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Research article

Multiple normalized solutions to Schrödinger systems with linear and nonlinear couplings

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Abstract: We establish the existence and multiplicity of normalized solutions to the coupled nonlinear Schrödinger system with linear and nonlinear couplings

$$\begin{cases} -u'' + \lambda u = \mu u^3 + \beta v^2 u + \kappa v & \text{in } \mathbb{R}, \\ -v'' + \lambda v = \mu v^3 + \beta u^2 v + \kappa u & \text{in } \mathbb{R}, \end{cases}$$

satisfying the total mass constraint

$$\int_{\mathbb{R}} (u^2 + v^2)\, \mathrm{d}x = m,$$

where the nonlinear coupling parameter $\beta = \mu > 0$. The system comes from the research on standing waves of coupled Gross–Pitaevskii equations for describing Bose–Einstein condensates. First, we show that, up to translations and sign symmetries, there exists exactly one class of normalized solutions when the linear coupling parameter $\kappa = 0$. The least energy level is also explicitly determined. Second, we prove that at least three nontrivial normalized solutions exist under suitable conditions on the linear coupling parameter $\kappa \neq 0$. We present a new approach based on transforming the Schrödinger system with both linear and nonlinear couplings into a Schrödinger system with purely nonlinear coupling. This seems to be the first result concerning the multiplicity of normalized solutions to the Schrödinger system with both linear and nonlinear couplings.

Keywords: nonlinear Schrödinger systems; linear and nonlinear couplings; normalized solutions; multiplicity; the total mass constraint

1. Introduction

In this paper, we are concerned with the existence and multiplicity of normalized solutions to the following coupled nonlinear Schrödinger system satisfying the total mass constraint:

$$\begin{cases} -u'' + \lambda u = \mu u^3 + \beta v^2 u + \kappa v & \text{in } \mathbb{R}, \\ -v'' + \lambda v = \mu v^3 + \beta u^2 v + \kappa u & \text{in } \mathbb{R}, \\ \int_{\mathbb{R}} (u^2 + v^2)\,\mathrm{d}x = m, \end{cases} \tag{1.1}$$

where $m > 0$ is prescribed, $\mu > 0$ is a constant, β is the nonlinear coupling parameter, and κ is the linear coupling parameter. Here $\lambda \in \mathbb{R}$ is unknown and has to be determined. If there exist $\lambda \in \mathbb{R}$ and $(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying (1.1), then (u, v) is called a *normalized solution* and λ is called the associated *Lagrange multiplier*.

When $\lambda > 0$ is prescribed, rather than the L^2-norm $\|u\|^2_{L^2(\mathbb{R})} + \|v\|^2_{L^2(\mathbb{R})}$, the unconstrained problem has been widely investigated in the past two decades. See [1–4] for the uniqueness of positive solutions; see [2, 3, 5–8] for the existence of least energy solutions and bound state solutions; see [9–12] for bifurcations from the synchronized solution branch with respect to β or κ; see [8, 13] for the symmetry and asymptotic behavior of least energy solutions.

The L^2-constrained problem (1.1) comes from the research on standing waves of the following coupled Gross–Pitaevskii equations for describing Bose–Einstein condensates:

$$\begin{cases} -\mathrm{i}\,\partial_t \Phi_1 = \partial_{xx}\Phi_1 + \mu\,|\Phi_1|^2\,\Phi_1 + \beta\,|\Phi_2|^2\,\Phi_1 + \kappa\Phi_2, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ -\mathrm{i}\,\partial_t \Phi_2 = \partial_{xx}\Phi_2 + \mu\,|\Phi_2|^2\,\Phi_2 + \beta\,|\Phi_1|^2\,\Phi_2 + \kappa\Phi_1, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \Phi_1(t, x) \to 0, \Phi_2(t, x) \to 0 \text{ as } |x| \to +\infty, \end{cases} \tag{1.2}$$

where $(\Phi_1(t, x), \Phi_2(t, x))$ is the complex valued macroscopic wave function, κ is the effective Rabi frequency to realize the internal atomic Josephson junction by a Raman transition, the parameters μ and β represent the intra-spaces and inter-species scattering length, respectively. A standing wave of (1.2) is a solution having the form

$$(\Phi_1, \Phi_2) = (e^{\mathrm{i}\lambda t} u(x), e^{\mathrm{i}\lambda t} v(x))$$

for some $\lambda \in \mathbb{R}$, where (u, v) solves (1.1). Also, (1.2) arises in the study of nonlinear optical fibers. Here Φ_1 and Φ_2 are two coupled electric field envelopes of the same wavelength but of different polarizations. The linear coupling κ is generated either by a twist applied to the fiber in the case of two linear polarizations, or by an elliptic deformation of the fiber's core in the case of circular polarizations. Note that the effects of the parameters on modulation instability growth rate, nonlinear modes generation, and modulated wave pattern in birefringent fibers have been deeply studied in [14, 15]. Interested readers can find additional details and applications in earlier studies [16–20] and recent studies [14, 15, 21].

It is well known that the total mass is conserved in time along trajectories of (1.2):

$$\int_{\mathbb{R}} \left(|\Phi_1(t, x)|^2 + |\Phi_2(t, x)|^2\right) \mathrm{d}x = \int_{\mathbb{R}} \left(|\Phi_1(0, x)|^2 + |\Phi_2(0, x)|^2\right) \mathrm{d}x, \quad \forall t \in \mathbb{R},$$

and that mass conservation is global in nature; see [22] for a detailed interpretation. This quantity is the total number of particles in Bose–Einstein condensates as discussed in [20]. Thus, from the physical point of view, it is important to consider the total mass constrained problem (1.1).

