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Research article

An alternating direction implicit scheme with graded time steps for a fractional evolution equation with a weakly singular kernel

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Abstract: This paper addresses the numerical solution of a fractional evolution equation with a weakly singular kernel. The temporal discretization is carried out by the Crank–Nicolson (CN) method based on graded time steps, while the spatial discretization is carried out using an alternating direction implicit (ADI) finite difference method. Stability and convergence rate are also discussed. The convergence rate is $O(k^2 + h_x^2 + h_y^2)$, where k is a parameter of the maximal time steps and h_x, h_y are the uniform grid steps in space. Three numerical examples are employed to demonstrate the errors and convergence behavior in practice.

Keywords: alternating direction implicit method; fractional evolution equation; graded mesh; finite difference; Crank–Nicolson scheme

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1. Introduction

This work is devoted to the analysis of the following two-dimensional initial-boundary value problem:

$$u_t + \int_0^t \beta(t-s)Au(x,y,s)ds = f(x,y,t), \qquad (x,y) \in \Omega, \qquad 0 < t \le T, \tag{1.1}$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \qquad 0 < t \le T, \tag{1.2}$$

$$u(x,y,0) = v(x,y), \qquad (x,y) \in \Omega. \tag{1.3}$$

Here, $u_t = \frac{\partial u}{\partial t}$ denotes the partial derivative of u with respect to time. The spatial domain is given by $\Omega = (0,1)^2$, with $\partial\Omega$ representing its boundary, and A denotes a linear operator that is self-adjoint and positive definite. The kernel $\beta(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is positive definite and exhibits singular behavior, where α is a fractional integration operator with $\alpha > 0$. Such equations are commonly used to model wave propagation phenomena.

In recent years, fractional differential equations (FDEs) have been increasingly employed to model problems arising in various fields, including economics [1, 2], meteorology and biology [3], physics [4], and chemistry [5]. A wide range of numerical techniques has been developed to study this class of fractional convolution equations, such as finite difference methods [4–7], finite element methods [8–10], compact difference methods [11], spectral methods [12], and orthogonal spline collocation methods [13–15]. In the present paper, we investigate the numerical solution of a class of fractional evolution equations with a weakly singular kernel. An alternating direction implicit (ADI) finite difference method is employed for the spatial discretization, while a variable time-step Crank–Nicolson (CN) method is used for the temporal discretization.

Due to the singular behavior of fractional derivatives, the convergence order of the numerical solution may deteriorate when standard numerical methods are applied to discretize such fractional evolution equations. To overcome this difficulty, several researchers have focused on employing novel iterative algorithms or variable time-step meshes [16–18]. Nonuniform mesh methods have been widely used by many researchers to solve various types of FDEs. The study presented in [19] employed a generalized CN scheme for temporal discretization on a nonuniform grid, together with linear finite element approximations in space, to study a fractional wave equation. The approach proposed in [20] used a Newton linear approximation on graded time grids to discretize a time-fractional quasi-linear diffusion equation. As reported in [21], Qiu et al. applied the CN scheme, together with graded meshes for temporal discretization, to handle a three-dimensional nonlocal evolution equation. The approach proposed in [22] applied an ADI Galerkin method on nonuniform grids for a nonlocal heat model. Liao et al. [23] proposed an adaptive time step that effectively captures the initial singular behavior of solutions. Motivated by the above studies, which demonstrate that graded meshes can efficiently handle the weak singularities in fractional derivatives, we adopt graded meshes in the present paper. For the time discretization, the CN method is applied on graded meshes, and its performance is investigated. The resulting discrete scheme will show that the numerical solution obtained by the CN method on graded meshes attains full-order convergence.

It is well known that the high-dimensional fractional derivative problems are challenging to approximate numerically. While standard methods perform efficiently for one-dimensional models, their extension to multidimensional problems becomes computationally prohibitive due to excessive storage requirements. In order to solve this problem, some researchers have proposed ADI schemes for the high-dimensional issues, such as [24–27]. Li et al. [24] studied a two-dimensional fractional diffusion-wave equation using an ADI Galerkin finite element method. In previous work [25], Khebchareon proposed an ADI difference method for an evolution equation with a positive-type memory term. The backward Euler formula, the CN formula, and the second-order differentiation formula were combined with an ADI finite element Galerkin method to solve a two-dimensional temporal derivative equation [26]. As reported in [27], Chen et al. applied a backward Euler ADI method for a three-dimensional fractional evolution equation. To reduce storage requirements, we design an ADI finite difference scheme in space that transforms the original two-dimensional problem into a sequence of uncoupled one-dimensional problems, thereby significantly reducing the computational complexity.

The main contributions are as follows:

- Since the convolution-integral term in a fractional derivative involves the solution at all previous time levels, the resulting discrete scheme depends on numerical solutions over the entire temporal history. Consequently, the inner product inequalities arising in the stability and convergence analysis contain contributions from each time level. To address this issue, a specialized quadrature rule of product-integration type is employed to discretize the fractional evolution term, yielding a discrete scheme whose coefficients can be bounded by expressions involving the maximum time step.

- To simplify the stability and convergence analysis, upper bounds for the relevant coefficients and perturbation terms are derived in the preliminary section.
- By adopting an ADI finite difference scheme, the two-dimensional problem is reduced to a sequence of one-dimensional problems, thereby decreasing the computational complexity in space.

To the best of our knowledge, no existing study has employed an ADI finite difference scheme combined with graded meshes to discretize fractional convolution equations with weakly singular kernels, such as (1.1)–(1.3). Therefore, in this paper, we investigate the discretization of Eps (1.1)–(1.3) by applying the CN method on graded time steps together with an ADI difference scheme.

The remainder of this paper is organized as follows. Section 2 introduces the notation and several preliminary results. The fully discrete scheme is formulated in Section 3. Stability and convergence rates are derived from the fully discrete scheme in Section 4. Section 5 presents a set of numerical experiments that corroborate the theoretical findings. Finally, concluding remarks are given in Section 6.

2. Preliminaries

Before deriving the numerical scheme, some necessary notations are defined, and several supporting lemmas are presented. Let $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ be a bounded spatial domain. We define $x_i = ih_x$ and $y_j = jh_y$ $(0 \le i \le M_x,\ 0 \le j \le M_y)$, where i, j, M_x, and M_y are positive integers and h_x and h_y denote the uniform spatial grid sizes in the x-direction and y-direction, respectively. Thus, $h_x = 1/M_x$ and $h_y = 1/M_y$. The discrete spatial grid is defined as $\Omega_h = \{(x_i, y_j) | 1 \le i \le M_x - 1,\ 1 \le j \le M_y - 1\}$, and its boundary is given by $\partial\Omega_h = \{(x_i, y_j) | i = 0,\ or\ i = M_x,\ or\ j = 0,\ or\ j = M_y\}$. The full grid is denoted by $\bar{\Omega}_h = \Omega_h \cup \partial\Omega_h$.

