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Research article

Quasi-resolving subcategories in triangulated categories

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Abstract: We have introduced the notion of quasi-resolving subcategories in a triangulated category with a proper class of triangles. We provided some criteria for computing the resolution dimension of objects relative to a quasi-resolving subcategory. We also provided a method to construct quasi-resolving subcategories from given ones.

Keywords: triangulated category; proper class of triangle; quasi-resolving subcategory; resolution dimension; Auslander-Buchweitz approximation

1. Introduction

In analogy to relative homological algebra in abelian categories, Beligiannis [1] introduced a relative version of homological algebra in a triangulated category by introducing the notion of proper classes of triangles. This theory has been studied widely. For example, Asadollahi and Salarian [2] studied the Gorenstein homological theory in a triangulated category. Furthermore, in a triangulated category, Ren and Liu [3] studied Gorenstein homological dimensions, and Yang and Wang [4] studied the stability of the Gorenstein category by constructing a proper resolution (resp., coproper coresolution). Recently, Fu et al. [5] introduced the notion of balanced pairs and established a bijection between balanced pairs and proper classes in a triangulated category. Ma et al. [6] studied the relationship among left Frobenius pairs, left (n-)cotorsion pairs and left (weak) Auslander-Buchweitz contexts.

Resolving subcategories and resolution dimensions play an important role in the study of relative homological theory in abelian categories and triangulated categories. In abelian categories, resolving subcategories and resolution dimensions are closely related to tilting theory (see [7]) and some homological conjectures (see [8]). In triangulated categories, Ma et al. [9] introduced the notions of (pre)resolving subcategories and homological dimensions relative to these subcategories, which give a parallel theory analogy to that of abelian categories in [8]. Then, Ma and Zhao [10] studied the properties of resolving subcategories and relative homological dimensions in triangulated categories relative to a resolving subcategory, and obtained the Auslander–Buchweitz approximation theory relative to resolving subcategories in triangulated categories, which gives a parallel theory analogy to that of abelian categories in [11]. On the other hand, Zhu [12] introduced the notion of quasi-resolving subcategories in an abelian category and generalized some results about resolving subcategories to quasi-resolving subcategories. More generally, Zhang et al. [13] introduced the notion of ext-quasi-resolving subcategories, which is a generalization of quasi-resolving subcategories. Cao and Wei [14] introduced and studied the Gorenstein quasi-resolving subcategories relative to a quasi-resolving subcategory in an abelian category.

In this paper, we study further properties of quasi-resolving subcategories and relative homological dimensions relative to a quasi-resolving subcategory in triangulated categories.

The paper is organized as follows. In Section 2, we give some terminology and some preliminary results. In Section 3, we introduce the notion of quasi-resolving subcategories of a triangulated category, and prove the following result. Let $\mathcal{T}$ be a triangulated category with enough ξ-projective and ξ-injective objects, where ξ is a proper class of triangles in $\mathcal{T}$. If $\mathcal{X}$ is a quasi-resolving subcategory of $\mathcal{T}$ and

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow M \longrightarrow 0$$

is a ξ-exact complex with all $X_i \in \mathcal{X}$, then there exists a ξ-exact complex

$$0 \longrightarrow B \longrightarrow X \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{X}$ and all $P_i \in \mathcal{P}_{\mathcal{X}}$ (Theorem 3.7). In Section 4, we study resolution dimensions relative to quasi-resolving subcategories and obtain Auslander-Buchweitz approximation triangles for objects having finite quasi-resolving resolution dimension (Proposition 4.9). We provide some criteria for computing the resolution dimension of objects relative to a quasi-resolving subcategory (Theorem 4.12). In Section 5, given a quasi-resolving subcategory $\mathcal{X}$ of $\mathcal{T}$, we construct a new quasi-resolving subcategory $\mathcal{GP}_{\mathcal{X}\mathcal{X}}(\xi)$ with a ξxt-injective ξ-cogenerator $\mathcal{X} \cap {}^{\perp}\mathcal{X}$. Then combining with Theorem 4.12, we obtain a generalization of [10, Proposition 5.5].

2. Preliminaries

Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an additive functor. All subcategories involved are full, additive, and closed under isomorphisms. One defines the category $\mathrm{Diag}(\mathcal{T}, \Sigma)$ as follows:

- An object of $\mathrm{Diag}(\mathcal{T}, \Sigma)$ is a diagram in $\mathcal{T}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

- A morphism in $\mathrm{Diag}(\mathcal{T}, \Sigma)$ between

$$X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} \Sigma X_i$$

where $i = 1, 2$ is a triple (α, β, γ) of morphisms in $\mathcal{T}$ such that the following diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & \Sigma X_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \end{array}$$

commutes.

Recall that a *triangulated category* is a triple $(\mathcal{T}, \Sigma, \Delta)$, where $\mathcal{T}$ is an additive category, $\Sigma : \mathcal{T} \to \mathcal{T}$ is an autoequivalence of $\mathcal{T}$ (called the *suspension functor*), and Δ is a full subcategory of $\mathrm{Diag}(\mathcal{T}, \Sigma)$ which is closed under isomorphisms and satisfies the axioms (T_1)–(T_4) in [1, Section 2.1] (also see [15]), where the axiom (T_4) is called the *octahedral axiom*. The elements in Δ are called *triangles*.

The following result is well known, which is useful in studying the properties of triangulated categories.

Proposition 2.1. ([1, Proposition 2.1]) *Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an autoequivalence of $\mathcal{T}$, and let Δ be a full subcategory of* $\mathrm{Diag}(\mathcal{T}, \Sigma)$*, which is closed under isomorphisms. Suppose that the triple $(\mathcal{T}, \Sigma, \Delta)$ satisfies all the axioms of a triangulated category except possibly the octahedral axiom. Then the following statements are equivalent.*

1) **Octahedral axiom**. *For any two morphisms $u : X \longrightarrow Y$ and $v : Y \longrightarrow Z$, there exists a commutative diagram:*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{u'} & Z' & \xrightarrow{u''} & \Sigma X \\
\| \downarrow & & \downarrow v & & \downarrow \alpha & & \downarrow \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{w} & Y' & \xrightarrow{w'} & \Sigma X \\
\downarrow u & & \| \downarrow & & \downarrow \beta & & \downarrow \Sigma u \\
Y & \xrightarrow{v} & Z & \xrightarrow{v'} & X' & \xrightarrow{v''} & \Sigma Y \\
\downarrow & & \downarrow 0 & & \downarrow (\Sigma u')v'' & & \downarrow \\
0 & \longrightarrow & \Sigma Z' & \xrightarrow{=} & \Sigma Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and the third column are triangles in Δ.

2) **Base change**. *For any triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in Δ and any morphism $\alpha : Z' \longrightarrow Z$, there exists the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & X' & \xrightarrow{=} & X' & \longrightarrow & 0 \\
\downarrow & & \downarrow \beta' & & \downarrow \beta & & \downarrow \\
X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\
\| \downarrow & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow & & \downarrow \gamma' & & \downarrow \gamma & & \downarrow \\
0 & \longrightarrow & \Sigma X' & \xrightarrow{=} & \Sigma X' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

3) **Cobase change**. *For any triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in Δ *and any morphism* $\beta : X \longrightarrow X'$, *there exists the following commutative diagram:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}Z' & \xrightarrow{=} & \Sigma^{-1}Z' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle -\Sigma^{-1}\gamma} & & \downarrow{\scriptstyle -\Sigma^{-1}\gamma'} & & \downarrow \\
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} & & \| \\
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow \\
0 & \longrightarrow & Z' & \xrightarrow{=} & Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

In the following, $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ is a triangulated category.

Definition 2.2. ([1]) A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called *split* if it is isomorphic to the triangle

$$X \xrightarrow{\binom{1}{0}} X \oplus Z \xrightarrow{(0,\ 1)} Z \xrightarrow{0} \Sigma X.$$

We use Δ_0 to denote the full subcategory of Δ consisting of all split triangles.

Definition 2.3. ([1]) Let ξ be a class of triangles in $\mathcal{T}$.

(1) ξ is said to be *closed under base change* (resp., *cobase change*) if for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in ξ and any morphism $\alpha : Z' \longrightarrow Z$ (resp., $\beta : X \longrightarrow X'$) as in Proposition 2.1(2) (resp., Proposition 2.1(3)), the triangle

$$X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X \ \text{(resp.,}\ \ X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X'\ )$$

is in ξ.

(2) ξ is said to be *closed under suspension* if for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in ξ and any $i \in \mathbb{Z}$ (the set of all integers), the triangle

$$\Sigma^i X \xrightarrow{(-1)^i \Sigma^i u} \Sigma^i Y \xrightarrow{(-1)^i \Sigma^i v} \Sigma^i Z \xrightarrow{(-1)^i \Sigma^i w} \Sigma^{i+1} X$$

is in ξ.

(3) ξ is called *saturated* if in the situation of base change as in Proposition 2.1(2), whenever the third vertical and the second horizontal triangles are in ξ, then the triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is in ξ.

Definition 2.4. ([1]) A class ξ of triangles in $\mathcal{T}$ is called *proper* if the following conditions are satisfied.

(1) ξ is closed under isomorphisms and finite coproducts, and $\Delta_0 \subseteq \xi$.
(2) ξ is closed under suspensions and is saturated.
(3) ξ is closed under base and cobase change.

In the following, assume that ξ is a proper class of triangles in $\mathcal{T}$.