When $\kappa = 0$, the existence of normalized solutions of (1.1) was first addressed by Cipolatti and Zumpichiatti [23] using the concentration-compactness principle. Subsequently, the case $\beta > \mu$ was studied by Cao et al. [24] via an approximation method. When the coupled nonlinear Schrödinger system involves external potentials, recent developments can be found in [20, 25–28], with further advances in [29,30]. For coercive potentials, Bao and Cai [20] established the existence and uniqueness of positive ground states. Guo et al. [25, 26] investigated the existence, uniqueness, and asymptotic properties of ground states. In case of bounded potentials, Liu and Yang [28] employed the linking theory to establish the existence of nonnegative normalized solutions. Guo et al. [27, 28] proved the existence and local uniqueness of normalized solutions by the finite dimensional reduction method. For the steep well potential with a unique global minimum point, Kong et al. [30] studied the existence and asymptotic properties of ground states. Naturally, the total mass constrained problem (1.1) is reminiscent of the double mass constrained problem:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = m_1 \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 \, \mathrm{d}x = m_2, \end{cases}$$

where $\mu_1, \mu_2, m_1, m_2 > 0$ are prescribed. Under various conditions on β, variational methods are widely used to explore normalized solutions for the above double mass constrained problem. In the mass subcritical case $N = 1$, the existence of ground states were established in [24, 31–33] by minimizing the corresponding constrained energy functional; a uniqueness result was proved in [34] for $\beta = \mu$ and $m_1 = m_2$ by means of Hirota's bilinearization method. In the mass supercritical case $N = 3$, the existence and multiplicity results were obtained in [35–39] using mini-max methods. It seems almost impossible for us to provide a complete list of references, so we refer the readers to the aforementioned papers and the references therein. Inspired by Frank et al. [34], our first goal is to give a complete classification of normalized solutions to the total mass constrained problem (1.1) for $\beta = \mu$.

When $\kappa \neq 0$, little is known about the existence of normalized solutions to (1.1), let alone the multiplicity. As far as we know, the only relevant results are presented in [20,40–42]. Bao and Cai [20] established the existence and uniqueness results for ground states of the total mass constrained problem with trapping potentials. Yun [42] studied the existence of normalized solutions to the double mass constrained problem with trapping potentials. Yun and Zhang [40, 41] considered the double mass constrained problem

$$\begin{cases} -\Delta u + \lambda_1 u = \mu u^3 + \beta v^2 u + \kappa(x) v & \text{in } \mathbb{R}^3, \\ -\Delta v + \lambda_2 v = \mu v^3 + \beta u^2 v + \kappa(x) u & \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} u^2 \, \mathrm{d}x = m_1 \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 \, \mathrm{d}x = m_2, \end{cases}$$

and proved the existence of positive, radially symmetric normalized solutions in two cases: $\kappa(x) = \kappa(|x|)$, $\kappa \in L^q\left(\mathbb{R}^3\right) \cap L^\infty\left(\mathbb{R}^3\right)$ with $q \in (3/2, \infty)$, $\beta > 0$, as well as $\kappa \in L^\infty\left(\mathbb{R}^3\right)$ and $\beta < 0$. The main techniques of [20, 40–42] depend critically on the compact embedding, and thus do not work if $N = 1$, κ is a nonzero constant, and external potentials are absent. The second goal of this paper is to establish the multiplicity of normalized solutions to (1.1) for $\beta = \mu$ in the absence of compactness. It is worthwhile to point out that even in the case where $\lambda > 0$ is prescribed, the multiplicity of solutions to (1.1) for $\beta = \mu$ has not yet been studied.

While significant progress has been achieved in studying normalized solutions to coupled nonlinear Schrödinger systems, it remains an open question as to the complete classification of normalized solutions to the total mass constrained problem (1.1) with purely nonlinear coupling ($\kappa = 0$ and $\beta = \mu$). Another important question is whether one can prove the multiplicity of normalized solutions to the total mass constrained problem (1.1) involving both linear and nonlinear couplings ($\kappa \neq 0$ and $\beta = \mu$). In this paper, we will answer these two questions affirmatively. Our findings will provide valuable insight into the structure of normalized solutions to the total mass constrained problem (1.1).

To formulate our results more precisely, let us introduce some notations first. A solution (u, v) is called *nonzero* if $u \not\equiv 0$ or $v \not\equiv 0$; a solution (u, v) is called *nontrivial* if $u \not\equiv 0$ and $v \not\equiv 0$; a solution (u, v) is called *semi-trivial* if (u, v) takes the form of $(u, 0)$ or $(0, v)$. As shown by Berestycki and Lions [43], up to translations and the sign $\pm$, the following scalar Schrödinger equation

$$-w'' + w = w^3, \quad w \in H^1(\mathbb{R}) \tag{1.3}$$

admits a unique solution

$$w(x) = \sqrt{2}\operatorname{sech} x. \tag{1.4}$$

In this paper, we focus on the case $\beta = \mu$. First, we employ variational methods and algebraic analysis to completely classify the normalized solutions to (1.1) when the linear coupling parameter $\kappa = 0$.

