



Research article

A note on the Cameron-Praeger conjecture

Ximin Wang¹, Zheng Huang^{1,*}, Xiaogang Zhang¹ and Weijun Liu²

¹ Department of Mathematics and Statistics, Hunan University of Science and Technology, Xiangtan 411201, China

² College of Education, Guangdong University of Science and Technology, Guangdong 523083, China

* **Correspondence:** Email: huangzhengmath@163.com.

Abstract: The Cameron-Praeger conjecture stands as a central problem at the intersection of group theory and combinatorial design, and has inspired sustained research by mathematicians worldwide for decades. In this paper, we took a step forward in the proof of the Cameron-Praeger conjecture. We studied a special case of the famous Cameron-Praeger conjecture in design theory and proved that there are no block-transitive 6 -($v, k, 2$) designs with k dividing v .

Keywords: Cameron-Praeger conjecture; block-transitive; Camina-Gagen condition; 6-designs; 3-homogeneous permutation groups

1. Introduction

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be an incidence structure. Here, $|\mathcal{P}| = v$, and each element of \mathcal{B} , called a block, is a k -subset of the point set \mathcal{P} . The structure \mathcal{D} is said to be a t -(v, k, λ) design precisely when every t -subset of \mathcal{P} is contained in exactly λ distinct blocks. If $\lambda = 1$, the design is further referred to as a *Steiner design*. We call a design *nontrivial* whenever $t < k < v$. Any permutation of \mathcal{P} mapping \mathcal{B} onto itself is referred to as an *automorphism* of \mathcal{D} . The set of all such automorphisms of \mathcal{D} forms a group under the composition of permutations, denoted $\text{Aut}(\mathcal{D})$. An automorphism group of a design is defined as any $G \leq \text{Aut}(\mathcal{D})$. We say that \mathcal{D} is G -block-transitive when G acts transitively on the blocks. Similarly, let s be a positive integer. Then \mathcal{D} is point s -transitive (respectively point s -homogeneous) if G acts s -transitively (respectively s -homogeneously) on the points. In the literature, it is also common to attribute these properties directly to the group G , saying that G is block-transitive, point s -transitive, or point s -homogeneous.

The classification research on G -block-transitive designs mainly focuses on the following three aspects.

- (1) Investigations into the upper bound of the parameter t in G -block-transitive designs.
- (2) Classification of G -block-transitive but point-imprimitive designs.
- (3) Classification of G -block-transitive point-primitive designs.

The research on the classification of G -block-transitive designs is quite extensive, with most conclusions focusing on 2-designs. In 1976, Clapham [1] aimed to classify the block-transitive 2- $(v, 3, 1)$ designs using 3/2-transitive permutation groups. In 1989, Delandtsheer and Doyen [2] initiated research on block-transitive 2-designs and established an upper bound for the number of points. In 1996, Camina [3] reduced line-transitive point-primitive linear spaces to two cases: Those with elementary abelian socles and those with nonabelian simple socles. Subsequently, in 2001, Camina and Praeger [4] extended this result to line-transitive point-quasiprimitive linear spaces. The latest advancements up to 2025 were comprehensively surveyed in a review article by Tian et al. [5], and thus are not repeated here. Additional information can be found in the review.

The upper bound of the parameter t when nontrivial G -block-transitive t -designs exist has long attracted extensive attention. One of the most significant results in this area was established by Camina and Praeger [6] in 1993. They proved that for a t -design \mathcal{D} ($t \geq 2$), if $G \leq \text{Aut}(\mathcal{D})$ is block-transitive, then G acts point $\lfloor \frac{t}{2} \rfloor$ -homogeneously on \mathcal{D} . According to this conclusion, when $t \geq 4$, all G -block-transitive t -designs are point 2-homogeneous, and thus also must be point-primitive. Therefore, the characterization of G -block-transitive t -designs with $t \geq 4$ relies on the classification of finite s -homogeneous groups. Using this conclusion, in the early 1990s, Cameron and Praeger further ruled out the existence of nontrivial G -block-transitive t -designs with $t \geq 8$ and proposed the following famous Cameron-Praeger conjecture (hereinafter referred to as the C-P conjecture).

Conjecture 1.1. (*C-P Conjecture*) *Nontrivial G -block-transitive 6-designs do not exist.*

As of now, the fundamental problem of whether G -block-transitive 7-designs exist has not been fully resolved in the field. Huber proved the nonexistence of nontrivial G -block-transitive 7- $(v, k, 1)$ designs in 2010. Gong and Liu proved the nonexistence of nontrivial G -block-transitive 7- (v, k, λ) designs with $\lambda \leq 5$, but for the general case of λ , there is still no complete answer so far. Since every nontrivial 7-design must be a 6-design, if the Cameron-Praeger conjecture can be proved to hold, the case of $t = 7$ will naturally be resolved. Since G -block-transitive and even flag-transitive 5-designs exist, research on the conjecture has become crucial for studying the upper bound of the parameter t in G -block-transitive designs. To date, the most significant progress has been made by Huber, who resolved the case of $\lambda = 1$.

Theorem 1.2. [7] *Let \mathcal{D} be a nontrivial 6- $(v, k, 1)$ design. Then a group $G \leq \text{Aut}(\mathcal{D})$ cannot be G -block-transitive, except possibly for the case of $G \cong \text{P}\Gamma\text{L}(2, p^e)$, where e is an odd prime power and $p = 2$ or 3.*

In 2014, Tan and Liu [8] conducted corresponding studies on this conjecture and proved the nonexistence of nontrivial G -block-transitive 6-designs with $k \leq 10^4$. In the past decade, there has been little research progress on the Cameron-Praeger conjecture.

In 1984, Camina and Gagen [9] considered block-transitive Steiner 2-designs with k dividing v and obtained the famous Camina-Gagen Theorem: Consider a nontrivial Steiner 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with $k \mid v$. If $G \leq \text{Aut}(\mathcal{D})$ is block-transitive on \mathcal{B} , \mathcal{D} is flag-transitive. In 2024, building on Huber's work [7], Huang and Liu [10, 11] established the nonexistence of nontrivial block-transitive t - $(v, k, 1)$

designs with k dividing v when $t = 4, 5, 6$. According to the findings above, it is natural to explore whether these results remain valid for non-Steiner designs. By summarizing the aforementioned work, we formulate the Camina-Gagen condition: $k \mid v$.

Because of the lack of significant breakthroughs in the C-P conjecture for many years, we considered the following weakened conjecture to facilitate progress.

Conjecture 1.3. (*Weakened C-P Conjecture*) *When one of the following additional conditions is imposed, nontrivial G -block-transitive 6-designs do not exist:*

- (i) $k \mid v$ (*Camina-Gagen condition*);
- (ii) *Block-primitive*.

In this article, we consider whether block-transitive $6-(v, k, 2)$ designs can exist when the Camina-Gagen condition is imposed, and obtain the following theorem:

Theorem 1.4. *There are no nontrivial G -block-transitive $6-(v, k, 2)$ designs with k dividing v .*

Cameron's theorems on the subgroups of $\text{PSL}(2, q)$ acting on the projective line [12], together with the categorized results of finite 3-homogeneous permutation groups provide the foundation for our analysis. Here, $\text{PSL}(2, q)$ stands for the projective special linear group of degree 2 over the finite field with q elements. Huber's articles [7, 13] also give us a lot of enlightenment.

2. Preliminaries

This section introduces several results on designs and groups. First, we present some well-known facts about designs.

Lemma 2.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial $t-(v, k, \lambda)$ design. Here, λ_2 denotes the number of blocks containing two distinct points. Then the following hold:*

- (i) $bk = vr$;
- (ii) $\binom{v}{t}\lambda = b\binom{k}{t}$;
- (iii) $r(k-1) = \lambda_2(v-1)$ for $t \geq 2$.

