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Research article

Global dynamics of an almost periodic diffusive single-species model with age structure

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Abstract: To understand the significant roles that seasonal variations and diffusion play in population growth, we proposed and investigated an almost periodic reaction-diffusion model incorporating age structure for a single species. The model was formulated as a reaction-diffusion system with a nonlocal term. Using the principal Lyapunov exponent λ^* as a threshold, we analyzed the global dynamics of this model. The results showed that the population will go extinct if $\lambda^* < 0$, but persist if $\lambda^* > 0$. We further established the existence of a globally attractive almost periodic solution when the population was uniformly persistent in a monotone case. The Nicholson blowflies model was investigated by means of numerical simulations, and the dependence of population development on maturation period, diffusion rate, and spatial environment was discussed.

Keywords: reaction-diffusion; almost periodic; single-species; maturation period; global dynamics

1. Introduction

In population biology, predicting population dynamics is one of the most important tasks. In particular, a meaningful work is to predict whether a population can persist over the long term [1]. Mathematical modeling provides a powerful way to analyze population development, and mathematical models have been widely used to investigate population dynamics [2, 3]. In early population models, individuals are assumed to be identical, and the birth rate is treated as a function of total population size. It is generally known that the life cycles of many species go through immature and mature periods, and reproduction depends solely on mature members [4, 5]. Accordingly, a considerable number of population models with age structure have been developed and applied to analyze the population dynamics [2, 6, 7]. Cooke et al. [4] considered the following growth model of adult population,

$$\frac{du(t)}{dt} = B(u(t-\tau))e^{-d_1\tau} - d_2u(t). \tag{1.1}$$

In model (1.1), $u(t)$ is the number of adult individuals, $B(u)$ denotes the birth rate function, τ is the maturation delay, and d_1 and d_2 represent the death rates of immature and mature stages, respectively. The observation in [4] shows that the maturation delay can qualitatively alter the dynamics when $B(u)$ is not monotone.

Empirical evidence indicates that changes in the natural environment are seasonal, and seasonal fluctuations have a substantial effect on population dynamics, making it necessary to incorporate seasonal variations into population models [8]. For example, the behavior and life cycles of mosquitoes are shaped by seasonal conditions, including humidity, temperature, rainfall, and day length [9]. The periodic population models have attracted the attention of many researchers over the past decades, and the age-structured population systems with periodic parameters have been proposed and analyzed in a considerable number of studies (see, e.g., [10–13]). A significant feature of a periodic system is that its parameters are periodic and admit a common period. In reality, although periodic parameters can be deliberately designed in controlled laboratory settings, exact periodicity is rarely observed in the natural environment [14, 15]. Even if natural variations are periodic, they may not always have a common period. As a generalization of periodic functions, almost periodic functions are better suited for modeling natural phenomena. From the theoretical aspect, although almost periodic systems inherit some characteristics from periodic ones, many research methods, such as the Poincaré map for periodic models, are not applicable to almost periodic ones. Several studies have investigated the dynamics of almost periodic population systems; see, e.g., [15, 16].

The aforementioned population models are formulated as ordinary differential equations, which means that the population is well mixed. From the perspective of space, a domain in which a population inhabits is often heterogeneous, and spatial diffusion plays an important role in the population dynamics [3]. It is therefore necessary to incorporate environmental heterogeneity and spatial diffusion into the population model. Up to now, some autonomous or periodic single-species reaction-diffusion systems have been adopted to investigate the population development (see, e.g., [17–20]). DeAngelis et al. [17] considered a logistic reaction-diffusion equation and investigated the properties of positive steady state. Jin and Zhao studied a nonlocal single-species model with diffusion and age structure in a periodic environment. Recently, Ruiz-Herrera and Touaoula [21] analyzed the global attractivity for a class of reaction-diffusion systems with delays in periodic situations, and applied their results to a periodic Nicholson blowfly equation with diffusion. For the almost periodic case, global attractivity in monotone and subhomogeneous systems has been investigated in [22], and the findings can be applied to logistic reaction-diffusion single-species models.

However, research on the global dynamics of almost periodic age-structured single-species reaction-diffusion models remains insufficient. This paper aims to explore the global dynamics of a single-species reaction-diffusion system with age structure and almost periodic parameters. Due to the incorporation of a spatial variable and time delay, the state space of our model differs from those in [16, 22]. The fundamental distinction between almost periodicity and periodicity undermines the applicability of the approach developed in [18] to our model. While the findings of [23] provide a reference point for analyzing threshold dynamics in delayed almost periodic reaction-diffusion models, the framework it presents does not extend to verifying the existence and attractiveness of almost periodic solutions.

The structure of this paper is as follows. In Section 2, we develop an almost periodic age-structured single species model containing a nonlocal delay term. In Section 3, the principal Lyapunov exponent

λ^* for this system is introduced, and the threshold behavior of the solution is presented based on λ^*. Furthermore, we prove that there is a globally attractive almost periodic solution in a monotone case when $\lambda^* > 0$. In Section 4, we investigate the Nicholson blowflies model by means of numerical simulations. A brief discussion section completes the paper.

2. Model formulation

Assume that (M, ϱ) is a metric space and $g \in C(\mathbb{R}, M)$. For any $\delta > 0$, if the set

$$L(g, \delta) = \{\theta \in \mathbb{R} : \varrho(g(s + \theta) - g(\theta)) < \delta, \forall s \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$, then g is called almost periodic. Let $P \subset \mathbb{R}^n$. If a function $h \in C(\mathbb{R} \times P, M) : (s, p) \mapsto h(s, p)$ is almost periodic in s for every $p \in P$, and h is uniformly continuous on $\mathbb{R} \times D$ for every compact set $D \subset P$, then it is called uniformly almost periodic [24, 25].

Let Ω represent the spatial region with smooth boundary $\partial\Omega$, where the population lives. We define $D(\alpha)$ as the diffusion rate for individuals aged α, and $d(t, \alpha, x)$ as the death rate at position x and time t for the same age group. We use $Q(t, \alpha, x)$ to represent the population density of age α at time t and position x. Assuming individuals walk randomly within the region, it then follows from the standard results for structured populations with diffusion (see, for example, [26]) that

$$\frac{\partial Q(t, \alpha, x)}{\partial t} + \frac{\partial Q(t, \alpha, x)}{\partial \alpha} = D(\alpha)\Delta Q(t, \alpha, x) - d(t, \alpha, x)Q(t, \alpha, x). \tag{2.1}$$

We use T to represent the length of maturation period, and let $I(t, x)$ and $M(t, x)$ denote the densities of immature and mature populations at time t and position x. Then,

$$M(t, x) = \int_T^\infty Q(t, \alpha, x)d\alpha, \ I(t, x) = \int_0^T Q(t, \alpha, x)d\alpha. \tag{2.2}$$