Theorem 1.1. *Assume $\beta = \mu > 0$ and $\kappa = 0$. Then for every $m > 0$, the following statements hold.*

(i) (1.1) *admits exactly one normalized solution, up to translations and sign symmetries, formed as (u, v, λ) if and only if the associated Lagrange multiplier $\lambda = m^2\mu^2/16$ and*

$$(u_\lambda(x), v_\lambda(x)) = \left(\sqrt{\frac{2\lambda}{\mu}}\cos\theta\operatorname{sech}(\sqrt{\lambda}x), \sqrt{\frac{2\lambda}{\mu}}\sin\theta\operatorname{sech}(\sqrt{\lambda}x)\right), \quad \theta \in \mathbb{R}. \tag{1.5}$$

(ii) *The least energy normalized solutions of* (1.1) *necessarily satisfy* (1.5) *and the least energy level is $-m^3\mu^2/96$.*

Remark 1.1. *Under the assumptions of Theorem 1.1, it is clear that the least energy normalized solutions to the total mass constrained problem* (1.1) *are not unique, which is quite different from the double mass constrained case; see Frank et al. [34, Theorem 8].*

Next, we establish the multiplicity of normalized solutions to (1.1) under suitable conditions on the linear coupling parameter $\kappa \neq 0$. In the case where $N = 1$ and $\kappa \neq 0$ is a constant, the classical constrained functional minimization method does not work. We have to confront two main difficulties: The Sobolev embedding $H^1_r(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ is not compact, and the linear coupling term in the constrained functional cannot be effectively controlled by the Gagliardo–Nirenberg inequality. Even if the above two difficulties were overcome, we would still encounter a new challenge arising from the total mass constraint: how to guarantee that each component of the normalized solution is nonzero.

We present a new approach when $\beta = \mu$: transforming the Schrödinger system (1.1) with both linear and nonlinear couplings into a Schrödinger system with the purely nonlinear coupling, and then reformulating the total mass constrained problem as a double mass constrained problem.

Theorem 1.2. *Assume $\beta = \mu > 0$ and $\kappa \neq 0$. Then for every $m > 0$, the following statements hold.*

(i) *For every* $\kappa \in \mathbb{R}\backslash\{0\}$, (1.1) *admits synchronized normalized solutions, up to translations and sign symmetries, formed as* (u, v, λ) *if and only if the associated Lagrange multiplier* $\lambda = m^2\mu^2/16 - \kappa$ *and* (u, v) *satisfies*

$$u_\lambda(x) = -v_\lambda(x) = \sqrt{\frac{\lambda+\kappa}{\mu}}\,\mathrm{sech}\,(\sqrt{\lambda+\kappa}x), \tag{1.6}$$

or the associated Lagrange multiplier $\lambda = m^2\mu^2/16 + \kappa$ *and* (u, v) *satisfies*

$$u_\lambda(x) = v_\lambda(x) = \sqrt{\frac{\lambda-\kappa}{\mu}}\,\mathrm{sech}\,(\sqrt{\lambda-\kappa}x). \tag{1.7}$$

(ii) *If* $\kappa \in \left(-m^2\mu^2/32, 0\right)$*, then* (1.1) *has a third normalized solution* (u, v)*, such that the associated Lagrange multiplier* $\lambda \in \left(m^2\mu^2/64, m^2\mu^2/32\right)$ *with* $\lambda > -\kappa$ *and up to translations, that satisfies*

$$u = \varphi_{\lambda_1,\lambda_2} + \psi_{\lambda_1,\lambda_2} \quad \text{and} \quad v = \varphi_{\lambda_1,\lambda_2} - \psi_{\lambda_1,\lambda_2},$$

where $\lambda_1 = \sqrt{\lambda-\kappa}$*,* $\lambda_2 = \sqrt{\lambda+\kappa}$*, and*

$$\begin{aligned} \varphi_{\lambda_1,\lambda_2}(x) &= \frac{\mu^{-1/2}\lambda_1\sqrt{\lambda_1^2-\lambda_2^2}\cosh\left[\lambda_2\left(x+\Delta x\right)\right]}{\lambda_1\cosh(\lambda_1 x)\cosh\left[\lambda_2\left(x+\Delta x\right)\right]-\lambda_2\sinh(\lambda_1 x)\sinh\left[\lambda_2\left(x+\Delta x\right)\right]}, \\ \psi_{\lambda_1,\lambda_2}(x) &= \frac{-\mu^{-1/2}\lambda_2\sqrt{\lambda_1^2-\lambda_2^2}\sinh(\lambda_1 x)}{\lambda_1\cosh(\lambda_1 x)\cosh\left[\lambda_2\left(x+\Delta x\right)\right]-\lambda_2\sinh(\lambda_1 x)\sinh\left[\lambda_2\left(x+\Delta x\right)\right]}, \end{aligned} \tag{1.8}$$

for arbitrary $\Delta x \in \mathbb{R}$.

(iii) *If* $\kappa \in \left(0, m^2\mu^2/32\right)$*, then* (1.1) *has a third normalized solution* (u, v)*, such that the associated Lagrange multiplier* $\lambda \in \left(m^2\mu^2/64, m^2\mu^2/32\right)$ *with* $\lambda > \kappa$ *and up to translations, that satisfies*

$$u = \varphi^{\lambda_1,\lambda_2} + \psi^{\lambda_1,\lambda_2} \quad \text{and} \quad v = \varphi^{\lambda_1,\lambda_2} - \psi^{\lambda_1,\lambda_2},$$

where $\lambda_1 = \sqrt{\lambda+\kappa}$*,* $\lambda_2 = \sqrt{\lambda-\kappa}$*, and*

$$\begin{aligned} \varphi^{\lambda_1,\lambda_2}(x) &= \frac{-\mu^{-1/2}\lambda_2\sqrt{\lambda_1^2-\lambda_2^2}\sinh(\lambda_1 x)}{\lambda_1\cosh(\lambda_1 x)\cosh\left[\lambda_2\left(x+\Delta x\right)\right]-\lambda_2\sinh(\lambda_1 x)\sinh\left[\lambda_2\left(x+\Delta x\right)\right]}, \\ \psi^{\lambda_1,\lambda_2}(x) &= \frac{\mu^{-1/2}\lambda_1\sqrt{\lambda_1^2-\lambda_2^2}\cosh\left[\lambda_2\left(x+\Delta x\right)\right]}{\lambda_1\cosh(\lambda_1 x)\cosh\left[\lambda_2\left(x+\Delta x\right)\right]-\lambda_2\sinh(\lambda_1 x)\sinh\left[\lambda_2\left(x+\Delta x\right)\right]}, \end{aligned} \tag{1.9}$$

for arbitrary $\Delta x \in \mathbb{R}$.