Definition 2.5. ([1, Definition 2.4] and [16, Definition 8]) Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle in ξ. Then the morphism u (resp., v) is called *ξ-proper monic* (resp., *ξ-proper epic*), and u (resp., v) is called the *hokernel* of v (resp., the *hocokernel* of u).

We use Hoker v to denote the hokernel of $v : Y \to Z$, and use Hocok u to denote the hocokernel of $u : X \to Y$.

Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$, and let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a triangle in ξ. We say that $\mathcal{X}$ is *closed under ξ-extensions* provided $Y \in \mathcal{X}$ whenever $X, Z \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under hokernels of ξ-proper epimorphisms* (resp., *hocokernels of ξ-proper monomorphisms*) provided $X \in \mathcal{X}$ (resp., $Z \in \mathcal{X}$) whenever $Y, Z \in \mathcal{X}$ (resp., $X, Y \in \mathcal{X}$).

Definition 2.6. ([1]) An object P (resp., I) in $\mathcal{T}$ is called *ξ-projective* (resp., *ξ-injective*) if for any triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ, the induced complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P, X) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P, Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P, Z) \longrightarrow 0$$

$$(\text{resp., } 0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z, I) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Y, I) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, I) \longrightarrow 0)$$

is exact. We use $\mathcal{P}(\xi)$ (resp., $\mathcal{I}(\xi)$) to denote the full subcategory of $\mathcal{T}$ consisting of ξ-projective (resp., ξ-injective) objects.

We say that $\mathcal{T}$ has *enough ξ-projective objects* if for any $M \in \mathcal{T}$, there exists a triangle

$$K \longrightarrow P \longrightarrow M \longrightarrow \Sigma K$$

in ξ with $P \in \mathcal{P}(\xi)$. Dually, we say that $\mathcal{T}$ has *enough ξ-injective objects* if for any $M \in \mathcal{T}$, there exists a triangle

$$M \longrightarrow I \longrightarrow K \longrightarrow \Sigma M$$

in ξ with $I \in \mathcal{I}(\xi)$.

Remark 2.7. ([10, Remark 2.7]) $\mathcal{P}(\xi)$ is closed under direct summands, hokernels of ξ-proper epimorphisms, and ξ-extensions. Dually, $\mathcal{I}(\xi)$ is closed under direct summands, hocokernels of ξ-proper monomorphisms, and ξ-extensions.

Definition 2.8. Let $\mathcal{E}$ be a subcategory of $\mathcal{T}$.

(1) A triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ is called $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, -)$-*exact* (resp., $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{E})$-*exact*) if for any $E \in \mathcal{E}$, the induced complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, X) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, Z) \longrightarrow 0$$
$$(\text{resp., } 0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z, E) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Y, E) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, E) \longrightarrow 0)$$

is exact.

(2) ([2]) A ξ-*exact* complex is a complex

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \tag{2.1}$$

in $\mathcal{T}$ such that for any $n \in \mathbb{Z}$, there exists a triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1} \tag{2.2}$$

in ξ and the differential d_n is defined as $d_n = g_{n-1} f_n$. A ξ-exact complex as (2.1) is called $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, -)$-*exact* (resp., $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{E})$-*exact*) if the triangle (2.2) is $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, -)$-exact (resp., $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{E})$-exact) for any $n \in \mathbb{Z}$.

From now on, assume that $\mathcal{T}$ is a triangulated category with enough ξ-projective and ξ-injective objects.

Let M be an object in $\mathcal{T}$. Beligiannis [1] defined the ξ-extension groups $\xi xt^n_\xi(-, M)$ to be the nth right ξ-derived functor of the functor $\mathrm{Hom}_{\mathcal{T}}(-, M)$, that is,

$$\xi xt^n_\xi(-, M) := \mathcal{R}^n_\xi \mathrm{Hom}_{\mathcal{T}}(-, M).$$

Remark 2.9. ([10, Remark 2.10]) Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a triangle in ξ. By [1, Corollary 4.12], there exists a long exact sequence

$$0 \longrightarrow \xi xt^0_\xi(Z, M) \longrightarrow \xi xt^0_\xi(Y, M) \longrightarrow \xi xt^0_\xi(X, M) \longrightarrow$$
$$\xi xt^1_\xi(Z, M) \longrightarrow \xi xt^1_\xi(Y, M) \longrightarrow \xi xt^1_\xi(X, M) \longrightarrow \cdots$$

of "ξxt" functors. For an object $N \in \mathcal{T}$, there exists a long exact sequence

$$0 \longrightarrow \xi xt^0_\xi(N, X) \longrightarrow \xi xt^0_\xi(N, Y) \longrightarrow \xi xt^0_\xi(N, Z) \longrightarrow$$
$$\xi xt^1_\xi(N, X) \longrightarrow \xi xt^1_\xi(N, Y) \longrightarrow \xi xt^1_\xi(N, Z) \longrightarrow \cdots$$

of "ξxt" functors.

Following Remark 2.9, we usually use the strategy of "dimension shifting", which is an important tool in relative homological theory of triangulated categories. We write

$$\mathcal{X}^{\perp} := \{M \in \mathcal{T} \mid \xi xt^i_{\xi}(X, M) = 0 \text{ for all } X \in \mathcal{X} \text{ and } i \geq 1\},$$

$$^{\perp}\mathcal{X} := \{M \in \mathcal{T} \mid \xi xt^i_{\xi}(M, X) = 0 \text{ for all } X \in \mathcal{X} \text{ and } i \geq 1\}.$$

For two subcategories $\mathcal{H}$ and $\mathcal{X}$ of $\mathcal{T}$, we say $\mathcal{H} \perp \mathcal{X}$ if $\mathcal{H} \subseteq {}^{\perp}\mathcal{X}$ (equivalently, $\mathcal{X} \subseteq \mathcal{H}^{\perp}$).

Taking $\mathcal{E} = \mathcal{P}(\xi)$ in [9, Definition 3.1], we have the following definition.

Definition 2.10. (cf. [9, Definition 3.1]) Let $\mathcal{H} \subseteq \mathcal{X}$ be two subcategories of $\mathcal{T}$. Then $\mathcal{H}$ is called a ξ-generator of $\mathcal{X}$ if for any $X \in \mathcal{X}$, there exists a triangle

$$Z \longrightarrow H \longrightarrow X \longrightarrow \Sigma Z$$

in ξ with $H \in \mathcal{H}$ and $Z \in \mathcal{X}$. Dually, $\mathcal{H}$ is called a ξ-cogenerator of $\mathcal{X}$ if for any $X \in \mathcal{X}$, there exists a triangle

$$X \longrightarrow H \longrightarrow Z \longrightarrow \Sigma X$$

in ξ with $H \in \mathcal{H}$ and $Z \in \mathcal{X}$. In particular, a ξ-cogenerator $\mathcal{H}$ is called *ξxt-injective* if $\mathcal{X} \perp \mathcal{H}$.

We need the following observation in the sequel.

Lemma 2.11. *(1) We are given a commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z' & \stackrel{=}{\longrightarrow} & Z' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}Z & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma Z' & \stackrel{=}{\longrightarrow} & \Sigma Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

(a) If the third vertical triangle and the triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

are in ξ, then so are the second vertical triangle and the triangle

$$X' \longrightarrow Y' \longrightarrow Z \longrightarrow \Sigma X'.$$

(b) If the second vertical triangle and the triangle

$$X' \longrightarrow Y' \longrightarrow Z \longrightarrow \Sigma X'$$

are in ξ, then so are the third vertical triangle and the triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

(2) We are given a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}Z' & \xrightarrow{=} & \Sigma^{-1}Z' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z' & \xrightarrow{=} & Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

(a) If the second horizontal triangle and the triangle

$$X' \longrightarrow Y \longrightarrow Z' \longrightarrow \Sigma X'$$

are in ξ*, then so are the third horizontal triangle and the triangle*

$$Y' \longrightarrow Z \longrightarrow Z' \longrightarrow \Sigma Y'.$$

(b) If the third horizontal triangle and the triangle

$$Y' \longrightarrow Z \longrightarrow Z' \longrightarrow \Sigma Y'$$

are in ξ*, then so are the second horizontal triangle and the triangle*

$$X' \longrightarrow Y \longrightarrow Z' \longrightarrow \Sigma X'.$$

Proof. Assertion (1) follows from [17, Propositions 2.4 and 2.7], and assertion (2) is a dual of (1). □

3. Quasi-resolving subcategories

Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. By an *$\mathcal{X}$-resolution* of M, we mean a ξ-exact complex

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with all $X_i \in \mathcal{X}$.

We write

$$\begin{aligned} \operatorname{res}^*(\mathcal{X}) &:= \{M \in \mathcal{T} \mid M \text{ has an } \mathcal{X}\text{-resolution}\}, \\ \mathcal{P}_{\mathcal{X}} &:= \mathcal{X} \cap \mathcal{P}(\xi). \end{aligned}$$

Now, we introduce the notion of a quasi-resolving subcategory in a triangulated category, which is an analogue of that in [12].

Definition 3.1. A subcategory $\mathcal{X}$ of $\mathcal{T}$ is called *quasi-resolving* if the following conditions are satisfied.

(1) $\mathcal{X}$ is closed under hokernels of ξ-proper epimorphisms.
(2) $\mathcal{X}$ is closed under ξ-extensions.
(3) $\mathcal{X} \subseteq \mathrm{res}^*(\mathcal{P}_{\mathcal{X}})$.

In fact, a quasi-resolving subcategory $\mathcal{X}$ is called a $\mathcal{P}(\xi)$-resolving subcategory in the sense of [9, Definition 3.2].

