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Research article

Delay-dependent stability of highly nonlinear neutral stochastic systems with non-differentiable delay

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Abstract: The aim of this paper is to investigate the delay-dependent stability of highly nonlinear hybrid neutral stochastic differential delay equations (NSDDEs). Departing from most existing studies, the system under consideration incorporates a time-varying delay that is not required to be differentiable. A novel decomposition scheme for the drift coefficient is introduced, relaxing the conventional restrictive Lipschitz condition on the delay component. By constructing appropriate Lyapunov functionals and employing an M-matrix approach, delay-dependent conditions are derived to ensure moment stability for the considered highly nonlinear NSDDEs. Finally, an example is given to demonstrate the effectiveness of our new theory.

Keywords: neutral stochastic differential equations; M-matrix; delay-dependent stability; non-differentiable delay; Lyapunov functional

1. Introduction

Neutral stochastic differential delay equations (NSDDEs) are often used to model systems whose evolution depends on not only the present state but also involves derivatives with delays [1–3]. Stability and control are the most basic issues in analyzing stochastic differential delay equations with Markovian switching (also known as hybrid SDDEs) [4–6]. It is well known that the stability criteria can be classified into two categories: delay independent and delay dependent. The delay independent stability criteria work for any size of delays, whereas the delay-dependent stability criteria take into account the size of delays and hence are generally less conservative than the delay-independent ones. However, most studies in delay-dependent stability criteria have primarily focused on systems whose coefficients are either linear or nonlinear but bounded by linear functions [7–9]. Fei et al. [10] were the first to establish delay-dependent criteria for highly nonlinear hybrid SDDEs whose coefficients do not satisfy the linear growth condition. Shen et al. [11] further investigated delay-dependent criterion for highly nonlinear hybrid NSDDEs. However, they all required drift coefficient to be globally Lipschitz continuous in the delay component, and this restrictive condition excludes many highly nonlinear hybrid SDDEs. To overcome this deficiency, Fei et al. [12] adopted a new decomposition method to study the almost inevitable stability of highly nonlinear stochastic delay differential equations. Later, Shen et al. [13] extended this decomposition to study delay-dependent stability of neutral functional stochastic differential equations and established criteria for exponential stability. Xu and Mao [14] adopted a new decomposition method to investigate delay-dependent stability of superlinear hybrid stochastic systems with general time-varying delays.

Despite significant advances in delay-dependent stability analysis for highly nonlinear stochastic systems, current theoretical frameworks still cannot address many important classes of highly nonlinear stochastic delay differential equations. Methodological and conclusion-level limitations also persist in the literature. For example, in most of the literature, delay function is supposed to be a constant or a differentiable function, but its derivative is less than 1 (see, e.g., [10–12]). This condition has been imposed only because of the mathematical technique used but might not be a natural feature of SDDE models in the real world. For example, sawtooth delays or piece-wise constant delays often occur in sampled-data controls or network-based controls. Moreover, in the study of NSDDEs, the neutral term is often assumed to be independent of the switching mode (see, e.g., [11, 13]). Those limitations may exclude many highly nonlinear hybrid SDDEs. In this paper, we will loosen those restrictive conditions to investigate the delay-dependent stability of highly nonlinear NSDDEs.

The key contributions of this work are summarized as follows:

1. The existence, uniqueness, and boundedness of solutions for highly nonlinear hybrid NSDDEs with non-differentiable delays are established.
2. The neutral term in this study is mode-dependent, necessitating the development of new mathematical techniques to address the challenges arising from the mode-sensitive neutral term and the non-differentiable delay.
3. A novel decomposition method is introduced to relax the restrictive conditions on the drift coefficient, thereby encompassing a significantly broader class of NSDDEs.

The structure of this article is arranged as follows. In Section 2, some preliminary definitions, assumptions, and two key lemmas are presented. The existence and uniqueness of solutions of highly nonlinear NSDDEs with non-differentiable delay are discussed in Section 3. The delay-dependent stability criteria of highly nonlinear NSDDEs are established in Section 4. An example is given to illustrate effectiveness of our theory in Section 5, while the conclusion is made in Section 6.

2. Preliminaries

Notations. Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\mathrm{trace}(A^T A)}$ be its trace norm. Let $\mathbb{R}_+ = [0, \infty)$. For $\tau > 0$, denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions ξ from $[-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\xi\| = \sup_{-\tau\le s\le 0} |\xi(s)|$. If a and b are both real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$

be an m-dimensional Brownian motion defined on the probability space. Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$, given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$ which are continuously twice differentiable in x and once in t. For such a function U, we will let $U_t = \frac{\partial U}{\partial t}$, $U_x = (\frac{\partial U}{\partial x_1}, \cdots, \frac{\partial U}{\partial x_n})$, and $U_{xx} = (\frac{\partial^2 U}{\partial x_k \partial x_l})_{n\times n}$.

Consider an m-dimensional hybrid NSDDE

$$\begin{aligned} d[x(t) - D(x(t-h(t)), r(t))] = & f(x(t), x(t-h(t)), t, r(t))dt \\ & + g(x(t), x(t-h(t)), t, r(t))dB(t) \end{aligned} \tag{2.1}$$

on $t \geq 0$ with the initial data

$$\{x(t) : -\tau \leq t \leq 0\} = \{\xi(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n), r(0) = r_0 \in S, \tag{2.2}$$

where the neutral term $D : \mathbb{R}^n \times S \to \mathbb{R}^n$, the drift coefficient $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$, and the diffusion coefficient $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}$ are all Borel measurable functions and $h(t) : \mathbb{R}_+ \to [0, \tau]$ is the system delay.

Numerous academic papers commonly impose a differentiability condition on the delay $h(t)$ when dealing with delay issues using mathematical techniques (see, e.g., [15, 16]). However, this approach appears overly stringent in many practical models. Therefore, this paper aims to explore a more general scenario and, accordingly, proposes the following hypothesis.

Assumption 1. Let $h(t)$ be a Borel measurable function satisfying the following property:

$$h^* := \lim_{\Delta\to 0^+} \sup(\sup_{s\geq -\tau} \frac{\mu(I_{s,\Delta})}{\Delta}) < \infty, \tag{2.3}$$

where $\mu(\cdot)$ denotes the Lebesgue measure on the real line and $I_{s,\Delta} = \{t \in \mathbb{R}_+ | t - h(t) \in [s, s+\Delta)\}$.

This condition serves as a key prerequisite for analyzing systems with non-differentiable delays and is essential for establishing solution uniqueness and stability. In this paper, we do not consider the case where $h(t) = 0$ for all $t \geq 0$. It is worth noting that when $h(t)$ is a differentiable global Lipschitz continuous function with Lipschitz constant $\hat{h} \in (0, 1)$, it also satisfies Assumption 1, where $h^* = \frac{1}{1-\hat{h}}$. When $h(t)$ is a left-limited-right-continuous piecewise constant function $h(t) = \sum_{k=0}^{\infty}(m_1 I_{[t_k, t_{k+\frac{1}{2}})}(t) + m_2 I_{[t_{k+\frac{1}{2}}, t_{k+1})}(t)), t \geq 0$, where $0 < m_1 < m_2 < \infty$, this function is not differentiable but satisfies Assumption 1.

Lemma 1. Suppose that Assumption 1 holds. Given $T > 0$, let $\Phi : [-\tau, T] \to \mathbb{R}_+$ be a continuous function. Then

$$\int_0^T \Phi(v - h(v))dv \leq h^* \int_{-\tau}^T \Phi(v)dv. \tag{2.4}$$

This lemma provides a method to address time delays under our new Assumption 1. For further details, see Lemma 2.2 in Dong et al. [17]. Note that in (2.4), the integral of $\Phi(v)$ on the right-hand side ranges from $-\tau$ to T due to the potential zero delay $h(t)$ in this paper. Moreover, we should point out that h^* given in Assumption 1 always satisfies $h^* \geq 1$.

To ensure the existence and uniqueness of solutions to stochastic differential equations, the following local Lipschitz condition and polynomial growth restriction are required.

Assumption 2. Both coefficients f and g are locally Lipschitz continuous. Also assume that there exist constants $q_1 > 1$ and non-negative constants L_j, $\hat{L}_j$ ($j = 1, 2, 3, 4$) such that

$$|f(x, y, t, i)| \leq L_1|x| + L_2|y| + L_3|x|^{q_1} + L_4|y|^{q_1}$$

and

$$|g(x, y, t, i)|^2 \leq \hat{L}_1|x|^2 + \hat{L}_2|y|^2 + \hat{L}_3|x|^{q_1+1} + \hat{L}_4|y|^{q_1+1}$$

for all $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$.

