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# Recollement ideals and recollements of Gorenstein defect categories

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**Abstract:** Let $R$ be an Artin algebra and $e \in R$ an idempotent. We study when the recollement ideal $ReR$ induces (left) recollement of Gorenstein defect categories. As a consequence of this phenomenon, concrete triangle-equivalences between Gorenstein defect categories (resp. stable categories of Gorenstein projective modules) of $R$ and $R/ReR$ (resp. $eRe$) are obtained, and reduction conditions for Gorensteinness (resp. CM-freeness) of $R$ and Gorenstein projective conjecture over $R$ are provided. Finally, some applications in the context of triangular matrix algebras are given.

**Keywords:** recollements; recollement ideals; Gorenstein projective modules; Gorenstein defect categories; Gorenstein projective conjecture

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## 1. Introduction

The notion of singularity categories was first introduced by Buchweitz [1], known at the time as the *stable derived category*. This notion was later revisited by Orlov [2] and was discovered to have a closed relation with the *homological mirror symmetry conjecture* due to the work of Kontsevich. For an Artin algebra $R$, recall that the singularity category $\mathbb{D}_{sg}(R)$ of $R$ is defined as the Verdier quotient of the bounded derived category of finitely generated left $R$-modules by the subcategory of perfect complexes, that is, the bounded homotopy category of finitely generated projective modules. A fundamental result by Buchweitz is known as the *Buchweitz theorem* ([1, Theorem 4.4.1]), which establishes a fully faithful triangle functor

$$F : \underline{\mathrm{Gproj}\, R} \to \mathbb{D}_{sg}(R),$$

where $\underline{\mathrm{Gproj}\,R}$ is the stable category of finitely generated Gorenstein projective left $R$-modules. Moreover, $F$ becomes a triangle-equivalence when $R$ is Gorenstein (i.e., the left and right self-injective dimensions of $R$ are finite).

Motivated by the Buchweitz theorem, Bergh, Jørgensen, and Oppermann [3] introduced the Verdier quotient $\mathbb{D}_{def}(R) := \mathbb{D}_{sg}(R)/\operatorname{Im} F$, and they called it the *Gorenstein defect category* of $R$. $\mathbb{D}_{def}(R)$ measures how far the algebra $R$ is from being Gorenstein. More precisely, $R$ is Gorenstein if and only if $\mathbb{D}_{def}(R)$ is trivial. It is shown in [4, 5] that $\mathbb{D}_{def}(R)$ could be also viewed as the certain Verdier quotient of the bounded derived category. Recently, the study of triangle-equivalences between singularity categories ([6–8]), between Gorenstein defect categories ([9]), and of other related topics ([10–13]) has attracted considerable attention.

The notion of recollements for triangulated categories was first introduced by Beilinson et al. [14] and then generalized to the level of abelian categories (see [8, 15]). This notion arises constantly in algebraic geometry and representation theory [8, 15–17]. For example, let $e \in R$ be an idempotent. It was shown in [8, 18] that the idempotent ideal $ReR$ induces a recollement

$$\begin{tikzcd}
\operatorname{mod} R/ReR \arrow[r, "i_e=R/ReR\otimes_{R/ReR}-"] & \operatorname{mod} R \arrow[l, bend right, "i_e^{\otimes}=R/ReR\otimes_R-"'] \arrow[l, bend left, "i_e^H=\operatorname{Hom}_R(R/ReR{,}-)"] \arrow[r, "s^e=eR\otimes_R-"] & \operatorname{mod} eRe \arrow[l, bend right, "s^e_{\otimes}=Re\otimes_{eRe}-"'] \arrow[l, bend left, "s^e_H=\operatorname{Hom}_{eRe}(eR{,}-)"]
\end{tikzcd}$$

of module categories. Due to [17, Theorem 1], the above recollement lifts to the level of bounded derived categories, that is, there exists the following recollement:

$$\begin{tikzcd}
\mathbb{D}^b(\operatorname{mod} R/ReR) \arrow[r, "\mathbb{D}(i_e)=R/ReR\otimes^{\mathbb{L}}_{R/ReR}-"] & \mathbb{D}^b(\operatorname{mod} R) \arrow[l, bend right, "\mathbb{D}(i_e^{\otimes})=R/ReR\otimes^{\mathbb{L}}_R-"'] \arrow[l, bend left, "\mathbb{D}(i_e^H)=\mathbb{R}\operatorname{Hom}_R(R/ReR{,}-)"] \arrow[r, "\mathbb{D}(s^e)=eR\otimes^{\mathbb{L}}_R-"] & \mathbb{D}^b(\operatorname{mod} eRe) \arrow[l, bend right, "\mathbb{D}(s^e_{\otimes})=Re\otimes^{\mathbb{L}}_{eRe}-"'] \arrow[l, bend left, "\mathbb{D}(s^e_H)=\mathbb{R}\operatorname{Hom}_{eRe}(eR{,}-)"]
\end{tikzcd} \qquad \text{(DR)}$$

if and only if $ReR$ is a recollement ideal. Here, for the notion of a recollement ideal, we refer the reader to Definition 3.1.

In [19], the authors studied when the recollement (DR) induces (left) recollement of singularity categories. Such research is based on the fact that the singularity category is the Verdier quotient of the bounded derived category by the subcategory of perfect complexes. As is known, the Gorenstein defect category could also be viewed as the certain Verdier quotient of the bounded derived category ([4, 5]). From this viewpoint, it is natural to ask: when does the recollement (DR) induce (left) recollement of Gorenstein defect categories? The main purpose of this paper is to study this question. To state our main results precisely, let us first introduce the following notion.

**Definition 1.1.** Let $A$, $B$ be two algebras and $\Phi : \operatorname{mod} A \to \operatorname{mod} B$ an additive functor. We call $\Phi$ *Gorenstein homologically finite* if $\operatorname{Gpd}_B \Phi(G) < \infty$ for any $G \in \operatorname{Gproj} A$.

The main results of this paper are as follows.

**Theorem 1.2.** *(see Theorems 3.4 and 3.9) Let ReR be a recollement ideal.*

*(1)* (DR) *induces the following left recollement of Gorenstein defect categories:*

$$\begin{tikzcd}
\mathbb{D}_{def}(R/ReR) \arrow[r, "\widetilde{\mathbb{D}(i_e)}"] & \mathbb{D}_{def}(R) \arrow[l, bend right, "\widetilde{\mathbb{D}(i_e^{\otimes})}"'] \arrow[r, "\widetilde{\mathbb{D}(s^e)}"] & \mathbb{D}_{def}(eRe) \arrow[l, bend right, "\widetilde{\mathbb{D}(s^e_{\otimes})}"']
\end{tikzcd}$$

*if and only if the functors* $i_e$ *and* $s^e$ *are Gorenstein homologically finite.*

*(2) Assume that* $eR \in \operatorname{proj} eRe$. *Then,* (DR) *induces the following recollement of Gorenstein defect categories:*

$$\begin{tikzcd}
\mathbb{D}_{def}(R/ReR) \arrow[r, "\widetilde{\mathbb{D}(i_e)}"] & \mathbb{D}_{def}(R) \arrow[l, bend right=30, "\widetilde{\mathbb{D}(i_e^{\otimes})}"'] \arrow[l, bend left=30, "\widetilde{\mathbb{D}(i_e^{H})}"] \arrow[r, "\widetilde{\mathbb{D}(s^e)}"] & \mathbb{D}_{def}(eRe) \arrow[l, bend right=30, "\widetilde{\mathbb{D}(s_{\otimes}^e)}"'] \arrow[l, bend left=30, "\widetilde{\mathbb{D}(s_H^e)}"]
\end{tikzcd}$$

*if and only if the functors* $i_e$, $s^e$, *and* $s_H^e$ *are Gorenstein homologically finite.*

Combining Theorem 1.2 with the main results in [19], we then get (left) recollement of stable categories of Gorenstein projective modules; see Propositions 3.5 and 3.10. These not only yield certain triangle-equivalences between Gorenstein defect categories (resp. stable categories of Gorenstein projective modules) of $R$ and $R/ReR$ (resp. $eRe$) but also supply reduction conditions for Gorensteinness (resp. CM-freeness) of $R$ (see Corollary 3.6). Additionally, the left recollement of stable categories of Gorenstein projective modules is helpful in reducing the Gorenstein projective conjecture of $R$ to its idempotent subalgebra $eRe$ and quotient algebra $R/ReR$ (see Proposition 3.8).