Remark 3.2. (1) If $\mathcal{X}$ is closed under hokernels of ξ-proper epimorphisms and closed under ξ-extensions, then so is $\mathcal{P}_{\mathcal{X}}$ by Remark 2.7. Moreover, $\mathcal{P}_{\mathcal{X}}$ is quasi-resolving. In general, $\mathcal{P}_{\mathcal{X}}$ is not resolving in the sense of [10, Definition 2.12]. In fact, if $\mathcal{P}_{\mathcal{X}}$ is resolving in the sense of [10, Definition 2.12], then $\mathcal{P}(\xi) \subseteq \mathcal{P}_{\mathcal{X}} \subseteq \mathcal{X}$, and thus $\mathcal{X}$ is resolving. If $\mathcal{T}$ has enough projectives and $\mathcal{P}(\xi) \subseteq \mathcal{X}$, then $\mathcal{X}$ is a resolving subcategory of $\mathcal{T}$, and $\mathrm{res}^*(\mathcal{P}_{\mathcal{X}}) = \mathcal{T}$. So each resolving subcategory of $\mathcal{T}$ is quasi-resolving, and hence

$$\{\text{resolving subcategories}\} \subsetneq \{\text{quasi-resolving subcategories}\}.$$

(2) In the following Proposition 3.5, we will show that if $\mathcal{X}$ is a quasi-resolving subcategory of $\mathcal{T}$, then $\mathrm{res}^*(\mathcal{X}) = \mathrm{res}^*(\mathcal{P}_{\mathcal{X}})$. However, if $\mathcal{X}$ is a resolving subcategory of $\mathcal{T}$, then $\mathrm{res}^*(\mathcal{X}) = \mathrm{res}^*(\mathcal{P}_{\mathcal{X}})= \mathrm{res}^*(\mathcal{P}(\xi))$. In general, $\mathcal{P}_{\mathcal{X}} \subseteq \mathcal{P}(\xi)$. This shows that the notion of a quasi-resolving subcategory has a comparative advantage over resolving subcategories.

Proposition 3.3. *Let $C \subseteq \mathcal{P}(\xi)$ be a subcategory of $\mathcal{T}$ closed under hokernels of ξ-proper epimorphisms and closed under ξ-extensions. Then $\mathrm{res}^*(C)$ is quasi-resolving and closed under hocokernels of ξ-proper monomorphisms.*

Proof. Since $C \subseteq \mathcal{P}(\xi) \cap \mathrm{res}^*(C) = \mathcal{P}_{\mathrm{res}^*(C)}$, we have

$$\mathrm{res}^*(C) \subseteq \mathrm{res}^*\left(\mathcal{P}_{\mathrm{res}^*(C)}\right).$$

Let

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \Sigma M_1$$

be a triangle in ξ.

Claim 1. If $M_1, M_3 \in \mathrm{res}^*(C)$, then $M_2 \in \mathrm{res}^*(C)$.

By assumption, we have the following ξ-exact complexes:

$$\cdots \longrightarrow C_n^{M_1} \longrightarrow C_{n-1}^{M_1} \longrightarrow \cdots \longrightarrow C_1^{M_1} \longrightarrow C_0^{M_1} \longrightarrow M_1 \longrightarrow 0$$

and

$$\cdots \longrightarrow C_n^{M_3} \longrightarrow C_{n-1}^{M_3} \longrightarrow \cdots \longrightarrow C_1^{M_3} \longrightarrow C_0^{M_3} \longrightarrow M_3 \longrightarrow 0$$

with all $C_i^{M_1},\ C_i^{M_3} \in C(\subseteq \mathcal{P}(\xi))$.

By using an argument similar to that in the proof of [1, Proposition 4.11], we consider the following commutative diagram:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_0^{M_1} & \xrightarrow{\binom{1}{0}} & C_0^{M_1} \oplus C_0^{M_3} & \xrightarrow{(0,\ 1)} & C_0^{M_3} & \xrightarrow{0} & \Sigma C_0^{M_1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & \Sigma M_1
\end{array}$$

and we get the following ξ-exact complex:

$$\cdots \longrightarrow C_n^{M_1} \oplus C_n^{M_3} \longrightarrow C_{n-1}^{M_1} \oplus C_{n-1}^{M_3} \longrightarrow \cdots \longrightarrow C_0^{M_1} \oplus C_0^{M_3} \longrightarrow M_2 \longrightarrow 0$$

in $\mathcal{T}$. Thus $M_2 \in \mathrm{res}^*(\mathcal{C})$.

Claim 2. If $M_2, M_3 \in \mathrm{res}^*(\mathcal{C})$, then $M_1 \in \mathrm{res}^*(\mathcal{C})$.

By assumption, we have the following ξ-exact complexes:

$$\cdots \longrightarrow C_n^{M_2} \longrightarrow C_{n-1}^{M_2} \longrightarrow \cdots \longrightarrow C_1^{M_2} \longrightarrow C_0^{M_2} \longrightarrow M_2 \longrightarrow 0$$

and

$$\cdots \longrightarrow C_n^{M_3} \longrightarrow C_{n-1}^{M_3} \longrightarrow \cdots \longrightarrow C_1^{M_3} \longrightarrow C_0^{M_3} \longrightarrow M_3 \longrightarrow 0$$

with all $C_i^{M_2},\ C_i^{M_3} \in \mathcal{C}(\subseteq \mathcal{P}(\xi))$. By [9, Theorem 3.7], we get a ξ-exact complex

$$\cdots \longrightarrow C_n^{M_3} \oplus C_{n-1}^{M_2} \longrightarrow C_{n-1}^{M_3} \oplus C_{n-2}^{M_2} \longrightarrow \cdots \longrightarrow C_2^{M_3} \oplus C_1^{M_2} \longrightarrow K \longrightarrow M_1 \longrightarrow 0$$

and a triangle

$$K \longrightarrow C_1^{M_3} \oplus C_0^{M_2} \longrightarrow C_0^{M_3} \longrightarrow \Sigma K$$

in ξ. It follows that $K \in \mathcal{C}$ by assumption. Thus $M_1 \in \mathrm{res}^*(\mathcal{C})$.

Claim 3. If $M_1, M_2 \in \mathrm{res}^*(\mathcal{C})$, then $M_3 \in \mathrm{res}^*(\mathcal{C})$.

By assumption, we have the following ξ-exact complexes:

$$\cdots \longrightarrow C_n^{M_1} \longrightarrow C_{n-1}^{M_1} \longrightarrow \cdots \longrightarrow C_1^{M_1} \longrightarrow C_0^{M_1} \longrightarrow M_1 \longrightarrow 0$$

and

$$\cdots \longrightarrow C_n^{M_2} \longrightarrow C_{n-1}^{M_2} \longrightarrow \cdots \longrightarrow C_1^{M_2} \longrightarrow C_0^{M_2} \longrightarrow M_2 \longrightarrow 0$$

with all $C_i^{M_1},\ C_i^{M_2} \in \mathcal{C}(\subseteq \mathcal{P}(\xi))$. By [9, Theorem 3.8], we have the following ξ-exact complex:

$$\cdots \longrightarrow C_n^{M_2} \oplus C_{n-1}^{M_1} \longrightarrow \cdots \longrightarrow C_2^{M_2} \oplus C_1^{M_1} \longrightarrow C_1^{M_2} \oplus C_0^{M_1} \longrightarrow C_0^{M_2} \longrightarrow M_3 \longrightarrow 0$$

in $\mathcal{T}$. Thus $M_3 \in \mathrm{res}^*(\mathcal{C})$. □

By Remark 3.2 and Proposition 3.3, we immediately have the following corollary.

Corollary 3.4. *Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ closed under hokernels of ξ-proper epimorphisms and closed under ξ-extensions. Then* $\mathrm{res}^*(\mathcal{P}_{\mathcal{X}})$ *is quasi-resolving and closed under hocokernels of ξ-proper monomorphisms.*

Let $M \in \mathcal{T}$. We use $\mathrm{add}\, M$ to denote the full subcategory of $\mathcal{T}$, whose objects are the direct summands of finite direct copies of M. For a ξ-projective object P, it is clear that $\mathrm{add}\, P$ is closed under hokernels of ξ-proper epimorphisms and closed under ξ-extensions, and then $\mathrm{res}^*(\mathrm{add}\, P)$ is a quasi-resolving subcategory of $\mathcal{T}$ by Corollary 3.4.

Proposition 3.5. *If $\mathcal{X}$ is a quasi-resolving subcategory of $\mathcal{T}$, then* $\mathrm{res}^*(\mathcal{X}) = \mathrm{res}^*(\mathcal{P}_{\mathcal{X}})$.

Proof. Clearly, $\mathrm{res}^*(\mathcal{X}) \supseteq \mathrm{res}^*(\mathcal{P}_{\mathcal{X}})$.