We denote the grid function $V = \{V_{i,j}\}$, $W = \{W_{i,j}\}$ (where $V_{i,j} = V(x_i, y_j)$, $W_{i,j} = W(x_i, y_j)$, and $1 \le i \le M_x - 1,\ 1 \le j \le M_y - 1$), and $V_{i,j} = W_{i,j} = 0$ on the boundary $\partial\Omega_h$. We denote

$$\delta_x^2 V_{i,j} = \frac{1}{h_x^2}(V_{i+1,j} - 2V_{i,j} + V_{i-1,j}), \qquad \delta_y^2 V_{i,j} = \frac{1}{h_y^2}(V_{i,j+1} - 2V_{i,j} + V_{i,j-1}),$$

$$\triangle_x^+ V_{i,j} = V_{i+1,j} - V_{i,j}, \qquad \triangle_y^+ V_{i,j} = V_{i,j+1} - V_{i,j},$$

$$\langle V, W\rangle = h_x h_y \sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} V_{i,j} W_{i,j} \quad and \quad \|V\|^2 = \langle V, V\rangle.$$

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T$ be a sequence of temporal grids, with the corresponding time step $k_n = t_n - t_{n-1}$ $(n = 1, 2, \cdots, N)$, where N is a positive integer. In this paper, we adopt the following graded temporal meshes:

$$t_n = (nk')^\gamma, \qquad n \ge 0 \qquad and \qquad \gamma \ge 1, \tag{2.1}$$

where $k' = T/N$ and γ is a grading parameter. The motivation for choosing $t_n = (nk')^\gamma$ is discussed in [18]. It is easy to see that when $\gamma = 1$, the meshes are uniform meshes. When $\gamma > 1$, the time levels

t_n are increasingly clustered near $t = 0$ as γ increases. Thereby, the graded meshes compensate for the singular behaviour near the initial time. The following elementary inequalities show the relationship between k_n and t_n:

$$\frac{\gamma}{2^{\gamma-1}} \le \frac{k_n}{k' t_n^{1-1/\gamma}} \le \gamma \qquad \text{for } n \ge 2. \tag{2.2}$$

Denote $I_n = [t_{n-1}, t_n]$. To derive the discrete scheme using the CN method, we employ the following quadrature rule of product-integration type:

$$q_{n-1/2}(\varphi) = \frac{1}{k_n} \int_{I_n} \int_0^t \beta(t-s) \bar{\varphi}^C(s) ds dt = w_{n1} k_1 \varphi^1 + \sum_{j=2}^{n} w_{nj} k_j \varphi^{j-1/2}, \tag{2.3}$$

where

$$w_{nj} = \frac{1}{k_n k_j} \int_{I_n} \int_{t_{j-1}}^{min(t,t_j)} \beta(t-s) ds dt \tag{2.4}$$

and

$$\bar{\varphi}^C(s) = \begin{cases} \varphi^1, & (t_0 < t < t_1), \\ \varphi^{j-1/2}, & (t_{j-1} < t < t_j), \end{cases} \tag{2.5}$$

with $\varphi^{j-1/2} = (\varphi^{j-1} + \varphi^j)/2$ and φ^j denoting the value of $\varphi(s)$ at the point t_j.

The following lemmas will be needed for the theoretical analysis.

Lemma 2.1 ([[28], Lemma 2.1]). Let $V_{i,j}$ be an arbitrary grid function. Then we have

$$\langle \delta_x^2 V, W \rangle = -\frac{h_y}{h_x} \sum_{i=0}^{M_x-1} \sum_{j=1}^{M_y-1} (\triangle_x^+ V_{i,j})(\triangle_x^+ W_{i,j}), \tag{2.6}$$

$$\langle \delta_y^2 V, W \rangle = -\frac{h_x}{h_y} \sum_{i=1}^{M_x-1} \sum_{j=0}^{M_y-1} (\triangle_y^+ V_{i,j})(\triangle_y^+ W_{i,j}) \qquad \textit{and} \tag{2.7}$$

$$\langle \delta_x^2 \delta_y^2 V, W \rangle = \frac{1}{h_x h_y} \sum_{i=0}^{M_x-1} \sum_{j=0}^{M_y-1} (\triangle_x^+ \triangle_y^+ V_{i,j})(\triangle_x^+ \triangle_y^+ W_{i,j}). \tag{2.8}$$

Lemma 2.2 ([28]). Let $V_{i,j}((x_i, y_j) \in \bar{\Omega}_h,\ 1 \le n \le N)$ be an arbitrary grid function. Then, it holds that

$$\langle \delta_x^2 \delta_y^2 [\partial_t V^n], V^n \rangle \ge \frac{1}{2} \partial_t \langle \delta_x^2 \delta_y^2 V^n, V^n \rangle, \tag{2.9}$$

where $[\partial_t V^n] = \frac{V^n - V^{n-1}}{k_n}$.

Lemma 2.3 ([18]). If β is a positive-definite operator, $\beta \in L_1(0,T)$, and $q_{n-1/2}$ is defined in Eq (2.3), then

$$Q_{N-1/2}(\varphi) = \sum_{n=1}^{N} k_n q_{n-1/2}(\varphi) \varphi^{n-1/2} \ge 0.$$

Lemma 2.4. Let $T = 1$, and define $t_n = (nk')^\gamma$, where $\gamma \geq 1$ and $k' = \frac{1}{N}$. By choosing $\beta(t) = \frac{1}{\Gamma(1/2)} t^{-1/2}$, we obtain

$$\begin{aligned}&(1) \sum_{i=1}^{N-1} k_i^4 w_{ii}^2 = \sum_{i=1}^{N-1} \frac{16}{9\Gamma(1/2)^2} k_i^3, \qquad \sum_{i=1}^{N} k_i^4 w_{ii}^2 \leq Ck^2 \qquad \textit{and} \\ &(2) k_{i+1}^3 - k_i^3 \leq 1,\end{aligned} \tag{2.10}$$

where C is a constant.

Proof. (1) For the first equality, by applying the definition of w_{ii} and $\beta(t)$, we can conclude that

$$\begin{aligned}\sum_{i=1}^{N-1} k_i^4 w_{ii}^2 &= \sum_{i=1}^{N-1} k_i^4 \Big(\frac{1}{k_i k_i} \int_{I_i} \int_{t_{i-1}}^{min(t,t_i)} \beta(t-s) ds dt\Big)^2 \\ &= \frac{1}{\Gamma(1/2)^2} \sum_{i=1}^{N-1} \Big(\int_{I_i} \int_{t_{i-1}}^{min(t,t_i)} (t-s)^{-1/2} ds dt\Big)^2 \\ &= \sum_{i=1}^{N-1} \frac{16}{9\Gamma(1/2)^2} k_i^3.\end{aligned} \tag{2.11}$$

Using the above equality, we have

$$\begin{aligned}\sum_{i=1}^{N} k_i^4 w_{ii}^2 &= \frac{16}{9\Gamma(1/2)^2} \sum_{i=1}^{N} k_i k_i^2 \\ &\leq \frac{16}{9\Gamma(1/2)^2} t_N^{2-\frac{2}{\gamma}} k^2 \sum_{i=1}^{N} k_i \\ &\leq Ck^2.\end{aligned}$$