As applications, our results are well applied to the triangular matrix algebra $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Consequently, (left) recollement of the Gorenstein defect category (resp. the stable category of Gorenstein projective modules) of $T$ relative to those of its corner algebras $A$ and $B$ are obtained; see Propositions 4.4 and 4.6 and Corollary 4.7 for details. Parts of this work generalize the corresponding results in [11–13, 20].

The paper is structured as follows: in Section 2, we define some notations and recall some basic definitions and relevant facts. In Section 3, we study the instance recollement (DR) induces (left) recollement of Gorenstein defect categories. In Section 4, we give some applications of our results in the context of triangular matrix algebra $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.

## 2. Preliminaries

Throughout, all algebras are Artin algebras over some commutative Artinian ring, and all subcategories are full additive and closed under isomorphisms. Let $R$ be an Artin algebra. Denote by $\operatorname{mod} R$ the category of finitely generated left $R$-modules, and by $\operatorname{proj} R$ its subcategory consisting of projective modules. Usually, we use ${}_RM$ (resp. $M_R$) to denote a left (resp. right) $R$-module $M$, and the projective dimension of ${}_RM$ (resp. $M_R$) will be denoted by $\operatorname{pd}_R M$ (resp. $\operatorname{pd} M_R$). For an additive category $\mathscr{A}$, the $*$-bounded homotopy category of $\mathscr{A}$ will be denoted by $\mathbb{K}^*(\mathscr{A})$, and if $\mathscr{A}$ is an abelian category, we will use $\mathbb{D}^*(\mathscr{A})$ to denote its $*$-bounded derived category, where $* \in \{\text{blank}, +, -, b\}$.

Let

$$X^\bullet = \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots$$

be a complex in $\operatorname{mod} R$. For any integer $n$, we set $Z^n(X^\bullet) = \operatorname{Ker} d^n$, $B^n(X^\bullet) = \operatorname{Im} d^{n-1}$, and $H^n(X^\bullet) = Z^n(X^\bullet)/B^n(X^\bullet)$. $X^\bullet$ is called *acyclic* (or *exact*) if $H^n(X^\bullet) = 0$ for any $n \in \mathbb{Z}$.

**Definition 2.1.** *([21–23])* (1) An acyclic complex $X^\bullet$ is called *totally acyclic* if each $X^i \in \operatorname{proj} R$ and $\operatorname{Hom}_R(X^\bullet, R)$ is acyclic. A module $M \in \operatorname{mod} R$ is *Gorenstein projective* if there exists some totally acyclic complex $X^\bullet$ such that $M \cong Z^0(X^\bullet)$. Denote by $\operatorname{Gproj} R$ the subcategory of $\operatorname{mod} R$ consisting of Gorenstein projective modules.

(2) Given a module $M \in \operatorname{mod} R$, the *Gorenstein projective dimension* $\operatorname{Gpd}_R M$ of $M$ is defined as $\operatorname{Gpd}_R M = \inf\{n :$ if there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$, where each $G_i \in \operatorname{Gproj} R\}$.

**Lemma 2.2.** *([5, Lemma 2.1]) Let $X^\bullet$ be an acyclic complex with each $X^i \in \operatorname{proj} R$ and $M$ a right $R$-module. If $\operatorname{pd} M_R < \infty$, then $M \otimes_R X^\bullet$ is acyclic. In this case, $\operatorname{Tor}_i^R(M, G) = 0$ for any $G \in \operatorname{Gproj} R$ and any integer $i \geq 1$.*

**Definition 2.3.** *([5])* A complex $X^\bullet \in \mathbb{D}^b(\operatorname{mod} R)$ is said to be *Gorenstein perfect* if $X^\bullet$ is isomorphic to some bounded complex of Gorenstein projective modules in $\mathbb{D}^b(\operatorname{mod} R)$. Denote by $\operatorname{Gperf}(R)$ the subcategory of $\mathbb{D}^b(\operatorname{mod} R)$ consisting of Gorenstein perfect complexes.

**Lemma 2.4.** *([5, Appendix A])*

(1) Let $M \in \operatorname{mod} R$. Then, $M \in \operatorname{Gperf}(R)$ if and only if $\operatorname{Gpd}_R M < \infty$.

(2) Let $X^\bullet \in \mathbb{D}^b(\operatorname{mod} R)$ be a bounded complex. If each $X^i$ has finite Gorenstein projective dimension, then $X^\bullet \in \operatorname{Gperf}(R)$.

(3) $\operatorname{Gperf}(R)$ is the smallest thick subcategory of $\mathbb{D}^b(\operatorname{mod} R)$ containing $\operatorname{Gproj} R$.

Denote by $F : \underline{\operatorname{Gproj} R} \to \mathbb{D}_{sg}(R)$ the Buchweitz embedding functor and by $\Pi_F : \mathbb{D}_{sg}(R) \to \mathbb{D}_{def}(R)$ the canonical quotient functor. From [4, Lemma 6.1] and [5, Theorem A.3], we have $\operatorname{Im} F = \operatorname{Gperf}(R)/\mathbb{K}^b(\operatorname{proj} R)$. Thus, $F$ induces a triangle-equivalence $\widehat{F} : \underline{\operatorname{Gproj} R} \to \operatorname{Gperf}(R)/\mathbb{K}^b(\operatorname{proj} R)$. Let $\iota : \operatorname{Gperf}(R)/\mathbb{K}^b(\operatorname{proj} R) \to \mathbb{D}^b(\operatorname{mod} R)/\mathbb{K}^b(\operatorname{proj} R)$ be the canonical embedding functor and $\pi_\iota : \mathbb{D}^b(\operatorname{mod} R)/\mathbb{K}^b(\operatorname{proj} R) \to \mathbb{D}^b(\operatorname{mod} R)/\operatorname{Gperf}(R)$ its induced quotient functor. Clearly, we have $F = \iota\widehat{F}$, and then $\pi_\iota F$ vanishes. Hence, there exists a unique triangle functor $\Theta : \mathbb{D}_{def}(R) \to \mathbb{D}^b(\operatorname{mod} R)/\operatorname{Gperf}(R)$ such that $\pi_\iota = \Theta\Pi_F$.