Conversely, let $M \in \mathrm{res}^*(\mathcal{X})$. Then there exists the following ξ-exact complex:

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0.$$

Consider the triangle

$$K_0 \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K_0 \tag{3.1}$$

in ξ. Since $\mathcal{X}$ is quasi-resolving, there exists a triangle

$$L_0 \longrightarrow P_0 \longrightarrow X_0 \longrightarrow \Sigma L_0 \tag{3.2}$$

in ξ with $P_0 \in \mathcal{P}_{\mathcal{X}}$ and $L_0 \in \mathcal{X}$. Applying base change to the triangle

$$\Sigma^{-1} M \longrightarrow K_0 \longrightarrow X_0 \longrightarrow M$$

along the morphism $P_0 \longrightarrow X_0$ yields the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & L_0 & \xrightarrow{=} & L_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M & \longrightarrow & W_0 & \longrightarrow & P_0 & \longrightarrow & M \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
\Sigma^{-1}M & \longrightarrow & K_0 & \longrightarrow & X_0 & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma L_0 & \xrightarrow{=} & \Sigma L_0 & \longrightarrow & 0.
\end{array}$$

Notice that the triangles (3.1) and (3.2) are in ξ, so we get the following triangles:

$$L_0 \longrightarrow W_0 \longrightarrow K_0 \longrightarrow \Sigma L_0 \tag{3.3}$$

and

$$W_0 \longrightarrow P_0 \longrightarrow M \longrightarrow \Sigma W_0$$

in ξ by Lemma 2.11(1)(a). Consider the triangle

$$K_1 \longrightarrow X_1 \longrightarrow K_0 \longrightarrow \Sigma K_1$$

in ξ. Applying base change to the triangle (3.3) along the morphism $X_1 \longrightarrow K_0$ yields the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1 & \xrightarrow{=} & K_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_0 & \longrightarrow & U & \longrightarrow & X_1 & \longrightarrow & \Sigma L_0 \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
L_0 & \longrightarrow & W_0 & \longrightarrow & K_0 & \longrightarrow & \Sigma L_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1 & \xrightarrow{=} & \Sigma K_1 & \longrightarrow & 0,
\end{array}$$

in which the second horizontal triangle is in ξ. Since $L_0, X_1 \in \mathcal{X}$, we have $U \in \mathcal{X}$. Since $K_1 \in \text{res}^*(\mathcal{X})$, we have $W_0 \in \text{res}^*(\mathcal{X})$.

Repeating this process for W_0, we get a triangle

$$W_1 \longrightarrow P_1 \longrightarrow W_0 \longrightarrow \Sigma W_1$$

in ξ with $P_1 \in \mathcal{P}_{\mathcal{X}}$ and $W_1 \in \text{res}^*(\mathcal{X})$. Continuing these steps, we obtain a ξ-exact complex

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$. Thus $M \in \text{res}^*(\mathcal{P}_{\mathcal{X}})$ and $\text{res}^*(\mathcal{X}) \subseteq \text{res}^*(\mathcal{P}_{\mathcal{X}})$. The proof is finished. □

Corollary 3.6. *If $\mathcal{X}$ is quasi-resolving in $\mathcal{T}$, then so is* $\text{res}^*(\mathcal{X})$*. Moreover,* $\text{res}^*(\mathcal{X})$ *is closed under hocokernels of ξ-proper monomorphisms.*

Proof. This follows from Corollary 3.4 and Proposition 3.5. □

Theorem 3.7. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$ and let*

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow M \longrightarrow 0$$

be a ξ-exact complex with all $X_i \in \mathcal{X}$. Then the following assertions hold.

(1) There exists a ξ-exact complex

$$0 \longrightarrow B \longrightarrow X \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{X}$ and all $P_i \in \mathcal{P}_{\mathcal{X}}$.

(2) If $\mathcal{X}$ has a ξ-cogenerator $\mathcal{H}$, then it holds that:

(i) There exists a ξ-exact complex

$$0 \to B \to H_{n-1} \to \cdots \to H_2 \to H_1 \to X \to M \to 0$$

with $X \in \mathcal{X}$ and all $H_i \in \mathcal{H}$.

(ii) For any $1 \leq i \leq n$, there exists a ξ-exact complex

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_{i+1} \longrightarrow X \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{X}$, all $H_j \in \mathcal{H}$ $(i+1 \leq j \leq n)$, and all $P_t \in \mathcal{P}_{\mathcal{X}}$ $(1 \leq t \leq i-1)$.

(iii) There exist a ξ-exact complex

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow W \longrightarrow 0$$

with all $H_i \in \mathcal{H}$ and a triangle

$$M \longrightarrow W \longrightarrow X \longrightarrow \Sigma M$$

in ξ with $X \in \mathcal{X}$.

(3) If $\mathcal{X}$ has a ξ-cogenerator $\mathcal{H}$ and $B \in \mathcal{X}$, then:

(i) There exists a ξ-exact complex

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow X \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{X}$ and all $H_i \in \mathcal{H}$.

(ii) For any $1 \leq i \leq n$, there exists a ξ-exact complex

$$0 \longrightarrow H_{n+1} \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_{i+1} \longrightarrow X \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{X}$, all $H_j \in \mathcal{H}$ $(i+1 \leq j \leq n+1)$, and all $P_t \in \mathcal{P}_{\mathcal{X}}$ $(1 \leq t \leq i-1)$.

(iii) There exist a ξ-exact complex

$$0 \longrightarrow H_{n+1} \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow W \longrightarrow 0$$

with all $H_i \in \mathcal{H}$ and a triangle

$$M \longrightarrow W \longrightarrow X \longrightarrow \Sigma M$$

in ξ with $X \in \mathcal{X}$.

Proof. (1) We proceed by induction on n. The case for $n = 1$ is true clearly. Following [9, Proposition 3.3], the case for $n = 2$ is true. Now suppose $n > 2$. By assumption, we get the following triangle:

$$B \longrightarrow X_n \longrightarrow K \longrightarrow \Sigma B$$

in ξ and the following ξ-exact complex:

$$0 \longrightarrow K \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow M \longrightarrow 0. \tag{3.4}$$

By the inductive hypothesis, we obtain from (3.4) the following ξ-exact complex:

$$0 \longrightarrow K \longrightarrow X' \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with $X' \in \mathcal{X}$ and all $P_i \in \mathcal{P}_{\mathcal{X}}$. This yields a ξ-exact complex

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow X' \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0.$$

Consider the following ξ-exact complexes:

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow X' \longrightarrow L \longrightarrow 0 \tag{3.5}$$

and

$$0 \longrightarrow L \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0. \tag{3.6}$$

From the ξ-exact complex (3.5), we get the following ξ-exact complex:

$$0 \longrightarrow B \longrightarrow X \longrightarrow P_{n-1} \longrightarrow L \longrightarrow 0 \tag{3.7}$$

with $X \in \mathcal{X}$ and $P_{n-1} \in \mathcal{P}_{\mathcal{X}}$. Now splicing (3.6) and (3.7) yields the desired ξ-exact complex.

(2) (i) It is a dual of (1).

(ii) By assumption, we get the following ξ-exact complexes:

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_{i+1} \longrightarrow K_i \longrightarrow 0 \tag{3.8}$$

and

$$0 \longrightarrow K_i \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow M \longrightarrow 0. \tag{3.9}$$

Applying the assertions (2)(i) and (1) to the ξ-exact complexes (3.8) and (3.9), respectively, we get the following ξ-exact complexes:

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_{i+2} \longrightarrow Y_2 \longrightarrow K_i \longrightarrow 0,$$

$$0 \longrightarrow K_i \longrightarrow Y_1 \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with $Y_1, Y_2 \in \mathcal{X}$, all $H_j \in \mathcal{H}$, and all $P_t \in \mathcal{P}_{\mathcal{X}}$. It induces the following ξ-exact complex:

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_{i+2} \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0.$$

Consider the following ξ-exact complexes:

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_{i+2} \longrightarrow T_2 \longrightarrow 0, \tag{3.10}$$

$$0 \longrightarrow T_2 \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow T_1 \longrightarrow 0, \tag{3.11}$$

$$0 \longrightarrow T_1 \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0. \tag{3.12}$$

Applying the assertion (2)(i) to (3.11) yields the following ξ-exact complex:

$$0 \longrightarrow T_2 \longrightarrow H_{i+1} \longrightarrow X \longrightarrow T_1 \longrightarrow 0 \tag{3.13}$$

with $H_{i+1} \in \mathcal{H}$ and $X \in \mathcal{X}$. Now splicing (3.10), (3.13), and (3.12) yields the desired ξ-exact complex.

(iii) By (2)(i), we get the following ξ-exact complex:

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_2 \longrightarrow X' \longrightarrow M \longrightarrow 0$$

with $X' \in \mathcal{X}$ and all $H_i \in \mathcal{H}$. Consider the following ξ-exact complex:

$$0 \longrightarrow B \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_2 \longrightarrow K \longrightarrow 0$$

and the triangle

$$K \longrightarrow X' \longrightarrow M \longrightarrow \Sigma K \tag{3.14}$$

in ξ. For $X' \in \mathcal{X}$, there exists a triangle

$$X' \longrightarrow H_1 \longrightarrow X \longrightarrow \Sigma X' \tag{3.15}$$

in ξ with $H_1 \in \mathcal{H}$ and $X \in \mathcal{X}$. Applying cobase change for the triangle

$$X' \longrightarrow M \longrightarrow \Sigma K \longrightarrow \Sigma X'$$

along the morphism $X' \longrightarrow H_1$ yields the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X & \xrightarrow{=} & \Sigma^{-1}X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & \Sigma K \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
K & \longrightarrow & H_1 & \longrightarrow & W & \longrightarrow & \Sigma K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X & \xrightarrow{=} & X & \longrightarrow & 0.
\end{array}$$

Notice that the triangles (3.14) and (3.15) are in ξ, so the triangles

$$K \longrightarrow H_1 \longrightarrow W \longrightarrow \Sigma K$$

and

$$M \longrightarrow W \longrightarrow X \longrightarrow \Sigma M$$

are in ξ by Lemma 2.11(2)(a). Thus we get the desired ξ-exact complex and the desired triangle.