Remark 1.2. (i) *It is worthwhile to note that Theorem 1.2 seems to be the first result concerning the multiplicity of normalized solutions for the Schrödinger system* (1.1) *with both linear and nonlinear couplings. Even for the fixed frequency problem (i.e.,* λ *is prescribed), the majority of literature focuses on the existence of solutions. To our knowledge, only Dai et al. [11] obtained the multiplicity of positive solutions when* $\beta \in (-\mu, \mu)$ *by means of the bifurcation techniques.*

(ii) *Theorem 1.2 implies that the third normalized solution to* (1.1) *is nonsynchronized.*
(iii) *Comparing with the purely nonlinear coupling case, Theorems 1.1 and 1.2 indicate that the linear coupling term enriches the solution set to the Schrödinger system* (1.1).

Remark 1.3. *Our approach based on the transformation offers two main advantages. First, we do not have to tackle the technical difficulties caused by the linear coupling terms. Second, the transformation allows us to make use of the methods and results about the double mass constrained Schrödinger systems, and thus avoids proving the nontriviality of each component encountered in directly addressing the total mass constrained Schrödinger systems.*

The normalized solutions of (1.1) derived in Theorems 1.1 and 1.2 are known as bound state solutions in quantum mechanics and can generate standing waves of coupled Gross–Pitaevskii equations (1.2) for describing two-component Bose–Einstein condensates. Our study reveals new pictures in Bose–Einstein condensates. It should be mentioned that the numerical investigation of nonlinear Schrödinger equations is another important subject that has attracted considerable attention. For instance, an optimal pointwise error estimate for the two-dimensional space fractional nonlinear Schrödinger equation was established for the first time in [21]; a linearized compact difference scheme suitable for long-time simulation was proposed in [44].

The remaining part of this paper is organized as follows. In Section 2, we present some preliminary results on the fixed frequency problem, which are of independent interest. Sections 3 and 4 are devoted to proving Theorems 1.1 and 1.2, respectively.

2. Preliminaries

In preparation for the proof of our main results, we need some auxiliary results.

Consider the fixed frequency problem:

$$\begin{cases} -u'' + \omega_1 u = \mu_1 u^3 + \beta v^2 u + \kappa v & \text{in } \mathbb{R}, \\ -v'' + \omega_2 v = \mu_2 v^3 + \beta u^2 v + \kappa u & \text{in } \mathbb{R}, \end{cases} \tag{2.1}$$

where ω_1 and ω_2 are positive constants.

We denote the fixed frequencies in the unconstrained problems as ω_1, ω_2, or ω to avoid any confusion with the Lagrange multiplier λ in the constrained problems.

Proposition 2.1. *Assume $\omega_1 = \omega_2 := \omega > 0$, $\beta = \mu_1 = \mu_2 := \mu > 0$, and $\kappa = 0$. Then* (2.1) *has no sign-changing solutions. Moreover, up to translations, all the nonzero solutions of* (2.1) *take exactly the form of*

$$(u(x), v(x)) = \left(\sqrt{\frac{2\omega}{\mu}} \cos\theta \operatorname{sech}(\sqrt{\omega} x), \sqrt{\frac{2\omega}{\mu}} \sin\theta \operatorname{sech}(\sqrt{\omega} x) \right), \quad \textit{where } \theta \in \mathbb{R}. \tag{2.2}$$

Proof. Suppose (u, v) is a nonzero solution of (2.1). We proceed in two cases.

Case 1. (u, v) is a semi-trivial solution.

Without loss of generality, we may assume $v = 0$. In this case, (2.1) becomes a scalar equation

$$-u'' + \omega u = \mu u^3, \quad u \in H^1(\mathbb{R}).$$

Then, from (1.3) and (1.4), we deduce that, up to translations and the sign ±, the above equation admits a unique solution:

$$u(x)=\sqrt{\frac{2\omega}{\mu}}\,\mathrm{sech}(\sqrt{\omega}x).$$

Therefore $(u,0)$ does not change sign and takes the form of (2.2) for some $\theta\in\{k\pi : k\in\mathbb{Z}\}$. Proceeding in the same way, we can prove $(0,v)$ does not change sign and takes the form of (2.2) for some $\theta\in\{k\pi+\pi/2 : k\in\mathbb{Z}\}$.

Case 2. (u,v) is a nontrivial solution.

In this case, using the fact that $\omega_1=\omega_2$, we derive from Tratnik and Sipe [16, Section III] (see also Frank et al. [34, Lemma 15]) that

$$u=cv \quad \text{for some constant } c\neq 0.$$

Thus, we can rewrite (2.1) as two uncoupled equations:

$$\begin{aligned}&-u''+\omega u=\frac{1}{c^2}\left(1+c^2\right)\mu u^3, \quad u\in H^1(\mathbb{R}),\\ &-v''+\omega v=\left(1+c^2\right)\mu v^3, \quad v\in H^1(\mathbb{R}).\end{aligned}$$

By (1.3) and (1.4), we then obtain, up to translations,

$$u=\pm c\sqrt{\frac{2\omega}{\mu}}\sqrt{\frac{1}{1+c^2}}\,\mathrm{sech}(\sqrt{\omega}x), \quad v=\pm\sqrt{\frac{2\omega}{\mu}}\sqrt{\frac{1}{1+c^2}}\,\mathrm{sech}(\sqrt{\omega}x).$$

Therefore, (u,v) does not change sign. In addition, it is clear that there exists some $\theta\in\mathbb{R}\backslash\{k\pi/2 : k\in\mathbb{Z}\}$ such that $\sqrt{c^2/(1+c^2)}=|\cos\theta|$. Then, one has $\sqrt{1/(1+c^2)}=|\sin\theta|$. Consequently, (u,v) takes the form of (2.2).