Corollary 2.2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial $t-(v, k, \lambda)$ design, and thus*

$$\lambda \binom{v-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}}$$

for each positive integer $s \leq t$

Theorem 2.3. [6] *Let \mathcal{D} be a t -design with $t \geq 2$. Then the following hold:*

- (i) *If $G \leq \text{Aut}(\mathcal{D})$ acts block-transitively on \mathcal{D} , it follows that G is point $\lfloor t/2 \rfloor$ -homogeneous on \mathcal{D} .*
- (ii) *If $G \leq \text{Aut}(\mathcal{D})$ acts flag-transitively on \mathcal{D} , it follows that G is point $\lfloor (t+1)/2 \rfloor$ -homogeneous on \mathcal{D} .*

Next, before proceeding to the proof of our main theorems, we present several lemmas, which will be utilized in Section 3 to establish our results. Let $q = p^e$ be a prime. Set $n = (q - 1, 2)$ and $H \leq \text{PSL}(2, q)$. For a positive integer l , to denote N_l use the number of orbits of H with size l . We consider the action of the subgroups of $\text{PSL}(2, q)$ on the projective line, then refer to [12].

Lemma 2.4. [12] *Let H be a cyclic group of order c , where $c \mid \frac{q \pm 1}{n}$. We obtain the following:*

- (i) *If c divides $\frac{q+1}{n}$, there are $\frac{q+1}{c}$ orbits of size c .*
- (ii) *If c divides $\frac{q-1}{n}$, there are 2 orbits of size 1 and $\frac{q-1}{c}$ orbits of size c .*

Lemma 2.5. [12] *Let H be a dihedral group of order $2c$, where $c \mid \frac{q \pm 1}{n}$. Then we have the following.*

- (i) *When $q \equiv 1 \pmod{4}$, we obtain the following:*
 - (a) *If c divides $\frac{q+1}{2}$, there are 2 orbits of size c and $\frac{q+1-2c}{2c}$ orbits of size $2c$.*
 - (b) *If c divides $\frac{q-1}{2}$, there are 2 orbits of size c , 1 orbit of size 2, and $\frac{q-1-2c}{2c}$ orbits of size $2c$. The only exception is when $c = 2$, in which case, there are 3 orbits of size 2 and $\frac{q-5}{4}$ orbits of size 4.*
- (ii) *When $q \equiv 1 \pmod{4}$, we obtain the following:*
 - (a) *If c divides $\frac{q+1}{2}$, there are $\frac{q+1}{2c}$ orbits of size $2c$.*
 - (b) *If c divides $\frac{q-1}{2}$, there are $\frac{q-1}{2c}$ orbits of size $2c$ and 1 orbit of size 2.*
- (iii) *When $q \equiv 0 \pmod{2}$, we obtain the following:*
 - (a) *If c divides $q + 1$, there is 1 orbit of size c and $\frac{q+1-c}{2c}$ orbits of size $2c$.*
 - (b) *If c divides $q - 1$, there is 1 orbit of size 2, 1 orbit of size c , and $\frac{q-1-c}{2c}$ orbits of size $2c$.*

Lemma 2.6. [12] *Let H be an elementary abelian group of order $\bar{q} \mid q$. Then we obtain $N_1 = 1$ and $N_{\bar{q}} = q/\bar{q}$.*

Lemma 2.7. [12] *Let $H = E \rtimes C$, where E is an elementary abelian group of order \bar{q} with \bar{q} dividing q , and C is a cyclic group of order c , such that $c \mid (\bar{q} - 1, q - 1)$. We obtain $N_{\bar{q}} = 1$, $N_1 = 1$ and $N_{c\bar{q}} = (q - \bar{q})/(c\bar{q})$.*

Lemma 2.8. [12] *Let $H = \text{PSL}(2, \bar{q})$ with $\bar{q}^m = q$ for some integer $m \geq 1$. If m is even and $N_{\bar{q}+1} = 1$, we obtain $N_{\bar{q}(\bar{q}-1)} = 1$, and all the remaining orbits are regular.*

Lemma 2.9. [12] *Let $H = \text{PGL}(2, \bar{q})$ with $\bar{q}^m = q$, and $m > 1$ being even. We obtain $N_{\bar{q}+1} = 1$, $N_{\bar{q}(\bar{q}-1)} = 1$, and all the remaining orbits are regular.*

Lemma 2.10. [12] *Let $H \cong A_4$. Then we have the following.*

- (i) *When $q \equiv 1 \pmod{4}$, we obtain the following:*
 - (a) *If 3 divides $\frac{q+1}{2}$, there is 1 orbit of size 6 and $\frac{q-5}{12}$ orbits of size 12.*
 - (b) *If 3 divides $\frac{q-1}{2}$, there are 2 orbits of size 4, 1 orbit of size 6, and $\frac{q-13}{12}$ orbits of size 12.*
 - (c) *If 3 divides q , there is 1 orbit of size 4, 1 orbit of size 6, and $\frac{q-9}{12}$ orbits of size 12.*

- (ii) When $q \equiv 3 \pmod{4}$, we obtain the following:
- (a) If 3 divides $\frac{q+1}{2}$, there are $\frac{q+1}{12}$ orbits of size 12.
 - (b) If 3 divides $\frac{q-1}{2}$, there are 2 orbits of size 4 and $\frac{q-7}{12}$ orbits of size 12.
 - (c) If 3 divides q , there is 1 orbit of size 4 and $\frac{q-3}{12}$ orbits of size 12.
- (iii) When $q = 2^e$, $e \equiv 0 \pmod{2}$, there is 1 orbit of size 1, 1 orbit of size 4, and $\frac{q-4}{12}$ orbits of size 12.

Lemma 2.11. [12] Let $H \cong S_4$. Then the following hold.

- (i) When $q \equiv 1 \pmod{8}$, we obtain the following:
- (a) If 3 divides $\frac{q+1}{2}$, there is 1 orbit of size 6, 1 orbit of size 12, and $\frac{q-17}{24}$ orbits of size 24.
 - (b) If 3 divides $\frac{q-1}{2}$, there is 1 orbit of size 6, 1 orbit of size 8, 1 orbit of size 12, and $\frac{q-25}{24}$ orbits of size 24.
 - (c) If 3 divides q , there is 1 orbit of size 4, 1 orbit of size 6, and $\frac{q-9}{24}$ orbits of size 24.
- (ii) when $q \equiv -1 \pmod{8}$, we obtain the following:
- (a) If 3 divides $\frac{q+1}{2}$, there are $\frac{q+1}{24}$ orbits of size 24.
 - (b) If 3 divides $\frac{q-1}{2}$, there is 1 orbit of size 8 and $\frac{q-7}{24}$ orbits of size 24.

Lemma 2.12. [12] Let $H \cong A_5$. Then the following hold.