(3) (i) We proceed by induction on n. When $n = 1$, since $B \in \mathcal{X}$, there exists a triangle

$$B \longrightarrow H \longrightarrow X' \longrightarrow \Sigma B$$

in ξ with $H \in \mathcal{H}$ and $X' \in \mathcal{X}$. Applying cobase change for the triangle $B \longrightarrow X_1 \longrightarrow M \longrightarrow \Sigma B$ along the morphism $B \longrightarrow H$ yields the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X' & \xrightarrow{=} & \Sigma^{-1}X' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M & \longrightarrow & B & \longrightarrow & X_1 & \longrightarrow & M \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
\Sigma^{-1}M & \longrightarrow & H & \longrightarrow & X & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X' & \xrightarrow{=} & X' & \longrightarrow & 0.
\end{array}$$

It follows that the triangles

$$X_1 \longrightarrow X \longrightarrow X' \longrightarrow \Sigma X_1$$

and

$$H \longrightarrow X \longrightarrow M \longrightarrow \Sigma H \tag{3.16}$$

are in ξ. Since $\mathcal{X}$ is closed under ξ-extensions, we have $X \in \mathcal{X}$. Then the triangle (3.16) is as desired.

Now suppose $n \geq 2$. Consider the ξ-exact complex

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow K_1 \longrightarrow 0$$

and the triangle

$$K_1 \longrightarrow X_1 \longrightarrow M \longrightarrow \Sigma K_1$$

in ξ. By the induction hypothesis, we get a ξ-exact complex

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow X_2' \longrightarrow K_1 \longrightarrow 0.$$

Then we get the following ξ-exact complex:

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow X_2' \longrightarrow X_1 \longrightarrow M \longrightarrow 0.$$

Consider the ξ-exact complexes

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow K \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow X_2' \longrightarrow X_1 \longrightarrow M \longrightarrow 0\ . \tag{3.17}$$

Applying (2)(i) to the ξ-exact complex (3.17) yields the following ξ-exact complex:

$$0 \longrightarrow K \longrightarrow H_1 \longrightarrow X \longrightarrow M \longrightarrow 0$$

with $H_1 \in \mathcal{H}$ and $X \in \mathcal{X}$. Then we get the desired ξ-exact complex

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow X \longrightarrow M \longrightarrow 0.$$

The proofs of (3)(ii and iii) are similar to that of (2)(ii and iii), respectively. □

Theorem 3.8. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$ and let*

$$0 \longrightarrow B \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow M \longrightarrow 0$$

be a ξ-exact complex with all $X_i \in \mathcal{X}$. Then the following assertions hold.

(1) There exist a ξ-exact complex

$$0 \longrightarrow L \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$ and a triangle

$$X \longrightarrow L \longrightarrow B \longrightarrow \Sigma X$$

in ξ with $X \in \mathcal{X}$.

(2) If $M \in \mathcal{X}$, then:

(i) There is a ξ-exact complex

$$0 \longrightarrow B \longrightarrow X \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0$$

with $X \in \mathcal{X}$ and all $P_i \in \mathcal{P}_{\mathcal{X}}$.

(ii) There is a ξ-exact complex

$$0 \longrightarrow T \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$ and a triangle

$$X \longrightarrow T \longrightarrow B \longrightarrow \Sigma X$$

in ξ with $X \in \mathcal{X}$.

Proof. (1) It is dual to Theorem 3.7(2)(iii).

(2) The assertions (i) and (ii) are dual to Theorem 3.7(3)(i and iii), respectively. □

4. Resolution dimensions relative to quasi-resolving subcategories

We first recall the following definition.

Definition 4.1. ([10, Definition 4.1]) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. The $\mathcal{X}$-resolution dimension $\mathcal{X}$-res.dim M of M is defined by

$$\mathcal{X}\text{- res.dim}\, M = \inf\{n \geq 0 \mid \text{ there exists a } \xi\text{-exact complex } 0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0 \text{ in } \mathcal{T} \text{ with all } X_i \in \mathcal{X}\}.$$

For a ξ-exact complex

$$\cdots \xrightarrow{f_{n+1}} X_n \longrightarrow \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \longrightarrow 0$$

with all $X_i \in \mathcal{X}$, the Hoker f_{n-1} is called an nth ξ-$\mathcal{X}$-syzygy of M, denoted by $\Omega^n_{\mathcal{X}}(M)$.

In the case of $\mathcal{X} = \mathcal{P}(\xi)$, we write ξ- pd $M := \mathcal{X}$- res.dim M and $\Omega^n(M) := \Omega^n_{\mathcal{P}(\xi)}(M)$. We use $\widehat{\mathcal{X}}$ to denote the full subcategory of $\mathcal{T}$ whose objects have finite $\mathcal{X}$-resolution dimension.

Lemma 4.2. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$ and $M \in \text{res}^*(\mathcal{X})$. If*

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M \longrightarrow 0$$

are ξ-exact complexes with all X_i, $Y_i \in \mathcal{X}$ for any $0 \leq i \leq n-1$, then $X_n \in \mathcal{X}$ if and only if $Y_n \in \mathcal{X}$.

Proof. By Proposition 3.5, we have $M \in \text{res}^*(\mathcal{P}_{\mathcal{X}})$. Then there exists a ξ-exact complex

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & P_n & & \longrightarrow & P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \\ & & \searrow & & \nearrow \\ & & & K_n & \end{array}$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$. Consider the following triangles:

$$K_1^{'} \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K_1^{'},$$

$$K_1^{''} \longrightarrow Y_0 \longrightarrow M \longrightarrow \Sigma K_1^{''}$$

in ξ. By using an argument similar to that in the proof of [1, Proposition 4.11], we get the following two ξ-exact complexes:

$$0 \longrightarrow K_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow X_{n-1} \oplus P_{n-2} \longrightarrow \cdots \longrightarrow X_2 \oplus P_1 \longrightarrow X_1 \oplus P_0 \longrightarrow X_0 \longrightarrow 0,$$

$$0 \longrightarrow K_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow Y_{n-1} \oplus P_{n-2} \longrightarrow \cdots \longrightarrow Y_2 \oplus P_1 \longrightarrow Y_1 \oplus P_0 \longrightarrow Y_0 \longrightarrow 0.$$

Set

$$X := \mathrm{Hoker}(X_{n-1} \oplus P_{n-2} \to X_{n-2} \oplus P_{n-3})$$

and

$$Y := \mathrm{Hoker}(Y_{n-1} \oplus P_{n-2} \to Y_{n-2} \oplus P_{n-3}).$$

Since $\mathcal{X}$ is quasi-resolving, we have that X and Y are in $\mathcal{X}$. Consider the following triangles:

$$K_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow X \longrightarrow \Sigma K_n$$

and

$$K_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow Y \longrightarrow \Sigma K_n$$

in ξ. We have that $X_n \oplus P_{n-1} \in \mathcal{X}$ if and only if $K_n \in \mathcal{X}$, and if and only if $Y_n \oplus P_{n-1} \in \mathcal{X}$.

On the other hand, from the following triangles

$$X_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow P_{n-1} \stackrel{0}{\longrightarrow} \Sigma X_n \quad \text{and} \quad Y_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow P_{n-1} \stackrel{0}{\longrightarrow} \Sigma Y_n$$

in ξ, we get that $X_n \in \mathcal{X}$ if and only if $X_n \oplus P_{n-1} \in \mathcal{X}$, and that $Y_n \in \mathcal{X}$ if and only if $Y_n \oplus P_{n-1} \in \mathcal{X}$. Thus $X_n \in \mathcal{X}$ if and only if $Y_n \in \mathcal{X}$. □

By Lemma 4.2, we have the following result.

Proposition 4.3. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. Then for any $m \geq 0$, the following statements are equivalent.*

(1) $\mathcal{X}$-$\mathrm{res.dim}\, M \leq m$.
(2) $\Omega^n_{\mathcal{P}_{\mathcal{X}}}(M) \in \mathcal{X}$ *for any* $n \geq m$.
(3) $\Omega^n_{\mathcal{X}}(M) \in \mathcal{X}$ *for any* $n \geq m$.

Let $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ be a triangle in ξ. The following comparison proposition shows that the resolution dimensions of any two terms may induce an upper bound for that of the third term.

Proposition 4.4. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

be a triangle in ξ. It holds that

(1) $\mathcal{X}$-res.dim $B \leq \max\{\mathcal{X}\text{- res.dim } A, \mathcal{X}\text{- res.dim } C\}$.
(2) $\mathcal{X}$-res.dim $A \leq \max\{\mathcal{X}\text{- res.dim } B, \mathcal{X}\text{- res.dim } C - 1\}$.
(3) $\mathcal{X}$-res.dim $C \leq \max\{\mathcal{X}\text{- res.dim } A + 1, \mathcal{X}\text{- res.dim } B\}$.

Proof. For any $M \in \mathcal{T}$, if $\mathcal{X}$-res.dim $M = m$, then there exists a ξ-exact complex

$$0 \longrightarrow P_m^M \longrightarrow P_{m-1}^M \longrightarrow \cdots \longrightarrow P_1^M \longrightarrow P_0^M \longrightarrow M \longrightarrow 0$$

in $\mathcal{T}$ with $P_i^M \in \mathcal{P}_{\mathcal{X}}$ for any $0 \leq i \leq m-1$ and $P_m^M \in \mathcal{X}$ by Proposition 4.3.

