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Research article

Notes on the equitable graph of type II of a finite group

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Abstract: Given a finite group G, the equitable graph of type II defined on G is an undirected graph with vertex set G, in which two distinct vertices a and b are adjacent if and only if either $0 < |o(a) - o(b)| \leq \min\{o(a), o(b)\}$ or one of a and b is equal to e, where $o(a)$ and $o(b)$ are the orders of a and b, respectively. In this paper, we observe that every equitable graph of type II of a finite group is a generalized lexicographic product and discuss finite groups G whose equitable graph of type II is C_n-free, where $n \geq 3$. As an application, we show that every equitable graph of type II of a finite group is perfect. Finally, we characterize the metric dimension of the equitable graph of type II of a finite group.

Keywords: equitable graph of type II; finite group; metric dimension

1. Introduction

Graphs associated with groups and other algebraic structures have been actively investigated, because studying these relationships can reveal theoretical properties shared between algebraic structures and their associated graphs. A pivotal concept with rings is the zero-divisor graph, which was initially proposed by Beck in [1]. Graphs associated with groups provide valuable insights and enable a combinatorial approach to group theory. A well-known graph associated with a group is the Cayley graph. This graph has a very long history. Moreover, graphs from algebraic structures have been actively investigated in the literature, since they have valuable applications (cf. [2]).

Let G be a finite group. We can define various graphs on G, such as, the commuting graph [3] and the power graph [4]. The order of an element is one of the most basic and important concepts in group theory. For example, this outstanding survey [5] summarized the research on the characterization of finite simple groups and the study of finite groups based on their "set of element orders" and "two orders" (the order of the group and the set of element orders).

Certainly, one can also define a class of graphs by a group and its element orders. For example, Hamzeh et al. [6] defined a class of graphs on groups G, which is called the *supergraph of the power* *graph* of a finite group and has vertex set G, in which two distinct vertices x, y are adjacent if $o(x)|o(y)$ or $o(y)|o(x)$, where $o(a)$ and $o(b)$ are the orders of a and b, respectively. Recently, the authors in [7] called this graph the *order graph* of a group and gave a characterization of all finite groups whose order graph is C_4-free. Given the growing importance of group-associated graphs in classifying groups and graphs, along with the key role of element orders in finite groups, Altassan et al. [8] proposed a novel graph based on variations in element orders within the group, which is called the equitable graph of type I of a group and is defined as follows:

Definition 1.1. *Given a group G, the equitable graph of type I defined on G, denoted by $\mathcal{E}_1(G)$, is an undirected simple graph with vertex set G, and two distinct vertices a and b are adjacent if*

$$|o(a) - o(b)| \leq \min\{o(a), o(b)\}.$$

Recently, Altassan et al. [9] introduced the equitable graph of type II of a group.

Definition 1.2. *Given a group G, the equitable graph of type II defined on G, denoted by $\mathcal{E}_2(G)$, is an undirected simple graph with vertex set G, in which two distinct vertices a and b are adjacent if either*

$$0 < |o(a) - o(b)| \leq \min\{o(a), o(b)\}$$

or one of a and b is e.

Thus, for the equitable graph of type I, elements of the same order are adjacent, and if the identity is not isolated, then it must be adjacent to an involution. However, for the equitable graph of type II, the identity is adjacent to every other element, and elements of the same order are not adjacent.

All graphs considered in our paper are simple. Let Γ be a graph, and let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and edge set of Γ, respectively. For two vertices $x, y \in V(\Gamma)$, the *distance* between x and y, denoted by $d_\Gamma(x, y)$, is the length of a shortest path from x to y in Γ. Moreover, if the situation is unambiguous, then we denote $d_\Gamma(x, y)$ simply by $d(x, y)$. If there is $z \in V(\Gamma)$ such that $d(x, z) \neq d(y, z)$, then z resolves x and y. For $S \subseteq V(\Gamma)$, S is called a *resolving set* of Γ provided that any pair of distinct vertices of Γ can be resolved by some vertex in S. The *metric dimension* of Γ, denoted by $\dim(\Gamma)$, is the minimum cardinality of a resolving set of Γ. The metric dimension was defined by Harary and Melter [10] and independently by Slater [11]. By [12], one can see that determining the metric dimension is NP-complete. Recently, the metric dimension of the intersection power graph of a group was studied in [13].

In [8], the authors investigated the connectedness and some graph-theoretic properties of equitable graphs of type I of various groups, particularly focusing on cyclic groups. In [9], the authors studied the properties of the equitable graphs of type II for specific classes of groups, including cyclic, dihedral, and dicyclic groups. In this paper, we show that the equitable graphs of type II is a generalized lexicographic product and discuss finite groups G whose equitable graph of type II is C_n-free, where $n \geq 3$. In particular, we show that every equitable graph of type II is perfect. This implies that we can get easily [9, Theorems 2.12–2.15]. Finally, we characterize the metric dimension of the equitable graph of type II of a finite group.