**Lemma 2.5.** *([5, Theorem A.5]) We have the following exact commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \underline{\operatorname{Gproj} R} \arrow[r, "F"] \arrow[d, "\widehat{F}"] & \mathbb{D}_{sg}(R) \arrow[r, "\Pi_F"] \arrow[d, equal] & \mathbb{D}_{def}(R) \arrow[r] \arrow[d, "\Theta"] & 0 \\
0 \arrow[r] & \operatorname{Gperf}(R)/\mathbb{K}^b(\operatorname{proj} R) \arrow[r, "\iota"] & \mathbb{D}^b(\operatorname{mod} R)/\mathbb{K}^b(\operatorname{proj} R) \arrow[r, "\pi_\iota"] & \mathbb{D}^b(\operatorname{mod} R)/\operatorname{Gperf}(R) \arrow[r] & 0
\end{tikzcd}$$

*with all vertical functors triangle-equivalences.*

**Definition 2.6.** *([14])* Let $\mathcal{T}, \mathcal{T}'$, and $\mathcal{T}''$ be triangulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is a diagram of triangulated categories and triangle functors

$$\begin{tikzcd}
\mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd} \tag{R}$$

which satisfies the following:

(R1) $(i^*, i_*)$, $(i_*, i^!)$, $(j_!, j^*)$, and $(j^*, j_*)$ are adjoint pairs;

(R2) $i_*$, $j_!$, and $j_*$ are fully faithful;

(R3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

By a *left recollement of triangulated categories*, we mean a diagram of triangulated categories and triangle functors consisting of the upper two rows of (R) which satisfy conditions (R1)–(R3), which involve only the functors $i^*, i_*, j_!$, and $j^*$.

If $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ in diagram (R) are abelian categories, and the six functors involved are additive functors, then we call (R) a *recollement of abelian categories*; see [8, 15] for details.

**Lemma 2.7.** *Let*

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \end{tikzcd} \tag{2.1}$$

*be a left recollement of triangulated categories. Assume that $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}''$ are thick subcategories of $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$, respectively. The following statements are equivalent:*

*(1) (2.1) restricts to the following left recollement:*

$$\begin{tikzcd} \mathcal{N}' \arrow[r, "i_*"] & \mathcal{N} \arrow[l, bend right, "i^*"'] \arrow[r, "j^*"] & \mathcal{N}''. \arrow[l, bend right, "j_!"'] \end{tikzcd} \tag{2.2}$$

*(2) (2.1) induces the following left recollement:*

$$\begin{tikzcd} \mathcal{T}'/\mathcal{N}' \arrow[r, "\overline{i_*}"] & \mathcal{T}/\mathcal{N} \arrow[l, bend right, "\overline{i^*}"'] \arrow[r, "\overline{j^*}"] & \mathcal{T}''/\mathcal{N}''. \arrow[l, bend right, "\overline{j_!}"'] \end{tikzcd} \tag{2.3}$$

*(3) $i^*(\mathcal{N}) \subseteq \mathcal{N}'$, $i_*(\mathcal{N}') \subseteq \mathcal{N}$, $j_!(\mathcal{N}'') \subseteq \mathcal{N}$, and $j^*(\mathcal{N}) \subseteq \mathcal{N}''$.*

*Proof.* To begin, we note that (1) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (3) are trivial. Hence, it suffices to show (3) $\Rightarrow$ (2). Assume that the conditions in (3) are satisfied; we will show $i^*(\mathcal{N}) = \mathcal{N}'$ and $j^*(\mathcal{N}) = \mathcal{N}''$. To do this, let $X' \in \mathcal{N}'$. Put $X = i_*X'$; it follows that $X \in \mathcal{N}$. Hence, $X' \cong i^*i_*(X') \cong i^*(X)$, and then $X' \in i^*(\mathcal{N})$. Thus, $i^*(\mathcal{N}) = \mathcal{N}'$ as desired. Similarly, one gets $j^*(\mathcal{N}) = \mathcal{N}''$. Consequently, by [12, Lemma 2.3], we get the left recollement (2.3). □

## 3. Recollement ideals and recollements of Gorenstein defect categories

In this section, assume that $R$ is an Artin algebra, and $e$ is an idempotent of $R$. We begin with the following definition.

**Definition 3.1.** *([17])* The idempotent ideal $ReR$ of $R$ is called a *recollement ideal* if the following conditions are satisfied:

(RI1) $ReR$ is a stratifying ideal, that is, $Re \otimes^{\mathbb{L}}_{eRe} eR \cong ReR$;

(RI2) $\operatorname{pd}_R ReR < \infty$;

(RI3) $\operatorname{pd} ReR_R < \infty$.

We remark that if the condition (RI1) is satisfied, then by [17, Theorem 1], the condition (RI2) is equivalent to $\mathrm{pd}_{eRe}\, eR < \infty$, and the condition (RI3) is equivalent to $\mathrm{pd}\, Re_{eRe} < \infty$. See also [16, Lemma 4.3(a) and Proposition 3.2(b)] for further details.

**Lemma 3.2.** *([17, Theorem 1]) The following assertions are equivalent.*

*(1) $ReR$ is a recollement ideal.*

*(2) We have the following recollement of bounded derived categories:*

$$\begin{tikzcd}[column sep=huge]
\mathbb{D}^b(\mathrm{mod}\, R/ReR) \arrow[r, "\mathbb{D}(i_e)=R/ReR\otimes^{\mathbb{L}}_{R/ReR}-"] & \mathbb{D}^b(\mathrm{mod}\, R) \arrow[l, bend right, "\mathbb{D}(i_e^{\otimes})=R/ReR\otimes^{\mathbb{L}}_{R}-"'] \arrow[l, bend left, "\mathbb{D}(i_e^{H})=\mathbb{R}\,\mathrm{Hom}_R(R/ReR,-)"] \arrow[r, "\mathbb{D}(s^e)=eR\otimes^{\mathbb{L}}_{R}-"] & \mathbb{D}^b(\mathrm{mod}\, eRe). \arrow[l, bend right, "\mathbb{D}(s^e_{\otimes})=Re\otimes^{\mathbb{L}}_{eRe}-"'] \arrow[l, bend left, "\mathbb{D}(s^e_{H})=\mathbb{R}\,\mathrm{Hom}_{eRe}(eR,-)"]
\end{tikzcd} \tag{DR}$$

From now on, let $ReR$ be a recollement ideal. In this case, the recollement (DR) of bounded derived categories always exists. By Lemma 2.5, we may view the Gorenstein defect category as the Verdier quotient of the bounded derived category by the subcategory of Gorenstein perfect complexes. We wonder whether and when (DR) induces a (left) recollement of Gorenstein defect categories.

We have the following fact.

**Lemma 3.3.** *Let $ReR$ be a recollement ideal. Then, both $i_e^{\otimes} : \mathrm{mod}\, R \to \mathrm{mod}\, R/ReR$ and $s^e_{\otimes} : \mathrm{mod}\, eRe \to \mathrm{mod}\, R$ preserve Gorenstein projective modules. Consequently, $i_e^{\otimes}$ and $s^e_{\otimes}$ are Gorenstein homologically finite.*

*Proof.* We only show that $i_e^{\otimes}$ preserves Gorenstein projective modules; the remaining proof involving $s^e_{\otimes}$ is similar. Let $M \in \mathrm{Gproj}\, R$. Then, there exists a totally acyclic complex $X^\bullet$ of $\mathrm{mod}\, R$ such that $M \cong Z^0(X^\bullet)$. It follows that $i_e^{\otimes}(X^\bullet) = R/ReR \otimes_R X^\bullet$; it is a complex of finitely generated projective $R/ReR$-modules. Because $\mathrm{pd}\, R/ReR_R < \infty$, it follows from Lemma 2.2 that $i_e^{\otimes}(X^\bullet)$ is acyclic. As $(i_e^{\otimes}, i_e)$ is an adjoint pair, one has

$$\mathrm{Hom}_{R/ReR}(i_e^{\otimes}(X^\bullet), R/ReR) \cong \mathrm{Hom}_R(X^\bullet, i_e(R/ReR)) \cong \mathrm{Hom}_R(X^\bullet, R/ReR).$$