(1) Assume $\mathcal{X}$-res.dim $A = m$ and $\mathcal{X}$- res.dim $C = n$. Without loss of generality, we assume $m \leq n$, and then we may assume $P_i^A = 0$ for any $i > m$. By using an argument similar to that in the proof of [1, Proposition 4.11], we get the following ξ-exact complex:

$$0 \longrightarrow P_n^A \oplus P_n^C \longrightarrow P_{n-1}^A \oplus P_{n-1}^C \longrightarrow \cdots \longrightarrow P_0^A \oplus P_0^C \longrightarrow B \longrightarrow 0$$

in $\mathcal{T}$, and thus $\mathcal{X}$-res.dim $B \leq n$.

(2) Assume $\mathcal{X}$-res.dim $B = m$ and $\mathcal{X}$-res.dim $C = n$. Without loss of generality, we assume $m \leq n-1$, and then we may assume $P_i^B = 0$ for any $i > m$. By [9, Theorem 3.7], we have the following ξ-exact complex:

$$0 \longrightarrow P_n^C \oplus P_{n-1}^B \longrightarrow P_{n-1}^C \oplus P_{n-2}^B \longrightarrow \cdots \longrightarrow P_2^C \oplus P_1^B \longrightarrow K \longrightarrow A \longrightarrow 0$$

and the following triangle:

$$K \longrightarrow P_1^C \oplus P_0^B \longrightarrow P_0^C \longrightarrow \Sigma K$$

in ξ. By Remark 3.2, we have $K \in \mathcal{P}_{\mathcal{X}}$ and $\mathcal{X}$-res.dim $A \leq n-1$.

(3) Assume $\mathcal{X}$-res.dim $A = m$ and $\mathcal{X}$-res.dim $B = n$. Without loss of generality, we assume $m+1 \leq n$, and then we may assume $P_i^A = 0$ for any $i > m$. By [9, Theorem 3.8], we have the following ξ-exact complex:

$$0 \longrightarrow P_n^B \oplus P_{n-1}^A \longrightarrow \cdots \longrightarrow P_2^B \oplus P_1^A \longrightarrow P_1^B \oplus P_0^A \longrightarrow P_0^B \longrightarrow C \longrightarrow 0$$

in $\mathcal{T}$, and thus $\mathcal{X}$-res.dim $C \leq n$. □

As an immediate consequence of Proposition 4.4, we get the following equivalent characterization of quasi-resolving subcategories.

Corollary 4.5. *Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ with $\mathcal{X} \subseteq \text{res}^*(\mathcal{P}_{\mathcal{X}})$. Then $\mathcal{X}$ is quasi-resolving if and only if for any triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

in ξ, it holds that

$$\mathcal{X}\text{- res.dim } B \leq \max\{\mathcal{X}\text{- res.dim } A, \mathcal{X}\text{- res.dim } C\}$$

and

$$\mathcal{X}\text{- res.dim } A \leq \max\{\mathcal{X}\text{- res.dim } B, \mathcal{X}\text{- res.dim } C - 1\}.$$

We have the following closure property for the subcategory $\widehat{\mathcal{X}}$.

Remark 4.6. Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$. Then $\widehat{\mathcal{X}}$ is closed under hokernels of ξ-proper epimorphisms, hocokernels of ξ-proper monomorphisms, and ξ-extensions by Proposition 4.4. On the other hand, we have

$$\begin{aligned}\widehat{\mathcal{X}} \subseteq \operatorname{res}^*(\mathcal{X}) &= \operatorname{res}^*(\mathcal{P}_{\mathcal{X}}) \quad \text{(by Proposition 3.5)}\\ &\subseteq \operatorname{res}^*(\mathcal{P}_{\widehat{\mathcal{X}}}).\end{aligned}$$

Thus $\widehat{\mathcal{X}}$ is a quasi-resolving subcategory of $\mathcal{T}$.

By Proposition 4.4, we also have the following corollary.

Corollary 4.7. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

be a triangle in ξ. Then the following assertions hold.

(1) If $C \in \mathcal{X}$, then $\mathcal{X}$-res.dim $A = \mathcal{X}$-res.dim B.
(2) If $B \in \mathcal{X}$, then either $A \in \mathcal{X}$ or $\mathcal{X}$-res.dim $A = \mathcal{X}$-res.dim $C - 1$.
(3) If $A \in \mathcal{X}$ and neither B nor C is in $\mathcal{X}$, then $\mathcal{X}$-res.dim $B = \mathcal{X}$-res.dim C.

Now we recall the following definition.

Definition 4.8. ([10, Definition 3.8]) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. A ξ-proper epimorphism $X \longrightarrow M$ is called a *right $\mathcal{X}$-approximation* of M if

$$\operatorname{Hom}_{\mathcal{T}}(\widetilde{X}, X) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(\widetilde{X}, M) \longrightarrow 0$$

is exact for any $\widetilde{X} \in \mathcal{X}$. In this case, there is a triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$$

in ξ.

The following result is an analogue of the Auslander-Buchweitz approximation (see [11, 18]).

Proposition 4.9. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let $\mathcal{H}$ be a ξ-cogenerator of $\mathcal{X}$. Then for any $M \in \mathcal{T}$ with $\mathcal{X}$- res.dim $M = m < \infty$, there exist two triangles*

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{4.1}$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M \tag{4.2}$$

in ξ with X, $X' \in \mathcal{X}$, $\mathcal{H}$- res.dim $K = m - 1$, and $\mathcal{H}$- res.dim $W = \mathcal{X}$- res.dim $W = m$ (if $m = 0$, this should be interpreted as $K = 0$).

In particular, if $\mathcal{X} \perp \mathcal{H}$, then the ξ-proper epimorphism $X \longrightarrow M$ is a right $\mathcal{X}$-approximation of M.

Proof. By the assertions (i) and (iii) in Theorem 3.7(3), there are two triangles (4.1) and (4.2) in ξ with $\mathcal{H}$- res.dim $K \leq m-1$ and $\mathcal{H}$- res.dim $W \leq m$. Notice that $\mathcal{X}$- res.dim $M = m$ by assumption, so $\mathcal{X}$- res.dim $K = m-1$ and $\mathcal{X}$- res.dim $W = m$ by Corollary 4.7. Since $\mathcal{X}$- res.dim $A \leq \mathcal{H}$- res.dim A for any $A \in \mathcal{T}$, we have $\mathcal{H}$- res.dim $K = m-1$ and $\mathcal{H}$- res.dim $W = \mathcal{X}$- res.dim $W = m$. □

Proposition 4.10. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let $\mathcal{H}$ be a ξ-cogenerator for $\mathcal{X}$. Then for any $M \in \widehat{\mathcal{X}}$, the following statements are equivalent.*

(1) $\mathcal{X}$-res.dim $M \leq m$.
(2) There exists a ξ-exact complex

$$0 \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_1 \longrightarrow X \longrightarrow M \longrightarrow 0$$

with all $H_i \in \mathcal{H}$ and $X \in \mathcal{X}$.
(3) There exists a triangle

$$M \longrightarrow W \longrightarrow X \longrightarrow \Sigma M$$

in ξ with $\mathcal{H}$-res.dim $W \leq m$ and $X \in \mathcal{X}$.
(4) For any $0 \leq i \leq m$, there exists a ξ-exact complex

$$0 \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_{i+1} \longrightarrow X \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $H_j \in \mathcal{H}$ $(i+1 \leq j \leq m)$, all $P_t \in \mathcal{P}_{\mathcal{X}}$ $(0 \leq t \leq i-1)$, and $X \in \mathcal{X}$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) By Corollary 4.7 and Proposition 4.9.
(1) $\Longleftrightarrow$ (4) By Theorem 3.7(3)(ii). □

The following result is a consequence of Proposition 4.10.

Corollary 4.11. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let $\mathcal{H}$ be a ξ-cogenerator for $\mathcal{X}$ such that $\mathcal{H} \subseteq \mathcal{P}_{\mathcal{X}}$. Then for any $M \in \widehat{\mathcal{X}}$, we have that $\mathcal{X}$-res.dim $M \leq m$ if and only if for any $0 \leq i \leq m$, there exists a ξ-exact complex*

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow X \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $P_t \in \mathcal{P}_{\mathcal{X}}$ and $X \in \mathcal{X}$.

We are now in a position to provide some criteria for computing the resolution dimension of objects.

Theorem 4.12. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let $\mathcal{H}$ be a ξxt-injective ξ-cogenerator for $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. Then for any $M \in \widehat{\mathcal{X}}$, the following statements are equivalent.*

(1) $\mathcal{X}$- res.dim $M \leq m$.
(2) $\Omega^n_{\mathcal{P}_{\mathcal{X}}}(M) \in \mathcal{X}$ *for any $n \geq m$.*
(3) $\Omega^n_{\mathcal{X}}(M) \in \mathcal{X}$ *for any $n \geq m$.*
(4) There exists a triangle

$$M \longrightarrow W \longrightarrow X \longrightarrow \Sigma M$$

in ξ with $\mathcal{H}$-res.dim $W \leq m$ and $X \in \mathcal{X}$.

(5) There exists a ξ-exact complex

$$0 \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_1 \longrightarrow X \longrightarrow M \longrightarrow 0$$

with all $H_i \in \mathcal{H}$ and $X \in \mathcal{X}$.

(6) For any $0 \leq i \leq m$, there exists a ξ-exact complex

$$0 \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_{i+1} \longrightarrow X \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $H_j \in \mathcal{H}$ $(i+1 \leq j \leq m)$, all $P_t \in \mathcal{P}_{\mathcal{X}}$ $(0 \leq t \leq i-1)$, and $X \in \mathcal{X}$.