(2) The second inequality follows directly from the definition of k_i:

$$k_{i+1} - k_i = \Big(\frac{i+1}{N}\Big)^\gamma - \Big(\frac{i}{N}\Big)^\gamma - \Big(\frac{i}{N}\Big)^\gamma + \Big(\frac{i-1}{N}\Big)^\gamma.$$

Supposing $f(x) = (x+1)^\gamma - 2x^\gamma + (x-1)^\gamma = [(x+1)^\gamma - x^\gamma] - [x^\gamma - (x-1)^\gamma]$, we have $k_{i+1} - k_i = \frac{1}{N^\gamma} f(i)$. Letting $g(x) = (x+1)^\gamma - x^\gamma$ and differentiating the function g, we can get

$$g'(x) = \gamma[(x+1)^{\gamma-1} - x^{\gamma-1}] > 0 \qquad \textit{and} \qquad (x \geq 1,\ \gamma \geq 1).$$

Therefore, the function $g(x)$ is monotonically increasing for $x \geq 1$ and $\gamma \geq 1$. It then follows that $g(x) \geq g(x-1)$, which implies $f(x) = g(x) - g(x-1) \geq 0$. Accordingly, we obtain the following conclusions: $k_{i+1} - k_i \geq k_i - k_{i-1} \geq 0$ and $k_i^3 \leq k_{i+1}^3$. Moreover,

$$\begin{aligned}[k_{i+1}^3 - k_i^3] - [k_i^3 - k_{i-1}^3] &= (k_{i+1} - k_i)(k_{i+1}^2 + k_{i+1}k_i + k_i^2) - (k_i - k_{i-1})(k_i^2 + k_i k_{i-1} + k_{i-1}^2) \\ &\geq (k_i - k_{i-1})(k_{i+1}^2 + k_{i+1}k_i + k_i^2) - (k_i - k_{i-1})(k_i^2 + k_i k_{i-1} + k_{i-1}^2) \\ &= (k_i - k_{i-1})[(k_{i+1}^2 - k_i^2) + (k_{i+1}k_i - k_i k_{i-1}) + (k_i^2 - k_{i-1}^2)]\end{aligned}$$

$$\geq \ 0.$$

Thus, the sequence $k_{n+1}^3 - k_n^3$ is monotonically increasing. Therefore, we can draw the following conclusion:

$$k_{i+1}^3 - k_i^3 \leq k_N^3 - k_{N-1}^3 \leq k_N^3 = (t_N - t_{N-1})^3 = [1 - (1 - \frac{1}{N})^\gamma]^3 \leq 1.$$

□

Remark 1. In this paper, C denotes a positive constant that may take different values in different estimates and is independent of the spatial and temporal discretization parameters.

Remark 2. Throughout this paper, the continuity of $\frac{\partial^4 u}{\partial x^4}$, $\frac{\partial^4 u}{\partial y^4}$, $\frac{\partial^4 u}{\partial x^2 \partial y^2}$, $\frac{\partial u}{\partial t}$, and $\frac{\partial^2 u}{\partial t^2}$ is assumed in the domain $\Omega \times (0, T)$. The corresponding regularity results can be found in Reference [15, 28]:

$$\|u_{tt}(\cdot, t)\| \leq C, \qquad \|u_{xxyy}(\cdot, t)\| \leq C.$$

Lemma 2.5. Let $V_{i,j}$ be an arbitrary grid function and $V_{ij}^n = 0((x_i, y_j) \in \partial\Omega_h,\ 1 \leq n \leq N)$. Then, one has

$$\|\delta_x^2 \delta_y^2 V^n\| \leq C \frac{\|V^n\|}{h_x^2 h_y^2}.$$

Proof. By applying the definition of $\|\delta_x^2 \delta_y^2 V^n\|$ and using the boundary condition $V_{ij}^n = 0$ on $\partial\Omega_h$, we can conclude that

$$\begin{aligned}
\|\delta_x^2 \delta_y^2 V^n\|^2 &= \sum_{i=1}^{M_x} \sum_{j=1}^{M_y} h_x h_y (\delta_x^2 \delta_y^2 V_{ij}^n)(\delta_x^2 \delta_y^2 V_{ij}^n) \\
&= \sum_{i=1}^{M_x} \sum_{j=1}^{M_y} \frac{h_x h_y}{(h_x^2 h_y^2)^2} [(V_{i+1,j+1}^n - 2V_{i+1,j}^n + V_{i+1,j-1}^n) - 2(V_{i,j+1}^n - 2V_{i,j}^n + V_{i,j-1}^n) \\
&\quad + (V_{i-1,j+1}^n - 2V_{i-1,j}^n + V_{i-1,j-1}^n)]^2 \\
&\leq C \frac{\|V\|^2}{(h_x^2 h_y^2)^2}.
\end{aligned}$$

In the derivation of the above inequality, Young's inequality is employed after multiplying the corresponding terms. □

3. Discretization by CN scheme and ADI difference scheme

In this section, problems (1.1)–(1.3) are approximated by applying the CN scheme for temporal discretization and an ADI finite difference approach in the spatial direction. Following the procedure described earlier, both sides of Eq (1.1) are integrated over the interval $[t_{n-1}, t_n]$ and then divided by k_n. The spatial discretization is handled using the ADI finite difference scheme, while the integro-differential term is discretized by the quadrature rule (2.3). The set $\{U_{ij}^n | (x_j, y_j) \in \Omega_h\}$ represents the numerical approximation of Eqs (1.1)–(1.3) at the grid point (x_i, y_j, t_n). Omitting the error terms, the resulting discrete scheme is given by

$$\frac{U_{ij}^n - U_{ij}^{n-1}}{k_n} = w_{n1}k_1(\delta_x^2 + \delta_y^2)U_{ij}^1 + \sum_{p=2}^{n} k_p w_{np}(\delta_x^2 + \delta_y^2)U_{ij}^{p-1/2} + f_{ij}^{n-1/2}, \tag{3.1}$$

$$U_{ij}^n = 0,\ (x_i, y_j) \in \partial\Omega_h, \qquad 1 \le n \le N \qquad and \tag{3.2}$$

$$U_{ij}^0 = v(x_i, y_j) \in \bar{\Omega}_h, \tag{3.3}$$

where $f_{ij}^{n-1/2} = \frac{f_{ij}^{n-1}+f_{ij}^n}{2}$ and $U_{ij}^{n-1/2} = \frac{U_{ij}^{n-1}+U_{ij}^n}{2}$.