This assumption serves as a standard condition in the analysis of highly nonlinear stochastic differential equations with delays and NSDDEs. This assumption is stricter and more structured than [10, 11], requiring the functions to vanish at the origin and explicitly separating different growth terms.

In addition to the constraints on the drift and diffusion coefficients, the Lipschitz continuity of the neutral term is a core assumption in the analysis of neutral-type stochastic systems. The precise assumption imposed on the neutral term is detailed below.

Assumption 3. There exist constants $\kappa_i \in (0, 1)$ such that

$$|D(u, i) - D(v, i)| \leq \kappa_i|u - v| \tag{2.5}$$

for all $u, v \in \mathbb{R}^n$ and $i \in S$. Moreover, we assume that $D(0, i) = 0$ for all $i \in S$ and let $\kappa = \max_{i \in S} \kappa_i$.

In the derivation of the main results, several inequalities are frequently invoked, which are listed as follows. The relevant inequalities are all mentioned in the literature [18, 19].

Lemma 2. Let Assumption 3 holds, $\lambda > 0$, $p \geq 1$, $i \in S$, and $a, b \geq 0$, then

$$(a + b)^p \leq (1 + \lambda)^{p-1}a^p + (1 + \lambda^{-1})^{p-1}b^p \tag{2.6}$$

and

$$(1 - \kappa)^{p-1}(|a|^p - \kappa|b|^p) \leq |a - D(b, i)|^p \leq (1 + \kappa)^{p-1}(|a|^p + \kappa|b|^p). \tag{2.7}$$

3. Global solution

Evidently, Assumption 2 alone is only sufficient to establish the existence of a unique maximal local solution for the hybrid NSDDEs. It fails to guarantee the uniqueness of a global solution. To address this limitation, the following conditions must be imposed.

Assumption 4. Let $\kappa h^* < 1$, and assume that, for all $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$, there exist non-negative constants $q \geq 2q_1$, $\alpha_i (i = 1, \cdots, 4)$ such that

$$(q\alpha_3 - \frac{q(q-2)\alpha_4}{q_1 + q - 1})(1-\kappa)^{q_1+q-2}(1-\kappa h^*) > \frac{q(q_1+1)\alpha_4 h^*}{q_1 + q - 1} \tag{3.1}$$

and

$$\begin{aligned}(x - D(y, i))^T f(x, y, t, i) &+ \frac{q-1}{2}|g(x, y, t, i)|^2 \\ &\leq \alpha_1 |x - D(y, i)|^2 + \alpha_2 |y|^2 - \alpha_3 |x - D(y, i)|^{q_1+1} + \alpha_4 |y|^{q_1+1}.\end{aligned} \tag{3.2}$$

It is easy to see that condition (3.2) is indeed more advanced than the conventional linear growth condition. This assumption incorporates hybrid switching dynamics and mode-dependent neutral delay terms, making it technically stronger and more adapted to our system class.

Theorem 1. Suppose that Assumptions 1–4 are satisfied. Then, for any given initial condition (2.2), the NSDDE (2.1) admits a unique global solution $x(t)$ for all $t \geq 0$ with the property that

$$\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^q < \infty, \quad \forall T > 0. \tag{3.3}$$

Proof. To make it more understandable, we will divide it into three steps.

Step 1. Fix the initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$ and $r_0 \in S$. By the local Lipschitz continuity of the system coefficients and Theorem 7.12 in the literature (see [20]), there exists a unique maximal local solution $x(t)$ on the interval $t \in [0, \sigma_e)$, where σ_e denotes the explosion time.

Let $k_0 > 0$ be sufficiently large for $k_0 > \|\xi\|$. For each integer $k \geq k_0$, define the stopping time $\sigma_k = \inf\left\{t \in [0, \sigma_e) \,\middle|\, |x(t)| \geq k\right\}$. Clearly, σ_k is non-decreasing in k and $\sigma_\infty := \lim_{k\to\infty} \sigma_k \leq \sigma_e$ a.s. To prove that $x(t)$ is a global solution, it suffices to show that $\sigma_\infty = \infty$ a.s.

Now, for any $k \geq k_0$, $t \in [0, T]$ and $T > 0$, set $\psi(t) = x(t) - D(x(t - h(t)), r(t))$, and we derive from the Itô formula and condition (3.2) that

$$\begin{aligned}&\mathbb{E}|\psi(t \wedge \sigma_k)|^q - |\psi(0)|^q \\ &\leq \mathbb{E}\int_0^{t\wedge\sigma_k} [q|\psi(s)|^{q-2}(|\psi(s)|^T f(x(s), x(s-h(s)), s, r(s)) + \frac{q-1}{2}|g(x(s), x(s-h(s)), s, r(s))|^2)]ds \\ &\leq \mathbb{E}\int_0^{t\wedge\sigma_k} q|\psi(s)|^{q-2}(\alpha_1|\psi(s)|^2 + \alpha_2|x(s-h(s))|^2 - \alpha_3|\psi(s)|^{q_1+1} + \alpha_4|x(s-h(s))|^{q_1+1})ds.\end{aligned} \tag{3.4}$$

By Lemma 2 and the Young inequality, we have

$$\begin{aligned}q|\psi(t)|^{q-2}(\alpha_1|\psi(t)|^2 &+ \alpha_2|x(t-h(t))|^2 - \alpha_3|\psi(t)|^{q_1+1} + \alpha_4|x(t-h(t))|^{q_1+1}) \\ &\leq \zeta_1|x(t)|^q + \zeta_2|x(t-h(t))|^q - \zeta_3|x(t)|^{q_1+q-1} + \zeta_4|x(t-h(t))|^{q_1+q-1},\end{aligned} \tag{3.5}$$

where

$$\zeta_1 = (q\alpha_1 + (q-2)\alpha_2)(1+\kappa)^{q-1}, \quad \zeta_2 = ((q\alpha_1 + (q-2)\alpha_2)(1+\kappa)^{q-1}\kappa + 2\alpha_2),$$
$$\zeta_3 = (q\alpha_3 - \frac{q(q-2)\alpha_4}{q_1+q-1})(1-\kappa)^{q_1+q-2}, \text{and } \zeta_4 = (q\alpha_3 - \frac{q(q-2)\alpha_4}{q_1+q-1})(1-\kappa)^{q_1+q-2}\kappa + \frac{q(q_1+1)\alpha_4}{q_1+q-1}.$$

Step 2. Combining (3.4) and (3.5) with Lemma 1 and condition (3.1), we derive

$$\begin{aligned}\mathbb{E}|\psi(t\wedge\sigma_k)|^q-|\psi(0)|^q &\le \mathbb{E}\int_{-\tau}^{0} h^*(\zeta_2|x(s)|^q+\zeta_4|x(s)|^{q_1+q-1})ds\\ &+\mathbb{E}\int_{0}^{t\wedge\sigma_k} \zeta_1|x(s)|^q+\zeta_2h^*|x(s)|^q-\zeta_3|x(s)|^{q_1+q-1}+\zeta_4h^*|x(s)|^{q_1+q-1}ds\\ &\le K_1t+\mathbb{E}\int_{-\tau}^{0} h^*(\zeta_2|x(s)|^q+\zeta_4|x(s)|^{q_1+q-1})ds,\end{aligned}$$

where $K_1=\sup_{u\ge 0}(-(\zeta_3-\zeta_4h^*)u^{q_1+q-1}+(\zeta_1+\zeta_2h^*)u^q)$. Note that $q_1+q-1>q$. Due to the properties of a univariate function, it is easy to obtain that the $K_1<\infty$. This means that for $t\in[0,T]$,

$$\mathbb{E}|\psi(t\wedge\sigma_k)|^q\le K_2(T), \tag{3.6}$$

where $K_2(T)=|\psi(0)|^q+K_1T+\mathbb{E}\int_{-\tau}^{0} h^*(\zeta_2|x(s)|^q+\zeta_4|x(s)|^{q_1+q-1})ds$ is finite. On the other hand, in light of the definition of σ_k, we derive that

$$\begin{aligned}|x(\sigma_k)-D(x(\sigma_k-h(\sigma_k)),r(\sigma_k))|I_{\{\sigma_k\le T\}} &\ge (|x(\sigma_k)|-|D(x(\sigma_k-h(\sigma_k)),r(\sigma_k))|)I_{\{\sigma_k\le T\}}\\ &\ge (|x(\sigma_k)|-\kappa|x(\sigma_k-h(\sigma_k))|)I_{\{\sigma_k\le T\}}\ge(1-\kappa)k\,I_{\{\sigma_k\le T\}},\end{aligned}$$

thus

$$[(1-\kappa)k]^q\mathbb{P}\{\sigma_k\le T\}\le K_2(T). \tag{3.7}$$

Letting $k\to\infty$, it follows that $\mathbb{P}\{\sigma_\infty\le T\}=0$, thereby yielding $\mathbb{P}\{\sigma_\infty>T\}=1$. Given the arbitrariness of $T\ge 0$, we conclude that $\mathbb{P}\{\sigma_\infty=\infty\}=1$ as desired.