Letting positive constants D_I and D_M be the diffusion rates of immature and mature individuals, one obtains

$$D(\alpha) = \begin{cases} D_I, & \alpha \in [0, T], \\ D_M, & \alpha \in [T, \infty]. \end{cases}$$

Moreover, we use d_I and d_M to denote the death rates of immature and mature classes, then for any $t \geq 0$ and $x \in \Omega$,

$$d(t, \alpha, x) = \begin{cases} d_I(t, x), & \alpha \in [0, T], \\ d_M(t, x), & \alpha \in [T, \infty]. \end{cases}$$

Integration of Eq (2.1) with respect to α from 0 to T together with (2.2) shows

$$\frac{\partial I(t, x)}{\partial t} = D_I\Delta I(t, x) + Q(t, 0, x) - Q(t, T, x) - d_I(t, x)I(t, x).$$

Similarly, integrating both sides of (2.1) from T to ∞, we have

$$\frac{\partial M(t, x)}{\partial t} = D_M\Delta M(t, x) + Q(t, T, x) - Q(t, \infty, x) - d_M(t, x)M(t, x).$$

From a biological perspective, we assume $Q(t, \infty, x) = 0$. In age-structured populations, where only mature individuals contribute to reproduction, that is, the birth rate depends only on M, we denote the per capita birth rate at time t and position x by the positive function $b(t, x, M)$. Then, the function for birth rate can be expressed as $B(t, x, M) = b(t, x, M)M$. It then follows that

$$Q(t, 0, x) = B(t, x, M(t, x)).$$

Moreover, the population is assumed to be active at all times within the region Ω, with the Neumann boundary condition applied. Then, the dynamics of the population evolution can be described by the following model,

$$\begin{cases} \dfrac{\partial I(t,x)}{\partial t} = D_I \Delta I(t,x) + B(t,x,M(t,x)) - Q(t,T,x) - d_I(t,x)I(t,x), \\ \dfrac{\partial M(t,x)}{\partial t} = D_M \Delta M(t,x) + Q(t,T,x) - d_M(t,x)M(t,x), \\ \dfrac{\partial I(t,x)}{\partial \upsilon} = \dfrac{\partial M(t,x)}{\partial \upsilon} = 0, \quad t > 0,\ x \in \partial\Omega, \end{cases} \tag{2.3}$$

where υ represents the outward unit normal vector on $\partial\Omega$.

Suppose that all those time-dependent coefficients in (2.3) are uniformly almost periodic. In addition, we make the following assumptions.

(A1) The map $b : \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and is Lipschitz continuous for M in bounded sets, uniformly for $t \in \mathbb{R}$ and $x \in \overline{\Omega}$.

(A2) $d_I(t, x)$ and $d_M(t, x)$ are nonnegative nontrivial and Hölder continuous on $\mathbb{R} \times \overline{\Omega}$, and for all $t \in \mathbb{R}$ and $x \in \overline{\Omega}$, $d_M(t, x) > d_0$ for some positive constant d_0.

(A3) $\frac{\partial b(t,x,M)}{\partial M} < 0$ and $\inf_{t\in\mathbb{R},x\in\overline{\Omega}} b(t, x, 0) > 0$, and there exists $K > 0$ such that for any $M > K$, $\sup_{t\in\mathbb{R},x\in\overline{\Omega}} b(t, x, M) < d_0$.

Note that Assumption (A3) shows that $b(t, x, M)$ is decreasing in M, which captures the density-dependent inhibitory effect of population size on per capita birth rate. This aligns with classical ecological theory, where increased intraspecific competition for limited resources (e.g., food, habitat, or mates) at higher population densities leads to reduced reproductive output. Furthermore, the constant K in Assumption (A3) can be viewed as the environmental carrying capacity.

To proceed further, we integrate along characteristics to determine $Q(t, T, x)$. Let $u(h, \alpha, x) = Q(\alpha + h, \alpha, x)$, $\forall h \geq 0$. We then obtain that for $\alpha \in [0, T]$,

$$\begin{cases} \dfrac{\partial u(h,\alpha,x)}{\partial \alpha} &= [\dfrac{\partial Q(t,\alpha,x)}{\partial t} + \dfrac{\partial Q(t,\alpha,x)}{\partial \alpha}]_{t=\alpha+h} \\ &= D(\alpha)\Delta Q(\alpha+h,\alpha,x) - d(\alpha+h,\alpha,x)Q(\alpha+h,\alpha,x) \\ &= D_I \Delta u(h,\alpha,x) - d_I(\alpha+h,x)u(h,\alpha,x), \\ u(h,0,x) &= Q(h,0,x) = B(h,x,M(h,x)). \end{cases} \tag{2.4}$$

Integrating Eq (2.4) (regarding z as a parameter), one has

$$u(h, \alpha, x) = \int_\Omega \Gamma(h + \alpha, h, x, y)B(h, y, M(h, y))dy,$$

where $\Gamma(t,r,x,y)$ $(x,y\in\Omega,\ t>r\geq 0)$ denotes the fundamental solution of operator $\partial_t - D_I\Delta - d_I(t,\cdot)$ with Neumann boundary (see, e.g., [27, Chapter 1]). Moreover, the dependence of $\Gamma(t,r,x,y)$ on $d_I(t,x)$ (see [27, Chapter 1]) shows that Γ is also uniformly almost periodic, due to the almost periodicity of d_I, in the sense that for any $\delta>0$, the set

$$T(\Gamma,\delta)=\bigcap_{x,y\in\Omega}\{\theta\in\mathbb{R}:|\Gamma(t+\theta,r+\theta,x,y)-\Gamma(t,r,x,y)|<\delta,\ \forall t>r\geq 0\}$$

is relatively dense in $\mathbb{R}$.

Note that $Q(t,\alpha,x)=u(t-\alpha,\alpha,x)$ for $t\geq\alpha$, we obtain

$$Q(t,\alpha,x)=\int_\Omega \Gamma(t,t-\alpha,x,y)B(t-\alpha,y,M(t-\alpha,y))dy.$$

Taking $\alpha=T$, then

$$Q(t,T,x)=\int_\Omega \Gamma(t,t-T,x,y)B(t-T,y,M(t-T,y))dy.$$

Thereby, model (2.3) can be reformulated as the following almost periodic system incorporating a nonlocal term,

$$\begin{cases}\dfrac{\partial I(t,x)}{\partial t} =D_I\Delta I(t,x)-\displaystyle\int_\Omega \Gamma(t,t-T,x,y)B(t-T,y,M(t-T,y))dy\\ \qquad\qquad + B(t,x,M(t,x))-d_I(t,x)I(t,x),\\ \dfrac{\partial M(t,x)}{\partial t} =D_M\Delta M(t,x)+\displaystyle\int_\Omega \Gamma(t,t-T,x,y)B(t-T,y,M(t-T,y))dy-d_M(t,x)M(t,x),\\ \dfrac{\partial I(t,x)}{\partial \upsilon} =\dfrac{\partial M(t,x)}{\partial \upsilon}=0,\quad t>0,\ x\in\partial\Omega.\end{cases}\tag{2.5}$$