Remark 2.1. *When $\omega_1=\omega_2>0$, $\beta=\mu_1=\mu_2>0$, and $\kappa=0$, Proposition 2.1 gives a complete classification of all the general solutions of* (2.1)*, and thus refines Wei and Yao [1, Theorem 1.2], which obtained the complete classification of all the positive solutions of* (2.1)

Proposition 2.2. *[45] Assume $\beta=\mu_1=\mu_2:=2\mu>0$, and $\kappa=0$.*

(i) *If $0<\omega_1<\omega_2$, then* (2.1) *has no positive solutions. Moreover, all the nontrivial solutions (u,v) of* (2.1) *satisfy*

$$u(x)=\frac{-\mu^{-1/2}\sqrt{\omega_1(\omega_2-\omega_1)}\sinh(\sqrt{\omega_2}x)}{\sqrt{\omega_2}\cosh(\sqrt{\omega_2}x)\cosh\left[\sqrt{\omega_1}(x+\Delta x)\right]-\sqrt{\omega_1}\sinh(\sqrt{\omega_2}x)\sinh\left[\sqrt{\omega_1}(x+\Delta x)\right]},$$

$$v(x)=\frac{\mu^{-1/2}\sqrt{\omega_2(\omega_2-\omega_1)}\cosh\left[\sqrt{\omega_1}(x+\Delta x)\right]}{\sqrt{\omega_2}\cosh(\sqrt{\omega_2}x)\cosh\left[\sqrt{\omega_1}(x+\Delta x)\right]-\sqrt{\omega_1}\sinh(\sqrt{\omega_2}x)\sinh\left[\sqrt{\omega_1}(x+\Delta x)\right]},$$

where $\Delta x\in\mathbb{R}$ is arbitrary.

(ii) *If* $\omega_1 > \omega_2 > 0$, *then* (2.1) *has no positive solutions. Moreover, all the nontrivial solutions* (u, v) *of* (2.1) *satisfy*

$$u(x) = \frac{\mu^{-1/2}\sqrt{\omega_1(\omega_1-\omega_2)}\cosh\left[\sqrt{\omega_2}\,(x+\Delta x)\right]}{\sqrt{\omega_1}\cosh(\sqrt{\omega_1}x)\cosh\left[\sqrt{\omega_2}\,(x+\Delta x)\right] - \sqrt{\omega_2}\sinh(\sqrt{\omega_1}x)\sinh\left[\sqrt{\omega_2}\,(x+\Delta x)\right]},$$

$$v(x) = \frac{-\mu^{-1/2}\sqrt{\omega_2(\omega_1-\omega_2)}\sinh(\sqrt{\omega_1}x)}{\sqrt{\omega_1}\cosh(\sqrt{\omega_1}x)\cosh\left[\sqrt{\omega_2}\,(x+\Delta x)\right] - \sqrt{\omega_2}\sinh(\sqrt{\omega_1}x)\sinh\left[\sqrt{\omega_2}\,(x+\Delta x)\right]},$$

where $\Delta x \in \mathbb{R}$ *is arbitrary.*

Remark 2.2. *Under the assumptions of Proposition 2.2, we deduce that* (2.1) *has a solution satisfying the following properties: One component is sign-changing that changes sign exactly once and another component does not change sign.*

Proposition 2.3. *Assume* $\omega_1 = \omega_2 := \omega$, $\beta = \mu_1 = \mu_2 := \mu > 0$, *and* $\kappa \neq 0$. *If* (u, v) *is a synchronized solution of* (2.1), *then either* $u = v$ *with* $\omega > \kappa$ *or* $u = -v$ *with* $\omega > -\kappa$.

Proof. Suppose (u, v) is a synchronized solution of (2.1). Notice that (2.1) has no semi-trivial solutions $(u, 0)$ or $(0, v)$. Then we may assume $u = cv$ for some constant $c \neq 0$. Substituting this into (2.1), we get

$$\begin{aligned} -u'' + \left(\omega - \frac{\kappa}{c}\right)u &= \mu\left(1 + \frac{1}{c^2}\right)u^3 \quad \text{in } \mathbb{R}, \\ -v'' + (\omega - \kappa c)\,v &= \mu(1 + c^2)v^3 \quad \text{in } \mathbb{R}. \end{aligned}$$

By the Pohozaev identity, we have

$$\omega - \frac{\kappa}{c} > 0 \quad \text{and} \quad \omega - \kappa c > 0.$$

Thus, from (1.3) and (1.4), we deduce that, up to translations,

$$\begin{aligned} u(x) &= \pm c\sqrt{\frac{2\,(\omega-\kappa/c)}{\mu\,(1+c^2)}}\operatorname{sech}(\sqrt{\omega-\kappa/c}\,x), \\ v(x) &= \pm\sqrt{\frac{2\,(\omega-\kappa c)}{\mu\,(1+c^2)}}\operatorname{sech}(\sqrt{\omega-\kappa c}\,x). \end{aligned} \tag{2.3}$$