Notice that $\mathrm{pd}\,_R R/ReR < \infty$; by [23, Theorem 2.20], we have $\mathrm{Hom}_R(X^\bullet, R/ReR)$, and hence, $\mathrm{Hom}_{R/ReR}(i_e^{\otimes}(X^\bullet), R/ReR)$ is acyclic. This implies that $i_e^{\otimes}(X^\bullet)$ is totally acyclic, and then $i_e^{\otimes}(M) \cong Z^0(i_e^{\otimes}(X^\bullet)) \in \mathrm{Gproj}\, R/ReR$. Therefore, $i_e^{\otimes}$ preserves Gorenstein projective modules. □

Let $X^\bullet$ be a complex of $\mathrm{mod}\, R$. The *length* $l(X^\bullet)$ of $X^\bullet$ is defined as the cardinal of the set $\{X^i \neq 0 | i \in \mathbb{Z}\}$. Let $n \in \mathbb{Z}$, denote by $X^\bullet_{\geqslant n}$ the complex with the $i$th component equal to $X^i$ whenever $i \geqslant n$ and to 0 elsewhere.

**Theorem 3.4.** *Let $ReR$ be a recollement ideal. Then,* (DR) *induces the following left recollement of Gorenstein defect categories:*

$$\begin{tikzcd}[column sep=huge]
\mathbb{D}_{def}(R/ReR) \arrow[r, "\widetilde{\mathbb{D}(i_e)}"] & \mathbb{D}_{def}(R) \arrow[l, bend right, "\widetilde{\mathbb{D}(i_e^{\otimes})}"'] \arrow[r, "\widetilde{\mathbb{D}(s^e)}"] & \mathbb{D}_{def}(eRe) \arrow[l, bend right, "\widetilde{\mathbb{D}(s^e_{\otimes})}"']
\end{tikzcd} \tag{3.1}$$

*if and only if $i_e$ and $s^e$ are Gorenstein homologically finite.*

*Proof.* For sufficiency, assume that $i_e$ and $s^e$ are Gorenstein homologically finite. Following Lemma 2.5, we view the Gorenstein defect category as a Verdier quotient of the bounded derived category by its subcategory consisting of Gorenstein perfect complexes. By Lemma 2.7, we only need to show that the four functors involved in the upper two rows of (DR) preserve the subcategories of Gorenstein perfect complexes.

To begin, we prove $\mathbb{D}(i_e)(\mathrm{Gperf}(R/ReR)) \subseteq \mathrm{Gperf}(R)$ and $\mathbb{D}(s^e)(\mathrm{Gperf}(R)) \subseteq \mathrm{Gperf}(eRe)$. To do this, take any $X^\bullet \in \mathrm{Gperf}(R/ReR)$; it follows that $X^\bullet \cong G^\bullet$ for some bounded complex $G^\bullet$ of Gorenstein projective $R/ReR$-modules. Because $i_e$ is exact, $\mathbb{D}(i_e)(X^\bullet) \cong \mathbb{D}(i_e)(G^\bullet) \cong i_e(G^\bullet)$. Notice that $i_e$ is Gorenstein homologically finite, we have that $i_e(G^\bullet)$ is a bounded complex with each degree being of finite Gorenstein projective dimension. From Lemma 2.4(2), we get $i_e(G^\bullet)$, and hence, $\mathbb{D}(i_e)(X^\bullet)$ is Gorenstein perfect. This means that $\mathbb{D}(i_e)(\mathrm{Gperf}(R/ReR)) \subseteq \mathrm{Gperf}(R)$. Similarly, we have $\mathbb{D}(s^e)(\mathrm{Gperf}(R)) \subseteq \mathrm{Gperf}(eRe)$.

Next, we prove $\mathbb{D}(i_e^{\otimes})(\mathrm{Gperf}(R)) \subseteq \mathrm{Gperf}(R/ReR)$ and $\mathbb{D}(s_{\otimes}^{e})(\mathrm{Gperf}(eRe)) \subseteq \mathrm{Gperf}(R)$. Let $G^\bullet$ be a bounded complex of Gorenstein projective $R$-modules. If $l(G^\bullet) = 1$, we may suppose $G^\bullet = G$ is the stalk complex concentrated in degree 0. Because $\mathrm{pd}\, R/ReR_R < \infty$, from Lemma 2.2, we get $\mathbb{D}(i_e^{\otimes})(G) \cong i_e^{\otimes}(G)$. By Lemma 3.3, $i_e^{\otimes}$ preserves Gorenstein projective modules. Then, we have that $\mathbb{D}(i_e^{\otimes})(G)$ is a stalk complex of Gorenstein projective module concentrated in degree 0. Hence, it follows from Lemma 2.4 that $\mathbb{D}(i_e^{\otimes})(G) \in \mathrm{Gperf}(R/ReR)$. Now, let $l(G^\bullet) = 2$. Suppose $G^n \neq 0$ and $G^i = 0$ for $i < n$. Then, we have the following triangle in $\mathbb{D}^b(\mathrm{mod}\, R)$:

$$G^n[-n-1] \to G^\bullet_{\geq n+1} \to G^\bullet \to G^n[-n].$$

Note that $l(G^n[-n-1]) = l(G^\bullet_{\geq n+1}) = 1$. Apply $\mathbb{D}(i_e^{\otimes})$ to the above triangle; we infer $\mathbb{D}(i_e^{\otimes})(G^\bullet) \in \mathrm{Gperf}(R/ReR)$ because $\mathrm{Gperf}(R/ReR)$ is a triangulated subcategory. Continuing inductively on the length $l(G^\bullet)$ of $G^\bullet$, we conclude that $\mathbb{D}(i_e^{\otimes})(G^\bullet) \in \mathrm{Gperf}(R/ReR)$. Hence, $\mathbb{D}(i_e^{\otimes})(\mathrm{Gperf}(R)) \subseteq \mathrm{Gperf}(R/ReR)$. By a similar argument, we also have $\mathbb{D}(s_{\otimes}^{e})(\mathrm{Gperf}(eRe)) \subseteq \mathrm{Gperf}(R)$.

For necessity, take any $G \in \mathrm{Gproj}\, R/ReR$. By Lemma 2.4(1), $G \in \mathrm{Gperf}(R/ReR)$. Hence, $G$ is a zero object in $\mathbb{D}_{def}(R/ReR)$. Note that $\widetilde{\mathbb{D}(i_e)}$ is the quotient functor of $\mathbb{D}(i_e)$. We deduce that $\widetilde{\mathbb{D}(i_e)}(G) \cong \mathbb{D}(i_e)(G) \cong i_e(G)$ is a zero object in $\mathbb{D}_{def}(R)$. Thus, $i_e(G) \in \mathrm{Gperf}(R)$. Following Lemma 2.4(1) again, $\mathrm{Gpd}_R\, i_e(G) < \infty$. Therefore, $i_e$ is Gorenstein homologically finite. Similarly, one can show that $s^e$ is also Gorenstein homologically finite. □

Let $ReR$ be a recollement ideal. By [19, Lemma 3.1], (DR) induces the following left recollement of singularity categories:

$$\begin{tikzcd}
\mathbb{D}_{sg}(R/ReR) \arrow[r, "\overline{\mathbb{D}(i_e)}"] & \mathbb{D}_{sg}(R) \arrow[l, bend right, "\overline{\mathbb{D}(i_e^{\otimes})}"'] \arrow[r, "\overline{\mathbb{D}(s^e)}"] & \mathbb{D}_{sg}(eRe). \arrow[l, bend right, "\overline{\mathbb{D}(s_{\otimes}^{e})}"']
\end{tikzcd} \tag{LR}$$

Thanks to the Buchweitz theorem, we obtain the following result.