Since $I(t,x)$ of model (2.5) is decoupled from the equation of $M(t,x)$, we only need to focus on studying the equation of mature class,

$$\begin{cases}\dfrac{\partial M(t,x)}{\partial t} = D_M\Delta M(t,x)+\displaystyle\int_\Omega \Gamma(t,t-T,x,y)B(t-T,y,M(t-T,y))dy-d_M(t,x)M(t,x),\\ \dfrac{\partial M(t,x)}{\partial \upsilon}=0,\quad t>0,\ x\in\partial\Omega.\end{cases}\tag{2.6}$$

Let $C_0:=C(\overline{\Omega},\mathbb{R})$ represent the space of continuous functions on $\overline{\Omega}$, with the norm $\|\cdot\|_{C_0}$ being the supremum norm, and let

$$C_0^+:=\{\phi\in C_0:\phi(x)\geq 0,\forall x\in\overline{\Omega}\}.$$

Obviously, $\mathrm{Int}(C_0^+)$, the interior of space C_0^+, is nonempty. For $T>0$, we define $X:=C([-T,0],C_0)$ equipped with the maximum norm $\|\psi\|:=\max_{s\in[-T,0]}\|\psi(s)\|_{C_0}$, $\forall\psi\in X$. Moreover, for some $f\in C([-T,L],C_0)$ $(L>0)$, $f_t\in X$ is defined by $f_t(s):=f(t+s)$, $\forall t\in[0,L)$, $s\in[-T,0]$.

Suppose that $U(t,s)$ represents the evolution family associated with

$$\frac{\partial M(t,x)}{\partial t}=D_M\Delta M(t,x)-d_M(t,x)M(t,x)$$

subject to the Neumann boundary condition. It is clear that $U(t, s)$ is compact and strongly positive. Define $F : [0, +\infty) \times X \to C_0$ by

$$F(t, \phi) = \int_{\Omega} \Gamma(t, t-T, \cdot, y) B(t-T, y, \phi(t-T, y)) dy, \ \forall t \geq 0, \ \phi \in X.$$

Then, (2.6) can be rewritten in the form of the following equation

$$\frac{dM(t)}{dt} = A(t)M(t) + F(t, M_t), \tag{2.7}$$

where $A(t)\varphi = D_M \Delta\varphi - d_M(t, \cdot)\varphi, \ \forall \varphi \in D(A(t))$, and

$$D(A(t)) = \{\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega}) : \frac{\partial \varphi(x)}{\partial \upsilon} = 0, \ x \in \partial\Omega\}.$$

For every $\phi \in X^+ := C([-T, 0], C_0^+)$, according to the abstract setting of [28], the solution of Eq (2.7) with initial data ϕ satisfies integral equation

$$\begin{cases} M(t, \phi) = U(t, 0)\phi(0) + \int_0^t U(t, r)F(r, M_r)dr, \ \forall t \geq 0, \\ M_0 = \phi. \end{cases}$$

The following lemma further indicates that solutions of Eq (2.6) exist globally on $[0, \infty)$.

Lemma 2.1. *For any $\phi \in X^+$, Eq* (2.6) *admits a unique mild solution $M(t, x; \phi)$ on* $[0, \infty)$ *satisfying $M_0(\cdot, \cdot, \phi) = \phi$. Furthermore, $M(t, x; \phi)$ is a classical solution when $t > T$.*

Proof. For any $(t, \phi) \in [0, +\infty) \times X^+$ and $\rho > 0$,

$$\phi(0) + \rho[F(t, \phi)] = \phi(0) + \rho \int_{\Omega} \Gamma(t, t-T, \cdot, y) B(t-T, y, \phi(t-T, y)) dy \geq \phi(0).$$

It yields that $\phi(0) + \rho[F(t, \phi)] \in C_0^+$, and hence,

$$\lim_{\rho \to 0^+} \mathrm{dist}(\phi(0) + \rho[F(t, \phi)], C_0^+) = 0, \quad \forall (t, \phi) \in [0, +\infty) \times X^+.$$

By [28, Corollary 4] with $S(t, s) = U(t, s)$ and $K = C_0^+$, Eq (2.6) admits a unique mild solution $M(t, x; \phi)$ satisfying $M_0(\cdot, \cdot, \phi) = \phi$ on its maximal interval of existence $[0, t_\phi)$ ($t_\phi \leq \infty$), and $M(t, \cdot; \phi) \in C_0^+$, $t \in [0, t_\phi)$. The analyticity of $U(t, s)$ about (t, s) yields that $M(t, x; \phi)$ is a classical solution when $t > T$.

Let $\phi^* = \max_{s \in [-T, 0], x \in \overline{\Omega}} \phi(s, x)$ and $K^* = \max\{\phi^*, K + 1\}$. Note that for any $K_0 > K$, Assumption (A3) implies

$$\begin{aligned}
&\max_{x \in \overline{\Omega}} \left\{ \int_{\Omega} \Gamma(t, t-T, x, y) B(t-T, y, K_0) dy - d_M(t, x) K_0 \right\} \\
=&\max_{x \in \overline{\Omega}} \left\{ \int_{\Omega} \Gamma(t, t-T, x, y) b(t-T, y, K_0) K_0 dy - d_M(t, x) K_0 \right\} \\
\leq&\max_{x \in \overline{\Omega}} \{ b(t-T, x, K_0) K_0 \} - d_0 K_0 \\
=&(\max_{x \in \overline{\Omega}} \{ b(t-T, x, K_0) \} - d_0) K_0 \\
<&0.
\end{aligned}$$

It then follows from the comparison principle that $M(t, x; \phi) \leq K^*$, $\forall t \geq 0$, $x \in \overline{\Omega}$. Thus, we have $t_\phi = \infty$. Consequently, Eq (2.6) admits a unique mild solution $M(t, x; \phi)$ on $[0, \infty)$ with initial data ϕ.

3. Global dynamics

To analyze the global dynamics of (2.6), we linearize Eq (2.6) at zero solution, obtaining the linear equation:

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} = D_M\Delta M(t,x) + \displaystyle\int_\Omega \Gamma(t,t-T,x,y)b(t-T,y,0)M(t-T,y)dy - d_M(t,x)M(t,x), \\ \dfrac{\partial M(t,x)}{\partial \upsilon} = 0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{3.1}$$

By the similar argument to that in [29, Section 4], Eq (3.1) has a unique mild solution $\mathcal{M}(t,x;\phi)$ with $\mathcal{M}_0(\cdot,\cdot;\phi)=\phi\in X$, and $\mathcal{M}(t,x;\phi)$ is a classical solution when $t>T$.