2. Preliminary results

For a graph Γ, a subset of $V(\Gamma)$ is called a *clique* of Γ if any two distinct vertices in this subset are adjacent. Certainly, a subset containing only one element also is a clique. We use $\omega(\Gamma)$ to denote the clique number of Γ. A *hole* in a graph Γ is an induced subgraph of Γ isomorphic to a cycle of length at least 4. An *antihole* is an induced subgraph H of Γ, such that $\overline{H}$ is a hole of $\overline{\Gamma}$. For a graph, if its every induced subgraph Δ satisfies $\omega(\Delta) = \chi(\Delta)$, then this graph is called a *perfect graph*, where $\chi(\Delta)$ is the chromatic number of Δ. Perfect graphs, introduced by Claude Berge in the 1960s (cf. [14]), have since become fundamental objects of study in graph theory, linear programming, and combinatorial optimization.

Every group considered in our paper is finite. Throughout our paper, G always denotes a finite group with identity element e. Take an element $g \in G$. We write $\langle g \rangle$ for the cyclic subgroup generated by g and $o(g)$ for the order of g. An element $g \in G$ is called an *involution* if $o(g) = 2$. As usual, the cyclic group of order n is denoted by $\mathbb{Z}_n$, and we use $\mathbb{Z}_n^k$ to denote the k-fold direct product of $\mathbb{Z}_n$. Furthermore, let S_n denote the symmetric group on n objects.

Given an integer $m \geq 2$, the *generalized quaternion group* of order $4m$, denoted by Q_{4m}, has a definition as follows:

$$Q_{4m} = \langle x, y : x^m = y^2, x^{2m} = y^4 = e, xy = yx^{-1} \rangle. \tag{2.1}$$

In the literature, Q_{4m} is also called a *dicyclic group*. Note that Q_{4m} is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{2m}$ and is a non-abelian group. Furthermore, it is easy to see that Q_{4n} has a unique involution, which is x^n,

$$Q_{4m} = \{y, xy, x^2y, \ldots, x^{2m-1}y\} \cup \langle x \rangle,$$

and $o(x^i y) = 4$ for each $0 \leq i \leq 2m - 1$.

Recall now the following elementary result.

Lemma 2.1. [15, Theorem 5.4.10 (ii)] *Let p be a prime. A p-group having a unique subgroup of order p is isomorphic to either a cyclic group or a generalized quaternion 2-group.*

By Definition 1.2, we can obtain the following observations.

Observation 2.2. *In $\mathcal{E}_2(G)$, the following hold:*

(a) $\{e, x\} \in E(\mathcal{E}_2(G))$ *for any* $x \in G \setminus \{e\}$;
(b) *for $x, y \in G \setminus \{e\}$ with $o(x) < o(y)$, x and y are adjacent if and only if $2o(x) \geq o(y)$;*
(c) *$\mathcal{E}_2(G)$ is complete if and only if $G \cong \mathbb{Z}_2$.*

We next give the definition of a generalized lexicographic product, which was first introduced by Sabidussi [16]. Given a graph $\mathcal{H}$ and a family of graphs $\mathbb{F}$ indexed by $V(\mathcal{H})$ as follows:

$$\mathbb{F} = \{\mathcal{F}_v : v \in V(\mathcal{H})\},$$

the *generalized lexicographic product* of $\mathcal{H}$ and $\mathbb{F}$, denoted by $\mathcal{H}[\mathbb{F}]$, is an undirected graph with vertex set $\{(v, w) : v \in V(\mathcal{H}), w \in V(\mathcal{F}_v)\}$ and edge set

$$\{\{(v_1, w_1), (v_2, w_2)\} : \{v_1, v_2\} \in E(\mathcal{H}) \text{ or } v_1 = v_2 \text{ and } \{w_1, w_2\} \in E(\mathcal{F}_{v_1})\}.$$

For finite group G, we define a relation on G, say $\equiv$, as follows:

$$x \equiv y \Leftrightarrow o(x) = o(y).$$

It is readily seen that $\equiv$ is an equivalence relation. Denote by $\overline{x}$ the $\equiv$-class containing $x \in G$. Notice that $\overline{e} = \{e\}$. Write

$$\overline{G} = \{\overline{x} : x \in G\}.$$

We next define a graph, denoted by $\mathcal{L}_G$, which has vertex set $\overline{G}$ and, for two distinct $\overline{x}, \overline{y} \in \overline{G}$, $\{\overline{x}, \overline{y}\} \in E(\mathcal{L}_G)$ if and only if $1 \le |o(x) - o(y)| \le \min\{o(x), o(y)\}$ or one of x and y is e. For any $\overline{x} \in \overline{G}$, let $\mathbf{E}_x$ be the empty graph with vertex set $\overline{x}$. Write

$$\mathbb{E} = \{\mathbf{E}_x : \overline{x} \in V(\mathcal{L}_G)\}.$$

Theorem 2.3. *Given a group G, we have that $\mathcal{E}_2(G) = \mathcal{L}_G[\mathbb{E}]$.*

We use the following example to illustrate Theorem 2.3.