- (i) When $q \equiv 1 \pmod{4}$, we obtain the following:
- (a) If $q = 5^e$, $e \equiv 1 \pmod{2}$, there is 1 orbit of size 6 and $\frac{q-5}{60}$ orbits of size 60.
 - (b) If $q = 5^e$, $e \equiv 0 \pmod{2}$, there is 1 orbit of size 6, 1 orbit of size 20, and $\frac{q-25}{60}$ orbits of size 60.
 - (c) If 15 divides $\frac{q+1}{2}$, there is 1 orbit of size 30 and $\frac{q-29}{60}$ orbits of size 60.
 - (d) If 3 divides $\frac{q+1}{2}$ and 5 divides $\frac{q-1}{2}$, there is 1 orbit of size 12, 1 orbit of size 30, and $\frac{q-41}{60}$ orbits of size 60.
 - (e) If 3 divides $\frac{q-1}{2}$ and 5 divides $\frac{q+1}{2}$, there is 1 orbit of size 20, 1 orbit of size 30, and $\frac{q-49}{60}$ orbits of size 60.
 - (f) If 15 divides $\frac{q-1}{2}$, there is 1 orbit of size 12, 1 orbit of size 20, 1 orbit of size 30, and $\frac{q-61}{60}$ orbits of size 60.
 - (g) If 3 divides q and 5 divides $\frac{q+1}{2}$, there is 1 orbit of size 10 and $\frac{q-9}{60}$ orbits of size 60.
 - (h) If 3 divides q and 5 divides $\frac{q-1}{2}$, there is 1 orbit of size 10, 1 orbit of size 12, and $\frac{q-21}{60}$ orbits of size 60.
- (ii) When $q \equiv 3 \pmod{4}$, we obtain the following:
- (a) If 15 divides $\frac{q+1}{2}$, there are $\frac{q+1}{60}$ orbits of size 60.
 - (b) If 3 divides $\frac{q+1}{2}$ and 5 divides $\frac{q-1}{2}$, there is 1 orbit of size 12 and $\frac{q-11}{60}$ orbits of size 60.
 - (c) If 3 divides $\frac{q-1}{2}$ and 5 divides $\frac{q+1}{2}$, there is 1 orbit of size 20 and $\frac{q-19}{60}$ orbits of size 60.
 - (d) If 1 divides $\frac{q-1}{2}$, there is 1 orbit of size 20, 1 orbit of size 12, and $\frac{q-31}{60}$ orbits of size 60.

3. The proof of main theorems

This section presents the proof of the central theorems of this paper.

Assume that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a nontrivial 6 -($v, k, 2$) design and $G \leq \text{Aut}(\mathcal{D})$ acting block-transitively on \mathcal{D} . This hypothesis remains in force for the whole section. Our starting point is Theorem 2.3(i). This result permits us to restrict attention exclusively to finite permutation groups that are 3-homogeneous. In this setting, G is necessarily one of two distinct categories.

(A) Affine type: There is a regular $T \trianglelefteq G$. T is an elementary abelian group of order $v = 2^d$. We identify G with a group of affine transformations on the vector space $V = V(d, 2)$. Each such transformation is of the form

$$x \mapsto x^g + u$$

where $g \in G_0$ and $u \in V$. The possibilities are listed below.

- (1) G is isomorphic to $\text{AFL}(1, 32)$, $\text{AFL}(1, 8)$, and $\text{AGL}(1, 8)$.
- (2) G_0 is isomorphic to $\text{SL}(d, 2)$, with $d \geq 2$.
- (3) G_0 is isomorphic to A_7 , with $v = 2^4$.

(B) Almost simple type: There is a simple $N \trianglelefteq G$, and $N \leq G \leq \text{Aut}(N)$. Specifically, the possibilities for N and the degree v are explicitly listed below.

- (1) M_v with $v \in \{11, 12, 22, 23, 24\}$;
- (2) M_{11} with $v = 12$;
- (3) A_v with $v \leq 5$;
- (4) $\text{PSL}(2, q)$ with $v = q + 1$ and $q = p^e > 3$.

We proceed to analyze each of these cases in turn.

Proposition 3.1. *There are no nontrivial 6 -($v, k, 2$) designs that are G -block-transitive with $k \mid v$ for a group G of **affine type**.*

Proof. Our analysis proceeds by exhaustively treating all the situations mentioned above. First, we handle the scenario where $G \cong \text{AFL}(1, 32)$, $\text{AFL}(1, 8)$, and $\text{AGL}(1, 8)$.

When $v = 8$, the block size necessarily satisfies $k = 7$, which immediately fails to meet the condition $k \mid v$. When $v = 32$, we are aware that $6 < k < 32$. For each value of k with $6 < k < 29$, $29 \mid b$. It follows from the block-transitive property that 29 divides $|G|$, which is a contradiction. For each value of k with $29 \leq k \leq 32$, $k = 32$ is the unique value satisfying $k \mid v$, yielding only the trivial design.

Next, we consider the scenario where $G_0 \cong \text{SL}(d, 2)$.

In this scenario, we have $d > 2$, and then $v = 2^d > 6$. If $d = 3$, this reduces to the case $v = 8$, which was already resolved above. Therefore, we suppose that $d > 3$. In the affine space $AG(d, 2)$, any six points that are noncoplanar and distinct must span an affine subspace of at least three dimensions. We define e_1, e_2, e_3 as the standard basis vectors in the vector space $V = V(d, 2)$ and use \mathcal{E} to denote the three-dimensional vector subspace they generate. Thus the pointwise stabilizer of \mathcal{E} in $SL(d, 2)$ acts transitively on V/\mathcal{E} . Consider the two unique blocks, designated B_1 and B_2 , which contain the specific six points $\{e_1, e_2, e_1 + e_2, e_3, e_2 + e_3, 0\}$. If B_1 includes some points outside \mathcal{E} , for any $y \in V/\mathcal{E}$, we have $y \in B_1 \cup B_2$. Hence $V/\mathcal{E} \subset B_1 \cup B_2$, implying that $k \geq 6 + \frac{v-8}{2} = \frac{v}{2} + 2$. But this lower bound on k conflicts with the condition $k \mid v$. Indeed, when $v = 2^d$ and k satisfies the inequality above together with the divisibility constraint, no integer solution exists for $d > 3$. Hence this subcase yields no feasible design.

Finally, let us take the scenario where $G_0 \cong A_7$.

For $v = 2^4$, the block size necessarily satisfies $k = 8$. Applying Lemma 2.1, we deduce that the number of blocks b is divisible by 3. Block-transitivity then requires that 13 divides $|G|$. However, $|A_7| = \frac{7!}{2}$, which is not a multiple of 13. This contradiction eliminates the possibility.

Since every conceivable affine-type realization of G leads either to a violation of the condition $k \mid v$ or to an inconsistency with block-transitivity, we conclude that no nontrivial 6-designs exist under this category. \square

Proposition 3.2. *There are no nontrivial 6-(v, k, λ) designs that are G -block-transitive with $k \mid v$ for an almost simple group G other than $N \cong \text{PSL}(2, q)$.*

Proof. We now turn to the almost simple case for G . Given the complexity involved, the case of $\text{PSL}(2, q)$ will be treated separately later in this paper. It is worth mentioning that in the present subsection, we extend the conclusion to the cases where the parameter λ is not prescribed to a specific value. The argument proceeds by case distinction on the socle N .

First, we consider the scenario where $N = M_v$ with $v \in \{11, 12, 22, 23, 24\}$.

Without loss of generality, we replace \mathcal{D} with its complementary design, which allows us to impose the standing assumption $k \leq \frac{v}{2}$. Given this and $k \geq 7$, we can see the impossibility of M_{11} and M_{12} immediately. Moreover, $b = |\mathcal{B}|$ satisfies $bk(k-5)(k-4)(k-3)(k-2)(k-1) = \lambda v(v-5)(v-4)(v-3)(v-2)(v-1)$, and the block-transitivity of G then implies that b divides $|G|$. If $v = 22, 23, 24$ or $G = \text{Aut}(M_{22})$, 19 must divide b , and then 19 must divide $|G|$. However, none of them possess an element of order 19, which is a contradiction.

Now consider, the scenario where $N = M_{11}$ with $v = 12$.

Following the same pattern as above, we again impose $k \leq 6$. Therefore, when $t = 6$, the arithmetic conditions detect that the prime 7 must divide b and, consequently, 7 divides $|G|$. Yet M_{11} and its automorphism group have orders that are not divisible by 7. This eliminates the possibility.

Finally, take the scenario where $N = A_v$ with $v \geq 5$.

Given that \mathcal{D} is nontrivial and $k \geq 7$, this forces $v \geq 8$. Consequently, G is 6-transitive on \mathcal{P} . However, none of the designs listed in Theorem 3 of [14] are 6-designs. Hence this entire family is excluded. \square

Now we consider $N \cong \text{PSL}(2, q)$, $q = p^e > 3$, and $v = q + 1$. Therefore, $\mathcal{P} = GF(q) \cup \{\infty\}$, $\text{Aut}(N) = \text{P}\Gamma\text{L}(2, q)$, and the order of G is given by $\frac{a}{n}q(q-1)(q+1)$ where $a \mid ne$. As $v \geq 8$, we may

suppose that $q \geq 7$. We will discuss this topic by examining four distinct cases.