Set $E_{ij}^n = U_{ij}^n - U_{ij}^{n-1}$ and we can rewrite Eq (3.1) as

$$\begin{aligned} E_{ij}^n - \frac{1}{2}k_n^2 w_{nn}(\delta_x^2 + \delta_y^2)E_{ij}^n &= k_n k_1 w_{n1}(\delta_x^2 + \delta_y^2)U_{ij}^1 + k_n \sum_{p=2}^{n-1} k_p w_{np}(\delta_x^2 + \delta_y^2)U_{ij}^{p-1/2} \\ &\quad + k_n^2 w_{nn}(\delta_x^2 + \delta_y^2)U_{ij}^{n-1} + k_n f_{ij}^{n-1/2} \\ &\equiv F_{ij}^n. \end{aligned} \tag{3.4}$$

The discrete scheme given below is derived by incorporating the small term $\frac{k_n^4}{4}w_{nn}^2(\delta_x^2\delta_y^2)E_{ij}^n$ into the left-hand side of Eq (3.4):

$$E_{ij}^n - \frac{1}{2}k_n^2 w_{nn}(\delta_x^2 + \delta_y^2)E_{ij}^n + \frac{1}{4}k_n^4 w_{nn}^2(\delta_x^2\delta_y^2)E_{ij}^n = F_{ij}^n, \tag{3.5}$$

$$U_{ij}^n = 0,\ (x_i, y_j) \in \partial\Omega_h, \qquad 1 \le n \le N \qquad and \tag{3.6}$$

$$U_{ij}^0 = v(x_i, y_j) \in \bar{\Omega}_h. \tag{3.7}$$

Suppose that I is the identical operator, and we can rewrite the above scheme as follows:

$$(I - \frac{1}{2}k_n^2 w_{nn}\delta_x^2)(I - \frac{1}{2}k_n^2 w_{nn}\delta_y^2)E_{ij}^n = F_{ij}^n.$$

If we denote

$$E_{ij}^* = (I - \frac{1}{2}k_n^2 w_{nn}\delta_y^2)E_{ij}^n, \qquad 0 \le i \le M_x - 1, \tag{3.8}$$

then the original problem is decomposed into two decoupled one-dimensional problems. The first one is Eq (3.8), which is associated with the x direction. The second one is given by

$$(I - \frac{1}{2}k_n^2 w_{nn}\delta_x^2)E_{ij}^* = F_{ij}^n, \qquad 1 \le j \le M_y - 1. \tag{3.9}$$

It is easy to see that this equation corresponds to the y direction. First, we can use Eq (3.9) to obtain the solution of E_{ij}^* at each time level I_n $(1 \le n \le N)$. Then we can use Eq (3.8) to get the solution of E_{ij}^n. Last, we can get the numerical solution $\{U_{ij}^n|(x_j, y_j) \in \Omega_h\}$ by using the definition of E_{ij}^n.

The stability analysis and the study of convergence rates are carried out in the following section.

4. Theory analysis for the ADI CN scheme

4.1. Stability analysis

We establish the following stability result for the ADI CN scheme.

Theorem 4.1. Let $\{U_{ij}^n | (x_i, y_j) \in \Omega_h, 0 \le n \le N\}$ be the numerical solutions of Eqs (3.5)–(3.7), and β is a positive-definite operator. Then it holds that

$$\|U^N\| \le \|U^0\| + 2\sum_{n=1}^{N} k_n \|f^{n-1/2}\| + \frac{4}{9\Gamma(1/2)^2}\sum_{i=1}^{N-1}(k_{i+1}^3 - k_i^3)\|\delta_x^2\delta_y^2 U^i\| + \frac{4}{9\Gamma(1/2)^2} k_1^3 \|\delta_x^2\delta_y^2 U^0\|. \tag{4.1}$$

Proof. Note that Eq (3.5) can be rewritten as

$$U_{ij}^n - U_{ij}^{n-1} + \frac{1}{4}k_n^5 w_{nn}^2(\delta_x^2\delta_y^2)[\partial_t U_{ij}^n] = w_{n1}k_1k_n(\delta_x^2 + \delta_y^2)U_{ij}^1 + k_n\sum_{p=2}^{n-1} k_p w_{np}(\delta_x^2 + \delta_y^2)U_{ij}^{p-1/2} + k_n f_{ij}^{n-1/2}. \tag{4.2}$$

Applying a summation over all interior grid points after weighting the above equation by $h_x h_y(U_{ij}^n + U_{ij}^{n-1})$, one obtains

$$\begin{aligned}\|U^n\|^2 - \|U^{n-1}\|^2 + \frac{1}{4}k_n^4 w_{nn}^2\langle(\delta_x^2\delta_y^2)(U^n - U^{n-1}), U^n + U^{n-1}\rangle &= w_{n1}k_1k_n\langle(\delta_x^2 + \delta_y^2)U^1, U^n + U^{n-1}\rangle \\ &+ k_n\sum_{p=2}^{n-1} k_p w_{np}\langle(\delta_x^2 + \delta_y^2)\frac{U^{p-1} + U^p}{2}, U^n + U^{n-1}\rangle + k_n\langle f^{n-1/2}, U^n + U^{n-1}\rangle,\end{aligned}$$

i.e.,

$$\begin{aligned}\|U^n\|^2 - \|U^{n-1}\|^2 \quad + \quad &\frac{1}{4}k_n^4 w_{nn}^2\langle(\delta_x^2\delta_y^2)(U^n - U^{n-1}), U^n + U^{n-1}\rangle = k_n\langle q_{n-1/2}((\delta_x^2 + \delta_y^2)U), U^{n-1/2}\rangle \\ &+ k_n\langle f^{n-1/2}, U^{n-1/2}\rangle.\end{aligned} \tag{4.3}$$

Using the discrete Green formula, we have

$$\langle(\delta_x^2\delta_y^2)(U^n - U^{n-1}), U^n + U^{n-1}\rangle = \langle(\delta_x\delta_y)(U^n - U^{n-1}), (\delta_x\delta_y)(U^n + U^{n-1})\rangle = \|\delta_x\delta_y U^n\|^2 - \|\delta_x\delta_y U^{n-1}\|^2. \tag{4.4}$$

Substituting Eq (4.4) into Eq (4.3) and summing n from 1 to N, Eq (4.3) can be rewritten as

$$\begin{aligned}\|U^N\|^2 - \|U^0\|^2 \quad + \quad &\frac{1}{4}\sum_{n=1}^{N} k_n^4 w_{nn}^2(\|\delta_x\delta_y U^n\|^2 - \|\delta_x\delta_y U^{n-1}\|^2) = \sum_{n=1}^{N} k_n\langle q_{n-1/2}((\delta_x^2 + \delta_y^2)U, U^{n-1/2}\rangle \\ &+ \sum_{n=1}^{N} k_n\langle f^{n-1/2}, U^{n-1/2}\rangle.\end{aligned} \tag{4.5}$$

