront solutions is a natural extension of the research on the existence of traveling wavefront solutions. In our future work, we will use the upper-lower solution method and comparison principle combined with the weighted energy method to conduct in-depth theoretical research on the monotonicity, uniqueness, and stability of traveling wavefront solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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