Step 3. Taking the limit as $k\to\infty$ in (3.6) gives

$$\sup_{0\le t\le T}\mathbb{E}|\psi(t)|^q\le K_2(T).$$

With the aid of Lemma 2, we have that

$$\begin{aligned}\sup_{0\le t\le T}\mathbb{E}|x(t)|^q &\le \kappa\sup_{0\le t\le T}\mathbb{E}|x(t-h(t))|^q+\frac{\sup_{0\le t\le T}\mathbb{E}|\psi(t)|^q}{(1-\kappa)^{q-1}}\\ &\le \kappa\sup_{0\le t\le T}\mathbb{E}|x(t-h(t))|^q+\frac{K_2(T)}{(1-\kappa)^{q-1}}\le\kappa\sup_{-\tau\le t\le T}\mathbb{E}|x(t)|^q+\frac{K_2(T)}{(1-\kappa)^{q-1}}\\ &\le \kappa\|\xi\|^q+\kappa\sup_{0\le t\le T}\mathbb{E}|x(t)|^q+\frac{K_2(T)}{(1-\kappa)^{q-1}},\end{aligned}$$

and this immediately yields that

$$\sup_{0\le t\le T}\mathbb{E}|x(t)|^q\le\frac{\kappa}{1-\kappa}\|\xi\|^q+\frac{K_2(T)}{(1-\kappa)^q}. \tag{3.8}$$

This implies the required assertion (3.3) immediately. □

Theorem 1 proves the existence, uniqueness, and moment boundedness of the global solution. The moment bound given in Eq (3.3) provides the foundation for the subsequent stability analysis.

4. Delay-dependent stability

Based on the results obtained in Section 3, this paper further turns to the core research objective, namely establishing stability criteria dependent on time delays. This section will introduce the corresponding technical tools and methodological framework.

In the paper of Shen et al. [13], the core hypothesis is that the function f can be decomposed into two components. Although this assumption provides some guidance in certain specific scenarios, it also reveals certain limitations. Therefore, in the discussion of this paper, we will adopt a new decomposition method [14] to study delay-dependent stability of NSDDEs.

Assumption 5. Assume that the drift coefficient f can be decomposed as

$$f(x, y, t, i) = f_1(x, y, t, i) + f_2(x, y, t, i). \tag{4.1}$$

Furthermore, we can find four constants $r \in [0, \frac{q_1-1}{2}]$, $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, and $\epsilon_3 \geq 0$. Among them, ϵ_1, ϵ_2, and ϵ_3 do not all equal zero simultaneously, such that

$$|f_1(x, y, t, i) - f_1(x, x, t, i)| \leq (\epsilon_1 + \epsilon_2|x|^r + \epsilon_3|y|^r)|x - y| \tag{4.2}$$

for all $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$.

It is easy to see that when $\epsilon_2 = \epsilon_3 = 0$ or $r = 0$, condition (4.2) becomes condition (3.7) in [10] and condition (3.6) in [11]. This decomposition achieves the separation of nonlinear terms, facilitating refined analysis. For example, if $f(x, y, t, i) = x + 2y^2 - 2x^3$, then $f_1(x, y, t, i) = 2y^2$ with $r = 1, \epsilon_1 = 0, \epsilon_2 = 2, \epsilon_3 = 2$, and $f_2(x, y, t, i) = x - 2x^3$.

Based on this decomposition, the drift coefficient can be rewritten as:

$$f(x, y, t, i) = (f_1(x, y, t, i) - f_1(x, x, t, i)) + (f_1(x, x, t, i) + f_2(x, y, t, i)). \tag{4.3}$$

For the deviation term, it follows from condition (4.2) that

$$\begin{aligned}
&|f_1(x, y, t, i) - f_1(x, x, t, i)| \\
&\leq \epsilon_1|x - y| + \epsilon_2 \sqrt{|x|^{2r}(|x| + |y|)}\sqrt{|x - y|} + \epsilon_3 \sqrt{|y|^{2r}(|x| + |y|)}\sqrt{|x - y|} \\
&\leq (\epsilon_1 + \frac{\epsilon_2 + \epsilon_3}{2\delta_1})|x - y| + \frac{\epsilon_2(4r + 1) + \epsilon_3}{4r + 2}\delta_1|x|^{2r+1} + \frac{\epsilon_2 + \beta_3(4r + 1)}{4r + 2}\delta_1|y|^{2r+1},
\end{aligned} \tag{4.4}$$

where $\delta_1 > 0$ is a tunable parameter. For the residual term, the following assumption is given.

Assumption 6. Suppose there exist real constants $a_i, \bar{a}_i$, positive constants $c_i, \bar{c}_i$, and non-negative constants $b_i, \bar{b}_i, d_i, \bar{d}_i$ such that

$$\begin{aligned}
&(x - D(y, i))^T(f_1(x, x, t, i) + f_2(x, y, t, i)) + \frac{1}{2}|g(x, y, t, i)|^2 \\
&\qquad \leq a_i|x - D(y, i)|^2 + b_i|y|^2 - c_i|x - D(y, i)|^{q_1+1} + d_i|y|^{q_1+1}
\end{aligned} \tag{4.5}$$

and

$$(x - D(y, i))^T(f_1(x, x, t, i) + f_2(x, y, t, i)) + \frac{q_1}{2}|g(x, y, t, i)|^2$$

$$\leq \bar{a}_i|x - D(y,i)|^2 + \bar{b}_i|y|^2 - \bar{c}_i|x - D(y,i)|^{q_1+1} + \bar{d}_i|y|^{q_1+1} \tag{4.6}$$

for all $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$, while both

$$A = -2\text{diag}(a_1, a_2, \ldots, a_N) - Q,\ \bar{A} = -(q_1 + 1)\text{diag}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N) - Q$$

are non-singular M-matrices.

The M-matrix methodology offers a distinct advantage in analyzing the stability of nonlinear stochastic systems by more effectively capturing and quantifying the influence of system coefficients.

The interplay between Assumptions 5 and 6 form the backbone of our analysis. Next, we leverage these conditions to construct a Lyapunov functional and establish the stability framework for neutral-type stochastic delay systems.

Define $(\eta_1, \eta_2, \ldots, \eta_N)^T := A^{-1}(1, 1, \ldots, 1)^T$ and $(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_N)^T := \bar{A}^{-1}(1, 1, \ldots, 1)^T$. Since A and $\bar{A}$ are non-singular M-matrices, all components η_i and $\bar{\eta}_i$ are positive. We now define Lyapunov function $U : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}_+$ by

$$U(x, t, i) = \eta_i|x|^2 + \bar{\eta}_i|x|^{q_1+1}, (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S \tag{4.7}$$

and the function $LU : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$ by

$$\begin{aligned}
&LU(x - D(y,i), y, t, i)\\
&= U_t(x - D(y,i), t, i) + U_x(x - D(y,i), t, i)(f_1(x, x, t, i) + f_2(x, y, t, i))\\
&+ \frac{1}{2}\text{trace}[g^T(x, y, t, i)U_{xx}(x - D(y,i), t, i)g(x, y, t, i)] + \sum_{j=1}^{N}\gamma_{ij}U(x - D(y,i), t, j).
\end{aligned}$$

By definition of $U(x, t, i)$, we have

$$\begin{aligned}
&LU(x - D(y,i), y, t, i) = 2\eta_i[(x - D(y,i))^T(f_1(x, x, t, i) + f_2(x, y, t, i)) + \frac{1}{2}|g(x, y, t, i)|^2]\\
&+ (q_1 + 1)\bar{\eta}_i|x - D(y,i)|^{q_1-1}[(x - D(y,i))^T(f_1(x, x, t, i) + f_2(x, y, t, i)) + \frac{q_1}{2}|g(x, y, t, i)|^2]\\
&+ \sum_{j=1}^{N}\gamma_{ij}\eta_j|x - D(y,i)|^2 + \sum_{j=1}^{N}\gamma_{ij}\bar{\eta}_j|x - D(y,i)|^{q_1+1}.
\end{aligned}$$