The third term on the left-hand side of the above equation is equal to

$$\sum_{n=1}^{N} k_n^4 w_{nn}^2(\|\delta_x\delta_y U^n\|^2 - \|\delta_x\delta_y U^{n-1}\|^2) \quad = \quad k_N^4 w_{NN}^2\|\delta_x\delta_y U^N\|^2 + \sum_{n=1}^{N-1}(k_i^4 w_{ii}^2 - k_{i+1}^4 w_{i+1,i+1}^2)\|\delta_x\delta_y U^i\|^2$$

$$-k_1^4 w_{11}^2 \|\delta_x \delta_y U^0\|^2. \tag{4.6}$$

Combining Eqs (4.5) and (4.6), we can obtain

$$\begin{aligned}\|U^N\|^2 - \|U^0\|^2 + \frac{k_N^4}{4} w_{NN}^2 \|\delta_x\delta_y U^N\|^2 &= \frac{1}{4}\sum_{n=1}^{N-1}(k_{i+1}^4 w_{i+1,i+1}^2 - k_i^4 w_{ii}^2)\|\delta_x\delta_y U^i\|^2 + \frac{1}{4}k_1^4 w_{11}^2\|\delta_x\delta_y U^0\|^2\\ &+ \sum_{n=1}^{N} k_n\langle q_{n-1/2}((\delta_x^2+\delta_y^2)U, U^{n-1/2}\rangle + \sum_{n=1}^{N} k_n\langle f^{n-1/2}, U^{n-1/2}\rangle. \end{aligned}\tag{4.7}$$

From Lemma 2.3, it follows that $Q_{N-1/2}(\varphi) = \sum_{n=1}^{N} k_n q_{n-1/2}(\varphi)\varphi^{n-1/2} \geq 0$. Applying Lemma 2.1, we obtain

$$\begin{aligned}\sum_{n=1}^{N} k_n\langle q_{n-1/2}((\delta_x^2+\delta_y^2)U), U^{n-1/2}\rangle &= \sum_{n=1}^{N} k_n \sum_{p=1}^{n} k_p w_{np}\langle(\delta_x^2+\delta_y^2)U^{p-1/2}, U^{n-1/2}\rangle\\ &= -\sum_{n=1}^{N} k_n \sum_{p=1}^{n} k_p w_{np}\Big[\sum_{i=0}^{M_x-1}\sum_{j=1}^{M_y-1}\frac{h_y}{h_x}(\triangle x^+ U_{ij}^{p-1/2})(\triangle x^+ U_{ij}^{n-1/2})\\ &\quad + \sum_{i=1}^{M_x-1}\sum_{j=0}^{M_y-1}\frac{h_x}{h_y}(\triangle y^+ U_{ij}^{p-1/2})(\triangle y^+ U_{ij}^{n-1/2})\Big]\\ &= -\sum_{i=0}^{M_x-1}\sum_{j=1}^{M_y-1}\frac{h_y}{h_x}Q_{N-1/2}(\triangle x^+ U_{ij}) - \sum_{i=1}^{M_x-1}\sum_{j=0}^{M_y-1}\frac{h_x}{h_y}Q_{N-1/2}(\triangle y^+ U_{ij})\\ &\leq 0. \end{aligned}\tag{4.8}$$

Due to Lemma 2.4, it is easy to see that $k_{i+1}^4 w_{i+1,i+1}^2 - k_i^4 w_{ii}^2 = \frac{16}{9\Gamma(1/2)^2}(k_{i+1}^3 - k_i^3)$. Substituting Eq (4.8) into Eq (4.7) and using $\|\delta_x\delta_y U^i\|^2 = \langle \delta_x^2\delta_y^2 U^i, U^i\rangle$, we can obtain

$$\begin{aligned}\|U^N\|^2 \quad - \quad \|U^0\|^2 &\leq \frac{1}{4}\frac{16}{9\Gamma(1/2)^2}\sum_{n=1}^{N-1}(k_{i+1}^3 - k_i^3)\|\delta_x^2\delta_y^2 U^i\|\cdot\|U^i\|\\ &+\frac{1}{4}k_1^4 w_{11}^2\|\delta_x^2\delta_y^2 U^0\|\cdot\|U^0\| + \sum_{n=1}^{N} k_n\|f^{n-1/2}\|\|U^{n-1/2}\|.\end{aligned}$$

So,

$$\|U^N\| \leq \|U^M\| \leq \|U^0\| + \frac{4}{9\Gamma(1/2)^2}\sum_{n=1}^{N-1}(k_{i+1}^3 - k_i^3)\|\delta_x^2\delta_y^2 U^i\| + \frac{4}{9\Gamma(1/2)^2}k_1^3\|\delta_x^2\delta_y^2 U^0\| + 2\sum_{n=1}^{N} k_n\|f^{n-1/2}\|,$$

which completes the proof. □

Theorem 4.2. If we denote $\{U_{ij}^n | (x_i, y_j) \in \Omega_h, 0 \leq n \leq N\}$ as the set of numerical solutions of Eqs (3.5)–(3.7), supposing that $\frac{1}{h_x h_y} \leq C'$ (where C' is a constant), then it holds that

$$\|U^N\| \leq C(\|U^0\| + \sum_{n=1}^{N} k_n\|f^{n-1/2}\| + k_1^3\|\delta_x^2\delta_y^2 U^0\|).$$

Proof. From Lemma 2.4, it follows that $\sum_{i=1}^{N-1}(k_{i+1}^3 - k_i^3)\|\delta_x^2\delta_y^2U^i\| \leq \sum_{i=1}^{N-1}\|\delta_x^2\delta_y^2U^i\|$. Applying Lemma 2.5 and the discrete Grönwall inequality, Theorem 4.1 can be bounded as follows:

$$\begin{aligned}
\|U^N\| &\leq \|U^0\| + 2\sum_{n=1}^{N}k_n\|f^{n-1/2}\| + \frac{4}{9\Gamma(1/2)^2}\sum_{i=1}^{N-1}\|\delta_x^2\delta_y^2U^i\| + \frac{4}{9\Gamma(1/2)^2}k_1^3\|\delta_x^2\delta_y^2U^0\| \\
&\leq \|U^0\| + 2\sum_{n=1}^{N}k_n\|f^{n-1/2}\| + \frac{4}{9\Gamma(1/2)^2}k_1^3\|\delta_x^2\delta_y^2U^0\| + \frac{4}{9\Gamma(1/2)^2}\sum_{i=1}^{N-1}\frac{\|U^i\|}{h_x^2h_y^2} \\
&\leq \|U^0\| + 2\sum_{n=1}^{N}k_n\|f^{n-1/2}\| + \frac{4}{9\Gamma(1/2)^2}k_1^3\|\delta_x^2\delta_y^2U^0\| + \frac{4}{9\Gamma(1/2)^2}C'^2\sum_{i=1}^{N-1}\|U^i\| \\
&\leq \|U^0\| + 2\sum_{n=1}^{N}k_n\|f^{n-1/2}\| + \frac{4}{9\Gamma(1/2)^2}k_1^3\|\delta_x^2\delta_y^2U^0\| + C\sum_{i=1}^{N-1}\|U^i\| \\
&\leq C(\|U^0\| + \sum_{n=1}^{N}k_n\|f^{n-1/2}\| + k_1^3\|\delta_x^2\delta_y^2U^0\|),
\end{aligned}$$

which completes the proof. □

4.2. Convergence analysis

Theorem 4.3. Suppose that u_{ij}^n is the exact solution of problems (1.1)–(1.3) and U_{ij}^n is the numerical solution of Eqs (3.5)–(3.7) at point (x_i, y_j, t_n). Denote that $e_{ij}^n = U_{ij}^n - u_{ij}^n$. Assume that the exact solution satisfies