(7) $\xi xt_{\xi}^{n}(M, H) = 0$ for any $n > m$ and $H \in \mathcal{H}$.

(8) $\xi xt_{\xi}^{n}(M, L) = 0$ for any $n > m$ and $L \in \widehat{\mathcal{H}}$.

(9) M admits a right $\mathcal{X}$-approximation $\varphi : X \to M$, where φ is ξ-proper epic, such that $\mathcal{H}\text{-}\operatorname{res.dim} \operatorname{Hoker} \varphi \leq m-1$.

(10) There are two triangles

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M$$

in ξ such that $X, X' \in \mathcal{X}$, $\mathcal{H}\text{-}\operatorname{res.dim} K \leq m-1$, and $\mathcal{H}\text{-}\operatorname{res.dim} W = \mathcal{X}\text{-}\operatorname{res.dim} W \leq m$.

Proof. By Propositions 4.3 and 4.10, we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) and (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6), respectively.

(1) $\Longrightarrow$ (7) Suppose $\mathcal{X}\text{-}\operatorname{res.dim} M \leq m$. Then there is a ξ-exact complex

$$0 \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with all $X_i \in \mathcal{X}$. Since $\mathcal{H}$ is a ξxt-injective ξ-cogenerator of $\mathcal{X}$, we have $\xi xt_{\xi}^{i\geq 1}(X_i, H) = 0$ for any $H \in \mathcal{H}$. So $\xi xt_{\xi}^{n}(M, H) \cong \xi xt_{\xi}^{n-m}(X_m, H) = 0$ for any $n > m$.

The implication (7) $\Longrightarrow$ (8) follows from [10, Lemma 3.9], and the implication (8) $\Longrightarrow$ (7) is clear.

(7) $\Longrightarrow$ (1) Since $M \in \widehat{\mathcal{X}}$, there is a triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$$

in ξ with $\mathcal{H}\text{-}\operatorname{res.dim} K < \infty$ and $X \in \mathcal{X}$ by Proposition 4.9. Since $\xi xt_{\xi}^{i\geq 1}(X, H) = 0$ for any $H \in \mathcal{H}$, we have $\xi xt_{\xi}^{i}(K, H) \cong \xi xt_{\xi}^{i+1}(M, H)$ for any $i \geq 1$. So $\xi xt_{\xi}^{i\geq m}(K, H) = 0$ by assumption. Since $\mathcal{H}\text{-}\operatorname{res.dim} K < \infty$, there is a ξ-exact complex

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_0 \longrightarrow K \longrightarrow 0$$

with all $H_i \in \mathcal{H}$. Then

$$\xi xt_{\xi}^{i}(\Omega_{\mathcal{H}}^{m-1}(K), H) \cong \xi xt_{\xi}^{i+m-1}(K, H) = 0$$

for any $i \geq 1$ and $H \in \mathcal{H}$, which means $\Omega_{\mathcal{H}}^{m-1}(K) \in {}^{\perp}\mathcal{H}$. Notice that $\mathcal{H}\text{-}\operatorname{res.dim} \Omega_{\mathcal{H}}^{m-1}(K) < \infty$, so $\Omega_{\mathcal{H}}^{m-1}(K) \in \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H}$. It follows from [10, Lemma 3.12] that $\Omega_{\mathcal{H}}^{m-1}(K) \in \mathcal{H}$. Thus $\mathcal{H}\text{-}\operatorname{res.dim} K \leq m-1$ and $\mathcal{X}\text{-}\operatorname{res.dim} M \leq m$.

By Proposition 4.9, we have (9) $\Longleftrightarrow$ (1) $\Longleftrightarrow$ (10). □

The following result gives a sufficient condition such that the $\mathcal{X}$-resolution dimension and the $\mathcal{H}$-resolution dimension of an object in $\mathcal{T}$ are identical.

Proposition 4.13. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let $\mathcal{H}$ be a ξxt-injective ξ-cogenerator for $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. For any $M \in \mathcal{T}$, if $\mathcal{H}\text{-}\operatorname{res.dim} M < \infty$, then $\mathcal{X}\text{-}\operatorname{res.dim} M = \mathcal{H}\text{-}\operatorname{res.dim} M$.*

Proof. It is trivial that $\mathcal{X}\text{-}\operatorname{res.dim} M \leq \mathcal{H}\text{-}\operatorname{res.dim} M$. Suppose $\mathcal{H}\text{-}\operatorname{res.dim} M = n < \infty$. Then $\mathcal{X}\text{-}\operatorname{res.dim} M \leq n$. Now suppose $\mathcal{X}\text{-}\operatorname{res.dim} M = m$. If $m < n$, then we consider the following ξ-exact complex:

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0$$

with all $H_i \in \mathcal{H}$. By Theorem 4.12, we have $\xi xt_{\xi}^{k>m}(M, H) = 0$ for any $H \in \mathcal{H}$. Then

$$\xi xt_{\xi}^{i}(\Omega_{\mathcal{H}}^{m}(M), H) \cong \xi xt_{\xi}^{i+m}(M, H) = 0$$

for any $i \geq 1$, and hence $\Omega_{\mathcal{H}}^{m}(M) \in {}^{\perp}\mathcal{H}$. Notice that $\mathcal{H}\text{-}\operatorname{res.dim} \Omega_{\mathcal{H}}^{m}(M) < \infty$, so $\Omega_{\mathcal{H}}^{m}(M) \in \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H}$. It follows from [10, Lemma 3.12] that $\Omega_{\mathcal{H}}^{m}(M) \in \mathcal{H}$. Thus $\mathcal{H}\text{-}\operatorname{res.dim} M = n \leq m$, which is a contradiction. Then $m \geq n$, and thus $\mathcal{X}\text{-}\operatorname{res.dim} M = \mathcal{H}\text{-}\operatorname{res.dim} M$. □

Corollary 4.14. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$, and let $\mathcal{H}$ be a ξxt-injective ξ-cogenerator for $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. For any $X \in \mathcal{X}$, if $\mathcal{H}\text{-}\operatorname{res.dim} X < \infty$, then $X \in \mathcal{H}$.*

Proof. By Proposition 4.13, we have $\mathcal{H}\text{-}\operatorname{res.dim} X = \mathcal{X}\text{-}\operatorname{res.dim} X = 0$, and thus $X \in \mathcal{H}$. □

5. A construction

In this section, we will construct a new quasi-resolving subcategory from a given quasi-resolving subcategory. The following notion is a generalization of that of $\mathcal{GP}_{\mathcal{X}}(\xi)$-Gorenstein categories in [10].

Definition 5.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. A *complete $\mathcal{P}_{\mathcal{X}}\mathcal{X}$-resolution* of M is a $\operatorname{Hom}_{\mathcal{T}}(-, \mathcal{X})$-exact ξ-exact complex

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

in $\mathcal{T}$ with all $P_i \in \mathcal{P}_{\mathcal{X}}$ and $X^i \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$ such that both

$$K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow \Sigma K_1 \ \text{ and } \ M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M$$

are corresponding triangles in ξ. The *relative Gorenstein category* $\mathcal{GP}_{\mathcal{X}\mathcal{X}}(\xi)$ is defined as

$$\mathcal{GP}_{\mathcal{X}\mathcal{X}}(\xi) := \{M \in \mathcal{T} \mid M \text{ admits a complete } \mathcal{P}_{\mathcal{X}}\mathcal{X}\text{-resolution}\}.$$

Remark 5.2.

(1) Since $\mathcal{P}_{\mathcal{X}} = \mathcal{X} \cap \mathcal{P}(\xi) \subseteq \mathcal{X} \cap {}^{\perp}\mathcal{X}$, we have $P_0 \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$. Then both

$$K_2 \longrightarrow P_1 \longrightarrow K_1 \longrightarrow \Sigma K_2 \ \text{ and } \ K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow \Sigma K_1$$

are the corresponding triangles in ξ, and thus $K_1 \in \mathcal{GP}_{\mathcal{X}\mathcal{X}}(\xi)$.

(2) If $M \in \mathcal{GP}_{\mathcal{XX}}(\xi)$, then $\xi xt^0_\xi(M, X) \cong \mathrm{Hom}_\mathcal{T}(M, X)$ and $\xi xt^1_\xi(M, X) = 0$ for any $X \in \mathcal{X}$. In fact, the following ξ-exact complex:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $\mathcal{P}_i \in \mathcal{P}_\mathcal{X}$ is a ξ-projective resolution of M (see [1]), which is $\mathrm{Hom}_\mathcal{T}(-, \mathcal{X})$-exact.

(3) (i) If $\mathcal{P}(\xi) \subseteq \mathcal{X}$, then $\mathcal{GP}_{\mathcal{XX}}(\xi)$ coincides with $\mathcal{GP}_\mathcal{X}(\xi)$ defined in [10].
(ii) If $\mathcal{X} = \mathcal{P}(\xi)$, then $\mathcal{X} \cap {}^\perp\mathcal{X} = \mathcal{P}(\xi)$, and thus $\mathcal{GP}_{\mathcal{XX}}(\xi)$ coincides with $\mathcal{GP}(\xi)$ defined in [2].

In the following result, we provide a method to construct new quasi-resolving subcategories from given ones, which generalizes the result in [10, Theorem 5.3].