Proposition 3.3. *Let $G = N \cong \text{PSL}(2, q)$, $q = p^e > 3$, $v = q + 1$, in which case, there are no nontrivial G -block-transitive 6 - $(v, k, 2)$ designs with $k \mid v$.*

Proof. First, by Lemma 2.1, we obtain

$$|\text{PSL}(2, q)_B|(q-4)(q-3)(q-2) = \frac{(k-5)(k-4)(k-3)(k-2)(k-1)k}{2n}. \quad (3.1)$$

If $|\text{PSL}(2, q)_B| = 1$, substituting this value into Eq (3.1) yields

$$(q-4)(q-3)(q-2) = k \frac{(k-5)(k-4)(k-3)(k-2)(k-1)}{2n}.$$

From this equation, we obviously obtain the divisibility relation $k \mid (q-4)(q-3)(q-2)$. It follows that $k \mid 60$ since $k \mid (q+1)$. Furthermore, the possible values of k are 10, 12, 15, 20, 30, 60, all of which lead to contradictions when substituted.

Hence, we may assume that $|\text{PSL}(2, q)_B| > 1$. Since \mathcal{D} is G -block-transitive and $\text{PSL}(2, q)_B$ fixes B , B must consist of some complete orbits of $\text{PSL}(2, q)_B$ acting on \mathcal{P} . Given the classification of subgroups of $\text{PSL}(2, q)$ (see Lemmas 2.4–2.10), we examine the following scenarios.

- (1) $\text{PSL}(2, q)_B \cong C_c$, where C_c is a cyclic group of order $c \geq 2$
 - (a) If $c \mid \frac{q+1}{n}$, $k = cz$;
 - (b) If $c \mid \frac{q-1}{n}$, $k = cz$, $k = cz + 1$ or $k = cz + 2$.
- (2) $\text{PSL}(2, q)_B \cong D_{2c}$, where D_{2c} is a dihedral group of order $2c$, $c \geq 3$
 - (a) When $q \equiv 1 \pmod{4}$,
 - (i) If $c \mid \frac{q+1}{2}$, $k = 2cz$ ($z > 1$) or $k = 2cz + c$;
 - (ii) If $c \mid \frac{q-1}{2}$, $k = 2cz + 2$ or $k = 2cz + c + 2$.
 - (b) When $q \equiv 3 \pmod{4}$,
 - (i) If $c \mid \frac{q+1}{2}$, $k = 2cz$ ($z > 1$);
 - (ii) If $c \mid \frac{q-1}{2}$, $k = 2cz + 2$.
 - (c) When $q \equiv 0 \pmod{2}$,
 - (i) If $c \mid (q+1)$, $k = 2cz$ or $k = 2cz + c$;
 - (ii) If $c \mid (q-1)$, $k = 2cz + 2$ or $k = 2cz + c + 2$.
- (3) $\text{PSL}(2, q)_B \cong E_{\bar{q}}$, where $E_{\bar{q}}$ is an elementary abelian group of order $\bar{q} \mid q$, and $k = \bar{q}z + 1$.
- (4) $\text{PSL}(2, q)_B \cong E \rtimes C$, where E is an elementary abelian group of order $\bar{q} \mid q$, C is a cyclic group of order $c \geq 2$, such that $c \mid (\bar{q} - 1, q - 1)$ and $k = c\bar{q}z + 1$, $k = c\bar{q}z + \bar{q} + 1$.
- (5) $\text{PSL}(2, q)_B \cong \text{PSL}(2, \bar{q})$ where $\bar{q}^m = q$ and $m \geq 1$. For an even m , the possible values of k are $\frac{\bar{q}^3 - \bar{q}}{n}z$, $\frac{\bar{q}^3 - \bar{q}}{n}z + \bar{q} + 1$, $\frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} - 1)\bar{q}$, and $\frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} - 1)\bar{q} + \bar{q} + 1$. For an odd m , either $k = \frac{\bar{q}^3 - \bar{q}}{n}z + \bar{q} + 1$ or $k = \frac{\bar{q}^3 - \bar{q}}{n}z$.

(6) $\text{PSL}(2, q)_B \cong \text{PGL}(2, \bar{q})$ where $\bar{q}^m = q$, with $m > 1$ being even, then $k = (\bar{q}^3 - \bar{q})z$, $k = (\bar{q}^3 - \bar{q})z + \bar{q} + 1$, $k = (\bar{q}^3 - \bar{q})z + (\bar{q} - 1)\bar{q}$, or $k = (\bar{q}^3 - \bar{q})z + (\bar{q} - 1)\bar{q} + \bar{q} + 1$.

(7) $\text{PSL}(2, q)_B \cong A_4$ with an even q , and $k = 12$.

Next, we will discuss each of the cases listed above individually and eliminate their possibilities.

(1): $|\text{PSL}(2, q)_B| = c$.

For $k = cz$, substituting this value into Eq (3.1) yields

$$(q-4)(q-3)(q-2) = z \frac{(k-5)(k-4)(k-3)(k-2)(k-1)}{2n}. \quad (3.2)$$

Obviously, $z \mid (q-4)(q-3)(q-2)$, and we also have $z \mid k$, and thus $z \mid 60$. In general, we consider $c \mid \frac{q+1}{n}$.

If q is odd, $n = 2$, and $c \mid \frac{q+1}{2}$, by Eq (3.2), we derive

$$\begin{aligned} c \mid \frac{(q-4)(q-3)(q-2)}{6} + 10 \\ = \frac{cz^2}{24}(c^4z^4 - 15c^3z^3 + 85c^2z^2 - 225cz + 274) - 5z + 10. \end{aligned}$$

We can get $24 \mid z^2$ and $c \mid (10 - 5z)$, and immediately $z = 12$ and $c \in \{5, 10, 25, 50\}$. Substituting these possible values into Eq (3.2) yields no value of q that satisfies the integer condition.

If q is even, $n = 1$, and $c \mid (q + 1)$, from Eq (3.2), we derive

$$c \mid (q-4)(q-3)(q-2) + 60 = \frac{cz}{2}(c^4z^4 - 15c^3z^3 + 85c^2z^2 - 225cz + 274) + 60 - 60z.$$

Then $c \mid (60 - 60z)$ with an even z , and $c \mid (120 - 120z)$ with an odd z . For each value of z , we can obtain all possible values of c and then derive the corresponding k .

Substituting the specific values obtained above into Eq (3.2) can rule out every possibility one by one.

For $k = cz + 1$, this will only occur in Scenario (b), and $c \mid \frac{q-1}{n}$. By Eq (3.2), we can get

$$2n(q-4)(q-3)(q-2) = kz(k-5)(k-4)(k-3)(k-2).$$

Immediately $k \mid 60$. Substituting the values of k into the equation, there is no qualifying q .

For $k = cz + 2$, which only arises in Scenario (b), by Eq (3.2), we can get

$$2n(q-4)(q-3)(q-2) = kz(k-5)(k-4)(k-3)(k-1).$$

When both k and q are odd, $k \mid (q + 1, 2(q-4)(q-3)(q-2))$. In this case, $k \mid 120$. Apart from that, it is consistent with the case where $k = cz + 1$. There is no qualifying q .

On the basis of the analysis above, this case is impossible.

(2): $|\text{PSL}(2, q)_B| = 2c$.

First, we research the possibility of $k = 2cz$. By Eq (3.1), we derive

$$2n(q-4)(q-3)(q-2) = z(k-5)(k-4)(k-3)(k-2). \quad (3.3)$$

We immediately see that $z \mid 60$.