We define the hull of d_M as $H(d_M)=\mathrm{cls}\{d_M\cdot\theta : \theta\in\mathbb{R}\}$ under the compact open topology, where $(d_M\cdot\theta)(t,x)=d_M(t+\theta,x)$. Define the hulls $H(\Gamma)$ and $H(b)$ of Γ and b in the same way. Let $\xi=(d_M,\Gamma,b)$, and we use $H(\xi)$ to denote the hull of ξ. For any $\kappa=(\bar{d}_M,\bar{\Gamma},\bar{b})\in H(\xi)$, the map $\varsigma:\mathbb{R}\times H(\xi)\to H(\xi)$, $(\theta,\kappa)\mapsto\kappa\cdot\theta$, defined as

$$(\kappa\cdot\theta)(t,x,y)=(\bar{d}_M(t+\theta,x),\bar{\Gamma}(t+\theta,t-T+\theta,x,y),\bar{b}(t+\theta,x,\cdot)),$$

generates an almost periodic and minimal flow ([30, Section VI.C]). We consider the following equation,

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} =D_M\Delta M(t,x) + \displaystyle\int_\Omega \bar{\Gamma}(t,t-T,x,y)\bar{b}(t-T,y,0)M(t-T,y)dy - \bar{d}_M(t,x)M(t,x), \\ \dfrac{\partial M(t,x)}{\partial \upsilon} =0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{3.2}$$

Let $\mathcal{M}(t,x;\varphi,\kappa)$ denote the solution of Eq (3.2) satisfying initial condition $\mathcal{M}_0(\varphi,\kappa)=\varphi\in X$, and we define $\mathcal{M}_t(\varphi,\kappa)(s,x):=\mathcal{M}(t+s,x;\varphi,\kappa)$, $\forall t\geq 0$, $s\in[-T,0]$, $x\in\overline{\Omega}$. Accordingly, we define the following skew-product semiflow,

$$\begin{aligned} \Pi:\mathbb{R}_+\times X\times H(\xi)&\to X\times H(\xi),\\ (t,\varphi,\kappa)&\mapsto(\mathcal{M}_t(\varphi,\kappa),\kappa\cdot t). \end{aligned}$$

Let $\Phi(t,\kappa)\varphi=\mathcal{M}_t(\varphi,\kappa)$, $\forall\varphi\in X$. Define λ_κ as

$$\lambda_\kappa=\limsup_{t\to\infty}\frac{\ln\|\Phi(t,\kappa)\|}{t},\quad \kappa\in H(\xi),$$

which is said to be the Lyapunov exponent of Eq (3.2). The number

$$\lambda^*=\sup_{\kappa\in H(\xi)}\lambda_\kappa,$$

is said to be the principal (upper) Lyapunov exponent of Eq (3.2). It then follows from [23, Theorem 2.2 and Lemma 2.3] that the following properties hold.

Lemma 3.1. *Equation* (3.1) *has a solution of the form* $\mathcal{M}(t,x)=e^{\int_0^t a(\theta)d\theta}\tilde{\mathcal{M}}(t,x)$*, where* $a\in C(\mathbb{R},\mathbb{R})$ *and* $\tilde{\mathcal{M}}\in\mathrm{Int}(C(\mathbb{R},C_0^+))$ *are almost periodic in t. Furthermore,*

$$\lambda^*=\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\theta)d\theta.$$

Lemma 3.2. *Let* $\mathcal{M}(t,x;\varphi)$ *represent the solution of Eq* (3.1) *with* $\mathcal{M}_0(\cdot,\cdot;\varphi)=\varphi\in\mathrm{Int}(X^+)$. *Then, for any* $x_0\in\overline{\Omega}$,

$$\lambda^*=\lim_{t\to\infty}\frac{\ln\mathcal{M}(t,x_0;\varphi)}{t}.$$

Now we prove that the sign of the principal Lyapunov exponent λ^* determines whether the population exhibits uniform persistence or faces global extinction.

Theorem 3.3. *Assume that* (A1)–(A3) *hold. If* $\lambda^*<0$, *then for any* $\varphi\in X^+$,

$$\lim_{t\to\infty}M(t,\cdot;\varphi)=0.$$

Proof. Since $B(t,x,M)=b(t,x,M)M$ and Assumption (A3) yields that $b(t,x,0)\geq b(t,x,M)$ for any $M>0$, it follows that

$$\begin{aligned}\frac{\partial M(t,x)}{\partial t}=&D_M\Delta M(t,x)+\int_\Omega\Gamma(t,t-T,x,y)B(t-T,y,M(t-T,y))dy-d_M(t,x)M(t,x)\\=&D_M\Delta M(t,x)+\int_\Omega\Gamma(t,t-T,x,y)b(t-T,y,M(t-T,y))M(t-T,y)dy\\&-d_M(t,x)M(t,x)\\\leq&D_M\Delta M(t,x)+\int_\Omega\Gamma(t,t-T,x,y)b(t-T,y,0)M(t-T,y)dy-d_M(t,x)M(t,x).\end{aligned}$$

Consider

$$\begin{cases}\dfrac{\partial M(t,x)}{\partial t}=D_M\Delta M(t,x)+\displaystyle\int_\Omega\Gamma(t,t-T,x,y)b(t-T,y,0)M(t-T,y)dy-d_M(t,x)M(t,x),\\\dfrac{\partial M(t,x)}{\partial \upsilon}=0,\quad t>0,\ x\in\partial\Omega.\end{cases}\tag{3.3}$$

Lemma 3.1 shows that Eq (3.3) has a solution with the form $\mathcal{M}(t,x)=e^{\int_0^t a(\theta)d\theta}\tilde{\mathcal{M}}(t,x)$, where $\tilde{\mathcal{M}}\in\mathrm{Int}(C(\mathbb{R},C_0^+))$ and $a\in C(\mathbb{R},\mathbb{R})$ are almost periodic functions. Note that we can choose a number K_0 such that $\varphi(s,x)\leq\mathcal{M}(s,x)$, $\forall s\in[-T,0]$, $x\in\overline{\Omega}$. Hence, the comparison principle shows that

$$M(t,x;\varphi)\leq\mathcal{M}(t,x),\ \forall t\geq0,\ x\in\overline{\Omega}.$$

By Lemma 3.1 again, furthermore,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\theta)d\theta=\lambda^*<0.$$

This indicates that $\lim\limits_{t\to\infty}\mathcal{M}(t,x)=0$ uniformly for $x\in\overline{\Omega}$. Hence, $\lim\limits_{t\to\infty}M(t,\cdot;\varphi)=0$.

In the following theorem, we prove the uniform persistence by using the skew-product semiflows approach.