$$t\|Au'(t)\| + t^2\|Au''(t)\| \leq Ct^{\sigma-1} \qquad \text{and} \qquad t\|f'(t)\| + t^2\|f''(t)\| \leq Ct^{\sigma-1} \qquad \text{for } 0 < t < T,$$

where σ is a positive constant for $t > 0$. If $\frac{1}{h_xh_y} \leq C'$, then as k, h_x, and h_y tend to be zero independently, we have

$$\|e^N\| \leq C\|e^0\| + Ck_1^3\|\delta_x^2\delta_y^2e^0\| + C(h_x^2 + h_y^2) + C_{\gamma,\sigma,T}M\begin{cases} k^{\gamma\sigma} & \text{if } 1 \leq \gamma < 2/\sigma, \\ k^2log(t_n/t_1) & \text{if } \gamma = 2/\sigma, \\ k^2 & \text{if } \gamma \geq 2/\sigma, \end{cases} \tag{4.9}$$

where C is a generic constant independent of k, h_x, and h_y and $C_{\gamma,\sigma,T}$ is a constant that depends on γ, σ, and T. Here, $k = \max_n k_n$ denotes the maximum graded time steps.

Proof. The discrete scheme (3.5) can be rewritten in the following form:

$$U_{ij}^n - U_{ij}^{n-1} + \frac{1}{4}k_n^4w_{nn}^2(\delta_x^2\delta_y^2)(U_{ij}^n - U_{ij}^{n-1}) = k_1k_nw_{11}(\delta_x^2 + \delta_y^2)U_{ij}^1 + k_n\sum_{p=2}^{n}k_pw_{np}(\delta_x^2 + \delta_y^2)U_{ij}^{p-1/2} + k_nf_{ij}^{n-1/2}.$$

Meanwhile, the exact solution u_{ij}^n of Eqs (1.1)–(1.3) satisfies

$$u_{ij}^n - u_{ij}^{n-1} - \int_{t_{n-1}}^{t_n}\int_0^t\beta(t-s)\Delta u(x_i, y_j, s)dsdt = \int_{t_{n-1}}^{t_n}f(x_i, y_j, t)dt.$$

By comparing the above two equations, we can obtain the relationship between them:

$$\begin{aligned} e_{ij}^{n}-e_{ij}^{n-1} &+ \frac{1}{4}k_n^4 w_{nn}^2(\delta_x^2\delta_y^2)(e_{ij}^n-e_{ij}^{n-1})-\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)[(\delta_x^2+\delta_y^2)\bar{U}_{ij}^C(s)-(\delta_x^2+\delta_y^2)\bar{u}_{ij}^C(s)]dsdt \\ &= k_n\eta^{n-1/2}, \end{aligned} \tag{4.10}$$

where

$$\bar{\varphi}_{ij}^C(s)=\begin{cases}\varphi_{ij}^1, & (t_0<t<t_1),\\ \varphi_{ij}^{j-1/2}, & (t_{j-1}<t<t_j)\end{cases}$$

and $\eta^{n-1/2}=\eta_1^{n-1/2}+\eta_2^{n-1/2}+\eta_3^{n-1/2}+\eta_4^{n-1/2}$, with

$$\eta_1^{n-1/2}=f_{ij}^{n-1/2}-\frac{1}{k_n}\int_{t_{n-1}}^{t_n} f_{ij}(t)dt,$$

$$\eta_2^{n-1/2}=-\frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)[\Delta u_{ij}(s)-\Delta\bar{u}_{ij}^C(s)]dsdt,$$

$$\eta_3^{n-1/2}=\frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)[(\delta_x^2+\delta_y^2)\bar{u}_{ij}^C(s)-\Delta\bar{u}_{ij}^C(s)]dsdt,$$

$$\eta_4^{n-1/2}=-\frac{1}{4}k_n^3 w_{nn}^2(\delta_x^2\delta_y^2)(u_{ij}^n-u_{ij}^{n-1}).$$

With the help of Theorem 4.2 and the initial condition $e^0=0$, Eq (4.10) has a similar structure as Eq (4.2). Therefore, we can get the following inequality:

$$\|e^N\|\le C(\|e^0\|+\sum_{n=1}^{N}k_n\|\eta^{n-1/2}\|+k_1^3\|\delta_x^2\delta_y^2 e^0\|). \tag{4.11}$$

Next, we consider the magnitude of the error $\eta^{n-1/2}$. Since

$$\begin{aligned} |k_n^5 w_{nn}^2(\delta_x^2\delta_y^2)[\partial_t u^n]| &\le |k_n^4 w_{nn}^2(\delta_x^2\delta_y^2)u^n|+|k_n^4 w_{nn}^2(\delta_x^2\delta_y^2)u^{n-1}| \\ &\le Ck_n^4 w_{nn}^2|u_{xxyy}(\zeta,\eta,t)| \\ &\le Ck_n^4 w_{nn}^2, \end{aligned}$$

we have $\sum_{n=1}^N k_n\|\eta_4^{n-1/2}\|\le Ck_n^4 w_{nn}^2\le Ck^2$. Consider the term of $\eta_3^{n-1/2}$. According to the error analysis of the finite difference method, the error of the following expression is given:

$$|(\delta_x^2+\delta_y^2)\bar{u}_{ij}^C-\Delta\bar{u}_{ij}^C|\le C(h_x^2+h_y^2),$$

so we can get the following inequality:

$$\|\eta_3^{n-1/2}\|\le C(h_x^2+h_y^2)\frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)dsdt\le C(h_x^2+h_y^2)\|\beta\|_{L_1(I_n)}. \tag{4.12}$$

Applying the result from the demonstration of Corollary 3.4 in [19], we have

$$\|\eta_1^{n-1/2}\| \le C_{\sigma,\gamma} M k^{\gamma\sigma} + C_\gamma M k^2 t_j^{\sigma-2/\gamma-1} k_j, \tag{4.13}$$

where $C_{\sigma,\gamma}$ is a constant independent of σ and γ and C_γ is a constant independent of γ. It is easy to see that $\|\eta_2^{n-1/2}\| \le \|\eta_{21}^{n-1/2}\| + \|\eta_{22}^{n-1/2}\|$ according to [18], where

$$\eta_{21}^{n-1/2} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)[Au_{ij}(s) - A\check{u}(s)]dsdt,\ \eta_{22}^{n-1/2} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)[A\check{u}(s) - A\bar{u}_{ij}(s)]dsdt$$

and

$$\check{u}(s) = \begin{cases} u(t_1), & (t_0 < t < t_1), \\ k_i^{-1}[(t_i - t)u(t_{i-1}) + (t - t_{i-1})u(t_i), & (t_{i-1} < t < t_i), \qquad \text{and } i \ge 2. \end{cases}$$