On the other hand, since $v = 1/cu$, we have

$$v(x) = \frac{1}{c}u(x) = \pm\sqrt{\frac{2\,(\omega-\kappa/c)}{\mu\,(1+c^2)}}\operatorname{sech}(\sqrt{\omega-\kappa/c}\,x),$$

which, together with (2.3), implies $\kappa/c = \kappa c$. Hence $c = \pm 1$, so that $\omega > \kappa$ and $u = v$ or $\omega > -\kappa$ and $u = -v$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1 (i). The Pohozaev identity of (1.1) is

$$\frac{1}{2}\int_{\mathbb{R}}\left(|u'|^2+|v'|^2\right)\mathrm{d}x+\frac{\mu}{4}\int_{\mathbb{R}}\left(u^4+2u^2v^2+v^4\right)\mathrm{d}x-\frac{\lambda}{2}\int_{\mathbb{R}}\left(u^2+v^2\right)\mathrm{d}x=0.$$

Multiplying the first equation of (1.1) by u and the second equation of (1.1) by v, and integrating, shows that

$$\int_{\mathbb{R}}|u'|^2\mathrm{d}x+\lambda\int_{\mathbb{R}}u^2\mathrm{d}x=\mu\int_{\mathbb{R}}\left(u^4+u^2v^2\right)\mathrm{d}x,$$
$$\int_{\mathbb{R}}|v'|^2\mathrm{d}x+\lambda\int_{\mathbb{R}}v^2\mathrm{d}x=\mu\int_{\mathbb{R}}\left(u^2v^2+v^4\right)\mathrm{d}x.$$

Thus,

$$\lambda m=\lambda\int_{\mathbb{R}}\left(u^2+v^2\right)\mathrm{d}x=\frac{3\mu}{4}\int_{\mathbb{R}}\left(u^4+2u^2v^2+v^4\right)\mathrm{d}x>0,$$

which implies that the Langrange multiplier $\lambda>0$. We then apply Proposition 2.1 to obtain that (1.1) admits normalized solutions (u,v) if and only of (u,v) takes the form of (1.5) up to translations. By straightforward computations, we have

$$\begin{aligned}
m=&\int_{\mathbb{R}}\left(u_\lambda^2+v_\lambda^2\right)\mathrm{d}x\\
=&\int_{\mathbb{R}}\left(\sqrt{\frac{2\lambda}{\mu}}\cos\theta\,\mathrm{sech}(\sqrt{\lambda}x)\right)^2\mathrm{d}x+\int_{\mathbb{R}}\left(\sqrt{\frac{2\lambda}{\mu}}\sin\theta\,\mathrm{sech}(\sqrt{\lambda}x)\right)^2\mathrm{d}x\\
=&\frac{2\lambda}{\mu}\int_{\mathbb{R}}\mathrm{sech}^2(\sqrt{\lambda}x)\,\mathrm{d}x\\
=&\frac{2\sqrt{\lambda}}{\mu}\int_{\mathbb{R}}\mathrm{sech}^2x\,\mathrm{d}x=\frac{4\sqrt{\lambda}}{\mu}.
\end{aligned}$$

This gives $\lambda=m^2\mu^2/16$.

Proof of Theorem 1.1 (ii). It is well known that normalized solutions of (1.1) can be characterized as critical points of the following energy functional:

$$E(u,v)=\frac{1}{2}\int_{\mathbb{R}}\left(|u'|^2+|v'|^2\right)\mathrm{d}x-\frac{\mu}{4}\int_{\mathbb{R}}\left(u^4+2u^2v^2+v^4\right)\mathrm{d}x,$$

under the total mass constraint

$$S:=\left\{(u,v)\in H^1(\mathbb{R})\times H^1(\mathbb{R}):\|u\|_{L^2(\mathbb{R})}^2+\|v\|_{L^2(\mathbb{R})}^2=m\right\}.$$

Recall from Theorem 1.1 (i) that $\lambda=m^2\mu^2/16$. Using the facts

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{sech}(\sqrt{\lambda}x)\right|^2=\lambda\left(\mathrm{sech}^2(\sqrt{\lambda}x)-\mathrm{sech}^4(\sqrt{\lambda}x)\right),$$
$$\int_{\mathbb{R}}\mathrm{sech}^2x\,\mathrm{d}x=2\quad\text{and}\quad\int_{\mathbb{R}}\mathrm{sech}^4x\,\mathrm{d}x=\frac{4}{3},$$

straightforward calculations reveal

$$\begin{aligned}
E(u_\lambda, v_\lambda) =& \frac{1}{2}\int_{\mathbb{R}} \frac{2\lambda^2}{\mu}\left(\operatorname{sech}^2(\sqrt{\lambda}x) - \operatorname{sech}^4(\sqrt{\lambda}x)\right)\mathrm{d}x - \frac{\mu}{4}\int_{\mathbb{R}} \frac{4\lambda^2}{\mu^2}\operatorname{sech}^4(\sqrt{\lambda}x)\mathrm{d}x \\
=& \frac{\lambda^2}{\mu}\int_{\mathbb{R}} \left(\operatorname{sech}^2(\sqrt{\lambda}x) - 2\operatorname{sech}^4(\sqrt{\lambda}x)\right)\mathrm{d}x \\
=& \frac{\lambda\sqrt{\lambda}}{\mu}\int_{\mathbb{R}} \left(\operatorname{sech}^2 x - 2\operatorname{sech}^4 x\right)\mathrm{d}x \\
=& -\frac{2\lambda\sqrt{\lambda}}{3\mu} = -\frac{m^3\mu^2}{96},
\end{aligned} \tag{3.1}$$

where (u_λ, v_λ) is given in (1.5). From (3.1) and Theorem 1.1 (i), we deduce that

$$\inf_{(u,v)\in S} E(u,v) = -\frac{m^3\mu^2}{96},$$

and the set of minimizers for $\inf_{(u,v)\in S} E(u,v)$ consists of solutions defined by (1.5).