Theorem 3.4. *Assume that* (A1)–(A3) *hold. If* $\lambda^*>0$, *then for any* $\varphi\in X^+$ *with* $\varphi(0,\cdot)\not\equiv0$, *there is a* $\sigma>0$ *such that*

$$\liminf_{t\to\infty}\min_{x\in\overline{\Omega}}M(t,x;\varphi)\geq\sigma.$$

Proof. Define

$$Y := \{\varphi \in X^+ : \varphi(0,\cdot) \not\equiv 0\},$$
$$\partial Y := X^+ \backslash Y = \{\varphi \in X^+ : \varphi(0,\cdot) \equiv 0\}$$

and

$$U = X^+ \times H(\xi), \quad U_0 = Y \times H(\xi), \quad \partial U_0 = \partial Y \times H(\xi).$$

Obviously, ∂U_0 and U_0 are relatively closed and open in U, respectively.

For any $\kappa = (\bar{d}_M, \bar{\Gamma}, \bar{b}) \in H(\xi)$, we consider

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} = D_M \Delta M(t,x) + \displaystyle\int_\Omega \bar{\Gamma}(t, t-T, x, y)\bar{B}(t-T, y, M(t-T, y))dy - \bar{d}_M(t,x)M(t,x), \\ \dfrac{\partial M(t,x)}{\partial \upsilon} = 0, \quad t > 0,\ x \in \partial\Omega, \end{cases} \tag{3.4}$$

where $\bar{B}(t-T, y, M(t-T,y)) = \bar{b}(t-T, y, M(t-T,y))M(t-T,y)$. Let $M(t,x;\varphi,\kappa)$ represent the solution of (3.4) with $M_0(\cdot,\cdot;\varphi,\kappa) = \varphi \in X^+$, and let Π_t be the continuous skew-product semiflow generated by $M(t,x;\varphi,\kappa)$ as follows,

$$\Pi_t : U \to U,$$
$$(\varphi,\kappa) \mapsto (M_t(\varphi,\kappa), \kappa\cdot t).$$

Here, $M_t(\varphi,\kappa)(x,s) = M(t+s, x;\varphi,\kappa)$, $\forall t \geq 0$, $s \in [-T, 0]$, and $x \in \overline{\Omega}$. Clearly, $\Pi_t(U_0) \in U_0$, $\forall t \geq 0$. The proof of Lemma 2.1 yields that Π_t is point dissipative. By [31, Theorem 2.1.8], we see that Π_t is compact for any $t > T$. Hence, Π_t is completely continuous. According to [32, Theorem 3.7.2], $\Pi_t : U \to U$ has a global attractor Q. Let

$$M_\partial := \{(\varphi,\kappa) \in \partial U_0 : \Pi_t(\varphi,\kappa) \in \partial U_0,\ \forall t \geq 0\},$$

which represents the maximal positively invariant set of Π_t in ∂U_0. Define $\mathcal{K} := \{(\hat{0},\kappa) : \kappa \in H(\xi)\}$, where $\hat{0} \in X$ and $\hat{0}(s,x) = 0$, $\forall s \in [-T, 0]$, $x \in \overline{\Omega}$. It is clear that $\mathcal{K}$ is isolated invariant and compact, and that no subset of $\mathcal{K}$ forms a cycle for Π_t in ∂U_0. Furthermore, the following observation holds.

Claim. There exists a constant $\eta > 0$ such that for any $(\varphi,\kappa) \in U_0$,

$$\limsup_{t\to\infty} d(\Pi_t(\varphi,\kappa), \mathcal{K}) \geq \eta.$$

Assume, by contradiction, for some $(\varphi^0,\kappa^0) = (\varphi^0, \tilde{d}_M, \tilde{\Gamma}, \tilde{b}) \in U_0$,

$$\limsup_{t\to\infty} d(\Pi_t(\varphi^0,\kappa^0), \mathcal{K}) < \eta.$$

Then, we can find $t_1 > 0$ such that $d(\Pi_t(\varphi^0,\kappa^0), \mathcal{K}) < \eta$, $\forall t \geq t_1$, and hence, $M(t,x;\varphi^0,\kappa^0) \leq \eta$, $\forall t \geq t_1$, $x \in \overline{\Omega}$. Then, $M(t,x;\varphi^0,\kappa^0)$ satisfies

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} \geq D_M \Delta M(t,x) + \displaystyle\int_\Omega \tilde{\Gamma}(t, t-T, x, y)\tilde{b}(t-T, y, \eta)M(t-T, y)dy \\ \qquad\qquad - \tilde{d}_M(t,x)M(t,x),\ t \geq t_1 + T,\ x \in \Omega, \\ \dfrac{\partial w(t,x)}{\partial \upsilon} = 0, \quad t \geq t_1 + T,\ x \in \partial\Omega. \end{cases}$$

Consider the following equation with a parameter $\eta > 0$,

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} =D_M \Delta M(t,x) + \displaystyle\int_{\Omega} \tilde{\Gamma}(t, t-T, x, y)\tilde{b}(t-T, y, \eta)M(t-T, y)dy - \tilde{d}_M(t,x)M(t,x), \\ \dfrac{\partial w(t,x)}{\partial \upsilon} =0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{3.5}$$

Note that $\lambda^* > 0$, and the continuous dependence of the principal Lyapunov exponent on parameters indicates that we can select $\eta > 0$ small enough such that $\lambda^*_\eta > 0$, where λ^*_η represents the principal Lyapunov exponent of (3.5). According to Lemma 3.1, Eq (3.5) exists a solution of the form $\mathcal{M}(t, x; \kappa^0) = e^{\int_0^t p(\theta)d\theta}\tilde{\mathcal{M}}(t, x; \kappa^0)$, where $p(t)$ and $\tilde{\mathcal{M}}(t, x; \kappa^0)$ are almost periodic functions. Lemma 3.1 further shows that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t p(\theta)d\theta = \lambda^*_\eta > 0.$$

Recall $(\varphi^0, \kappa^0) \in U_0$, and it follows that $M(t, x; \varphi^0, \kappa^0) > 0$ for $t > 0$. Hence, there exist $t_2 > 0$ and $\delta > 0$ such that $M(t_2 + s, x; \varphi^0, \kappa^0) \geq \delta\mathcal{M}(t_2 + s, x, \kappa^0)$, $\forall s \in [-T, 0]$, $x \in \overline{\Omega}$. Using the comparison principle,

$$M(t, x; \varphi^0, \kappa^0) \geq \delta\mathcal{M}(t, x; \kappa^0) = \delta e^{\int_0^t p(\theta)d\theta}\tilde{\mathcal{M}}(t, x; \kappa^0), \quad \forall t \geq t_3,\ x \in \overline{\Omega},$$

where $t_3 = \max\{t_1 + T, t_2\}$. Based on the boundedness of $\tilde{\mathcal{M}}(t, x; \kappa^0)$ and

$$\lim_{t\to\infty} e^{\int_0^t p(\theta)d\theta} = \infty,$$

we get $\lim_{t\to\infty} M(t, x; \varphi^0, \kappa^0) = \infty$, a contradiction.