By virtue of Assumption 6 and the Young inequality, it can be deduced that

$$\begin{aligned}
&LU(x - D(y,i), y, t, i) \leq -|x - D(y,i)|^2 + 2\eta_i b_i|y|^2\\
&- ((q_1 + 1)\bar{\eta}_i\bar{c}_i - \frac{q_1^2 - 1}{2q_1}\bar{\eta}_i\bar{d}_i)|x - D(y,i)|^{2q_1} + \frac{(q_1 + 1)^2}{2q_1}\bar{\eta}_i\bar{d}_i|y|^{2q_1}\\
&- (2\eta_i c_i + 1 - (q_1 - 1)\bar{\eta}_i\bar{b}_i)|x - D(y,i)|^{q_1+1} + (2\eta_i d_i + 2\bar{\eta}_i\bar{b}_i)|y|^{q_1+1}.
\end{aligned} \tag{4.8}$$

To analyze the delay-dependent stability in this study, we primarily adopt the Lyapunov functional approach. For $t \geq 0$, we define a segment $\bar{x}_t = \{x(t + s) \mid -2\tau \leq s \leq 0\}$ and construct the following Lyapunov functional:

$$V(\bar{x}_t, t, r(t)) = U(x(t), t, r(t)) + \int_{-2\tau}^{0}\int_{t+s}^{t}(\rho_1|x(v)|^2 + \rho_2|x(v)|^{2q_1} + \rho_3|x(v)|^{q_1+1})dvds, \tag{4.9}$$

where ρ_j $(j = 1, 2, 3)$ are positive constants to be determined later. It can be shown that $V(\bar{x}_t, t, r(t))$ is an Itô process for $t \geq 0$, with its Itô differential given by

$$dV(\bar{x}_t, t, r(t)) = \mathcal{L}Vdt + dM(t), \tag{4.10}$$

where the infinitesimal generator $\mathcal{L}V$ is expressed as

$$\begin{aligned}
&\mathcal{L}V(\bar{x}_t, t, r(t)) = LU(x(t) - D(x(t-h(t)), r(t)), x(t-h(t)), t, r(t)) \\
&+ U_x(x(t) - D(x(t-h(t)), r(t)), t, r(t))(f_1(x(t), x(t-h(t)), t, r(t)) - f_1(x(t), x(t), t, r(t))) \\
&+ 2\rho_1\tau|x(t)|^2 + 2\rho_2\tau|x(t)|^{2q_1} + 2\rho_3\tau|x(t)|^{q_1+1} - \rho_1\int_{t-2\tau}^{t}|x(v)|^2dv \\
&- \rho_2\int_{t-2\tau}^{t}|x(v)|^{2q_1}dv - \rho_3\int_{t-2\tau}^{t}|x(v)|^{q_1+1}dv,
\end{aligned} \tag{4.11}$$

where $M(t)$ denotes a continuous martingale that vanishes at $t = 0$.

Before delving into the stability analysis, two indispensable lemmas for the subsequent proof are first presented.

Lemma 3. Let all conditions in Theorem 1 hold and $0 < \kappa < \frac{1}{2h^*}$. Then, for any $t \geq 2\tau$, we have

$$\begin{aligned}
\int_0^t \mathbb{E}|x(s) - x(s-h(s))|^2ds &\leq C + (G_1\tau + G_2)\int_0^t\int_{s-2\tau}^{s}\mathbb{E}|x(u)|^2duds \\
&+ (G_3\tau + G_4\tau)\int_0^t\int_{s-2\tau}^{s}\mathbb{E}|x(u)|^{2q_1}duds \\
&+ (G_5\tau + G_6)\int_0^t\int_{s-2\tau}^{s}\mathbb{E}|x(u)|^{q_1+1}duds + \frac{8\bar{\gamma}\tau\kappa h^*}{1-2\kappa h^*}\int_0^t\mathbb{E}|x(s)|^2ds,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
&G_1 = 2\Lambda(L_1^2 + L_2^2h^*), G_2 = \Lambda(\hat{L}_1 + \hat{L}_2h^*), G_3 = 2\Lambda L_3^2, G_4 = 2\Lambda L_4^2h^*, \\
&G_5 = 2\Lambda(L_1L_3 + \frac{L_1L_4 + L_2L_3q_1}{q_1+1} + L_2L_4h^* + \frac{L_2L_3 + L_1L_4q_1}{q_1+1}h^*), \\
&G_6 = \Lambda(\hat{L}_3 + \hat{L}_4h^*), \Lambda = \frac{2}{(1-\kappa)(1-2\kappa h^*)}, \text{and } \bar{\gamma} = \max_{i\in S}\{-\gamma_{ii}\}.
\end{aligned}$$

Additionally, unless otherwise specified, all constants throughout the article are denoted by C.

Proof. To simplify the presentation and avoid unnecessary complexity, let $R(t) = \int_{t-h(t)}^{t} f(x(u), x(u-h(u)), u, r(u))\,du + \int_{t-h(t)}^{t} g(x(u), x(u-h(u)), u, r(u))\,dB(u)$, the processing method for the neutral term based on Theorem 3.4 of Shen et al. [11]. We can easily get it from Eq (2.1):

$$\begin{aligned}
&\mathbb{E}|x(s) - x(s-h(s))|^2 \leq (1 + \frac{1}{\lambda})\mathbb{E}|R(s)|^2 \\
&+ 2(1+\lambda)\mathbb{E}|D(x(s-h(s)), r(s)) - D(x(s-h(s)), r(s-h(s)))|^2 \\
&+ 2(1+\lambda)\mathbb{E}|D(x(s-h(s)), r(s-h(s))) - D(x(s-2h(s)), r(s-h(s)))|^2
\end{aligned}$$

$$\leq 2(1+\frac{1}{\lambda})\mathbb{E}\int_{s-h(s)}^{s} W(u)du$$
$$+ 2(1+\lambda)\mathbb{E}|D(x(s-h(s)), r(s)) - D(x(s-h(s)), r(s-h(s)))|^2$$
$$+ 2(1+\lambda)\kappa^2\mathbb{E}\,|x(s-h(s)) - x(s-2h(s))|^2\,,$$

where $W(t) = \tau|f(x(t), x(t-h(t)), t, r(t))|^2 + |g(x(t), x(t-h(t)), t, r(t))|^2$. Moreover, by Assumption 3 and using the asymptotic properties of the Markov chain (see, e.g, Li et al. [21]; Lemma 3.4), we have

$$\begin{aligned}
&\mathbb{E}|D(x(s-h(s)), r(s)) - D(x(s-h(s)), r(s-h(s)))|^2\\
&= \mathbb{E}[\mathbb{E}|D(x(s-h(s)), r(s)) - D(x(s-h(s)), r(s-h(s)))|^2|\mathcal{F}_{s-h(s)}]\\
&\leq \mathbb{E}[4\kappa^2|x(s-h(s))|^2\mathbb{E}(I_{r(s)\neq r(s-h(s))}|\mathcal{F}_{s-h(s)})]\\
&= \mathbb{E}[4\kappa^2|x(s-h(s))|^2\mathbb{E}(\sum_{j\in S} I_{r(s-h(s))=j}I_{r(s)\neq j}|\mathcal{F}_{s-h(s)})]\\
&= \mathbb{E}[4\kappa^2|x(s-h(s))|^2\sum_{j\in S} I_{r(s-h(s))=j}\times\mathbb{P}(r(s)\neq j|r(s-h(s))=j)]\\
&\leq \mathbb{E}[4\kappa^2|x(s-h(s))|^2(1-e^{-\bar{\gamma}\tau})]\\
&\leq 4\bar{\gamma}\tau\kappa^2\mathbb{E}|x(s-h(s))|^2.
\end{aligned}$$