4. Proof of Theorem 1.2

Proof of Theorem 1.2 (i). *Sufficiency*. It is not difficult to verify the sufficiency by direct computations. Indeed, if $\lambda = m^2\mu^2/16 - \kappa$ and (u,v) satisfies (1.6), then we have

$$\begin{aligned}
& -u_\lambda'' + \lambda u_\lambda - \mu u_\lambda^3 - \mu v_\lambda^2 u_\lambda - \kappa v_\lambda \\
=& -u_\lambda'' + \lambda u_\lambda - \mu u_\lambda^3 - \mu u_\lambda^2 u_\lambda + \kappa u_\lambda \\
=& -u_\lambda'' + (\lambda+\kappa)u_\lambda - 2\mu u_\lambda^3,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
& -v_\lambda'' + \lambda v_\lambda - \mu v_\lambda^3 - \mu u_\lambda^2 v_\lambda - \kappa u_\lambda \\
=& -(-u_\lambda)'' + \lambda(-u_\lambda) - \mu(-u_\lambda)^3 - \mu u_\lambda^2(-u_\lambda) - \kappa u_\lambda \\
=& u_\lambda'' - (\lambda+\kappa)u_\lambda + 2\mu u_\lambda^3.
\end{aligned} \tag{4.2}$$

On the other hand, from (1.3) and (1.4), we deduce that u_λ solves

$$-u_\lambda'' + (\lambda+\kappa)u_\lambda = 2\mu u_\lambda^3 \quad \text{in } \mathbb{R},$$

which, together with (4.1) and (4.2), implies that (u_λ, v_λ) satisfies

$$\begin{cases}
-u_\lambda'' + \lambda u_\lambda = \mu u_\lambda^3 + \beta v_\lambda^2 u_\lambda + \kappa v_\lambda & \text{in } \mathbb{R}, \\
-v_\lambda'' + \lambda v_\lambda = \mu v_\lambda^3 + \beta u_\lambda^2 v_\lambda + \kappa u_\lambda & \text{in } \mathbb{R}.
\end{cases} \tag{4.3}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}} \left(u_\lambda^2 + v_\lambda^2\right)\mathrm{d}x =& 2\int_{\mathbb{R}} u_\lambda^2 \mathrm{d}x \\
=& \frac{2(\lambda+\kappa)}{\mu}\int_{\mathbb{R}} \operatorname{sech}^2(\sqrt{\lambda+\kappa}x)\,\mathrm{d}x
\end{aligned}$$

$$=\frac{2\sqrt{\lambda+\kappa}}{\mu}\int_{\mathbb{R}}\operatorname{sech}^2 x\,\mathrm{d}x$$
$$=\frac{4\sqrt{\lambda+\kappa}}{\mu}=\frac{4\sqrt{m^2\mu^2/16-\kappa+\kappa}}{\mu}=m,$$

we conclude from (4.3) that (u_λ, v_λ), defined by (1.6), is a synchronized normalized solution to (1.1). In a similar way, we can prove that (u_λ, v_λ) defined by (1.7) is a synchronized normalized solution to (1.1).

Necessity. According to Proposition 2.3, we obtain that either $u = v$ with $\lambda > \kappa$ or $u = -v$ with $\lambda > -\kappa$. Furthermore, from (2.3) we deduce that either $u = v$ satisfies (1.7) or $u = -v$ satisfies (1.6). Since

$$\int_{\mathbb{R}}\left|\sqrt{\frac{\lambda+\kappa}{\mu}}\operatorname{sech}(\sqrt{\lambda+\kappa}x)\right|^2\mathrm{d}x=\frac{2\sqrt{\lambda+\kappa}}{\mu},$$
$$\int_{\mathbb{R}}\left|\sqrt{\frac{\lambda-\kappa}{\mu}}\operatorname{sech}(\sqrt{\lambda-\kappa}x)\right|^2\mathrm{d}x=\frac{2\sqrt{\lambda-\kappa}}{\mu},$$

and

$$\int_{\mathbb{R}}\left(u^2+v^2\right)\mathrm{d}x=m,$$

we verify that $\lambda = m^2\mu^2/16 + \kappa$ when $u = v$ and $\lambda = m^2\mu^2/16 - \kappa$ when $u = -v$.

Proof of Theorem 1.2 (ii) ***and*** (iii). Suppose (u, v) is a normalized solution of (1.1) with the associated Lagrange multiplier λ. By performing a change of variables

$$\varphi=\frac{1}{2}(u+v),\quad \psi=\frac{1}{2}(u-v),$$

(1.1) can be reduced to the form

$$\begin{cases}-\varphi''+(\lambda-\kappa)\varphi=2\mu\varphi^3+2\mu\psi^2\varphi & \text{in } \mathbb{R},\\ -\psi''+(\lambda+\kappa)\psi=2\mu\psi^3+2\mu\varphi^2\psi & \text{in } \mathbb{R},\\ \displaystyle\int_{\mathbb{R}}(\varphi^2+\psi^2)\,\mathrm{d}x=\frac{m}{2}.\end{cases}\tag{4.4}$$

Consider a related double mass constrained problem:

$$\begin{cases}-\varphi''+\ell_1\varphi=2\mu\varphi^3+2\mu\psi^2\varphi & \text{in } \mathbb{R},\\ -\psi''+\ell_2\psi=2\mu\psi^3+2\mu\varphi^2\psi & \text{in } \mathbb{R},\\ \displaystyle\int_{\mathbb{R}}\varphi^2\,\mathrm{d}x=\rho,\int_{\mathbb{R}}\psi^2\,\mathrm{d}x=\frac{m}{2}-\rho,\end{cases}\tag{4.5}$$

where $0 \le \rho \le m/2$.