Recall that $\mathcal{K}$ is isolated and invariant for Π_t in ∂U_0. The above claim yields that $\mathcal{K}$ is also isolated and invariant for Π_t in U. Let

$$W^s(\mathcal{K}) = \{(\varphi, \kappa) \in U : \omega(\varphi, \kappa) \neq \emptyset, \omega(\varphi, \kappa) \subset \mathcal{K}\}.$$

Then, the above claim indicates that $W^s(\mathcal{K}) \cap U_0 = \emptyset$. According to [22, Theorem 1.3.1 and Remarks 1.3.1 and 1.3.2], $\Pi_t : U \to U$ is uniformly persistent. Recall that Π_t is compact for all $t > T$ and hence, asymptotically smooth. It then follows from [33, Theorem 3.7 and Remark 3.10] that $\Pi_t : U_0 \to U_0$ has a global attractor Q_0.

Based on the persistence of skew-product semiflow, we finally establish the persistence of the solution. Note that $Q_0 \in U_0$ and $\Pi_t(Q_0) = Q_0$, and one has $Q_0 \in \mathrm{Int}(U)$. We construct a continuous function $g : U \to [0, \infty)$ given by

$$g(\varphi, \kappa) = \min_{x\in\overline{\Omega}}\{\varphi(0, x)\}, \quad \forall(\varphi, \kappa) \in U.$$

Obviously, for any $(\varphi, \kappa) \in Q_0$, $g(\varphi, \kappa) > 0$. Then, $\inf_{(\varphi,\kappa)\in Q_0} g(\varphi, \kappa) = \min_{(\varphi,\kappa)\in Q_0} g(\varphi, \kappa) > 0$ due to the compactness of Q_0. Therefore, for any $(\varphi, \kappa) \in U_0$, there is a $\sigma > 0$ so that

$$\liminf_{t\to\infty} M(t, \cdot; \varphi, \kappa) \geq \sigma.$$

We complete the proof.

Theorems 3.3 and 3.4 show that the uniform persistence and extinction of the population is determined by the sign of λ^*. When the population exhibits uniform persistence, an interesting problem is whether a globally attractive almost periodic solution exists. For this purpose, an assumption about the birth rate is needed.

(A4) $\frac{\partial B(t,x,M)}{\partial M} > 0$ for all $t \in \mathbb{R}$, $x \in \overline{\Omega}$, and $M \geq 0$.

An example of $B(t,x,M)$ is the birth function $(\frac{A(t,x)}{M} + c(t,x))M$ (see, e.g., [4]), where $A(t,x)$ and $c(t,x)$ are positive uniformly almost periodic functions.

Theorem 3.5. *Assuming* (A1)–(A4) *hold. If* $\lambda^* > 0$*, then Eq* (2.6) *has a unique positive almost periodic solution* $M^*(t,x)$*, and* $\lim_{t\to\infty} \|M(t,\cdot;\phi) - M^*(t,\cdot)\|_{C_0} = 0$ *for any* $\phi \in X^+ \setminus \{\hat{0}\}$.

Proof. The skew-product semiflow defined in the proof of Theorem 3.4 is still used here. We first prove that $M_t(\phi,\kappa)$ is strongly monotone for every $(t,\kappa) \in (2T,\infty)\times H(\xi)$, that is, $M_t(\phi,\kappa) \gg M_t(\varphi,\kappa)$ if $\phi > \varphi$. Consider the following initial value problem under the Neumann boundary condition (regarding z as a parameter),

$$\begin{cases} \dfrac{\partial u(h,\alpha,x)}{\partial \alpha} =D_I\Delta u(h,\alpha,x) - d_I(\alpha+h,x)u(h,\alpha,x), \\ \quad u(h,0,x) =Q(h,x),\ Q(h,\cdot) \in C_0^+. \end{cases} \tag{3.6}$$

By the similar arguments to those in (2.4), we have that $\mathcal{F}(h,\alpha,x,Q) := \int_\Omega \Gamma(h+\alpha,h,x,y)Q(h,y)dy$ is the solution of (3.6). Let $\mathbb{F}(h,\alpha,Q) := \mathcal{F}(h,\alpha,\cdot,Q(h,\cdot))$, and the standard comparison theorem implies that $\mathbb{F}(h,\alpha,\cdot)$ is strongly monotone on C_0^+ for each h,α. For any $M(h,\cdot) \in C_0^+$, we define $\hat{\mathbb{F}}(h,\alpha,M) := \int_\Omega \Gamma(h+\alpha,h,\cdot,y)B(h,y,M(h,y))dy$. Note that Assumption (A4) yields $B(h,x,M_1) > B(h,x,M_2)$ for any $M_1 > M_2$ and that $\hat{\mathbb{F}}(h,\alpha,M)$ is a special case of $\mathbb{F}(h,\alpha,Q)$, where $Q(h,x) = B(h,x,M(h,x))$. Thus, the strong monotonicity of $\mathbb{F}(h,\alpha,Q)$ indicates that $\hat{\mathbb{F}}(h,\alpha,\cdot)$ is strongly monotone on C_0^+ for each h,α. Choosing $h+\alpha = t$ and $\alpha = T$, then

$$\hat{\mathbb{F}}(t-T,T,M) = \int_\Omega \Gamma(t,t-T,\cdot,y)B(t-T,y,M(t-T,y))dy,$$

which is strongly monotone on C_0^+ for each $t > 0$. Furthermore, for any $\phi,\varphi \in X^+$ with $\phi > \varphi$, if there exists an $s_0 \in [0,T]$ such that $\phi(s_0,\cdot) > \varphi(s_0,\cdot)$, it then follows from the integral version of (3.4) that $M(s_0+T,\cdot;\phi,\kappa) \gg M(s_0+T,\cdot;\varphi,\kappa)$. Hence, $M(t,\cdot;\phi,\kappa) \gg M(t,\cdot;\varphi,\kappa)$ for all $t \geq s_0+T$. Thus, $M_t(\phi,\kappa)$ is strongly monotone for every $(t,\kappa) \in (2T,\infty)\times H(\xi)$.