When q is odd, $k > 2c$, so the possible values of z are 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Note that the condition $c \mid \frac{q+1}{2}$ holds in this situation. By Eq (3.3), we derive

$$c \mid \frac{cz^2}{6}(8c^4z^4 - 56c^3z^3 + 170c^2z^2 - 225cz + 137) - 5z + 10.$$

We can get $6 \mid z^2$ and $c \mid (10 - 5z)$, and immediately $z = 6, 12, 30, 60$. For each possible value of z , the corresponding possible values of c and k are as follows:

- $z = 6, k = 12c, c \mid 20$, and thus $c = 4, 5, 10, 20$;
- $z = 12, k = 24c, c \mid 50$, and thus $c = 5, 10, 25, 50$;
- $z = 30, k = 60c, c \mid 140$, and thus $c = 4, 5, 7, 10, 14, 20, 28, 35, 70, 140$;
- $z = 60, k = 120c, c \mid 290$, and thus $c = 5, 10, 29, 58, 145, 290$.

When q is even, the possible values of z are 1, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Note that the condition $c \mid q + 1$ holds in this situation. By Eq (3.3), we obtain

$$c \mid cz^2(16c^4z^4 - 112c^3z^3 + 340c^2z^2 - 450cz + 274) + 60 - 30z.$$

Therefore, $c \mid (60 - 30z)$. For each possible value of z , the corresponding possible values of c and k are as follows:

- $z = 1, k = c, c \mid 30$, and thus $c = 10, 15, 30$;
- $z = 3, k = 6c, c \mid 30$, and thus $c = 3, 5, 6, 10, 15, 30$;
- $z = 4, k = 8c, c \mid 60$, and thus $c = 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$;
- $z = 5, k = 10c, c \mid 90$, and thus $c = 3, 5, 6, 9, 10, 15, 18, 30, 45, 90$;
- $z = 6, k = 12c, c \mid 120$, and thus $c = 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$;
- $z = 10, k = 20c, c \mid 240$, and thus $c = 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 60, 80, 120, 240$;
- $z = 12, k = 24c, c \mid 300$, and thus $c = 3, 4, 5, 6, 10, 12, 15, 20, 25, 30, 50, 60, 75, 100, 150, 300$;
- $z = 15, k = 30c, c \mid 390$, and thus $c = 3, 5, 6, 10, 13, 15, 26, 30, 39, 65, 78, 130, 195, 390$;
- $z = 20, k = 40c, c \mid 540$, and thus $c = 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 27, 30, 36, 45, 54, 60, 90, 108, 135, 180, 270, 540$;
- $z = 30, k = 60c, c \mid 840$, and thus $c = 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40, 42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840$;
- $z = 60, k = 120c, c \mid 1740$, and thus $c = 3, 5, 6, 10, 12, 15, 20, 29, 30, 58, 60, 87, 116, 145, 174, 290, 348, 435, 580, 870, 1740$.

Similarly, substituting these possible values into Eq (3.3) yields a contradiction. Thus, no designs with $k = 2cz$ exist in this case.

Next, we examine the possibility of $k = 2cz + c = (2z + 1)c$. In this case, $q \equiv 1 \pmod{4}$ or $q \equiv 0 \pmod{2}$. It follows that

$$4n(q-4)(q-3)(q-2) = (2z+1)(k-5)(k-4)(k-3)(k-2)(k-1). \quad (3.4)$$

Therefore, $2z+1 \mid ((q-4)(q-3)(q-2), q+1)$. Note that because of $2z+1 \mid k$, we can obtain $2z+1 \mid 60$, which means that the values of z and the corresponding values of k can be summarized as shown in Table 1.

Table 1. The possible cases of z and k .

Parameter	Case 1	Case 2	Case 3
z	1	2	7
k	$3c$	$5c$	$15c$

For $z = 1$ and $k = 3c$.

When $q \equiv 0 \pmod{2}$, c is odd and $c \mid (q + 1)$ holds. We find that

$$c \mid (q - 4)(q - 3)(q - 2) + 60 = \frac{1}{4}(729c^5 - 2645c^4 + 6885c^3 - 6075c^2 + 2436c) + 15.$$

Since $8 \mid (k - 4)(k - 3)(k - 2)(k - 1)$, $c \mid 15$. Through calculation, the possible values of c and the corresponding values of k are summarized in Table 2.

Table 2. The possible cases of c and k .

Parameter	Case 1	Case 2	Case 3
c	3	5	15
k	9	15	45

If we substitute them into Eq (3.4), there is no qualifying q .

When $q \equiv 1 \pmod{4}$, c is odd and $c \mid \frac{q+1}{2}$ holds. We find that

$$\begin{aligned} c \mid \frac{(q - 4)(q - 3)(q - 2)}{2} + 30 \\ = \frac{1}{8}(729c^5 - 2645c^4 + 6885c^3 - 6075c^2 + 2436c) - 5. \end{aligned}$$

Therefore, $c = 5$ and $k = 15$. Substituting them into Eq (3.4), we find no valid q satisfying the conditions.

For $z = 2$ (so $k = 5c$) and $z = 7$ (so $k = 15c$), the argument is analogous to the previous case and will not be repeated here.

Next, we examine the case $k = 2cz + 2$. Likewise, we obtain $2n(q - 4)(q - 3)(q - 2) = zk(k - 5)(k - 4)(k - 3)(k - 1)$. Instantly, $k \mid 120$. After a concise analysis, this situation can be ruled out.

Finally, we research the possibility of $k = 2cz + 2 + c$. We can detect $4n(q - 4)(q - 3)(q - 2) = k(2z + 1)(k - 5)(k - 4)(k - 3)(k - 1)$. Instantly, $k \mid 240$. This situation can be ruled out through a straightforward analysis of each possible value of k .

According to the analysis above, this case is impossible.

$$(3): |\text{PSL}(2, q)_B| = \bar{q} < \frac{2}{n}k.$$

Under this assumption, $k = \bar{q}z + 1$. Substituting it into Eq (3.1), we get $(q - 4)(q - 3)(q - 2) = kz \frac{(k-5)(k-4)(k-3)(k-2)}{2n}$. From this equation, we obtain the divisibility relation $k \mid 60$. A simple analysis suffices to rule out this case.

$$(4): |\text{PSL}(2, q)_B| = c\bar{q}.$$

First, we research the possibility of $k = c\bar{q}z + 1$. If we substitute it into Eq (3.1), $(q - 4)(q - 3)(q - 2) = kz \frac{(k-5)(k-4)(k-3)(k-2)}{2n}$ holds. We also obtain $k \mid 60$. Therefore, by adopting a treatment similar to that for Case (3), its possibility can be excluded.

Next, we consider the possibility of $k = c\bar{q}z + \bar{q} + 1$. Since $c \mid (\bar{q} - 1)$, $c\bar{q} \mid (k - 1)(k - 2)$. From Eq (3.1), we see that $k \mid 120$ when k is even and q is odd. Except for this particular situation, $k \mid 60$. Both of these situations can be excluded by adopting a similar approach as described previously.

$$(5) \text{ and } (6): |\text{PSL}(2, q)_B| = \frac{\bar{q}^3 - \bar{q}}{n}, \bar{q}^m = q.$$

(a) $k = \frac{\bar{q}^3 - \bar{q}}{n}z$. When q is odd, we obtain $\bar{q} \mid k$, which contradicts $k \mid (q + 1)$. When q is even, k must be even but $v = q + 1$ is odd, yielding a contradiction.

(b) $k = \frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} + 1) = (\frac{\bar{q}^3 - \bar{q}}{n}z + 1)(\bar{q} + 1)$. In this situation, m is even. Therefore, $q + 1$ can be expressed as follows:

$$q + 1 = (\bar{q} + 1) \sum_{i=0}^{m-1} (-1)^i \bar{q}^i + 2$$

Since $k \mid (q + 1)$, $(\bar{q} + 1) \mid 2$ contradicts the condition $|\text{PSL}(2, q)_B| > 1$.