The norms of η_{21} and η_{22} satisfy the following inequalities, which are described in [19]:

$$\sum_{n=1}^{N} k_n \|\eta_{21}^{n-1/2}\| \le C t_N \Big(\int_0^{t_1} t\|Au^{'}(t)\|dt + \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \|Au^{''}(t)\|dt\Big) \quad \textit{and} \tag{4.14}$$

$$\sum_{n=1}^{N} k_n \|\eta_{22}^{n-1/2}\| \le C_T\Big(k_2 \int_{t_1}^{t_2} \|Au^{'}(t)\|dt + \sum_{j=2}^{N-1} \delta_j\Big), \tag{4.15}$$

where C_T is a constant independent of T and

$$\begin{aligned} \delta_j &= \|k_j A\Delta u^j - k_{j+1}A\Delta u^{j+1}\| + \|k_j A\Delta u^{j+1} - k_{j+1}A\Delta u^j\| + (k_{j+1} - k_j)\|A\Delta u^j\| + k_{j+1}^{-1}\|k_{j+1}^2 A\Delta u^{j+1} - k_j^2 A\Delta u^j\| \\ &\le 3k_j^2 \int_{t_{j-1}}^{t_{j+1}} \|Au^{''}(t)\|dt + 6k_j \int_{t_{j-1}}^{t_j^*} \|Au^{'}(t)\|dt + 3(k_{j+1} - k_j)\int_{t_{j-1}}^{t_{j+1}} \|Au^{'}(t)\|dt. \end{aligned} \tag{4.16}$$

Here, t_j^* denotes the midpoint of the interval $[t_{j-1}, t_{j+1}]$, and $\Delta u^j = u(t_j) - u(t_{j-1})$. By applying the mesh assumption, it follows that

$$\int_0^{t_1} t\|Au^{'}(t)\|dt + k_2 \int_{t_1}^{t_2} \|Au^{'}(t)\|dt \le C_{\sigma,\gamma} M k^{\gamma\sigma} \tag{4.17}$$

and

$$k_j^2 \int_{t_{j-1}}^{t_{j+1}} \|Au^{''}(t)\|dt + k_j \int_{t_{j-1}}^{t_j^*} \|Au^{'}(t)\|dt + (k_{j+1} - k_j)\int_{t_{j-1}}^{t_{j+1}} \|Au^{'}(t)\|dt \le C_{\sigma,\gamma} M k^2 t_j^{\sigma-2/\gamma-1} k_j. \tag{4.18}$$

The sum is estimated by

$$\sum_{j=2}^{n} t_j^{\sigma-2/\gamma-1} k_j \le C_{\gamma,\sigma} \begin{cases} t_1^{\sigma-2/\gamma}/(2/\gamma - \sigma) & \text{if } \gamma < 2/\sigma, \\ log(t_n/t_1) & \text{if } \gamma = 2/\sigma, \\ t_n^{\sigma-2/\gamma}/(\sigma - 2/\gamma) & \text{if } \gamma \ge 2/\sigma, \end{cases} \qquad t_1^{\sigma-2/\gamma}/(2/\gamma - \sigma) \le C_{\sigma,\gamma} k^{\gamma\sigma-2}. \tag{4.19}$$

Substituting inequalities (4.12)–(4.19) into (4.11) and using the boundedness of $\|\beta\|_{L_1(0,t_N)}$, we can obtain the convergence result. □

5. Numerical experiment

Numerical simulations of three examples are presented in this section. The L_2 errors and the corresponding convergence rates are computed. Examples 1 and 2 are selected from Reference [28]. In both examples, the CN scheme on graded meshes $t_n = (nk')^\gamma$ ($k' = T/N$) is employed. In this paper, we set $\beta(t) = \frac{t^{-1/2}}{\Gamma(1/2)}$, $T = 1$, and $\Omega = (0,1) \times (0,1)$.

In the numerical implementation of the ADI finite difference scheme, a uniform spatial mesh with grid sizes $h_x = h_y = h$ is employed. The L_2 errors and the corresponding convergence rates are then calculated using the following formulas:

$$e(k',h) = \max_{0\le m\le N} \|U^m - u^m\| \qquad \textit{and}$$

$$\text{Convergence} \quad \text{rate } \epsilon_1 \approx \frac{\log(e(2k',h)/e(k',h))}{\log(2k'/k')} \qquad \textit{and} \qquad \epsilon_2 \approx \frac{\log(e(k',2h)/e(k',h))}{\log(2h/h)}.$$

The focus of this paper is twofold. First, the ADI scheme is employed to decompose the discrete system into two one-dimensional problems, thereby reducing the computational complexity. Second, a variable time-step method is adopted to decrease the weak singularity in the temporal direction and to achieve full-order accuracy. Consequently, in the numerical experiments, in order to avoid redundancy, the spatial convergence order is reported only for Example 1. For the remaining two examples, the spatial convergence behavior is omitted, and only the numerical results in the time direction are presented.

Example 1. We focus on the following nonhomogeneous problem:

$$u_t - \frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}\Delta u(x,y,s)ds = f(x,y,t), \qquad 0 < x,y < 1, \qquad 0 \le t \le T,$$

$$u(0,y,t)) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0, \qquad 0 < t < T \qquad \textit{and}$$

$$u(x,y,0) = \sin(\pi x)\sin(\pi y), \qquad 0 \le x,y \le 1.$$

The function f is chosen as $f(x,y,t) = \frac{2t^{1/2}}{\sqrt{\pi}}(2\pi^2\sin(\pi x)\sin(\pi y) - \sin(2\pi x)\sin(2\pi y)) - 4\pi^2t^2\sin(2\pi x)\sin(2\pi y)$, so that the exact solution is given by

$$u(x,y,t) = \sin(\pi x)\sin(\pi y) - \frac{4t^{3/2}}{3\sqrt{\pi}}\sin(2\pi x)\sin(2\pi y).$$

Thus, for $t > 0$, the exact solution satisfies

$$t\|Au'(t)\| + t^2\|Au''(t)\| \le Ct^{3/2} \qquad \text{and} \qquad t\|f'(t)\| + t^2\|f''(t)\| \le Ct^{1/2}.$$

Considering the CN scheme, the parameter σ in Theorem 4.3 is chosen to be $\sigma = 3/2$. According to the theoretical analysis in Theorem 4.3, the following results are obtained: For $1 \le \gamma = 1 < 2/\sigma$, the convergence rate is $O(k^{3/2})$. For $\gamma = 2/\sigma = 4/3$, the convergence rate is $O(k^2)$. For $\gamma = 3 \ge 2/\sigma$, the mesh is over-graded, and the convergence rate is $O(k^2)$.