Next, we claim that $M_t(\phi,\kappa)$ is strictly subhomogeneous for any $t > T$ in the sense that $M_t(\lambda\phi,\kappa) > \lambda M_t(\phi,\kappa)$ for any $\phi \gg 0$ and $\lambda \in (0,1)$. In view of $\phi \gg 0$, it is not difficult to see that $M(t,x;\phi,\kappa) \gg 0$ for any $t > 0$ and $x \in \overline{\Omega}$. Note that (A3) yields that the function $B(t,x,M)$ is strictly subhomogeneous in M, then for any $\lambda \in (0,1)$,

$$\begin{cases} \dfrac{\partial(\lambda M(t,x))}{\partial t} =D_M\Delta(\lambda M(t,x)) + \lambda\displaystyle\int_\Omega \bar{\Gamma}(t,t-T,x,y)\bar{B}(t-T,y,M(t-T,y))dy \\ \qquad\qquad - \bar{d}_M(t,x)(\lambda M(t,x)) \\ \qquad\qquad <D_M\Delta(\lambda M(t,x)) + \displaystyle\int_\Omega \bar{\Gamma}(t,t-T,x,y)\bar{B}(t-T,y,\lambda M(t-T,y))dy \\ \qquad\qquad - \bar{d}_M(t,x)(\lambda M(t,x)) \\ \dfrac{\partial(\lambda M(t,x))}{\partial \upsilon} =0, \quad t > 0,\ x \in \partial\Omega. \end{cases}$$

Hence, $\lambda M(t, x; \phi, \kappa)$ is a lower solution of (3.4) with $\lambda M(s, x; \phi, \kappa) = \lambda\phi(s, x)$ for $s \in [-T, 0]$ and $x \in \overline{\Omega}$. Then, $M(t, x; \lambda\phi, \kappa) \geq \lambda M(t, x; \phi, \kappa)$ for $t \geq 0$, and $M(t, x; \lambda\phi, \kappa) \not\equiv \lambda M(t, x; \phi, \kappa)$ for $0 < t < T$, where $M(t, x; \lambda\phi, \kappa)$ is the solution of (3.4) with $M(s, x; \lambda\phi, \kappa) = \lambda\phi(s, x)$ for $s \in [-T, 0]$ and $x \in \overline{\Omega}$. It follows that $M_t(\lambda\phi, \kappa) > \lambda M_t(\phi, \kappa)$ for $t > T$, that is, $M_t(\phi, \kappa)$ is strictly subhomogeneous for any $t > T$.

Taking $\phi^0 \in X^+ \setminus \{\hat{0}\}$. By [22, Theorem 2.3.5 and Remarks 2.3.2 and 2.3.3] and Theorem 3.4, for any $\phi \in X^+ \setminus \{\hat{0}\}$, $\lim_{t\to\infty} \|M(t, \cdot; \phi, \kappa) - M(t, \cdot; \phi^*, \kappa)\|_{C_0} = 0$, where $(\phi^*, \kappa) \in \omega(\phi^0, \kappa)$. Since $\Pi_t : \omega(\phi, \kappa) \to \omega(\phi, \kappa)$ is an almost periodic minimal flow, $\Pi_t(\phi^*, \kappa) = (M(t, \cdot; \phi^*, \kappa), \kappa \cdot t)$ is almost periodic [30, Lemma VI.9]. Hence, $M(t, \cdot; \phi^*, \kappa)$ is a unique, positive, and almost periodic solution of Eq (2.6). This completes the proof.

4. Numerical simulation

In this section, we focus on the Nicholson blowflies model and conduct a numerical analysis of how its population dynamics depend on the maturation period, diffusion rate, and spatial environment.

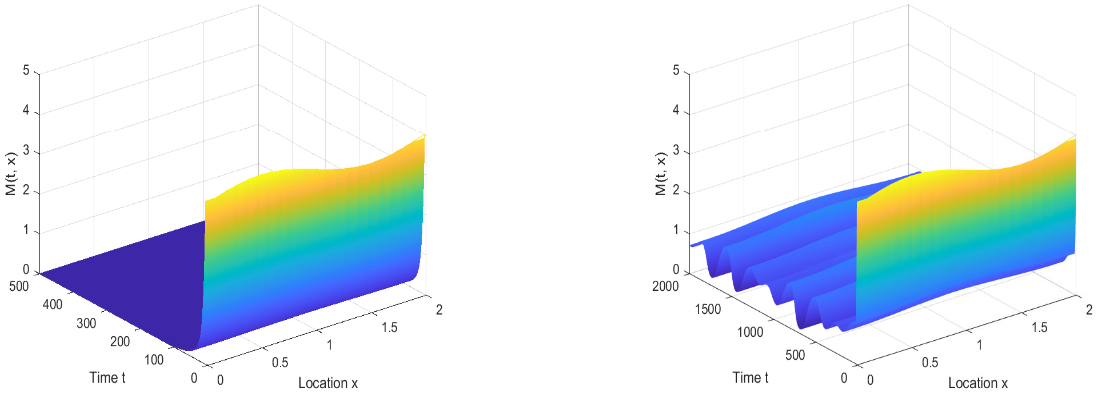
The Nicholson blowflies model is a classical time-delay differential equation model used to describe population dynamics with overlapping generations, particularly the population changes of the Australian sheep blowfly (Lucilia cuprina). Proposed by Gurney et al. [34] in 1980, it has become a significant framework in mathematical ecology due to its ability to generate complex dynamical behaviors. In [16], an almost periodic stage-structured ordinary differential Nicholson model was analyzed using numerical simulation techniques.

We focus on a one-dimensional region $\Omega = (0, 2)$. Assume that the birth rate is spatially homogeneous. Let constant r denote the average number of eggs laid by an adult blowfly and q be the strength of reproductive density dependence, then the typical birth rate function for blowflies can be expressed as $b(t, M) = r(t)e^{-q(t)M}$ [13,34]. Moreover, we assume the death rate of immature individuals to be constant and that of mature individuals to be temporally homogeneous. It then follows that the model of mature individuals is governed by

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} = D_M \Delta M(t,x) - d_M(x) M(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(t, t-T, x, y) r(t-T) e^{-q(t-T)M(t-T,y)} M(t-T, y) dy, \\ \dfrac{\partial M(t,x)}{\partial \upsilon} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{4.1}$$

Corresponding to Eq (3.1), the linearized equation of Eq (4.1) at zero solution is

$$\begin{cases} \dfrac{\partial M(t,x)}{\partial t} = D_M \Delta M(t,x) - d_M(x) M(t,x) + r(t-T) \displaystyle\int_\Omega \Gamma(t, t-T, x, y) M(t-T, y) dy, \\ \dfrac{\partial M(t,x)}{\partial \upsilon} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases}$$



(a) (b)

Figure 1. The evolution of population (a) $\lambda^* = -0.05$; (b) $\lambda^* = 0.06$.

In view of [13,35], we take $T = 6$, $r(t) = c(1 + 0.3(\sin(\frac{2\pi t}{365}) + \cos(\frac{\sqrt{2}\pi t}{365})))$, $q(t) = 1$, and $d_I(t) = 0.1$, where c is a positive constant. Given the diversity of spatial environment, one takes $d_M(x) = 0.133 + 0.1\cos(\pi x)$. The calculation of the principal Lyapunov exponent is derived by Lemma 3.2. Taking $\phi \in \mathrm{Int}(X^+)$, it follows that the value $\lambda^*_T = \frac{\ln \mathcal{M}(T, x_0; \phi)}{T}$ (T large enough) gives an approximation to λ^*.