Example 2.4. *Let $G = S_3$. Then $\overline{G} = \{\{(1)\}, \{(12), (13), (23)\}, \{(123), (123)\}\}$. As a result, $\mathcal{L}_G$ is a complete graph of order 3. Now $\mathcal{E}_2(G)$ is shown in Figure 1, where both $\overline{(13)}$ and $\overline{(123)}$ are independent sets.*

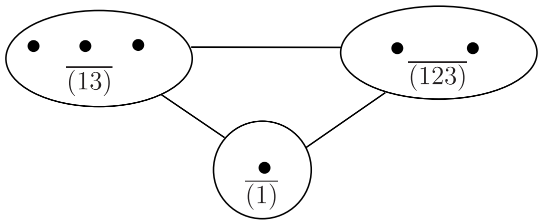

Figure 1. The equitable graph of type II of S_3.

The following result characterizes the maximal cliques of $\mathcal{E}_2(G)$.

Proposition 2.5. *Let G be a group, C a subset of G, and $d = \min\{o(g) : g \in C \setminus \{e\}\}$. Then C is a maximal clique of $\mathcal{E}_2(G)$ if and only if*

$$C = \{x \in G : d \le o(x) \le 2d\} \cup \{e\}, \tag{2.2}$$

where $o(x) \neq o(y)$ for distinct $x, y \in C$.

Proof. Suppose first that (2.2) holds and $o(x) \neq o(y)$ for distinct $x, y \in C$. Then by Observation 2.2(b), it is easy to see that C is a maximal clique of $\mathcal{E}_2(G)$. For the converse, suppose that C is a maximal clique of $\mathcal{E}_2(G)$. Clearly, $e \in C$ and $o(x) \neq o(y)$ for distinct $x, y \in C$. Note that $d = \min\{o(g) : g \in C \setminus \{e\}\}$. Let $u \in C$ with $o(u) = d$. If there exists $x \in G$ such that $o(x) > 2d$, then x and u are not adjacent, so $x \notin C$. It follows that (2.2) holds. □

We use the following example to illustrate Proposition 2.5.

Example 2.6. *Let $G = \mathbb{Z}_{2\cdot3\cdot5} = \langle g\rangle$. Then $\pi_e(G) = \{1, 2, 3, 5, 6, 10, 15, 30\}$. By Proposition 2.5, the maximal cliques of $\mathcal{E}_2(G)$ are*

$$\{e, g^{15}, g^{10}\}, \{e, g^{10}, g^6, g^5\}, \{e, g^6, g^5, g^3\}, \{e, g^5, g^3\}, \{e, g^3, g^2\}, \{e, g^2, g\}.$$

Thus, it is easy to see that $\omega(\mathcal{E}_2(G)) = 4$.

3. C_n-free equitable graphs of type II

In this section, we shall discuss finite groups G whose equitable graph of type II is C_n-free, where $n \ge 3$. In particular, we show that every equitable graph of type II is perfect. The first result follows from Observation 2.2 and the definition of an equitable graph of type II.

Proposition 3.1. *For a finite group G, let $\pi_e(G) = \{1, d_1, d_2, \ldots, d_t\}$ with $d_i < d_{i+1}$ for each $1 \le i \le t-1$. Then the following statements are equivalent:*

(a) *$\mathcal{E}_2(G)$ is a star;*
(b) *$\mathcal{E}_2(G)$ is a tree;*
(c) *$\mathcal{E}_2(G)$ is C_3-free;*
(d) *$2d_i < d_{i+1}$ for each $1 \le i \le t-1$.*

In the following, let Ψ denote the set of all finite groups G satisfying the following two conditions:

(a) $\pi_e(G) = \{1, d_1, d_2, \ldots, d_t\}$ with $t \ge 1$ and $d_i < d_{i+1}$ for each $1 \le i \le t-1$;
(b) $2d_i < d_{i+1}$ for each $1 \le i \le t-1$.

If a group belongs to Ψ, then we call this group a Ψ-*group*.

Example 3.2. (a) *Let p be a prime. For a p-group G, G is a Ψ-group if and only if either $p \ge 3$ or $G \cong \mathbb{Z}_2^m$, where m is a positive integer;*
(b) *$\mathbb{Z}_{3q}$ is a Ψ-group, if $q \ge 7$ is a prime;*
(c) *$\mathbb{Z}_{3\cdot 7\cdot q}$ is a Ψ-group, if $q \ge 42$ is a prime.*

Proposition 3.3. *$\mathcal{E}_2(G)$ is C_4-free if and only if one of the following holds:*

(a) *G is a Ψ-group;*
(b) *$G \cong \mathbb{Z}_4$;*
(c) *$G \cong Q_8$.*

Proof. If $\mathcal{E}_2(G)$ is C_3-free, then $\mathcal{E}_2(G)$ is a star by Proposition 3.1, and so $\mathcal{E}_2(G)$ is C_4-free. Now $\mathcal{E}_2(\mathbb{Z}_4)$ and $\mathcal{E}_2(Q_8)$ are shown in Figure 2, where $\mathbb{Z}_4 = \langle a \rangle$ and $Q_8 = \langle x, y \rangle$ as presented in (2.1). Thus, if $G \cong \mathbb{Z}_4$ or Q_8, then $\mathcal{E}_2(G)$ is C_4-free.