Theorem 5.3. *If $\mathcal{X}$ is a quasi-resolving subcategory of $\mathcal{T}$, then $\mathcal{GP}_{\mathcal{XX}}(\xi)$ is also a quasi-resolving subcategory of $\mathcal{T}$.*

Proof. Let $P \in \mathcal{P}_\mathcal{X}$. Consider the following ξ-exact complex:

$$\cdots \longrightarrow 0 \overset{0}{\longrightarrow} P \overset{\mathrm{id}_P}{\longrightarrow} P \overset{0}{\longrightarrow} 0 \longrightarrow \cdots$$

in $\mathcal{T}$. Clearly, it is $\mathrm{Hom}_\mathcal{T}(-, \mathcal{X})$-exact. In particular,

$$0 \overset{0}{\longrightarrow} P \overset{\mathrm{id}_P}{\longrightarrow} P \overset{0}{\longrightarrow} 0 \text{ and } P \overset{\mathrm{id}_P}{\longrightarrow} P \overset{0}{\longrightarrow} 0 \overset{0}{\longrightarrow} \Sigma P$$

are corresponding triangles in ξ. Since $P \in \mathcal{P}_\mathcal{X} \subseteq \mathcal{X} \cap {}^\perp\mathcal{X}$, we have $\mathcal{P}_\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{XX}}(\xi)$. So

$$\mathcal{P}_\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{XX}}(\xi) \cap \mathcal{P}(\xi) = \mathcal{P}_{\mathcal{GP}_{\mathcal{XX}}(\xi)}.$$

It follows that $\mathcal{GP}_{\mathcal{XX}}(\xi) \subseteq \mathrm{res}^*(\mathcal{P}_{\mathcal{GP}_{\mathcal{XX}}(\xi)})$.

By using an argument similar to that in the proof of [9, Theorem 4.3(1)], we get that $\mathcal{GP}_{\mathcal{XX}}(\xi)$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms. Thus $\mathcal{GP}_{\mathcal{XX}}(\xi)$ is quasi-resolving. □

Lemma 5.4. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$ satisfying $\mathcal{X} \cap {}^\perp\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{XX}}(\xi)$. Then $\mathcal{X} \cap {}^\perp\mathcal{X}$ is a ξxt-injective ξ-cogenerator for $\mathcal{GP}_{\mathcal{XX}}(\xi)$ and is closed under hokernels of ξ-proper epimorphisms.*

Proof. Let $M \in \mathcal{GP}_{\mathcal{XX}}(\xi)$. Then there exists a $\mathrm{Hom}_\mathcal{T}(-, \mathcal{X})$-exact triangle

$$M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M \tag{5.1}$$

in ξ with $X^0 \in \mathcal{X} \cap {}^\perp\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{XX}}(\xi)$. There exists a $\mathrm{Hom}_\mathcal{T}(-, \mathcal{X})$-exact triangle

$$K'_1 \longrightarrow P'_0 \longrightarrow X^0 \longrightarrow \Sigma K'_1 \tag{5.2}$$

in ξ with $P'_0 \in \mathcal{P}_\mathcal{X}$, so $K'_1 \in \mathcal{GP}_{\mathcal{XX}}(\xi)$ by Remark 5.2.

Applying base change to the triangle

$$\Sigma^{-1}K^1 \longrightarrow M \longrightarrow X^0 \longrightarrow K^1$$

along the morphism $P'_0 \longrightarrow X^0$ yields the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K'_1 & \xrightarrow{=} & K'_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}K^1 & \longrightarrow & U & \longrightarrow & P'_0 & \longrightarrow & K^1 \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
\Sigma^{-1}K^1 & \longrightarrow & M & \longrightarrow & X^0 & \longrightarrow & K^1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K'_1 & \xrightarrow{=} & \Sigma K'_1 & \longrightarrow & 0.
\end{array}$$

Notice that the triangles (5.1) and (5.2) are in ξ, so we have that the second vertical triangle in the above diagram and the triangle

$$U \longrightarrow P'_0 \longrightarrow K^1 \longrightarrow \Sigma U \tag{5.3}$$

are in ξ by Lemma 2.11(1)(a). Then $U \in \mathcal{GP}_{\mathcal{XX}}(\xi)$ since $\mathcal{GP}_{\mathcal{XX}}(\xi)$ are closed ξ-extensions. So there exists a $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{X})$-exact ξ-exact complex

$$\cdots \longrightarrow P'_2 \longrightarrow P'_1 \longrightarrow U \longrightarrow 0$$

with $P'_i \in \mathcal{P}_{\mathcal{X}}$ $(i \geq 1)$.

For any $\widetilde{X} \in \mathcal{X}$, applying the functor $\mathrm{Hom}_{\mathcal{T}}(-, \widetilde{X})$ to the triangle (5.1) yields the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(K^1, \widetilde{X}) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(X^0, \widetilde{X}) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(M, \widetilde{X}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & & & \\
0 & \longrightarrow & \xi xt^0_{\xi}(K^1, \widetilde{X}) & \longrightarrow & \xi xt^0_{\xi}(X^0, \widetilde{X}) & \longrightarrow & \xi xt^0_{\xi}(M, \widetilde{X}) & \longrightarrow & \xi xt^1_{\xi}(K^1, \widetilde{X}) & \longrightarrow & \xi xt^1_{\xi}(X^0, \widetilde{X})(= 0),
\end{array}$$

where the two isomorphisms follow from the assumption that X^0, $M \in \mathcal{GP}_{\mathcal{XX}}(\xi)$ and Remark 5.2(2). It follows that $\xi xt^1_{\xi}(K^1, \widetilde{X}) = 0$ and $\xi xt^0_{\xi}(K^1, \widetilde{X}) \cong \mathrm{Hom}_{\mathcal{T}}(K^1, \widetilde{X})$. Then the triangle (5.3) is $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{X})$-exact. So the following ξ-exact complex

$$\cdots \longrightarrow P'_2 \longrightarrow P'_1 \longrightarrow P'_0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

is a complete $\mathcal{P}_{\mathcal{X}}\mathcal{X}$-resolution and

$$U \longrightarrow P'_0 \longrightarrow K^1 \longrightarrow \Sigma U \quad \text{and} \quad K^1 \longrightarrow X^1 \longrightarrow K^2 \longrightarrow \Sigma M$$

are corresponding triangles in ξ, so $K^1 \in \mathcal{GP}_{\mathcal{XX}}(\xi)$. Then $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is a ξ-cogenerator of $\mathcal{GP}_{\mathcal{XX}}(\xi)$. Obviously, $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is a ξxt-injective ξ-cogenerator for $\mathcal{GP}_{\mathcal{XX}}(\xi)$.

It is easy to check that $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is closed under hokernels of ξ-proper epimorphisms. □

By Theorems 4.12 and 5.3, and Lemma 5.4, we have the following result. It generalizes [10, Proposition 5.5].

Proposition 5.5. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{T}$ satisfying $\mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{XX}}(\xi)$ and $M \in \mathcal{T}$. If $M \in \widehat{\mathcal{GP}_{\mathcal{XX}}(\xi)}$, then the following statements are equivalent for any $m \geq 0$.*

(1) $\mathcal{GP}_{\mathcal{XX}}(\xi)$- res.dim $M \leq m$.
(2) $\Omega^n_{\mathcal{P}_{\mathcal{GP}_{\mathcal{XX}}(\xi)}}(M) \in \mathcal{GP}_{\mathcal{XX}}(\xi)$ *for any* $n \geq m$.
(3) $\Omega^n_{\mathcal{GP}_{\mathcal{XX}}(\xi)}(M) \in \mathcal{GP}_{\mathcal{XX}}(\xi)$ *for any* $n \geq m$.
(4) There exists a triangle

$$M \longrightarrow W \longrightarrow X \longrightarrow \Sigma M$$

in ξ *with* $\mathcal{X} \cap {}^{\perp}\mathcal{X}$-res.dim $W \leq m$ *and* $X \in \mathcal{GP}_{\mathcal{XX}}(\xi)$.
(5) There exists a ξ*-exact complex*

$$0 \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_1 \longrightarrow X \longrightarrow M \longrightarrow 0$$

with all $H_i \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$ *and* $X \in \mathcal{GP}_{\mathcal{XX}}(\xi)$.
(6) For any $0 \leq i \leq m$, *there exists a* ξ*-exact complex*

$$0 \longrightarrow H_m \longrightarrow H_{m-1} \longrightarrow \cdots \longrightarrow H_{i+1} \longrightarrow X \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $H_j \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$, *all* $P_t \in \mathcal{P}_{\mathcal{GP}_{\mathcal{XX}}(\xi)}$, *and* $X \in \mathcal{GP}_{\mathcal{XX}}(\xi)$.
(7) $\xi xt^n_{\xi}(M, H) = 0$ *for any* $n > m$ *and* $H \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$.
(8) $\xi xt^n_{\xi}(M, L) = 0$ *for any* $n > m$ *and* $L \in \widehat{\mathcal{X} \cap {}^{\perp}\mathcal{X}}$.
(9) M *admits a right* $\mathcal{GP}_{\mathcal{XX}}(\xi)$*-approximation* $\varphi : X \to M$, *where* φ *is* ξ*-proper epic, such that* $\mathcal{X} \cap {}^{\perp}\mathcal{X}$- res.dim Hoker $\varphi \leq m - 1$.
(10) There exist two triangles

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M$$

in ξ *such that* $X, X' \in \mathcal{GP}_{\mathcal{XX}}(\xi)$, $\mathcal{X} \cap {}^{\perp}\mathcal{X}$- res.dim $K \leq m - 1$, *and* $\mathcal{X} \cap {}^{\perp}\mathcal{X}$- res.dim $W = \mathcal{GP}_{\mathcal{XX}}(\xi)$- res.dim $W \leq m$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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