Setting $\lambda = \frac{1}{\kappa} - 1$, it follows from Lemma 1 that

$$\begin{aligned}
\int_0^t \mathbb{E}|x(s)-x(s-h(s))|^2ds &\leq \frac{2}{1-\kappa}\mathbb{E}\int_0^t\int_{s-h(s)}^{s}\mathbb{E}W(u)\,du\,ds + 8\bar{\gamma}\tau\kappa h^*\int_0^t\mathbb{E}|x(s)|^2ds\\
&+ 2\kappa h^*\int_0^t\mathbb{E}|x(s)-x(s-h(s))|^2ds + 8\bar{\gamma}\tau\kappa h^*\int_{-\tau}^0\mathbb{E}|x(s)|^2ds\\
&+ 2\kappa h^*\int_{-\tau}^0\mathbb{E}|x(s)-x(s-h(s))|^2ds.
\end{aligned}$$

We can obtain

$$\int_0^t \mathbb{E}|x(s)-x(s-h(s))|^2ds \leq C + \Lambda\mathbb{E}\int_0^t\int_{s-h(s)}^{s}\mathbb{E}W(u)duds + \frac{8\bar{\gamma}\tau\kappa h^*}{1-2\kappa h^*}\int_0^t\mathbb{E}|x(s)|^2ds. \tag{4.13}$$

By Assumption 2, we can have

$$\begin{aligned}
W(t) \leq&(2L_1^2\tau+\hat{L}_1)|x(t)|^2 + (2L_2^2\tau+\hat{L}_2)|x(t-h(t))|^2 + 2L_3^2\tau|x(t)|^{2q_1} + 2L_4^2\tau|x(t-h(t))|^{2q_1}\\
&+ (2(L_1L_3 + \frac{L_1L_4+L_2L_3q_1}{q_1+1})\tau + \hat{L}_3)|x(t)|^{q_1+1}\\
&+ (2(L_2L_4 + \frac{L_2L_3+L_1L_4q_1}{q_1+1})\tau + \hat{L}_4)|x(t-h(t))|^{q_1+1}.
\end{aligned} \tag{4.14}$$

By applying Lemma 1 and substituting into (4.13), we can obtain (4.12), completing the proof. □

Lemma 4. Set three positive free parameters $\delta_1, \delta_2, \delta_3$, where δ_1 has been already given in (4.4). Then under Assumption 5, for any $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$, we have

$$U_x(x - D(y, i), t, i)(f_1(x, y, t, i) - f_1(x, x, t, i))$$

$$
\begin{aligned}
&\leq (\eta_M\delta_2 + \Pi_1\delta_1)|x - D(y,i)|^2 + \Pi_2\delta_1|x|^2 + \Pi_3\delta_1|y|^2 \\
&+ (\frac{q_1+1}{2}\bar{\eta}_M\delta_3 + \Pi_5\delta_1)|x - D(y,i)|^{2q_1} + \Pi_4\delta_1|x|^{2q_1} + \Pi_6\delta_1|y|^{2q_1} \\
&+ (\Pi_1 + \Pi_5)\delta_1|x - D(y,i)|^{q_1+1} + (\Pi_2 + \Pi_4)\delta_1|x|^{q_1+1} + (\Pi_3 + \Pi_6)\delta_1|y|^{q_1+1} \\
&+ (\epsilon_1 + \frac{\epsilon_2+\epsilon_3}{2\delta_1})^2(\frac{\eta_M}{\delta_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\delta_3})|x-y|^2, \qquad (4.15)
\end{aligned}
$$

where $\eta_M = \max_{i\in S}\eta_i, \bar{\eta}_M = \max_{i\in S}\bar{\eta}_i$, $\Pi_1 = \frac{\epsilon_2+\epsilon_3}{r+1}\eta_M$, $\Pi_2 = \frac{\epsilon_2(4r+1)+\epsilon_3}{2r+2}\eta_M$, $\Pi_3 = \frac{\epsilon_2+\epsilon_3(4r+1)}{2r+2}\eta_M$, $\Pi_4 = \frac{(q_1+1)(\epsilon_2(4r+1)+\epsilon_3)}{2(q_1+2r+1)}\bar{\eta}_M$, $\Pi_5 = \frac{q_1(q_1+1)(\epsilon_2+\epsilon_3)}{q_1+2r+1}\bar{\eta}_M$, and $\Pi_6 = \frac{(q_1+1)(\epsilon_2+\epsilon_3(4r+1))}{2(q_1+2r+1)}\bar{\eta}_M$.

Proof. Recalling the estimation of $f_1(x,y,t,i) - f_1(x,x,t,i)$ in (4.4), we have, for any $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$,

$$
\begin{aligned}
U_x(x - D(y,i),t,i)(f_1(x,y,t,i) - f_1(x,x,t,i)) &\leq (2\eta_M|x - D(y,i)| + (q_1+1)\bar{\eta}_M|x - D(y,i)|^{q_1}) \\
&((\epsilon_1 + \frac{\epsilon_2+\epsilon_3}{2\delta_1})|x-y| + \frac{\epsilon_2(4r+1)+\epsilon_3}{4r+2}\delta_1|x|^{2r+1} + \frac{\epsilon_2+\epsilon_3(4r+1)}{4r+2}\delta_1|y|^{2r+1}). \qquad (4.16)
\end{aligned}
$$

By the elementary inequality and the Young inequality, we can have

$$
\begin{aligned}
&U_x(x - D(y,i),t,i)(f_1(x,y,t,i) - f_1(x,x,t,i)) \\
&\leq \eta_M\delta_2|x - D(y,i)|^2 + \frac{q_1+1}{2}\bar{\eta}_M\delta_3|x - D(y,i)|^{2q_1} + \Pi_1\delta_1|x - D(y,i)|^{2r+2} \\
&+ \Pi_5\delta_1|x - D(y,i)|^{q_1+2r+1} + \Pi_2\delta_1|x|^{2r+2} + \Pi_3\delta_1|y|^{2r+2} + \Pi_4\delta_1|x|^{q_1+2r+1} \\
&+ \Pi_6\delta_1|y|^{q_1+2r+1} + (\epsilon_1 + \frac{\epsilon_2+\epsilon_3}{2\delta_1})^2(\frac{\eta_M}{\delta_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\delta_3})|x-y|^2,
\end{aligned}
$$

where $\Pi_1 - \Pi_6$ have been given before. Observing that $|x - D(y,i)|^{2r+2} \leq |x - D(y,i)|^2 + |x - D(y,i)|^{q_1+1}$ and $|x - D(y,i)|^{q_1+2r+1} \leq |x - D(y,i)|^{q_1+1} + |x - D(y,i)|^{2q_1}$, since $0 \leq r \leq \frac{q_1-1}{2}$, which is required in Assumption 5, we obtain inequality (4.15). □

Now, we introduce a method to determine the value of τ^*, that is, the upper bound of τ. Before proceeding, to keep the article concise and clear, we first present the following assumptions.

Assumption 7. Assume that the following three numbers $N_j (j = 1,2,3)$ are positive:

$$
\begin{aligned}
N_1 &= (1-\kappa)(1-h^*\kappa) - \omega_1 h^*, \\
N_2 &= (1-\kappa)^{2q_1-1}(1-h^*\kappa)\omega_2 - ((1-\kappa)^{2q_1-1}(1-h^*\kappa)\frac{q_1-1}{2q_1} + \frac{q_1+1}{2q_1}h^*)\omega_3, \\
\text{and } N_3 &= (1-\kappa)^{q_1}(1-h^*\kappa)(\omega_4+1) - ((1-\kappa)^{q_1}(1-h^*\kappa)\frac{q_1-1}{2} + h^*)\omega_6 - h^*\omega_5,
\end{aligned}
$$

where $\omega_1 = 2\max_{i\in S}\eta_i b_i$, $\omega_2 = (q_1+1)\min_{i\in S}\bar{\eta}_i\bar{c}_i$, $\omega_3 = (q_1+1)\max_{i\in S}\bar{\eta}_i\bar{d}_i$, $\omega_4 = 2\min_{i\in S}\eta_i c_i$, $\omega_5 = 2\max_{i\in S}\eta_i d_i$, and $\omega_6 = 2\max_{i\in S}\bar{\eta}_i\bar{b}_i$.