(c) $k = \frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} - 1)\bar{q} = (\bar{q} - 1)(\frac{(\bar{q}+1)\bar{q}}{n}z + \bar{q})$. Here, m is even. From $(\bar{q} - 1) \mid k$ and $\bar{q} - 1 \mid (q - 1)$, we derive $(\bar{q} - 1, q + 1) \mid 2$. When $\bar{q} = 2$, we note that $q + 1 = 2^m + 1$ is odd while $k = 6z + 2$ is even, which contradicts the condition $k \mid (q + 1)$. When $\bar{q} = 3$, we have $q + 1 = 3^m + 1 = 9^{m/2} + 1$ and $k = 3(4z + 1)$, so $3 \mid k$ but $q + 1 \equiv 1 \pmod{3}$, again contradicting $k \mid (q + 1)$. Therefore, this case can be ruled out.

(d) $k = \frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} - 1)\bar{q} + \bar{q} + 1$. We substitute it into Eq (3.1) and sort it out to obtain

$$(q - 4)(q - 3)(q - 2) = k\left(\frac{\bar{q}}{n}z + 1\right)\left(\frac{\bar{q}^2 - 1}{n}z + \bar{q}\right)\frac{(k - 5)(k - 4)(k - 3)}{2}.$$

Note that $(\frac{\bar{q}^2 - 1}{n}z + \bar{q})(\frac{\bar{q}}{n}z + 1)$ must be an integer. Thus, $k \mid 60$. Since $\bar{q} \geq 3$, $k \geq 22$. Therefore, $k = 30$ or 60 . There is no corresponding integer q , so this case can be ruled out.

(7): $\text{PSL}(2, q)_B \cong A_4$.

As $k = 12$, following a brief analysis, this possibility can be ruled out.

In summary, there are no G -block-transitive 6 - $(v, k, 2)$ designs satisfying $k \mid v$ whose automorphism group is $\text{PSL}(2, q)$. \square

Proposition 3.4. *There are no nontrivial 6 - $(v, k, 2)$ designs that are G -block-transitive with $k \mid v = q + 1$ when $N \cong \text{PSL}(2, q)$ and $N < G \leq \text{Aut}(N)$, $q = p^e > 3$ and $p > 3$.*

Proof. We first introduce the subgroup $G^* = G \cap (\text{PSL}(2, q) \rtimes \langle \tau_\alpha \rangle)$, where τ_α is an element of $\text{Sym}(\text{GF}(p^e) \cup \{\infty\}) \cong S_v$ with order e , which is induced by the Frobenius map $\alpha: x \mapsto x^p$ on $\text{GF}(p^e)$. The notation $\text{GF}(p^e)$ stands for the finite field (also known as a Galois field) with p^e elements. Applying Dedekind's modular law, we obtain the decomposition

$$G^* = \text{PSL}(2, q) \rtimes (G^* \cap \langle \tau_\alpha \rangle).$$

Let $\text{P}\Sigma\text{L}(2, q) = \text{PSL}(2, q) \rtimes \langle \tau_\alpha \rangle$. The stabilizer of three points $\{0, 1, \infty\}$ in this group is exactly $\langle \tau_\alpha \rangle$, and it is classically known that $\langle \tau_\alpha \rangle$ fixes $p + 1$ distinct points [15].

Here, $p > 3$ is a prime, so $p + 1 \geq 6$ holds. We select six fixed points to form the set S . Since $\langle \tau_\alpha \rangle$ fixes each point in S , it must map a block containing S to another block containing S . By the definition of a 6 - $(v, k, 2)$ design, these six points lie precisely in two blocks $B_1, B_2 \in \mathcal{B}$. Thus, the action of $\langle \tau_\alpha \rangle$ on

$\{B_1, B_2\}$ can only be of two types: $\langle \tau_\alpha \rangle$ fixes B_1 and B_2 , respectively, or $\langle \tau_\alpha \rangle$ acts transitively between B_1 and B_2 . A detailed discussion of these two cases follows below.

When $\langle \tau_\alpha \rangle$ fixes B_1 and B_2 respectively, $B \in \mathcal{B}$ exists such that $G^* \cap \langle \tau_\alpha \rangle \leq G^*$. Hence, we obtain

$$\begin{aligned} |B^{G^*}| &= [G^* : G_B^*] \\ &= [\text{PSL}(2, q) \rtimes (G^* \cap \langle \tau_\alpha \rangle) : \text{PSL}(2, q)_B \rtimes (G^* \cap \langle \tau_\alpha \rangle)] \\ &= [\text{PSL}(2, q) : \text{PSL}(2, q)_B] \\ &= |B^{\text{PSL}(2, q)}|. \end{aligned}$$

We find $|B^{G^*}| = |B^{\text{PSL}(2, q)}| = b$ when G^* is block-transitive. This implies that $\text{PSL}(2, q)$ is block-transitive on \mathcal{D} , a case we have already excluded. Therefore, we may assume that G^* is not block-transitive. In this scenario, the index of G^* in G is 2 and G^* partitions the block set into precisely two orbits of equal size. Thus, the orbit containing B satisfies $|B^{G^*}| = |B^{\text{PSL}(2, q)}| = \frac{b}{2}$. As $\text{PSL}(2, q)$ is a normal subgroup of G , it possesses an additional orbit of the same length in its action on the block set. Through analyzing the properties of the orbit where B is located, we obtain

$$|\text{PSL}(2, q)_B|(q-4)(q-3)(q-2) = k(k-5)(k-4)(k-3)(k-2)(k-1)\frac{1}{n}. \quad (3.5)$$

If $|\text{PSL}(2, q)_B| = 1$, $k \mid 60$. By substituting each possible value of k into Eq (3.5), all can be excluded. If $|\text{PSL}(2, q)_B| > 1$, then B must be a union of complete orbits of $\text{PSL}(2, q)_B$ acting on \mathcal{P} . We now analyze the orbits of the subgroups of $\text{PSL}(2, q)$ listed in Lemmas 2.4–2.12, considering each case in turn.

(1) If $\text{PSL}(2, q)_B \cong C_c$, a cyclic group of order $c \geq 2$, through Eq (3.5), we derive

$$(q-4)(q-3)(q-2)c = k(k-5)(k-4)(k-3)(k-2)(k-1)\frac{1}{n}.$$

First, if $c \mid \frac{q+1}{2}$, $k \mid 60c$, that is to say, $k = zc$, where $z \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$. As $c \mid \frac{q+1}{n}$, we derive

$$\begin{aligned} c \mid & \frac{(q-4)(q-3)(q-2)}{n} + \frac{60}{n} \\ &= \frac{cz^2(c^4z^4 - 15c^3z^3 + 85c^2z^2 - 225cz + 274)}{n^2} + \frac{60n - 120z}{n^2}. \end{aligned}$$

When z is odd, $c \mid 60n - 120z$; when z is even, $c \mid \frac{60n-120z}{n}$. For every value of z , substituting the respective values of k into Eq (3.5) yields a contradiction. Next, if $c \mid \frac{q-1}{n}$. As $c \mid (q+1)$, $c=2$, $k \mid 120$. Similarly, the discussion around Eq (3.5) allows us to rule out this case.

(2) If $\text{PSL}(2, q)_B \cong D_{2c}$, a dihedral group whose order is $2c$, $c \geq 3$, and $k = c, 2c, 3c$ or $6c$; also, by Eq (3.5), we have $k \mid 120c$. First, $c \mid \frac{q+1}{n}$. When $k = 2zc$, we have

$$\begin{aligned} c \mid & \frac{(q-4)(q-3)(q-2)}{n} + \frac{60}{n} \\ &= \frac{4cz^2(8c^4z^4 - 70c^3z^3 + 170c^2z^2 - 225cz + 137)}{n^2} + \frac{60n - 120z}{n^2}. \end{aligned}$$

Thus, $c \mid \frac{60n-120z}{n^2}$, and on the basis of the feasible values of z , we derive the corresponding (c, k) pairs and then substitute each pair into Eq (3.5) to eliminate the invalid cases.