If $\rho = 0$ or $m/2$, the double mass constrained problem (4.5) reduces to the single component Bose–Einstein condensate:

$$\begin{cases}-\phi''+\ell\phi=2\mu\phi^3 & \text{in } \mathbb{R},\\ \displaystyle\int_{\mathbb{R}}\phi^2\,\mathrm{d}x=m/2.\end{cases}\tag{4.6}$$

Cazenave and Lions [46, Theorem II.1] have shown that (4.6) has a normalized solution ϕ for some $\ell > 0$; see also Lions [47, Theorem I.2]. Then by (1.3) and (1.4), we have

$$\phi(x) = \sqrt{\frac{\ell}{\mu}} \operatorname{sech}(\sqrt{\ell}x). \tag{4.7}$$

It follows from $\|\phi\|^2_{L^2(\mathbb{R})} = m/2$ that $\ell = m^2\mu^2/16$. Hence, we deduce from (4.4) and (4.5) that the following assertions hold:

(i) In case $\rho = 0$, i.e., $u = -v$. For every $\kappa \neq 0$, let $\lambda = m^2\mu^2/16 - \kappa$. Then, (1.1) has a normalized solution $(\phi, -\phi)$, where ϕ is defined by (4.7).

(ii) In case $\rho = m/2$, i.e., $u = v$. For every $\kappa \neq 0$, let $\lambda = m^2\mu^2/16 + \kappa$. Then, (1.1) has a normalized solution (ϕ, ϕ), where ϕ is defined by (4.7).

Thus, we next assume $\rho \in (0, m/2)$. It follows from Nguyen and Wang [33, Theorem 2.1] that there exists a normalized solution $(\varphi_\rho, \psi_\rho)$ of (4.5) for some $\ell_{1,\rho} > 0$ and $\ell_{2,\rho} > 0$. In view of Frank et al. [34, Lemma 16], we have

$$\rho = \int_{\mathbb{R}} \phi^2 \,\mathrm{d}x = 2\mu^{-1}\sqrt{\ell_{1,\rho}} \quad \text{and} \quad \frac{1}{2}m - \rho = \int_{\mathbb{R}} \psi^2 \,\mathrm{d}x = 2\mu^{-1}\sqrt{\ell_{2,\rho}},$$

which implies

$$\ell_{1,\rho} = \frac{1}{4}\mu^2\rho^2 \quad \text{and} \quad \ell_{2,\rho} = \frac{1}{4}\mu^2\left(\frac{m}{2} - \rho\right)^2.$$

Consequently,

$$\begin{cases} \ell_{1,\rho} = \ell_{2,\rho} & \text{for } \rho = m/4, \\ \ell_{1,\rho} < \ell_{2,\rho} & \text{for } \rho \in (0, m/4), \\ \ell_{1,\rho} > \ell_{2,\rho} & \text{for } \rho \in (m/4, m/2). \end{cases} \tag{4.8}$$

Let

$$\kappa = \frac{\ell_{2,\rho} - \ell_{1,\rho}}{2} = \frac{1}{8}\mu^2\left[\left(\frac{m}{2} - \rho\right)^2 - \rho^2\right] \tag{4.9}$$

and

$$\lambda = \frac{\ell_{1,\rho} + \ell_{2,\rho}}{2} = \frac{1}{8}\mu^2\left[\rho^2 + \left(\frac{m}{2} - \rho\right)^2\right]. \tag{4.10}$$

Clearly,

$$\lambda - \kappa = \ell_{1,\rho} > 0 \quad \text{and} \quad \lambda + \kappa = \ell_{2,\rho} > 0. \tag{4.11}$$

Combining Proposition 2.2, (4.8), and (4.11), we deduce that

$$\begin{aligned} &\text{If } \rho \in (0, m/4), \text{ then } (\varphi_\rho, \psi_\rho) \text{ satisfies (1.9) and solves (4.4).} \\ &\text{If } \rho \in (m/4, m/2), \text{ then } (\varphi_\rho, \psi_\rho) \text{ satisfies (1.8) and solves (4.4).} \end{aligned} \tag{4.12}$$

Note that $\kappa = 0$ if $\rho = m/4$. Therefore, $\rho \in (0, m/4) \cup (m/4, m/2)$. By a direct computation, from (4.9) and (4.10), we deduce that

$$\begin{cases} \kappa \in \left(0, m^2\mu^2/32\right) & \text{for } \rho \in (0, m/4), \\ \kappa \in \left(-m^2\mu^2/32, 0\right) & \text{for } \rho \in (m/4, m/2), \end{cases} \tag{4.13}$$

and

$$\begin{cases} \lambda \in \left(m^2\mu^2/64, m^2\mu^2/32\right) & \text{for } \rho \in (0, m/4), \\ \lambda \in \left(m^2\mu^2/64, m^2\mu^2/32\right) & \text{for } \rho \in (m/4, m/2). \end{cases} \tag{4.14}$$

Then from (4.12), we deduce that for every κ satisfies (4.13), $(\varphi_\rho + \psi_\rho, \varphi_\rho - \psi_\rho)$ is a normalized solution of (1.1), where the associated Lagrange multiplier λ satisfies (4.11) and (4.14). Finally, we show that $(\varphi_\rho + \psi_\rho, \varphi_\rho - \psi_\rho)$ is not the synchronized normalized solution obtained in Theorem 1.2 (i). Indeed, if this is false, then $\varphi_\rho = 0$ or $\psi_\rho = 0$, which is impossible since $\rho \neq 0$ and $\rho \neq m/2$. Hence, $(\varphi_\rho + \psi_\rho, \varphi_\rho - \psi_\rho)$ is a third normalized solution of (1.1). The proof is complete.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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