**Proposition 3.5.** *Let ReR be a recollement ideal. Then,* (LR) *restricts to the following left recollement of stable categories of Gorenstein projective modules:*

$$\begin{tikzcd}
\underline{\mathrm{Gproj}\, R/ReR} \arrow[r] & \underline{\mathrm{Gproj}\, R} \arrow[l, bend right] \arrow[r] & \underline{\mathrm{Gproj}\, eRe} \arrow[l, bend right]
\end{tikzcd} \tag{3.2}$$

*if and only if $i_e$ and $s^e$ are Gorenstein homologically finite.*

*Proof.* For an arbitrary algebra $A$, by the Buchweitz theorem, $\underline{\mathrm{Gproj}}\,A$ could be viewed as a thick subcategory of $\mathbb{D}_{sg}(A)$ and $\mathbb{D}_{def}(A)$ as the Verdier quotient of $\mathbb{D}_{sg}(A)$ by $\underline{\mathrm{Gproj}}\,A$ (up to equivalences). Therefore, this conclusion follows from Lemma 2.7 and Theorem 3.4. □

An algebra $A$ is said to be *Cohen Macaulay free*, or simply *CM-free* ([24]), if every Gorenstein projective $A$-module is projective. We have the following results.

**Corollary 3.6.** *Let ReR be a recollement ideal. Assume that $i_e$ and $s^e$ are Gorenstein homologically finite; then, the following statements hold true.*

*(1) R is Gorenstein (resp. CM-free) if and only if $R/ReR$ and eRe are, as well.*

*(2) $\mathbb{D}(i_e)$ induces a triangle-equivalence $\mathbb{D}_{def}(R/ReR) \simeq \mathbb{D}_{def}(R)$ if and only if eRe is Gorenstein.*

*(3) $\mathbb{D}(s^e_{\otimes})$ induces a triangle-equivalence $\mathbb{D}_{def}(eRe) \simeq \mathbb{D}_{def}(R)$ if and only if $R/ReR$ is Gorenstein.*

*(4) $\mathbb{D}(i_e)$ induces a triangle-equivalence $\underline{\mathrm{Gproj}}\,R/ReR \simeq \underline{\mathrm{Gproj}}\,R$ if and only if eRe is CM-free.*

*(5) $\mathbb{D}(s^e_{\otimes})$ induces a triangle-equivalence $\underline{\mathrm{Gproj}}\,eRe \simeq \underline{\mathrm{Gproj}}\,R$ if and only if $R/ReR$ is CM-free.*

*Proof.* Because $ReR$ is a recollement ideal, and $i_e$ and $s^e$ are Gorenstein homologically finite, it follows from Theorem 3.4 and Proposition 3.5 that the left recollements (3.1) and (3.2) exist. Notice that, for an algebra $A$, $A$ is Gorenstein if and only if $\mathbb{D}_{def}(A) = 0$, whereas $A$ is CM-free if and only if $\underline{\mathrm{Gproj}}\,A = 0$. Hence, the assertions involving Gorensteinness follow from the left recollement (3.1), and the remaining assertions involving CM-freeness follow from the left recollement (3.2). □

For an algebra $A$, Luo and Huang [25] proposed the *Gorenstein projective conjecture*, which states that any $A$-module $M \in \mathrm{Gproj}\,A$ satisfying $\mathrm{Ext}^n_A(M, M) = 0$ for all $n > 0$ is necessarily projective. We note that this conjecture is still open and could be viewed as a special case of the Auslander-Reiten conjecture in the sense of [26]. Furthermore, both the Gorenstein projective conjecture and the Auslander-Reiten conjecture have close connections with other homological conjectures; see [25, 27] and the references therein for details.

Next, we will show that Proposition 3.5 is useful in the study of the Gorenstein projective conjecture. We need the following observation.

**Lemma 3.7.** *Let A and B be two algebras. Assume that $F : \underline{\mathrm{Gproj}}\,A \to \underline{\mathrm{Gproj}}\,B$ is a fully faithful triangle functor. If the Gorenstein projective conjecture holds for B, then it holds for A.*

*Proof.* Suppose the Gorenstein projective conjecture holds for $B$. For any $M, N \in \mathrm{Gproj}\,B$ and $i > 0$, it follows from [28, Lemma 5.2] that we have the following isomorphism:

$$\mathrm{Ext}^i_B(M, N) \cong \mathrm{Hom}_{\underline{\mathrm{Gproj}}\,B}(M, \Sigma^i N),$$

where $\Sigma^i$ denotes the $i$-shift functor in $\underline{\mathrm{Gproj}}\,B$. Now, take any $X \in \mathrm{Gproj}\,A$ with $\mathrm{Ext}^i_A(X, X) = 0$, $\forall i > 0$. It follows that $\mathrm{Ext}^i_B(F(X), F(X)) = 0$, $\forall i > 0$. Then, we get that $F(X)$ is projective by assumption. Because $F$ is fully faithful, we infer that $X$ is projective. Hence, the Gorenstein projective conjecture holds for $A$. □

**Proposition 3.8.** *Let ReR be a recollement ideal. Assume that $i_e$ and $s^e$ are Gorenstein homologically finite. Then the following statements hold true.*

*(1) The Gorenstein projective conjecture holds for R, which implies that it holds for eRe and $R/ReR$.*

*(2) If eRe is CM-free, then the Gorenstein projective conjecture holds for R if and only if it holds for R/ReR.*

*(3) If R/ReR is CM-free, then the Gorenstein projective conjecture holds for R if and only if it holds for eRe.*

*Proof.* (1) From Proposition 3.5, we get the left recollement (3.2). Then, by Lemma 3.7, we get this assertion.

(2) Because $eRe$ is CM-free, it follows from Corollary 3.6(4) that $\underline{\mathrm{Gproj}\,R/ReR}$ and $\underline{\mathrm{Gproj}\,R}$ are triangle-equivalent. Hence, this assertion follows from Lemma 3.7 directly.

(3) It is similar to (2). □

We have the following theorem.

**Theorem 3.9.** *Let ReR be a recollement ideal such that* $eR \in \mathrm{proj}\,eRe$*. Then,* (DR) *induces the following recollement of Gorenstein defect categories:*

$$\begin{tikzcd}
\mathbb{D}_{def}(R/ReR) \arrow[r, "\widetilde{\mathbb{D}(i_e)}"] & \mathbb{D}_{def}(R) \arrow[l, bend right=25, "\widetilde{\mathbb{D}(i_e^{\otimes})}"'] \arrow[l, bend left=25, "\widetilde{\mathbb{D}(i_e^{H})}"] \arrow[r, "\widetilde{\mathbb{D}(s^e)}"] & \mathbb{D}_{def}(eRe) \arrow[l, bend right=25, "\widetilde{\mathbb{D}(s_{\otimes}^e)}"'] \arrow[l, bend left=25, "\widetilde{\mathbb{D}(s_H^e)}"]
\end{tikzcd} \tag{3.3}$$

*if and only if* $i_e$*,* $s^e$ *and* $s_H^e$ *are Gorenstein homologically finite.*

*Proof.* For necessity, in view of Theorem 3.4, it remains to show that $s_H^e$ is Gorenstein homologically finite. Because (DR) induces the recollement (3.3), we get $\mathbb{D}(s_H^e)(\mathrm{Gperf}(eRe)) \subseteq \mathrm{Gperf}(R)$. Take any $M \in \mathrm{Gproj}\,eRe$. Because $eR \in \mathrm{proj}\,eRe$, $\mathbb{D}(s_H^e)(M) = \mathbb{R}\mathrm{Hom}_{eRe}(eR, M) \cong s_H^e(M) \in \mathrm{Gperf}(R)$. By Lemma 2.4(1), $\mathrm{Gpd}_R\, s_H^e(M) < \infty$. Hence, $s_H^e$ is Gorenstein homologically finite.