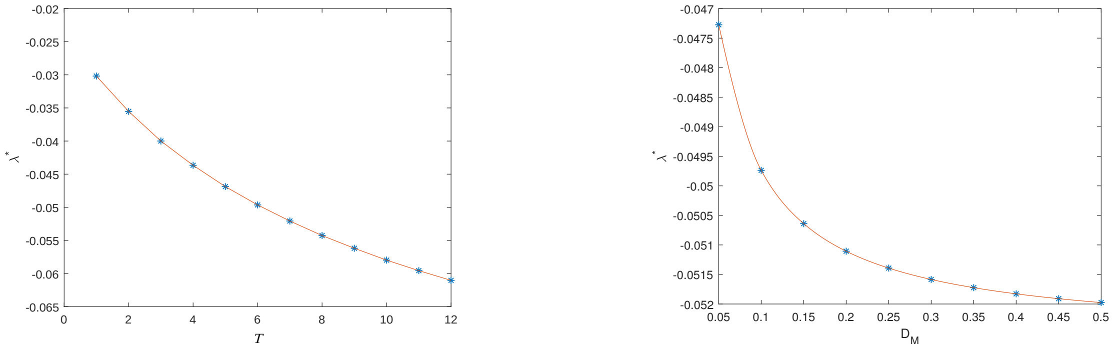
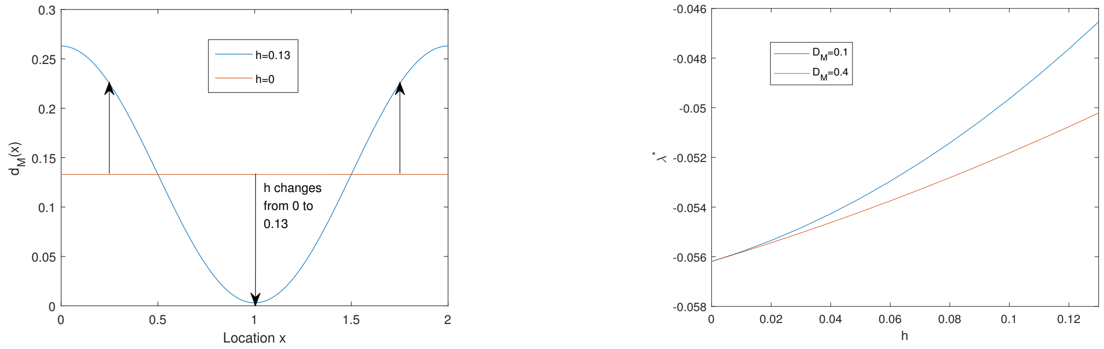


Figure 2. λ^* as functions of T and D_M.

Under these parameter settings, if one takes $c = 0.1$, we then get $\lambda^* = -0.05 < 0$. Figure 1(a) illustrates the change of the solution of system (4.1) with initial value

$$M(s, x) = 4 + 0.3\sin(\pi x), \quad \forall s \in [-6, 0], \; x \in [0, 2]. \tag{4.2}$$

It indicates that the population will go extinct in the region, and this observation aligns with the theoretical result in Theorem 3.3. Increasing c to 0.5, and others remain the same. Calculation shows that $\lambda^* = 0.06$. In such a case, Figure 1(b) plots the variation of a solution of system (4.1) with initial condition (4.2). The observation indicates that the population persists, aligning with the outcome stated in Theorem 3.4.



(a) (b)

Figure 3. The relation between λ^* and the spatial distribution of d_M.

Obviously, the development trend of the population depends entirely on the value of λ^*. It motivates us to investigate the relationship between λ^* and the relevant parameters. For the sake of convenience, we take $r(t) = 0.1(1 + 0.3(\sin(2\pi t) + \cos(\sqrt{2}\pi t)))$, and others remain unchanged. As shown in Figure 2, λ^* exhibits a decreasing trend with respect to both the maturation period length T and diffusion rate D_M. These observations suggest that shortening the maturation period can effectively enhance population persistence, likely because a reduced maturation duration accelerates individual maturation and reproductive output. Furthermore, lowering the diffusion rate also promotes population survival, which may cause by the spatial heterogeneity of death rate d_M.

In order to analyze the effect of spatial heterogeneity of d_M on λ^*. We take $d_M(x) = 0.133 + h \cdot \cos(\pi x)$, where $h \in [0, 0.13]$. Note that the integration $\int_0^2 d_M(x)dx$ remains unchanged as h changes from 0 to 0.13. When $h = 0$, the spatial distribution of death rate of mature individuals is uniform, but this uniformity diminishes with rising h (see Figure 3(a)). Figure 3(b) reveals that λ^* rises with h, and this trend becomes more pronounced as the diffusion rate D_M decreases. It seems to show that spatial heterogeneity of death rate d_M is beneficial to this population development. But it should be noted that this numerical simulation result was obtained under the assumption of a spatially uniform birth rate.

5. Discussion

In population biology, mathematical models have been widely used to investigate the dynamics of populations. Reaction-diffusion models constitute a pivotal class of mathematical frameworks for describing population dynamics, effectively capturing the interplay between spatial heterogeneity and individual diffusion in disease transmission processes. On the other hand, the living habits of many species are significantly shaped by seasonal changes such as temperature, humidity, and precipitation. Combining the effects of climate variations and environmental heterogeneity on population dynamics, we formulate and investigate an almost periodic reaction-diffusion model incorporating age structure for a single species. The almost periodicity of model parameters reflects the seasonal fluctuations of the environment. Because of the interaction of maturation delay, diffusion, and mobility, a remarkable feature of this model is that it contains a nonlocal term.

The principal Lyapunov exponent λ^*, a key threshold value, is introduced in this population model, which reflects the growth rate of the population. It indicates that λ^* is a critical value for the population's survival or decline. To be more exact, the population is eliminated when λ^* is negative, but persists when λ^* is positive. In the case of $\lambda^* > 0$, we further prove that Eq (2.6) admits a globally attractive almost periodic solution for a monotone case. In the numerical simulation, we numerically analyze the roles of maturation period, spatial heterogeneity, and diffusion rate in population development by a classical Nicholson blowflies model. Numerical simulations reveal that prolonging the length of the maturation period and increasing the diffusion rate of the mature population are unfavorable to the survival of the population (see Figure 2), but spatial heterogeneity is beneficial to it (see Figure 3). However, these observations have been derived only for a specific set of parameters. To illustrate general scenarios, additional research is necessary.

Although almost periodic functions can reflect seasonal variation, they are unable to capture the effects of environmental noise. Stochastic systems provide a powerful framework for analyzing the impact of stochastic disturbances; see, e.g., [36–38]. Therefore, a question arises as to whether the approach in this paper can be extended to stochastic settings. We leave this for further study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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