To compute the convergence rate in a temporal direction, we choose the spatial step h to be sufficiently small. Similarly, a sufficiently small time step is used when computing spatial convergence rates. Table 1 displays the errors and convergence rates ϵ_1 in the temporal direction at $h = 1/500$ by choosing $\gamma = 1,\ \gamma = 4/3$, and $\gamma = 3$, respectively. In Table 1, the numerical results show that the convergence rates are $O(k^{3/2})$ and $O(k^2)$ when $\gamma = 1$ and $\gamma = 4/3$, respectively, and the numerical results show that the convergence rate is over $O(k^2)$ when $\gamma = 3$. These data show that the numerical solution matches the theoretical analysis. By comparison with the numerical results reported in Reference [28], we note that the discretization in Reference [28] is based on uniform meshes. Consequently, when $h = 1/500$ and $N = 20,\ 40,\ 80,\ 160$, the observed convergence rates are 1.1656, 1.0395, and 1.0330. However, by employing graded meshes in this paper, the convergence order reaches 2 or even exceeds 2. In Reference [28], the minimum L_2 error is reported as $8.7962e-04$ when $N = 160$, whereas in the present paper, the minimum L_2 error is reduced to $7.96951e-04$ for the same value of N when $\gamma = 3$.

We observe that the convergence rates also match well with the theoretical convergence rates in the spatial direction in Table 2. In Table 2, we display the errors and convergence rates ϵ_2 in the spatial direction at $k = 1/1000$. By using the finite difference method, the convergence order in space is 2.

To evaluate the efficiency of the difference scheme, numerical results for the homogeneous problem are also presented.

Table 1. Errors and convergence rates in time when $\gamma = 1,\ \gamma = 4/3,$ *and* $\gamma = 3$ with $h = 1/500$ for Example 1.

	$\gamma = 1$		$\gamma = 4/3$		$\gamma = 3$	
N	*Error*	ϵ_1	*Error*	ϵ_1	*Error*	ϵ_1
20	1.77667e-02	–	4.14236e-02	–	9.41196e-03	–
40	6.93881e-03	1.35641	1.10280e-02	1.90928	1.39877e-03	2.75034
80	2.63213e-03	1.39846	2.81090e-03	1.97207	2.21484e-04	2.65888
160	9.77503e-04	1.42906	7.15204e-04	1.97461	3.77838e-05	2.55137

Table 2. Errors and convergence rates in space when $\gamma = 1,\ \gamma = 4/3,$ *and* $\gamma = 3$ with $k = 1/1000$ for Example 1.

	$\gamma = 1$		$\gamma = 4/3$		$\gamma = 3$	
M	*Error*	ϵ_2	*Error*	ϵ_2	*Error*	ϵ_2
5	5.51169e-02	–	5.51192e-02	–	5.51127e-02	–
10	1.30396e-02	2.07959	1.30413e-02	2.07947	1.30309e-02	2.08045
20	3.21348e-03	2.02070	3.21565e-03	2.01990	3.20909e-03	2.02170
40	7.98002e-04	2.00967	8.00557e-04	2.00603	7.96951e-04	2.00960

Example 2.

$$u_t = \frac{1}{2}\int_0^t (t-s)^{-1/2}\Delta u(x,y,s)ds, \qquad 0 < x,y < 1, \qquad 0 < t < T,$$

$$u(0,y,t)) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0, \qquad 0 < t \le T \qquad and$$

$$u(x,y,0) = \sin(\pi x)\sin(\pi y), \qquad 0 \le x,y \le 1.$$

The exact solution of this problem is $u(x,t) = M(\pi^{5/2}t^{3/2})\sin(\pi x)\sin(\pi y)$, where M denotes the entire function

$$M(z) = \sum_{n=0}^{\infty}(-1)^n\Gamma(\frac{3}{2}n+1)^{-1}z^n.$$

In Table 3, we present the maximal numbers of L_2 errors and convergence orders in the temporal direction for the CN scheme on graded time steps. Considering the data in Table 3, we can see the convergence rates are $O(k^{3/2})$ and $O(k^2)$ when $\gamma = 1$ and $\gamma \geq 4/3$, respectively. We note that, for both nonhomogeneous and homogeneous problems, the numerical results coincide with the theoretical analysis. Compared with the numerical results reported in Reference [28], we note that the convergence rates in Reference [28] are 0.8964, 0.9303, and 0.9730 when $h = 1/500$ and $N = 20,\ 40,\ 80,\ 160$. However, by employing graded meshes, the convergence order can reach second order for the homogeneous problem. In Reference [28], the minimum L_2 error is reported as 0.0027 when $N = 160$, whereas in the present paper, the minimum L_2 error is reduced to $9.50251e - 05$ for the same value of N when $\gamma = 3$.

Table 3. Errors and convergence rates in time when $\gamma = 1,\ \gamma = 4/3,$ *and* $\gamma = 3$ with $h = 1/500$ for Example 2.

	$\gamma = 1$		$\gamma = 4/3$		$\gamma = 3$	
N	*Error*	ϵ_1	*Error*	ϵ_1	*Error*	ϵ_1
20	2.22187e-03	–	8.31847e-04	–	4.94706e-03	–
40	8.74632e-04	1.34503	2.40878e-04	1.78801	1.43522e-03	1.78530
80	3.38517e-04	1.36945	6.34777e-05	1.92398	3.72884e-04	1.94447
160	1.28514e-04	1.39730	1.62956e-05	1.96177	9.50251e-05	1.97235

Example 3. We further consider an additional numerical example for which no explicit exact solution is available. Specifically, we consider the homogeneous problem in Example 2 with the initial condition $u(x, y, 0) = B * \exp(-((x - x_0)^2 + (y - y_0)^2)/(2 * (\rho)^2))$, which represents a Gaussian pulse function. For simplicity, we set $B = 1,\ x_0 = y_0 = 0$, and $\rho = 1$ to illustrate the temporal convergence behavior. The numerical solution obtained with $N = 1280$ and $h = \frac{1}{50}$ is regarded as the reference solution. Then numerical solutions with $N = 40,\ 80,\ 160,\ 320$ are compared against this reference solution. From Table 4, it can be observed that the proposed methods are also effective for this class of equations whose exact solutions are unknown. Moreover, the temporal convergence orders remain 3/2 and 2 for $\gamma = 1$ and $\gamma \geq 4/3$, respectively.

Table 4. Errors and convergence rates in time when $\gamma = 1,\ \gamma = 4/3,$ *and* $\gamma = 3$ with $h = 1/50$ for Example 3.

	$\gamma = 1$		$\gamma = 4/3$		$\gamma = 3$	
N	*Error*	ϵ_1	*Error*	ϵ_1	*Error*	ϵ_1
40	8.56640e-02	–	4.68134e-02	–	1.11338e-01	–
80	2.65970e-02	1.68743	1.19981e-02	1.96411	2.94067e-02	1.92072
160	8.37819e-03	1.66655	3.08778e-03	1.95817	7.55917e-03	1.95985
320	2.54985e-03	1.71623	7.73763e-04	1.99661	1.87810e-03	2.00896

6. Conclusions

This study develops an ADI-based numerical approach for fractional evolution equations featuring weakly singular kernels, where the time discretization is constructed on graded meshes. We choose the temporal meshes to be $t_n = (nk')^\gamma$, and we present the convergence orders for different γ. The maximal L_2 convergence order in time reaches 2. In subsequent work, we plan to investigate space-fractional derivative equations by constructing high-order accurate spatial schemes on nonuniform grids.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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