For sufficiency, assume that $i_e$, $s^e$, and $s_H^e$ are Gorenstein homologically finite. We view the Gorenstein defect category as a Verdier quotient of the bounded derived category by its subcategory consisting of Gorenstein perfect complexes. By the proof of Theorem 3.4, we have $\mathbb{D}(i_e^{\otimes})(\mathrm{Gperf}(R)) \subseteq \mathrm{Gperf}(R/ReR)$, $\mathbb{D}(i_e)(\mathrm{Gperf}(R/ReR)) \subseteq \mathrm{Gperf}(R)$, and $\mathbb{D}(s^e)(\mathrm{Gperf}(R)) \subseteq \mathrm{Gperf}(eRe)$. Now, we prove $\mathbb{D}(s_H^e)(\mathrm{Gperf}(eRe)) \subseteq \mathrm{Gperf}(R)$. To do this, take any bounded complex $X^\bullet$ of Gorenstein projective $eRe$-modules. We proceed by induction on the length $l(X^\bullet)$ of $X^\bullet$. If $l(X^\bullet) = 1$, we may suppose that $X^\bullet = X$ is a Gorenstein projective module concentrated in degree 0. Because $eR \in \mathrm{proj}\,eRe$, $\mathbb{D}(s_H^e)(X^\bullet) = \mathbb{R}\mathrm{Hom}_{eRe}(eR, X) \cong s_H^e(X)$. Notice that $s_H^e$ is Gorenstein homologically finite; we have $\mathrm{Gpd}_R\, s_H^e(X) < \infty$, and then $\mathbb{D}(s_H^e)(X^\bullet) \in \mathrm{Gperf}(R)$. Now, let $l(X^\bullet) = n \geq 2$, and assume the assertion holds true for any integer less than $n$. We may suppose $X^m \neq 0$ and $X^i = 0$ for $i < m$. Then, we have the following triangle in $\mathbb{D}^b(\mathrm{mod}\,eRe)$:

$$X^m[-m-1] \to X^\bullet_{\geq m+1} \to X^\bullet \to X^m[-m].$$

Apply the functor $\mathbb{D}(s_H^e)$ to it to obtain the following triangle in $\mathbb{D}^b(\mathrm{mod}\,R)$:

$$\mathbb{D}(s_H^e)(X^m)[-m-1] \to \mathbb{D}(s_H^e)(X^\bullet_{\geq m+1}) \to \mathbb{D}(s_H^e)(X^\bullet) \to \mathbb{D}(s_H^e)(X^m)[-m].$$

By the induction hypothesis, we have that both $\mathbb{D}(s_H^e)(X^m)[-m-1]$ and $\mathbb{D}(s_H^e)(X^\bullet_{\geq m+1})$ lie in $\mathrm{Gperf}(R)$. Hence, $\mathbb{D}(s_H^e)(X^\bullet) \in \mathrm{Gperf}(R)$. Therefore, $\mathbb{D}(s_H^e)(\mathrm{Gperf}(eRe)) \subseteq \mathrm{Gperf}(R)$.

Next, we claim $\mathbb{D}(i_e^{\otimes})(\mathrm{Gperf}(R)) = \mathrm{Gperf}(R/ReR)$ and $\mathbb{D}(s^e)(\mathrm{Gperf}(R)) = \mathrm{Gperf}(eRe)$. Indeed, let $Y^\bullet \in \mathrm{Gperf}(R/ReR)$. Apply $Z^\bullet = \mathbb{D}(i_e)(Y^\bullet)$ to obtain $Z^\bullet \in \mathrm{Gperf}(R)$. Notice that $Y^\bullet \cong \mathbb{D}(i_e^{\otimes})\mathbb{D}(i_e)(Y^\bullet) \cong \mathbb{D}(i_e^{\otimes})(Z^\bullet)$; we deduce $Y^\bullet \in \mathbb{D}(i_e^{\otimes})(\mathrm{Gperf}(R))$. Hence, we have $\mathbb{D}(i_e^{\otimes})(\mathrm{Gperf}(R)) = \mathrm{Gperf}(R/ReR)$. Similarly, $\mathbb{D}(s^e)(\mathrm{Gperf}(R)) = \mathrm{Gperf}(eRe)$.

Finally, it follows from [11, Proposition 2.5] that (DR) induces the recollement (3.3) of Gorenstein defect categories. This completes the proof. □

As a consequence of Theorem 3.9, we get the following result.

**Proposition 3.10.** *Let $ReR$ be a recollement ideal such that $eR \in \operatorname{proj} eRe$. Assume $\mathrm{pd}_R\, s^e_H(eRe) < \infty$; then,* (3.2) *can be extended to the following recollement of stable categories of Gorenstein projective modules:*

$$\begin{tikzcd} \underline{\mathrm{Gproj}\, R/ReR} \arrow[r] & \underline{\mathrm{Gproj}\, R} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \underline{\mathrm{Gproj}\, eRe} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \tag{3.4}$$

*if and only if $i_e$, $s^e$, and $s^e_H$ are Gorenstein homologically finite.*

*Proof.* Because $eR \in \operatorname{proj} eRe$, we have $\mathbb{R}\mathrm{Hom}_{eRe}(eR, eRe) \cong \mathrm{Hom}_{eRe}(eR, eRe) = s^e_H(eRe)$. As $\mathrm{pd}_R\, s^e_H(eRe) < \infty$, it follows that $\mathbb{R}\mathrm{Hom}_{eRe}(eR, eRe) \in \mathbb{K}^b(\operatorname{proj} R)$. By [19, Theorem 1.1], (DR) induces the following recollement of singularity categories:

$$\begin{tikzcd}[column sep=huge] \mathbb{D}_{sg}(R/ReR) \arrow[r, "\overline{\mathbb{D}(i_e)}"] & \mathbb{D}_{sg}(R) \arrow[l, bend right, "\overline{\mathbb{D}(i_e^{\otimes})}"'] \arrow[l, bend left, "\overline{\mathbb{D}(i_e^{H})}"] \arrow[r, "\overline{\mathbb{D}(s^e)}"] & \mathbb{D}_{sg}(eRe). \arrow[l, bend right, "\overline{\mathbb{D}(s^e_{\otimes})}"'] \arrow[l, bend left, "\overline{\mathbb{D}(s^e_H)}"] \end{tikzcd}$$

Following the Buchweitz theorem, the existence of the recollement (3.3) is equivalent to that of (3.4). Therefore, by Theorem 3.9, we have the recollement (3.4) if and only if $i_e$, $s^e$, and $s^e_H$ are Gorenstein homologically finite. □

## 4. Applications in triangular matrix algebras

In this section, $A$ and $B$ are two algebras, ${}_AM_B$ is an $A$-$B$-bimodule, and $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a triangular matrix algebra. By applying the results in Section 3, we will construct a (left) recollement of Gorenstein defect categories over $T$.