We now define a domain $\mathbb{A}$ on $\mathbb{R}_+^3$ by

$$
\mathbb{A} = \{(\delta_1,\delta_2,\delta_3) \mid \delta_1 > 0, \delta_2 > 0, \delta_3 > 0,
$$

$$(1-\kappa)(1-h^*\kappa)\eta_M\delta_2 + ((1-\kappa)(1-h^*\kappa)\Pi_1 + \Pi_2 + \Pi_3 h^*)\,\delta_1 < N_1,$$
$$(1-\kappa)^{2q_1-1}(1-h^*\kappa)\frac{q_1+1}{2}\bar{\eta}_M\delta_3 + \Big((1-\kappa)^{2q_1-1}(1-h^*\kappa)\Pi_5 + \Pi_4 + \Pi_6 h^*\Big)\delta_1 < N_2,$$
$$\text{and } ((1-\kappa)^{q_1}(1-h^*\kappa)(\Pi_1+\Pi_5) + \Pi_2 + \Pi_3 h^* + \Pi_4 + \Pi_6 h^*)\,\delta_1 < N_3\}$$

and four functions on $\mathbb{A}$ by

$$\varphi_1(\delta_1,\delta_2,\delta_3) = N_1 - (1-\kappa)(1-h^*\kappa)\eta_M\delta_2 - ((1-\kappa)(1-h^*\kappa)\Pi_1 + \Pi_2 + \Pi_3 h^*)\,\delta_1,$$
$$\varphi_2(\delta_1,\delta_2,\delta_3) = N_2 - (1-\kappa)^{2q_1-1}(1-h^*\kappa)\frac{q_1+1}{2}\bar{\eta}_M\delta_3 - \Big((1-\kappa)^{2q_1-1}(1-h^*\kappa)\Pi_5 + \Pi_4 + \Pi_6 h^*\Big)\delta_1,$$
$$\varphi_3(\delta_1,\delta_2,\delta_3) = N_3 - ((1-\kappa)^{q_1}(1-h^*\kappa)(\Pi_1+\Pi_5) + \Pi_2 + \Pi_3 h^* + \Pi_4 + \Pi_6 h^*)\,\delta_1,$$
$$\text{and } \varphi_4(\delta_1,\delta_2,\delta_3) = (\epsilon_1 + \frac{\epsilon_2+\epsilon_3}{2\delta_1})^2(\frac{\eta_M}{\delta_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\delta_3}).$$

Then τ^* is given by

$$\tau^* = \sup_{(\delta_1,\delta_2,\delta_3)\in\mathbb{A}} \varphi(\delta_1,\delta_2,\delta_3),$$

where

$$\varphi(\delta_1,\delta_2,\delta_3) = \frac{-(G_2 + \frac{4\bar{\gamma}\kappa h^*}{1-2\kappa h^*}) + \sqrt{(G_2 + \frac{4\bar{\gamma}\kappa h^*}{1-2\kappa h^*})^2 + 4G_1\frac{\varphi_1(\delta_1,\delta_2,\delta_3)}{2\varphi_4(\delta_1,\delta_2,\delta_3)}}}{2G_1} \wedge \sqrt{\frac{1}{G_3+G_4}\frac{\varphi_2(\delta_1,\delta_2,\delta_3)}{2\varphi_4(\delta_1,\delta_2,\delta_3)}} \wedge \frac{-G_6 + \sqrt{G_6{}^2 + 4G_5\frac{\varphi_3(\delta_1,\delta_2,\delta_3)}{2\varphi_4(\delta_1,\delta_2,\delta_3)}}}{2G_5}. \tag{4.17}$$

Under the condition that Assumption 5 specifies ϵ_1, ϵ_2, and ϵ_3 cannot be simultaneously zero, the strict positivity of $\varphi_4(\delta_1,\delta_2,\delta_3)$ implies that τ^* must be finite. Further analysis reveals that as $(\delta_1,\delta_2,\delta_3)$ approaches the boundary of $\mathbb{A}$, the continuous function $\varphi(\delta_1,\delta_2,\delta_3)$ decreases toward zero. Thus, there exists a point $(\hat{\delta_1},\hat{\delta_2},\hat{\delta_3})$ in $\mathbb{A}$ satisfying

$$\tau^* = \max_{(\delta_1,\delta_2,\delta_3)\in\mathbb{A}} \varphi(\delta_1,\delta_2,\delta_3) = \varphi(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3).$$

We can find that the continuous function φ tends to zero as $(\delta_1,\delta_2,\delta_3)$ approaches the boundary of $\mathbb{A}$. As a result, there exists $(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3) \in \mathbb{A}$ such that $\tau^* = \varphi(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3)$.

Subsequently, the free parameters δ_1, δ_2, and δ_3 are fixed as $\hat{\delta}_1$, $\hat{\delta}_2$, and $\hat{\delta}_3$, respectively. For notational simplicity, we denote $\hat{\varphi}_1 = \varphi_1(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3)$, $\hat{\varphi}_2 = \varphi_2(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3)$, $\hat{\varphi}_3 = \varphi_3(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3)$, and $\hat{\varphi}_4 = \varphi_4(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3)$.

Theorem 2. Under the conditions in Theorem 1, together with Assumptions 5 and 7, there is a positive number τ^* such that, for any initial data (2.2), the solution of hybrid NSDDE (2.1) has the properties that

$$\int_0^\infty \mathbb{E}|x(s)|^{2q_1} ds < \infty \tag{4.18}$$

and

$$\sup_{0\le t<\infty} \mathbb{E}|x(t)|^{q_1+1} < \infty \tag{4.19}$$

as long as $\tau < \tau^*$.

Proof. To make it more understandable, we will divide it into three steps.

Step 1. Arbitrarily fix the initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$ and $r_0 \in S$. We deduce that

$$\mathbb{E}V(\bar{x}_{t\wedge\sigma_k}, t, r(t \wedge \sigma_k)) = V(\bar{x}_0, 0, r(0)) + \mathbb{E}\int_0^{t\wedge\sigma_k} \mathcal{L}V(\bar{x}_s, s, r(s))ds.$$

Taking the limit as $k \to \infty$ and invoking the Fatou lemma, the dominated convergence theorem yields that

$$\mathbb{E}V(\bar{x}_t, t, r(t)) = V(\bar{x}_0, 0, r(0)) + \mathbb{E}\int_0^{t} \mathcal{L}V(\bar{x}_s, s, r(s))ds. \tag{4.20}$$

By inequalities (4.8), (4.15) and Assumption 7, it can be readily verified from (4.11) that

$$\begin{aligned}\mathcal{L}V(\bar{x}_t,t, r(t)) \le& -(1 - \eta_M\hat{\delta}_2 - \Pi_1\hat{\delta}_1)|x(t) - D(x(t - h(t), r(t))|^2\\ &+ (\Pi_2\hat{\delta}_1 + 2\rho_1\tau)|x(t)|^2 + (\Pi_3\hat{\delta}_1 + \omega_1)|x(t - h(t))|^2\\ &- (\omega_2 - \frac{q_1-1}{2q_1}\omega_3 - \frac{q_1+1}{2}\bar{\eta}_M\hat{\delta}_3 - \Pi_5\hat{\delta}_1)|x(t) - D(x(t - h(t), r(t))|^{2q_1}\\ &+ (\Pi_4\hat{\delta}_1 + 2\rho_2\tau)|x(t)|^{2q_1} + (\Pi_6\hat{\delta}_1 + \frac{q_1+1}{2q_1}\omega_3)|x(t - h(t))|^{2q_1}\\ &- (\omega_4 + 1 - \frac{q_1-1}{2}\omega_6 - (\Pi_1 + \Pi_5)\hat{\delta}_1)|x(t) - D(x(t - h(t), r(t))|^{q_1+1}\\ &+ ((\Pi_2 + \Pi_4)\hat{\delta}_1 + 2\rho_3\tau)|x(t)|^{q_1+1} + ((\Pi_3 + \Pi_6)\hat{\delta}_1 + \omega_5 + \omega_6)|x(t - h(t))|^{q_1+1}\\ &+ (\epsilon_1 + \frac{\epsilon_2+\epsilon_3}{2\delta_1})^2(\frac{\eta_M}{\delta_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\delta_3})|x(t) - x(t - h(t))|^2 - \rho_1\int_{t-2\tau}^{t}|x(v)|^2dv\\ &- \rho_2\int_{t-2\tau}^{t}|x(v)|^{2q_1}dv - \rho_3\int_{t-2\tau}^{t}|x(v)|^{q_1+1}dv.\end{aligned}$$