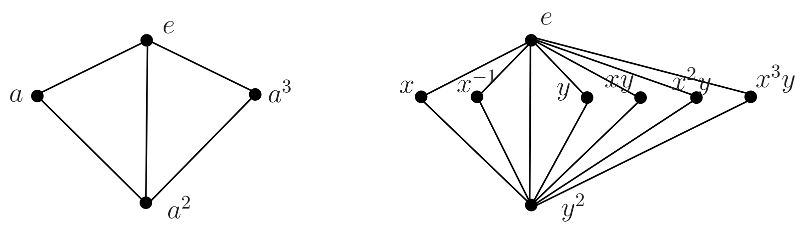

Figure 2. The equitable graphs of type II $\mathcal{E}_2(\mathbb{Z}_4)$ and $\mathcal{E}_2(Q_8)$.

Suppose first that $\mathcal{E}_2(G)$ is C_4-free. Assume that $\mathcal{E}_2(G)$ is not C_3-free. Then it suffices to prove that $G \cong \mathbb{Z}_4$ or Q_8. By Proposition 3.1, we have that there exist $x, y \in G \setminus \{e\}$ such that $\{x, y\} \in E(\mathcal{E}_2(G))$. Without loss of generality, let $o(x) < o(y)$. Then $2o(x) \ge o(y)$. Note that $o(x) \ge 2$. It follows that if $o(x) > 2$, then this subgraph of $\mathcal{E}_2(G)$ induced by $\{x, x^{-1}, y, y^{-1}\}$ is isomorphic to C_4, which is impossible. Thus, we conclude that $o(x) = 2$. Similarly, we can obtain that x is the unique involution of G. This means that $x \in Z(G)$, the center of G. If G has an element z of order q, where q is an odd prime, then xz has order $2q$, and so the subgraph induced by $\{z, z^{-1}, xz, (xz)^{-1}\}$ is isomorphic to C_4, which is a contradiction. As a result, G is a 2-group, and we must have $o(y) = 4$. If G has an element w of order 8, then the subgraph induced by $\{y, y^{-1}, w, w^{-1}\}$ is isomorphic to C_4, a contradiction. Thus, G has no elements of order 8. Now, it follows from Lemma 2.1 that G is either a cyclic group or a generalized quaternion group. If G is cyclic, since G has no elements of order 8, we have $G = \langle y \rangle \cong \mathbb{Z}_4$, as desired. In the following, thus, we can assume G is a generalized quaternion group. By (2.1), we see that $|G| = 8$, and so $G \cong Q_8$, as desired. □

Proposition 3.4. *For any integer $n \geq 5$, $\mathcal{E}_2(G)$ is C_n-free.*

Proof. Suppose, for a contradiction, that $\mathcal{E}_2(G)$ contains an induced subgraph Δ isomorphic to C_n, where $n \geq 5$. Let

$$V(\Delta) = \{a_1, a_2, \ldots, a_n\},\ \ E(\Delta) = \{\{a_1, a_n\}, \{a_i, a_{i+1}\} : i = 1, 2, \ldots, n-1\}.$$

Clearly, $a_i \neq e$ for any $1 \leq i \leq n$. Take $a_k \in V(\Delta)$ satisfying

$$o(a_k) = \min\{o(a_i) : 1 \leq i \leq n\}.$$

Then without loss of generality, we may assume that $a_{k-1}, a_k, a_{k+1} \in V(\Delta)$. Since $\{a_k, a_{k-1}\}, \{a_k, a_{k+1}\} \in E(\Delta)$, it follows from Observation 2.2(b) that $2o(a_k) \geq o(a_{k-1})$ and $2o(a_k) \geq o(a_{k+1})$. Note that a_{k-1} and a_{k+1} are not adjacent in $\mathcal{E}_2(G)$. Thus, by Proposition 2.5, we have that $o(a_{k-1}) = o(a_{k+1})$. Since $n \geq 5$, we have that there exists $a_t \in V(\Delta) \setminus \{a_k\}$ such that a_t is adjacent to a_{k-1} in $\mathcal{E}_2(G)$. This is impossible, since a_t is also adjacent to a_{k+1} in $\mathcal{E}_2(G)$. □

Proposition 3.5. *For any integer $n \geq 5$, $\mathcal{E}_2(G)$ is $\overline{C_n}$-free.*

Proof. For the sake of contradiction, suppose that $\mathcal{E}_2(G)$ contains an induced subgraph Δ isomorphic to $\overline{C_n}$, where $n \geq 5$. Let

$$V(\Delta) = \{a_1, a_2, \ldots, a_n\},\ \ E(\overline{\Delta}) = \{\{a_1, a_n\}, \{a_i, a_{i+1}\} : i = 1, 2, \ldots, n-1\}. \tag{3.1}$$