Recall that a left $T$-module is identified with a triple $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$, where $X \in \operatorname{mod} A$, $Y \in \operatorname{mod} B$, and $\phi : M \otimes_B Y \to X$ is an $A$-morphism. If there is no possible confusion, we shall omit the morphism $\phi$ and write $\begin{pmatrix} X \\ Y \end{pmatrix}$ for short. A $T$-morphism $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \to \begin{pmatrix} X' \\ Y' \end{pmatrix}_{\phi'}$ will be identified with a pair $\begin{pmatrix} f \\ g \end{pmatrix}$, where $f \in \mathrm{Hom}_A(X, X')$ and $g \in \mathrm{Hom}_B(Y, Y')$ such that the diagram

$$\begin{tikzcd} M \otimes_B Y \arrow[r, "\phi"] \arrow[d, "1\otimes g"'] & X \arrow[d, "f"] \\ M \otimes_B Y' \arrow[r, "\phi'"] & X' \end{tikzcd}$$

is commutative.

A sequence $0 \to \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\phi_1} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\phi_2} \xrightarrow{\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\phi_3} \to 0$ in $\operatorname{mod} T$ is exact if and only if $0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0$ and $0 \to Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \to 0$ are exact in $\operatorname{mod} A$ and $\operatorname{mod} B$, respectively. Indecomposable projective $T$-modules are exactly $\begin{pmatrix} P \\ 0 \end{pmatrix}$ and $\begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}$, where $P$ runs over indecomposable projective $A$-modules, and $Q$ runs over indecomposable projective $B$-modules. We refer the reader to [27, 29] for more details.

Denote by $e_B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ the idempotent of $T$. It is known that $A \cong T/Te_BT$ and $B \cong e_BTe_B$ as algebras. Furthermore, by [16, Example 3.4], we get that $Te_BT$ is a stratifying ideal of $T$. Because $e_BT \cong B$ as left $B$-modules, $e_BT$ is projective as a left $B$-module. Furthermore, $Te_B \cong M \oplus B$ as right $B$-modules. Hence, we get that $Te_BT$ is a recollement ideal if and only if $\operatorname{pd} M_B < \infty$.

**Definition 4.1.** *([13, Definition 1.1])* An $A$-$B$-bimodule $M$ is said to be *compatible* if the following two conditions are satisfied:

**(C1)** $M \otimes_B -$ carries every acyclic complex of projective $B$-modules to an acyclic $A$-complex.

**(C2)** $\operatorname{Hom}_A(-, M)$ carries every totally acyclic $A$-complex to an acyclic $B$-complex. In this case, we call $M$ left compatible for simplicity.

We need the following facts.

**Lemma 4.2.** *([13, Theorem 1.4]) Let* $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a triangular matrix algebra such that* ${}_AM_B$ *is compatible. Then,* $\begin{pmatrix} X \\ Y \end{pmatrix}_{\phi} \in \operatorname{Gproj} T$ *if and only if* $Y \in \operatorname{Gproj} B$*, and* $\phi : M \otimes_B Y \to X$ *is an injective A-morphism with* $\operatorname{Coker} \phi \in \operatorname{Gproj} A$.

**Lemma 4.3.** *([9, Corollary 4.2]) Let* $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a triangular matrix algebra such that* ${}_AM_B$ *is compatible. The following statements hold true.*

*(1)* $\operatorname{Gpd}_T \begin{pmatrix} X \\ 0 \end{pmatrix} = \operatorname{Gpd}_A X$.

*(2) Assume that* ${}_AM \otimes_B - : \operatorname{mod} B \to \operatorname{mod} A$ *is Gorenstein homologically finite. Then,* $\operatorname{Gpd}_T \begin{pmatrix} 0 \\ Y \end{pmatrix} < \infty$ *if and only if* $\operatorname{Gpd}_B Y < \infty$.

**Proposition 4.4.** *(Compare to [12, Proposition 3.6]) Let* $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a triangular matrix algebra such that M is left compatible. If* $\operatorname{pd} M_B < \infty$*, then we have the following left recollement of Gorenstein defect categories:*

$$\begin{tikzcd} \mathbb{D}_{def}(A) \arrow[r, "\widetilde{\mathbb{D}(i_{e_B})}"] & \mathbb{D}_{def}(T) \arrow[l, bend right, "\widetilde{\mathbb{D}(i_{e_B}^{\otimes})}"'] \arrow[r, "\widetilde{\mathbb{D}(s^{e_B})}"] & \mathbb{D}_{def}(B). \arrow[l, bend right, "\widetilde{\mathbb{D}(s_{\otimes}^{e_B})}"'] \end{tikzcd} \tag{4.1}$$

*In this case, we have the following left recollement of stable categories of Gorenstein projective modules:*

$$\begin{tikzcd} \underline{\mathrm{Gproj}\,A} \arrow[r] & \underline{\mathrm{Gproj}\,T} \arrow[l, bend right] \arrow[r] & \underline{\mathrm{Gproj}\,B} \arrow[l, bend right] . \end{tikzcd} \tag{4.2}$$

*Consequently, the Gorenstein projective conjecture holds for $T$ implies that it holds for $A$ and $B$.*

*Proof.* Because $\mathrm{pd}\, M_B < \infty$, it follows that $Te_BT$ is a recollement ideal. Furthermore, as $M$ is left compatible, by the proof of [13, Proposition 1.3], we get that ${}_AM_B$ is compatible. Note that $i_{e_B}(G) \cong \begin{pmatrix} G \\ 0 \end{pmatrix}$ for any $G \in \mathrm{Gproj}\, A$. From Lemma 4.2, we have $i_{e_B}(G) \in \mathrm{Gproj}\, T$. This implies that $i_{e_B}$ is Gorenstein homologically finite. Similarly, one gets that $s^{e_B}$ is also Gorenstein homologically finite. Then, by Theorem 3.4 and Proposition 3.5, we obtain the left recollements (4.1) and (4.2), respectively. Finally, in view of (4.2), it follows from Lemma 3.7 that the Gorenstein projective conjecture holds for $T$, which implies that it holds for $A$ and $B$. □

As a consequence of Proposition 4.4, we get the following.

**Corollary 4.5.** *Let* $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a triangular matrix algebra such that $M$ is left compatible, and* $\mathrm{pd}\, M_B < \infty$. *Then, the folowing statements hold true.*

*(1) $T$ is Gorenstein (resp. CM-free) if and only if $A$ and $B$ are, as well.*

*(2) $\widetilde{\mathbb{D}(i_{e_B})} : \mathbb{D}_{def}(A) \to \mathbb{D}_{def}(T)$ is a triangle-equivalence if and only if $B$ is Gorenstein.*

*(3) $\widetilde{\mathbb{D}(s^{e_B}_{\otimes})} : \mathbb{D}_{def}(B) \to \mathbb{D}_{def}(T)$ is a triangle-equivalence if and only if $A$ is Gorenstein.*

*(4) We have a triangle-equivalence $\underline{\mathrm{Gproj}\, A} \simeq \underline{\mathrm{Gproj}\, T}$ provided that $B$ is CM-free. In this case, the Gorenstein projective conjecture holds for $T$ if and only if it holds for $A$.*

*(5) We have a triangle-equivalence $\underline{\mathrm{Gproj}\, B} \simeq \underline{\mathrm{Gproj}\, T}$ provided that $A$ is CM-free. In this case, the Gorenstein projective conjecture holds for $T$ if and only if it holds for $B$.*

The following theorem unifies [12, Theorem 3.12] and [20, Claim 3].