By Lemmas 1 and 2, we obtain that

$$\begin{aligned}\mathbb{E}\int_0^t \mathcal{L}V(\bar{x}_s, s, r(s))ds \le& C - \Xi_1\mathbb{E}\int_0^t|x(s)|^2ds - \Xi_2\mathbb{E}\int_0^t|x(s)|^{2q_1}ds - \Xi_3\mathbb{E}\int_0^t|x(s)|^{q_1+1}ds\\ &+ (\epsilon_1 + \frac{\epsilon_2+\epsilon_3}{2\hat{\delta}_1})^2(\frac{\eta_M}{\hat{\delta}_2} + \frac{q_1+1}{2}\frac{\bar{\eta}_M}{\hat{\delta}_3})\mathbb{E}\int_0^t|x(s) - x(s - h(s))|^2ds - \rho_1\mathbb{E}\int_0^t\int_{s-2\tau}^{s}|x(v)|^2dvds\\ &- \rho_2\mathbb{E}\int_0^t\int_{s-2\tau}^{s}|x(v)|^{2q_1}dvds - \rho_3\mathbb{E}\int_0^t\int_{s-2\tau}^{s}|x(v)|^{q_1+1}dvds,\end{aligned} \tag{4.21}$$

where $\Xi_1 = (1 - \eta_M\hat{\delta}_2 - \Pi_1\hat{\delta}_1)(1 - \kappa)(1 - h^*\kappa) - \Pi_2\hat{\delta}_1 - 2\rho_1\tau - (\Pi_3\hat{\delta}_1 + \omega_1)h^*$, $\Xi_2 = (\omega_2 - \frac{q_1-1}{2q_1}\omega_3 - \frac{q_1+1}{2}\bar{\eta}_M\hat{\delta}_3 - \Pi_5\hat{\delta}_1)(1 - \kappa)^{2q_1-1}(1 - h^*\kappa) - \Pi_4\hat{\delta}_1 - 2\rho_2\tau - (\Pi_6\hat{\delta}_1 + \frac{q_1+1}{2q_1}\omega_3)h^*$, and $\Xi_3 = \omega_4 + 1 - \frac{q_1-1}{2}\omega_6 - (\Pi_1 + \Pi_5)\hat{\delta}_1)(1 - \kappa)^{q_1}(1 - h^*\kappa) - (\Pi_2 + \Pi_4)\hat{\delta}_1 - 2\rho_3\tau - ((\Pi_3 + \Pi_6)\hat{\delta}_1 + \omega_5 + \omega_6)h^*$.

Substituting (4.21) into (4.20), recalling the definitions of conditions N_1, N_2, and N_3 in Assumption 7, and by leveraging the conditions given in the aforementioned method for determining the time delay τ^*, we have

$$\begin{aligned}\mathbb{E}V(\bar{x}_t, t, r(t)) \leq C &- (\hat{\varphi}_1 - 2\rho_1\tau)\mathbb{E}\int_0^t |x(s)|^2 ds - (\hat{\varphi}_2 - 2\rho_2\tau)\mathbb{E}\int_0^t |x(s)|^{2q_1} ds \\ &- (\hat{\varphi}_3 - 2\rho_3\tau)\mathbb{E}\int_0^t |x(s)|^{q_1+1} ds + \hat{\varphi}_4\mathbb{E}\int_0^t |x(s) - x(s-h(s))|^2 ds \\ &- \rho_1\mathbb{E}\int_0^t\int_{t-2\tau}^t |x(v)|^2 dvds - \rho_2\mathbb{E}\int_0^t\int_{t-2\tau}^t |x(v)|^{2q_1} dvds \\ &- \rho_3\mathbb{E}\int_0^t\int_{t-2\tau}^t |x(v)|^{q_1+1} dvds. \end{aligned} \tag{4.22}$$

For $t \in [0, 2\tau]$,

$$\begin{aligned}\int_0^t \mathbb{E}|x(s) - x(s-h(s))|^2 ds &\leq \int_0^{2\tau} 2(\mathbb{E}|x(s)|^2 + \mathbb{E}|x(s-h(s))|^2) ds \\ &\leq \int_0^{2\tau} 4 \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2 ds \leq 8\tau \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2,\end{aligned}$$

then for any $t \geq 0$, according to Lemma 3 , we obtain that

$$\begin{aligned}\int_0^t \mathbb{E}|x(s) - x(s-h(s))|^2 ds &\leq (C + 8\tau \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2)s \\ &+ (G_1\tau + G_2)\int_0^t\int_{s-2\tau}^s \mathbb{E}|x(u)|^2 dud + (G_3\tau + G_4\tau)\int_0^t\int_{s-2\tau}^s \mathbb{E}|x(u)|^{2q_1} duds \\ &+ (G_5\tau + G_6)\int_0^t\int_{s-2\tau}^s \mathbb{E}|x(u)|^{q_1+1} duds + \frac{8\bar{\gamma}\tau\kappa h^*}{1-2\kappa h^*}\int_0^t \mathbb{E}|x(s)|^2 ds.\end{aligned} \tag{4.23}$$

Substitute (4.23) into (4.22), it follows that

$$\begin{aligned}\mathbb{E}V(\bar{x}_t, t, r(t)) &\leq (C + 8\tau \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2)\hat{\varphi}_4 - (\hat{\varphi}_1 - 2\rho_1\tau - \frac{8\bar{\gamma}\tau\kappa h^*}{1-2\kappa h^*}\hat{\varphi}_4)\mathbb{E}\int_0^t |x(s)|^2 ds \\ &- (\hat{\varphi}_2 - 2\rho_2\tau)\mathbb{E}\int_0^t |x(s)|^{2q_1} ds - (\hat{\varphi}_3 - 2\rho_3\tau)\mathbb{E}\int_0^t |x(s)|^{q_1+1} ds \\ &+ ((G_1\tau + G_2)\hat{\varphi}_4 - \rho_1)\int_0^t\int_{s-2\tau}^s \mathbb{E}|x(u)|^2 duds \\ &+ ((G_3\tau + G_4\tau)\hat{\varphi}_4 - \rho_2)\int_0^t\int_{s-2\tau}^s \mathbb{E}|x(u)|^{2q_1} duds \\ &+ ((G_5\tau + G_6)\hat{\varphi}_4 - \rho_3)\int_0^t\int_{s-2\tau}^s \mathbb{E}|x(u)|^{q_1+1} duds,\end{aligned}$$

letting

$$\rho_1 = (G_1\tau + G_2)\,\hat{\varphi}_4, \rho_2 = (G_3\tau + G_4\tau)\,\hat{\varphi}_4, \text{and } \rho_3 = (G_5\tau + G_6)\,\hat{\varphi}_4,$$

and those give that

$$\mathbb{E}V(\bar{x}_t, t, r(t)) \leq (C + 8\tau \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2)\hat{\varphi}_4 - (\hat{\varphi}_1 - 2(G_1\tau^2 + G_2\tau + \frac{4\bar{\gamma}\tau\kappa h^*}{1-2\kappa h^*})\hat{\varphi}_4)\mathbb{E}\int_0^t |x(s)|^2 ds$$
$$- (\hat{\varphi}_2 - 2(G_3\tau^2 + G_4\tau^2)\hat{\varphi}_4)\mathbb{E}\int_0^t |x(s)|^{2q_1} ds - (\hat{\varphi}_3 - 2(G_5\tau^2 + G_6\tau)\hat{\varphi}_4)\mathbb{E}\int_0^t |x(s)|^{q_1+1} ds.$$

Step 2. Recalling (4.17) and leveraging quadratic function properties, when $\tau < \tau^*$, it can be deduced that

$$G_1\tau^2 + G_2\tau + \frac{4\bar{\gamma}\tau\kappa h^*}{1-2\kappa h^*} < \frac{\hat{\varphi}_1}{2\hat{\varphi}_4}, G_3\tau^2 + G_4\tau^2 < \frac{\hat{\varphi}_2}{2\hat{\varphi}_4}, \text{and } G_5\tau^2 + G_6\tau < \frac{\hat{\varphi}_3}{2\hat{\varphi}_4}.$$

This immediately yields that

$$\bar{\eta}_M \mathbb{E}|x(t) - D(x(t-h(t), r(t))|^{q_1+1} \leq (C + 8\tau \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2)\hat{\varphi}_4$$

and

$$K_3\mathbb{E}\int_0^t |x(s)|^{2q_1} ds \leq (C + 8\tau \sup_{-\tau\leq v\leq 2\tau} \mathbb{E}|x(v)|^2)\hat{\varphi}_4,$$

where $K_3 = \hat{\varphi}_2 - 2(G_3\tau^2 + G_4\tau^2)\hat{\varphi}_4$ is a positive constant. Finally, letting $t \to \infty$ implies the assertion (4.18), and

$$\sup_{0\leq t<\infty} \mathbb{E}|x(t) - D(x(t-h(t), r(t))|^{q_1+1} < \infty. \tag{4.24}$$