Then without loss of generality, let $a_2 \in V(\Delta)$ with $o(a_2) = \min\{o(a_i) : 1 \leq i \leq n\}$. It follows that $\{a_2, a_n\}, \{a_2, a_{n-1}\} \in E(\Delta)$. Since $\{a_n, a_{n-1}\} \notin E(\Delta)$, we deduce that $o(a_n) = o(a_{n-1})$. Notice that $n \geq 5$ and $\{a_n, a_{n-2}\} \in E(\Delta)$. We conclude that $\{a_{n-1}, a_{n-2}\} \in E(\Delta)$, so this is a contradiction by (3.1). □

By the strong perfect graph theorem [17], a graph is perfect if and only if it contains no odd holes or odd antiholes. Thus, by Propositions 3.4 and 3.5, we have the following result, which is [9, Theorem 2.10].

Corollary 3.6. *For every finite group G, $\mathcal{E}_2(G)$ is a perfect graph.*

As a result of Corollary 3.6, we have $\omega(\mathcal{E}_2(G)) = \chi(\mathcal{E}_2(G))$. Now, by Proposition 2.5, we can get easily [9, Theorems 2.12 and 2.14]. In fact, if G is a cyclic group of order 2^k, where $k \geq 2$ or a generalized quaternion group of order 4×2^t where $t \geq 1$, then

$$\{1, 2, 4\} \subseteq \pi_e(G) \subseteq \{1, 2, 4, 8, 16, \ldots\},$$

and so $\chi(\mathcal{E}_2(G)) = 3$ by Proposition 2.5. Similarly, one also can easily obtain [9, Theorems 2.13 and 2.15].

4. Metric dimension of an equitable graph of type II

This section will characterize the metric dimension of the equitable graph of type II of a finite group. Recall first that the equivalence relation $\equiv$ on G, is defined as follows:

$$x \equiv y \Leftrightarrow o(x) = o(y).$$

$\overline{x}$ denotes the $\equiv$-class containing $x \in G$. The *open neighborhood* of x in a graph Γ is

$$N_\Gamma(x) = \{y \in V(\Gamma) : d(y, x) = 1\},$$

and the *closed neighborhood* of x in Γ is

$$N_\Gamma[x] = \{y \in V(\Gamma) : d(y, x) \leq 1\}.$$

If the situation is unambiguous, we denote $N_\Gamma(x)$ and $N_\Gamma[x]$ simply by $N(x)$ and $N[x]$, respectively.

We next define the relation $\approx$ on G as follows:

$$x \approx y \Leftrightarrow N[x] = N[y] \text{ or } N(x) = N(y) \text{ in } \mathcal{E}_2(G), \text{ where } x, y \in G.$$

Notice that $N[x] = N(x) \cup \{x\}$. This relation in a general graph was studied by Hernando et al. [18]. By [18, Lemma 2.6], we see that $\approx$ is an equivalence relation on G. We use $\widehat{x}$ to denote the $\approx$-class containing $x \in G$. Write $\widehat{G} = \{\widehat{x} : x \in G\}$. For distinct $a, b \in G$, we define

$$R\{a, b\} = \{x \in G : d(a, x) \neq d(b, x)\}$$

as the set of vertices resolving a and b in $\mathcal{E}_2(G)$. Note that $\{a, b\} \subseteq R\{a, b\}$, since $d(a, a) = 0$ and $d(a, b) = 1$ or 2.

Let Φ denote the set of all finite groups G satisfying the following three conditions:

(a) G has a unique involution z;
(b) $|\widehat{z}| = 1$;
(c) there exist $x, y \in G \setminus \{z\}$ such that

$$\widehat{x} \neq \widehat{y}, \quad \{x, y\} \notin E(\mathcal{E}_2(G)), \quad R\{x, y\} = \{x, y, z\}. \tag{4.1}$$

A group is called a Φ-group if this group belongs to Φ. Note that if G is a Φ-group, then $|\widehat{G}| \geq 3$ by (c) of the above definition. In fact, there are many Φ-groups; see Lemma 4.5.

The following theorem is our main theorem of this section.

Theorem 4.1. *Let G be a finite group with $|\widehat{G}| = n$. Then*

$$\dim(\mathcal{E}_2(G)) = \begin{cases} |G| + 1 - n, & \text{if } G \text{ is a } \Phi\text{-group;} \\ |G| - n, & \text{otherwise.} \end{cases}$$

Before proving Theorem 4.1, we first present several results.

Lemma 4.2. *For distinct $x, y \in G$, $N[x] = N[y]$ in $\mathcal{E}_2(G)$ if and only if G is isomorphic to one of $\mathbb{Z}_2, \mathbb{Z}_4$ and Q_8, and $\{x, y\} = \{e, a\}$, where a is the unique involution of G.*

Proof. By Figure 2 and Observation 2.2(c), if $G \cong \mathbb{Z}_2, \mathbb{Z}_4$ or Q_8 and $\{x, y\} = \{e, a\}$, where a is the unique involution of G, then $N[x] = G = N[y]$, as desired.