When $k = c(2z + 1)$, we derive

$$(q - 4)(q - 3)(q - 2) = (2z + 1)(k - 5)(k - 4)(k - 3)(k - 2)(k - 1).$$

Therefore, $2z + 1 \mid (q + 1, (q - 4)(q - 3)(q - 2))$; that is to say, $z = 1, 2, 7$. Respectively, $k = 3c, 5c$, and $15c$. We substitute each value into Eq (3.5) and rule out the case.

Next, $c \mid \frac{q-1}{n}$. When $k = 2cz + 2$, we derive

$$(q - 4)(q - 3)(q - 2) = kz(k - 1)\frac{(k - 5)(k - 4)(k - 3)}{n}.$$

From this equation, we obtain the divisibility relation $k \mid 60$. We can easily deduce its impossibility.

When $k = 2cz + c + 2$, we derive

$$(q - 4)(q - 3)(q - 2) = k(k - 1)(2z + 1)\frac{(k - 5)(k - 4)(k - 3)}{n}.$$

From this equation, we obtain the divisibility relation $k \mid 60$. It can be ruled out similarly.

(3) If $\text{PSL}(2, q)_B \cong E_{\bar{q}}$, an elementary abelian group of order $\bar{q} \mid q$, Eq (3.5) implies

$$\bar{q}(q - 4)(q - 3)(q - 2) = kz\frac{(k - 5)(k - 4)(k - 3)(k - 2)}{n}.$$

As $(k, \bar{q}) = 1$, we can also obtain $k \mid 60$. It can be ruled out similarly.

(4) If $\text{PSL}(2, q)_B \cong E \rtimes C$, where E is an elementary abelian group of order $\bar{q} \mid q$, C is a cyclic group of order $c \geq 2$, such that $c \mid (\bar{q} - 1, q - 1)$. In this case, $k = zc\bar{q} + 1$ or $k = zc\bar{q} + \bar{q} + 1$. Inserting these values into Eq (3.5) leads to $k \mid 60$. An analogous argument reveals that this case is also excluded.

(5) $\text{PSL}(2, q)_B \cong \text{PSL}(2, \bar{q})$ where $\bar{q}^m = q$ and $m \geq 1$. The possible values of k depend on the parity of m : For an even m , $k = \frac{\bar{q}^3 - \bar{q}}{n}z + \bar{q} + 1$, $k = \frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} - 1)\bar{q}$, $k = \frac{\bar{q}^3 - \bar{q}}{n}z + (\bar{q} - 1)\bar{q} + \bar{q} + 1$, or $k = \frac{\bar{q}^3 - \bar{q}}{n}z$. For an odd m , $k = \frac{\bar{q}^3 - \bar{q}}{n}z + \bar{q} + 1$ or $k = \frac{\bar{q}^3 - \bar{q}}{n}z$. We use arguments analogous to those in the case of $G = N$ to rule out these possibilities one by one; the details are omitted for brevity.

(6) If $\text{PSL}(2, q)_B \cong \text{PGL}(2, \bar{q})$ where $\bar{q}^m = q$ and $m > 1$ is even, then $k = z\bar{q}(\bar{q} - 1)(\bar{q} + 1) + \bar{q} + 1$ or $z\bar{q}(\bar{q} - 1)(\bar{q} + 1) + \bar{q}(\bar{q} - 1) + \bar{q} + 1$. Similar to the previous cases, substituting into Eq (3.5) can rule out the possibilities.

(7) If $\text{PSL}(2, q)_B \cong A_4, S_4$, or A_5 . There are several possibilities of k that can be excluded through some detailed discussions centered around Eq (3.5).

When $\langle \tau_\alpha \rangle$ acts transitively between B_1 and B_2

$$\begin{aligned} |B^{G^*}| &= [G^* : G_B^*] \\ &= [\text{PSL}(2, q) \rtimes (G^* \cap \langle \tau_\alpha \rangle) : \text{PSL}(2, q)_B \rtimes \frac{(G^* \cap \langle \tau_\alpha \rangle)}{2}] \end{aligned}$$

$$\begin{aligned}
&= [\text{PSL}(2, q) : \frac{\text{PSL}(2, q)_B}{2}] \\
&= 2|B^{\text{PSL}(2, q)}|.
\end{aligned}$$

If $G^* \leq \text{Aut}(\mathcal{D})$ is block-transitive, we derive $|B^{G^*}| = 2|B^{\text{PSL}(2, q)}| = b$ and may continue our analysis as in the case where $\langle \tau_\alpha \rangle$ fixes B_1, B_2 . Therefore, we suppose that $G^* \leq \text{Aut}(\mathcal{D})$ is not block-transitive on \mathcal{D} . In this situation, the index of G^* in G is 2 and $|B_1^{G^*}| = 2|B_1^{\text{PSL}(2, q)}| = \frac{b}{2}$. By analyzing the properties of the orbit of B_1 we obtain

$$|\text{PSL}(2, q)_{B_1}|(q-4)(q-3)(q-2) = k \frac{2}{n} (k-5)(k-4)(k-3)(k-2)(k-1).$$

By adapting the arguments used for the case where $\langle \tau_\alpha \rangle$ fixes B_1, B_2 , we can rule out this possibility. \square

Proposition 3.5. *There are no nontrivial G -block-transitive 6 - $(v, k, 2)$ designs with $k \mid v = q + 1$ when $N \cong \text{PSL}(2, q)$ and $N < G \leq \text{Aut}(N)$, $q = p^e > 3$, and $p = 3$.*

Proof. $\text{Aut}(N) = \text{P}\Gamma\text{L}(2, q) = \text{PGL}(2, q) \rtimes \langle \tau_\alpha \rangle$. Since $\text{PGL}(2, q)$ is a sharp 3-transitive group, $\text{P}\Gamma\text{L}(2, q)_{0,1,\infty} = \langle \tau_\alpha \rangle$. Set $G^* = G \cap (\text{PSL}(2, q) \rtimes \langle \tau_\alpha \rangle)$ and express it in the form of

$$G^* = \text{PSL}(2, q) \rtimes (G^* \cap \langle \tau_\alpha \rangle).$$

If $\langle \tau_\alpha \rangle \leq \text{P}\Sigma\text{L}$ holds for a certain appropriate block, $\text{PSL}(2, q)$ acts transitively on the set of blocks, which has already been considered in case where $G = N$. Hence, we assume $\langle \tau_\alpha \rangle \not\leq \text{P}\Sigma\text{L}(2, q)_B$. Let s denote the smallest prime factor of $e = |\langle \tau_\alpha \rangle|$. Set the group $H := (\text{P}\Sigma\text{L}(2, q)_{0,1,\infty})^s$, and note that $H \trianglelefteq \langle \tau_\alpha \rangle$ is equipped with $p^s + 1$ different fixed points. For a 6 - $(v, k, 2)$ design, any six distinct points lie in two different blocks B_1, B_2 . When H fixes B_1, B_2 , respectively, there is a unique suitable block $B \in \mathcal{B}$ satisfying $G \cap H \leq G_B^*$. Furthermore, for $\langle \tau_\alpha \rangle \not\leq \text{P}\Sigma\text{L}(2, q)_B$, $G_B^* = \text{PSL}(2, q) \rtimes (G \cap H)$. Therefore, G is one of four following groups:

- (A) $\text{PSL}(2, q) \rtimes (G \cap H)$;
- (B) $\text{PSL}(2, q) \rtimes \langle \tau_\alpha \rangle$;
- (C) $\text{PGL}(2, q) \rtimes (G \cap H)$;
- (D) $\text{P}\Gamma\text{L}(2, q)$.