**Proposition 4.6.** *Let* $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a triangular matrix algebra such that $M$ is left compatible. If* $\mathrm{pd}\, M_B < \infty$, *then we have the following recollement of Gorenstein defect categories:*

$$\begin{tikzcd}[column sep=huge] \mathbb{D}_{def}(A) \arrow[r, "\widetilde{\mathbb{D}(i_{e_B})}"] & \mathbb{D}_{def}(T) \arrow[l, bend right=40, "\widetilde{\mathbb{D}(i^{\otimes}_{e_B})}"'] \arrow[l, bend left=40, "\widetilde{\mathbb{D}(i^{H}_{e_B})}"] \arrow[r, "\widetilde{\mathbb{D}(s^{e_B})}"] & \mathbb{D}_{def}(B) \arrow[l, bend right=40, "\widetilde{\mathbb{D}(s^{e_B}_{\otimes})}"'] \arrow[l, bend left=40, "\widetilde{\mathbb{D}(s^{e_B}_{H})}"] \end{tikzcd} \tag{4.3}$$

*if and only if ${}_AM \otimes_B -$ is Gorenstein homologically finite.*

*Proof.* To begin, by the foregoing proof in Proposition 4.4, we obtain that ${}_AM_B$ is compatible, and the functors $i_{e_B}$ and $s^{e_B}$ are Gorenstein homologically finite.

Because $\mathrm{pd}\, M_B < \infty$, $Te_BT$ is a recollement ideal. Notice that $e_BT \cong B$ as left $B$-modules; it follows that $e_BT$ is projective as a left $B$-module. In view of Theorem 3.9, it suffices to show that ${}_AM \otimes_B -$ is

Gorenstein homologically finite if and only if $s_H^{e_B}$ is, as well. To do this, take any $Y \in \operatorname{Gproj} B$. From [30, Section 2], we have $s_H^{e_B}(Y) \cong \begin{pmatrix} 0 \\ Y \end{pmatrix}$. Consider the following exact sequence of $T$-modules:

$$0 \to \begin{pmatrix} M \otimes_B Y \\ 0 \end{pmatrix} \to \begin{pmatrix} M \otimes_B Y \\ Y \end{pmatrix} \to \begin{pmatrix} 0 \\ Y \end{pmatrix} \to 0.$$

From it, we get $\operatorname{Gpd}_T s_H^{e_B}(Y) < \infty$ if and only if $\operatorname{Gpd}_T \begin{pmatrix} M \otimes_B Y \\ 0 \end{pmatrix} < \infty$. By Lemma 4.3(1), one has $\operatorname{Gpd}_T \begin{pmatrix} M \otimes_B Y \\ 0 \end{pmatrix} = \operatorname{Gpd}_A M \otimes_B Y$. Hence, $\operatorname{Gpd}_T s_H^{e_B}(Y) < \infty$ if and only if $\operatorname{Gpd}_A M \otimes_B Y < \infty$. Therefore, we conclude ${}_AM \otimes_B -$ is Gorenstein homologically finite if and only if $s_H^{e_B}$ is, as well. □

**Corollary 4.7.** *(Compare to [13, Theorem 3.5]) Let* $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a triangular matrix algebra such that* $\operatorname{pd}_A M$ *and* $\operatorname{pd} M_B$ *are finite. Then, we have the following recollement of stable categories of Gorenstein projective modules:*

$$\begin{tikzcd}
\underline{\operatorname{Gproj} A} \arrow[r] & \underline{\operatorname{Gproj} T} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \underline{\operatorname{Gproj} B} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd} \tag{4.4}$$

*if and only if* ${}_AM \otimes_B -$ *is Gorenstein homologically finite.*

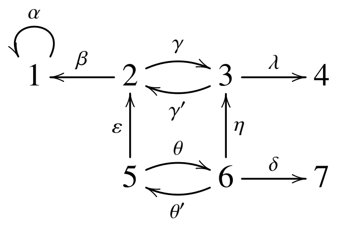
*Proof.* Because $\operatorname{pd}_A M$ is finite, it follows from [13, Proposition 1.3] that $M$ is left compatible. Notice that $\operatorname{pd} M_B$ is also finite; from [19, Theorem 4.1], we have the following recollement of singularity categories:

$$\begin{tikzcd}
\mathbb{D}_{sg}(A) \arrow[r, "\overline{\mathbb{D}(i_{e_B})}"] & \mathbb{D}_{sg}(T) \arrow[l, bend right, "\overline{\mathbb{D}(i_{e_B}^{\otimes})}"'] \arrow[l, bend left, "\overline{\mathbb{D}(i_{e_B}^{H})}"] \arrow[r, "\overline{\mathbb{D}(s^{e_B})}"] & \mathbb{D}_{sg}(B). \arrow[l, bend right, "\overline{\mathbb{D}(s_{\otimes}^{e_B})}"'] \arrow[l, bend left, "\overline{\mathbb{D}(s_H^{e_B})}"]
\end{tikzcd} \tag{4.5}$$

For sufficiency, let ${}_AM \otimes_B -$ be Gorenstein homologically finite. Then, by Proposition 4.6, the recollement (4.3) of Gorenstein defect categories exists. Combining (4.3) with (4.5) and applying the Buchweitz theorem, we obtain the recollement (4.4).

For necessity, assume the recollement (4.4) exists. Combining (4.4) with (4.5), we have the recollement (4.3) of Gorenstein defect categories. Hence, by Proposition 4.6, we deduce that ${}_AM \otimes_B -$ is Gorenstein homologically finite. □

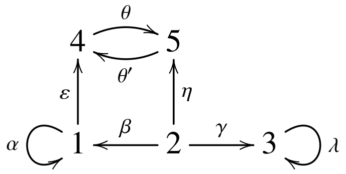
**Example 4.8.** (1) Let $T$ be a finite dimensional $k$-algebra over a field $k$ given by the following quiver:



with relations $\{\alpha^2, \gamma\gamma', \gamma'\gamma, \lambda\gamma, \lambda\eta, \theta\theta', \theta'\theta, \delta\theta, \gamma\varepsilon - \eta\theta, \gamma'\eta - \varepsilon\theta'\}$. Let $e_i$ be the idempotent of $T$ corresponding to the vertex $i$, and put $e = e_1 + e_2 + e_3 + e_4$. Denote $A = eTe$ and $B = (1-e)T(1-e)$. It follows that $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with $M = eT(1-e)$. Clearly, $M_B$ is projective, and $\mathrm{pd}_A\, M = 1 < \infty$. Note that $B$ is of radical square zero but not self-injective. Following [24], $B$ is CM-free. Hence, we obtain that ${}_AM \otimes_B -$ is Gorenstein homologically finite. By Proposition 4.6, we get the following recollement of Gorenstein defect categories:

$$\begin{tikzcd}
\mathbb{D}_{def}(A) \arrow[r] & \mathbb{D}_{def}(T) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbb{D}_{def}(B). \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(2) Let $T$ be a finite dimensional $k$-algebra over a field $k$ given by the following quiver:



with relations $\{\alpha^2, \alpha\beta, \lambda^2, \theta\theta', \theta'\theta, \varepsilon\beta - \theta'\eta\}$. Let $e_i$ be the idempotent of $T$ corresponding to the vertex $i$, and put $e = e_4 + e_5$. Denote $A = eTe$ and $B = (1-e)T(1-e)$. Then, $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with $M = eT(1-e)$. It is easy to check that ${}_AM$ and $M_B$ are projective. Moreover, we have that $A$ is self-injective. Consequently, ${}_AM \otimes_B -$ is Gorenstein homologically finite. From Corollary 4.7, we get the following recollement of stable categories of Gorenstein projective modules:

$$\begin{tikzcd}
\underline{\mathrm{Gproj}\, A} \arrow[r] & \underline{\mathrm{Gproj}\, T} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \underline{\mathrm{Gproj}\, B}. \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

## Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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