Conversely, suppose that $N[x] = N[y]$. Then x and y are adjacent in $\mathcal{E}_2(G)$, so x^{-1} is also adjacent to y. It follows that $x^{-1} \in N[x]$. Since $o(x^{-1}) = o(x)$, we have that $x^{-1} = x$, which implies that $x^2 = e$. Similarly, we also can obtain $y^2 = e$. Note now that x and y are adjacent in $\mathcal{E}_2(G)$. It follows that $o(x) \neq o(y)$, and this forces that $\{x, y\} = \{e, a\}$, where a is an involution of G. We conclude that $N[a] = N[e] = G$, meaning that G has a unique involution which is a. Moreover, we have that $\pi_e(G) \subseteq \{1, 2, 4\}$. Now Lemma 2.1 implies that $G \cong \mathbb{Z}_2, \mathbb{Z}_4$ or Q_8. □

Corollary 4.3. *In $\mathcal{E}_2(G)$, we have:*

(a) $\overline{x} \subseteq \widehat{x}$*;*
(b) $|\widehat{e}| \leq 2$*, where the equality holds if and only if G is isomorphic to one of* $\mathbb{Z}_2, \mathbb{Z}_4$ *and* Q_8*;*
(c) $\widehat{e}$ *is a clique. Moreover, for $x \in G$, if $e \notin \widehat{x}$, then $\widehat{x}$ is an independent set.*

Lemma 4.4. *Taking $x \in G \setminus \{e\}$, we have the following:*

(a) *If $|\widehat{x}| = 1$, then G has a unique involution, which is x. The converse does not hold.*
(b) $|\widehat{G}| = 1$ *if and only if $\mathcal{E}_2(G)$ is complete, which in turn is true if and only if G is isomorphic to* $\mathbb{Z}_2$*.*
(c) $|\widehat{G}| = 2$ *if and only if G is isomorphic to one group of* $\Psi \setminus \{\mathbb{Z}_2\}$, $\mathbb{Z}_4$ *and* Q_8*.*

Proof. (a) By Corollary 4.3(a), if $|\widehat{x}| = 1$ for $x \in G \setminus \{e\}$, then x must be an involution, and G must have a unique involution, which is x, as desired. For the converse, let $G = \mathbb{Z}_8 = \langle g \rangle$. Then g^4 is the unique involution of G. It is easy to see that $N(g^4) = \{e, g^2, g^6\} = N(g)$. In fact, we have that $\widehat{g^4} = \{g^4, g, g^3, g^5, g^7\}$.

(b) Note that in $\mathcal{E}_2(G)$, $N[e] = G$. Thus, $|\widehat{G}| = 1$ if and only if $\widehat{e} = G$, which in turn is true if and only if $\mathcal{E}_2(G)$ is complete. Now the required result follows from Observation 2.2(c).

(c) Clearly, if $G \in \Psi \setminus \{\mathbb{Z}_2\}$, then $\widehat{G} = \{\widehat{e}, \widehat{x}\}$ with $\widehat{x} = G \setminus \{e\}$. Now by Figure 2, the sufficiency holds.

Conversely, suppose that $|\widehat{G}| = 2$. Let $\widehat{G} = \{\widehat{e}, \widehat{x}\}$. Assume that $|\widehat{e}| = 1$. Then $\widehat{x} = G \setminus \{e\}$. By Corollary 4.3(c), we have that $\widehat{x}$ is an independent set. It follows that $\mathcal{E}_2(G)$ is a star. Now by (b), we have $G \neq \mathbb{Z}_2$. As a result, $G \in \Psi \setminus \{\mathbb{Z}_2\}$ by Proposition 3.1, as desired. In the following, assume that $|\widehat{e}| \geq 2$. Now by (b) and Corollary 4.3(b), we deduce that $|\widehat{e}| = 2$ and G is isomorphic to one of $\mathbb{Z}_4$ and Q_8, as wanted. □

Lemma 4.5. *$\mathbb{Z}_n$ is a Φ-group if and only if $n = 2^4 q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, where $37 \leq q_1 < q_2 < \cdots < q_r$ are primes and α_i is a positive integer for each $1 \leq i \leq r$.*

Proof. Suppose $G = \mathbb{Z}_n$ and $n = 2^4 q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, where $37 \leq q_1 < q_2 < \cdots < q_r$ are primes and α_i is a positive integer for each $1 \leq i \leq r$. Take $x, y \in G$ with $o(x) = 4$ and $o(y) = 16$. It is easy to verify that (4.1) holds. Thus, G is a Φ-group.