If G is (A) or (B), $G = G^*$. According to Lemma 2.1, we have

$$2|\text{PSL}(2, q)_B||G \cap H|(q-4)(q-3)(q-2) = k|G \cap \langle \tau_\alpha \rangle|(k-5)(k-4)(k-3)(k-2)(k-1).$$

If G is (C), $\text{PSL}(2, q)$ is block-transitive, which is analogous to the situation $G = N$. If G is (D), we detect

$$2|\text{PSL}(2, q)_B|(q-4)(q-3)(q-2) = ks(k-5)(k-4)(k-3)(k-2)(k-1). \quad (3.6)$$

On the one hand, as $k \mid (q + 1)$, $3 \nmid (q + 1)$, and k is even; on the other hand, B is the union of several orbits of $\text{PSL}(2, q)_B$ acting on the projective line. According to Lemmas 2.4–2.12, the possible

isomorphism types of the group $\text{PSL}(2, q)_B$ can be exhaustively divided into seven distinct scenarios. This case analysis is highly analogous to the preceding arguments we applied. Except for a few minor modifications to account for differences in the detail, each of these scenarios can be systematically analyzed and discussed with reference to Eq (3.6). By using a method analogous to that used in the case where $G = N$, we can similarly establish the impossibility of each scenario. When H acts transitively on $\{B_1, B_2\}$, we have $B_1^H = \{B_1, B_2\}$, so $|H_{B_1}| = \frac{|H|}{2}$. Define H' such that $|H'| = \frac{|G \cap H|}{2}$.

In this case, there is an appropriate and unique block B_1 satisfying $H' \leq G_{B_1}^* = \text{PSL}(2, q)_B \rtimes H'$. Therefore, G is one of six following groups.

(A) $\text{PSL}(2, q) \rtimes (G \cap H)$;

(B) $\text{PSL}(2, q) \rtimes \langle \tau_\alpha \rangle$;

(C) $\text{PGL}(2, q) \rtimes (G \cap H)$;

(D) $\text{P}\Gamma\text{L}(2, q)$;

(E) $\text{PSL}(2, q) \rtimes H'$;

(F) $\text{PGL}(2, q) \rtimes H'$.

Cases (A)–(D) follow from the previous discussion. If G is Case (E) or Case (F), the following equation holds:

$$|\text{PSL}(2, q)_B|(q-4)(q-3)(q-2) = k(k-5)(k-4)(k-3)(k-2)(k-1). \quad (3.7)$$

By adapting the arguments used for the case of $G = N$, an analysis of Eq (3.7) allows us to rule out this scenario. □

Proposition 3.6. *There are no nontrivial G -block-transitive 6 -($v, k, 2$) designs D with $k \mid v = q + 1$ when $N \cong \text{PSL}(2, q)$ and $N < G \leq \text{Aut}(N)$, $q = p^e > 3$, and $p = 2$.*

Proof. We divide the discussion into two cases: $e > 2$ and $e \leq 2$. When $e > 2$, making detailed modifications according to the treatment approach of the case where $p = 3$ can rule out its possibility. When $e \leq 2$, we note that $\text{Aut}(N) = \text{P}\Gamma\text{L}(2, q) = \text{PGL}(2, q) \rtimes \langle \tau_\alpha \rangle$, and $G_B = \text{PSL}(2, q)_B \rtimes \langle \tau_\alpha \rangle_B$. By Lemma 2.1, we have

$$2n|\text{PSL}(2, q)_B|(q-4)(q-3)(q-2) = k(k-5)(k-4)(k-3)(k-2)(k-1), \quad (3.8)$$

or

$$4n|\text{PSL}(2, q)_B|(q-4)(q-3)(q-2) = k(k-5)(k-4)(k-3)(k-2)(k-1). \quad (3.9)$$

Similarly, an analysis of Eqs (3.8) and (3.9) allows us to rule out this case. □

Combining Propositions 3.1–3.6, we arrive at the conclusion that no nontrivial G -block-transitive 6- $(v, k, 2)$ designs with $k \mid v$ exist. We finalize the proof of Theorem 1.4.

Remark. It is worth noting that in the preceding proof, the condition $\lambda = 2$ was only invoked in a limited number of specific cases. In most instances, our arguments did not rely on this constraint, which opens up the possibility of generalizing our findings to a broader class of designs. In future work, we plan to extend this line of research by systematically investigating the existence of such designs under the completely unconstrained scenario where the parameter λ is not fixed to 2.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by NSFC (Nos.12401020 and 12471022), Hunan Provincial Natural Science Foundation (2026JJ60324), and the Innovation and Entrepreneurship Training Program for College Students in Hunan Province (S202510534089).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. P. C. Clapham, Steiner triple systems with block-transitive automorphism groups, *Discrete Math.*, **14** (1976), 121–131. [https://doi.org/10.1016/0012-365X\(76\)90055-8](https://doi.org/10.1016/0012-365X(76)90055-8)
2. A. Delandtsheer, J. Doyen, Most block-transitive designs are point-primitive, *Geom. Dedicata*, **29** (1989), 307–310. <https://doi.org/10.1007/bf00572446>
3. A. R. Camina, The socle of automorphism groups of linear spaces, *Bull. London Math. Soc.*, **28** (1996), 269–272. <https://doi.org/10.1112/blms/28.3.269>
4. A. R. Camina, C. E. Praeger, Line-transitive, point quasiprimitive automorphism groups of finite linear spaces are affine or almost simple, *Aequ. Math.*, **61** (2001), 221–232. <https://doi.org/10.1007/s000100050174>
5. D. Tian, Z. Huang, W. Liu, S. Zhou, A survey of groups and designs, *Adv. Math.*, **54** (2025), 265–291.
6. P. J. Cameron, C. E. Praeger, Block-transitive t -designs, II: large t , in *Finite Geometries and Combinatorics*, (1993), 103–120. <https://doi.org/10.1017/cbo9780511526336.012>
7. M. Huber, On the Cameron-Praeger conjecture, *J. Comb. Theory Ser. A.*, **117** (2010), 196–203. <https://doi.org/10.1016/j.jcta.2009.04.004>
8. Q. Tan, W. Liu, J. Chen, On block-transitive 6- (v, k, λ) designs with k at most 10000, *Algebra Colloq.*, **21** (2014), 231–234. <https://doi.org/10.1142/S1005386714000182>

9. A. R. Camina, T. M. Gagen, Block transitive automorphism groups of designs, *J. Algebra*, **86** (1984), 549–554. [https://doi.org/10.1016/0021-8693\(84\)90047-4](https://doi.org/10.1016/0021-8693(84)90047-4)
10. Z. Huang, W. Liu, L. Feng, On the existence of G -block-transitive Steiner designs with k dividing v , *Queast. Math.*, **47** (2024), 321–339. <https://doi.org/10.2989/16073606.2023.2225756>
11. Z. Huang, W. Liu, L. Feng, Camina-Gagen theorem for Steiner 5-designs, *Sci. Sin. Math.*, **53** (2023), 651–666. <https://doi.org/10.1360/ssm-2022-0106>
12. P. J. Cameron, H. R. Maimani, G. R. Omid, B. Tayfeh-Rezaie, 3-Designs from $\text{PSL}(2, q)$, *Discrete Math.*, **306** (2006), 3063–3073. <https://doi.org/10.1016/j.disc.2005.06.041>
13. M. Huber, A census of highly symmetric combinatorial designs, *J. Algebraic Comb.*, **26** (2007), 453–476. <https://doi.org/10.1007/s10801-007-0065-4>
14. W. M. Kantor, Homogeneous designs and geometric lattices, *J. Comb. Theory Ser. A*, **38** (1985), 66–74. [https://doi.org/10.1016/0097-3165\(85\)90022-6](https://doi.org/10.1016/0097-3165(85)90022-6)
15. P. Dembowski, *Finite Geometries*, 1st edition, Springer, 1997. <https://doi.org/10.1007/978-3-642-62012-6>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)