Conversely, suppose that $\mathbb{Z}_n$ is a Φ-group, and let $G = \mathbb{Z}_n$. Then $|\widehat{G}| \geq 3$. Now let $|\widehat{z}| = 1$, where z is the unique involution of G, and let $x, y \in G \setminus \{z\}$ satisfy (4.1). Then by Corollary 4.3 and Lemma 4.4, we have that $|G| \geq 6$. Note that e is adjacent to any other vertex in $\mathcal{E}_2(G)$. Since $R\{x, y\} = \{x, y, z\}$, we have that in $\mathcal{E}_2(G)$, one of x, y must be adjacent to z, and the other is not adjacent to z. Thus, from $o(x) < o(y)$, it follows that x and z are adjacent, hence $o(x) = 3$ or 4.

We divide our proof into two cases.

Case 1. $o(x) = 3$.

Then $o(y) \geq 7$. If $o(y)$ is odd, then G has an element w of order $2o(y)$, so w is adjacent to y and is not adjacent to x, which implies $w \in R\{x, y\}$, contrary to (4.1). We conclude that $o(y)$ is even. If $o(y) \geq 14$, then clearly, an element u of order $o(y)/2$ belongs to $R\{x, y\}$; this also contradicts (4.1). It follows that $o(y) = 8, 10$, or 12. If $o(y) = 8$, then an element of order 12 belongs to $R\{x, y\}$; if $o(y) = 10$, then an element of order 15 belongs to $R\{x, y\}$; if $o(y) = 12$, then an element of order 4 belongs to $R\{x, y\}$. Thus, it follows from (4.1) that every case above is not valid.

Case 2. $o(x) = 4$.

Then $o(y) \geq 9$. It is similar to the proof of Case 1. We see that $o(y) = 10, 12, 14$, or 16. If $o(y) = 10$, then an element of order 20 belongs to $R\{x, y\}$; if $o(y) = 12$, then an element of order 3 belongs to $R\{x, y\}$; if $o(y) = 14$, then an element of order 28 belongs to $R\{x, y\}$. Note that $R\{x, y\} = \{x, y, z\}$ in (4.1). As a result, we have that $o(y) = 16$. Clearly, G has no elements of order 32. It follows that $n = 2^4 q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, where $3 \leq q_1 < q_2 < \cdots < q_r$ are primes and α_i is a positive integer for each $1 \leq i \leq r$. If $q_1 \leq 31$, then one element of order $4q_1, 2q_1$, or q_1 must belong to $R\{x, y\}$, which is impossible. Consequently, we have that $37 \leq q_1$, as desired. □

Remark 4.6. *Similar to the proof of Lemma* 4.5*, one can obtain that Q_{4m} is a Φ-group, where $m = 2^3 q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, $37 \leq q_1 < q_2 < \cdots < q_r$ are primes, and α_i is a positive integer for each $1 \leq i \leq r$.*

Proposition 4.7. *Let G be a non-Φ-group with $|\widehat{G}| \geq 3$, and let $\{x_1, x_2, \ldots, x_r\}$ be a system of representatives for the ≈-classes of G. Then*

$$S = G \setminus \{x_1, x_2, \ldots, x_r\}$$

is a resolving set of $\mathcal{E}_2(G)$.

Proof. It follows from Lemmas 4.2 and 4.4 that for two distinct $x, y \in G$, $x \approx y$ if and only if $N(x) = N(y)$ in $\mathcal{E}_2(G)$. So, every ≈-class is an independent set of $\mathcal{E}_2(G)$, and $\widehat{e} = \{e\}$ by Corollary 4.3(b). Now, assume that a and b are two distinct elements of $\{x_1, x_2, \ldots, x_r\}$. It suffices to show that there exists at least one $s \in S$ such that $s \in R\{a, b\}$. Note that $\widehat{a} \neq \widehat{b}$.

Assume that one of a and b is equal to e. Without loss of generality, let $a = e$. If $|\widehat{b}| \geq 2$, then choose $b' \in \widehat{b} \setminus \{b\}$. We can deduce $d(b, b') = 2$ since $\widehat{b}$ is an independent set of $\mathcal{E}_2(G)$, which implies $b' \in S \cap R\{a, b\}$, as desired. In the following, we assume that $|\widehat{b}| = 1$. By Lemma 4.4(a), we conclude that b is the unique involution of G. Since $\widehat{e} = \{e\}$, we have $N[b] \neq G$, so there exists $u \in G$ such that $u \notin N(b)$. Clearly, $o(u) \geq 3$, $\widehat{u} \neq \widehat{b}$, and $|\widehat{u}| \geq 2$. Then we may choose element $u' \in S \cap \widehat{u}$ with $o(u') = o(u)$, and then $u' \in R\{a, b\}$, as desired.

Next, we assume that $a \neq e$ and $b \neq e$. We divide our proof into two cases.

Case 1. $\{a, b\} \in E(\mathcal{E}_2(G))$.

By Lemma 4.4(a), we must have that one of $\widehat{a}$ and $\widehat{b}$ has a size of at least 2. Without loss of generality, let $|\widehat{b}| \geq 2$. Then taking $b' \in \widehat{b} \cap S$, we have that $d(a, b') = 1$ and $d(b, b') = 2$, which implies that $b' \in R\{a, b\}$, as desired.

Case 2. $\{a, b\} \notin E(\mathcal{E}_2(G))$.

By Lemma 4.2 and $\widehat{a} \neq \widehat{b}$, we have that $N(a) \neq N(b)$. Thus, there exists at least one vertex, say w, such that one of a and b is adjacent to w, and the other is not adjacent to w. If such w can only be an involution and $|\widehat{w}| = 1$, then by the definition of a Φ-group, we have that G is a Φ-group, which is impossible. As a result, we may assume that $|\widehat{w}| \geq 2$. Taking $w' \in \widehat{w} \cap S$, we have that $w' \in R\{a, b\}$, as desired. □

By the proof of Proposition 4.7, the following result holds.

Proposition 4.8. *Let G be a Φ-group with $|\widehat{G}| \geq 3$, and let $\{x_1, x_2, \ldots, x_r\}$ be a system of representatives for the $\approx$-classes of G, where $|\widehat{x_1}| = 1$. Then*

$$S = G \setminus \{x_2, \ldots, x_r\}$$

is a resolving set of $\mathcal{E}_2(G)$.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Suppose that $|\widehat{G}| = 1$. Then we have that $\mathcal{E}_2(G)$ is complete by Lemma 4.4(b). Thus, it follows from [19, Theorem 3] that $\dim(\mathcal{E}_2(G)) = |G| - |\widehat{G}| = |G| - 1$, as desired. Now suppose that $|\widehat{G}| = 2$. Note that $G \neq \mathbb{Z}_2$. If $\mathcal{E}_2(G)$ is a star, by [19, Theorem 4], we have that $\dim(\mathcal{E}_2(G)) = |G| - 2$, as desired. Moreover, by Figure 2, if $G \cong \mathbb{Z}_4$ or Q_8, then $\dim(\mathcal{E}_2(G)) = |G| - 2$, as desired. Thus, in view of Lemma 4.4(c), in the following, we may assume that $|\widehat{G}| \geq 3$.

Now take a resolving set S of $\mathcal{E}_2(G)$ with size $\dim(\mathcal{E}_2(G))$. If there exist two distinct vertices $x, y \in G$ such that $\widehat{x} = \widehat{y}$ and $x, y \in G \setminus S$, clearly, no element of S can resolve x and y, which is impossible. It follows that if $\widehat{g} \in \widehat{G}$ with $|\widehat{g}| \geq 2$, then $\widehat{g}$ must have $|\widehat{g}| - 1$ elements belonging to S. This means that

$$\dim(\mathcal{E}_2(G)) = |S| \geq |G| - n. \tag{4.2}$$

If G is not a Φ-group, then Proposition 4.7 implies that $\dim(\mathcal{E}_2(G)) \leq |G| - n$, and so $\dim(\mathcal{E}_2(G)) = |G| - n$ by (4.2), as desired. Finally, suppose that G is a Φ-group. Then $\dim(\mathcal{E}_2(G)) \neq |G| - n$ by (4.1) in the definition of a Φ-group. Also, from Proposition 4.8, it follows that $\dim(\mathcal{E}_2(G)) \leq |G| + 1 - n$. As a result, we conclude that $\dim(\mathcal{E}_2(G)) = |G| + 1 - n$, as desired. □

Note that every dihedral group has at least 3 involutions. Thus, the following result is valid by Theorem 4.1.

Corollary 4.9. *If G is an odd order group or a dihedral group, then*

$$\dim(\mathcal{E}_2(G)) = |G| - n,$$

where $n = |\widehat{G}|$.

If G is a p-group with prime $p \geq 3$, then $\mathcal{E}_2(G)$ is a star, so $\dim(\mathcal{E}_2(G)) = |G| - 2$ by Theorem 4.1. Finally, we determine $\dim(\mathcal{E}_2(G))$ for any 2-group G.

Example 4.10. *Let P be a finite 2-group of order n. Then*

$$\dim(\mathcal{E}_2(P)) = \begin{cases} 1, & \text{if } P \cong \mathbb{Z}_2; \\ n-2, & \text{if } \exp(P) = 2 \text{ or } P \cong \mathbb{Z}_4; \\ n-3, & \text{if } \exp(P) = 8 \text{ or } G \text{ is non-cyclic with } \exp(P) = 4; \\ n-4, & \text{if } P \cong \mathbb{Z}_{16}; \\ n-5, & \text{if } G \text{ is non-cyclic with } \exp(P) = 16; \\ n-m-1, & \text{if } \exp(P) = 2^m \text{ where } m \geq 5. \end{cases}$$

Proof. Note that if $\exp(P) = 8$, then $|\widehat{P}| = 3$. Moreover, if $\exp(P) = 2^m$, where $m \geq 5$, then $|\widehat{P}| = m + 1$. Thus, the desired result follows from Lemma 4.5 and Theorem 4.1. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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