Step 3. It can be obtained from (2.7) that

$$\sup_{0\leq t<\infty} \mathbb{E}|x(t)|^{q_1+1} \leq \sup_{0\leq t<\infty} \frac{1}{(1-\kappa)^{q_1}}\mathbb{E}|\psi(t)|^{q_1+1} + \sup_{0\leq t<\infty} \kappa\mathbb{E}|x(t-h(t))|^{q_1+1}$$
$$\leq \sup_{0\leq t<\infty} \frac{1}{(1-\kappa)^{q_1}}\mathbb{E}|\psi(t)|^{q_1+1} + \kappa(\sup_{-\tau\leq t\leq 0} \mathbb{E}|x(t)|^{q_1+1} + \sup_{0\leq t<\infty} \mathbb{E}|x(t)|^{q_1+1}),$$

which implies that

$$\sup_{0\leq t<\infty} \mathbb{E}|x(t)|^{q_1+1} \leq \sup_{0\leq t<\infty} \frac{1}{(1-\kappa)^{q_1+1}}\mathbb{E}|\psi(t)|^{q_1+1} + \frac{\kappa}{1-\kappa}\sup_{-\tau\leq t\leq 0} \mathbb{E}|\xi(t)|^{q_1+1}.$$

Based on (4.24), we know that

$$\sup_{0\leq t<\infty} \mathbb{E}|x(t)|^{q_1+1} < \infty.$$

This completes the proof. □

In the following theorem, we demonstrate that hybrid NSDDE (2.1) is also moment stable.

Theorem 3. All conditions of Theorem 1 are satisfied. If it is further assumed that $p \geq 2$ and $p-1 \leq q_1$, then for any initial data (2) and $\tau < \tau^*$, the solution $x(t)$ of the hybrid NSDDE (1) satisfies

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0, \tag{4.25}$$

that is, the solution is pth moment stable.

By Lemma 1, we can use the same analysis as in the proof of Theorem 3.6 in Shen et al. [11] to show this theorem, so we omit it. Moreover, if $h(t) = \tau$ and $D(x(t-h(t)), r(t)) = D(x(t-h(t)))$, then we can get Theorem 3.6 in Shen et al. [11].

5. An example

This section uses a scalar mixed NSDDE example to verify the theoretical validity. Although the example is in scalar form, its validation method is universal.

Example 1. Consider a scalar hybrid NSDDE

$$\begin{aligned} d[x(t) - D(x(t-h(t)), r(t))] = {} & f(x(t), x(t-h(t)), t, r(t))dt \\ & + g(x(t), x(t-h(t)), t, r(t))dB(t), \end{aligned} \tag{5.1}$$

where $B(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \tag{5.2}$$

In this context, the coefficients f and g are defined by $f(x, y, t, i) = \hat{b}_i e^{-0.1|y|} y - \hat{a}_i x - \hat{c}_i x^3$ and $g(x, y, t, i) = \hat{d}_i y$, for $x, y \in \mathbb{R}$, where $\hat{a}_1 = 0.8, \hat{b}_1 = -2, \hat{c}_1 = 7, \hat{d}_1 = 0.2, \hat{a}_2 = 1.2, \hat{b}_2 = -3, \hat{c}_2 = 6, \hat{d}_2 = 0.3$, and $D(y, i) = 0.1y$.

To demonstrate the impact of time delay on stability more intuitively, we consider two scenarios with constant time delay: $h(t) \equiv 0.001$ and $h(t) \equiv 3$. In the case of $h(t) \equiv 0.001$, let the initial data $\xi(t) = 3 + 0.1sin(t)$ for $t \in [-0.001, 0]$ and $r_0 = 2$. In the case of $h(t) \equiv 3$, let the initial condition $\xi(t) = 3 + 0.1sin(t)$ for $t \in [-3, 0]$ and $r_0 = 2$. The simulation results are shown in Figures 1 and 2. Figure 1 indicates that the hybrid NSDDE (5.1) exhibits stability when $h(t) \equiv 0.001$, whereas Figure 2 demonstrates that the NSDDE becomes unstable when $h(t) \equiv 3$.

From this, we infer that the hybrid NSDDE (5.1) tends to be stable as the time delay decreases. Our theory aims to clarify the critical value of this delay. Let us consider the delay function $h(t) : \mathbb{R}_+ \to [0, \tau]$ meeting Assumption 1 and $h^* = 1.25$. It is easy to show that Assumption 2 is satisfied with $q_1 = 3, L_1 = 1.2, L_2 = 3, L_3 = 7, L_4 = 0, \hat{L}_1 = 0, \hat{L}_2 = 0.09, \hat{L}_3 = 0$, and $\hat{L}_4 = 0$. Then we have

$$\begin{aligned} & (x - D(y, i))^T f(x, y, t, i) + \frac{q-1}{2}|g(x, y, t, i)|^2 \\ \leq & \begin{cases} -0.24|x - 0.1y|^2 + 1.14|y|^2 - 5.4233|x - 0.1y|^4 + 0.5303|y|^4, & i = 1, \\ 0.36|x - 0.1y|^2 + 1.785|y|^2 - 4.6485|x - 0.1y|^4 + 0.4545|y|^4, & i = 2. \end{cases} \end{aligned} \tag{5.3}$$

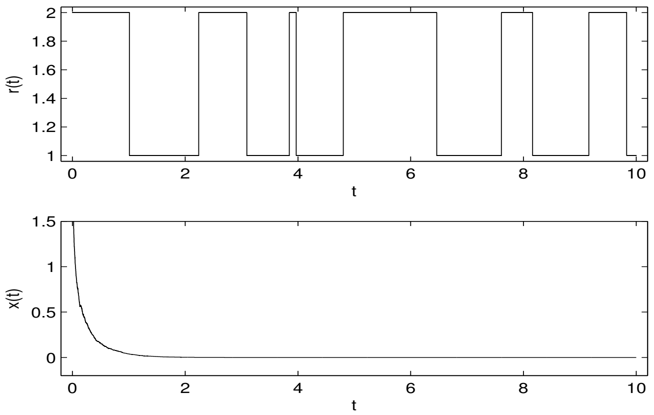

Figure 1. The computer simulation of the sample paths of the Markov chain and NSDDE (5.1) with constant delay $h(t) \equiv 0.001$, $\xi(t) = 3 + 0.1sin(t)$ for $t \in [-0.001, 0]$, and $r_0 = 2$, using the truncated Euler-Maruyama method with step size of 10^{-4}.

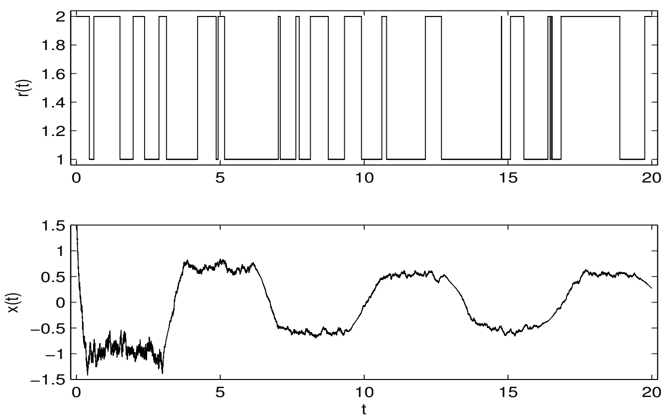

Figure 2. The computer simulation of the sample paths of the Markov chain and NSDDE (5.1) with constant delay $h(t) \equiv 3$, $\xi(t) = 3 + 0.1sin(t)$ for $t \in [-3, 0]$, and $r_0 = 2$, using the truncated Euler-Maruyama method with step size of 10^{-4}.

We can choose $q = 6$ (satisfying $q \geq 2q_1$). To ensure the validity of Eq (3.2) in Assumption 4, we select the parameters $\alpha_1 = 0.36$, $\alpha_2 = 1.785$, $\alpha_3 = 4.6485$, and $\alpha_4 = 0.5303$. Through calculations, we can obtain $(q\alpha_3 - \frac{q(q-2)\alpha_4}{q_1+q-1})(1-\kappa)^{q_1+q-2}(1-\kappa h^*) = 11.0068$ and $\frac{q(q_1+1)\alpha_4 h^*}{q_1+q-1} = 1.9886$, which indicates that Eq (3.1) holds. Next, we can decompose f as (4.1) with

$$f_1(x, y, t, i) = \hat{b}_i e^{-0.1|y|} y \text{ and } f_2(x, y, t, i) = -\hat{a}_i x - \hat{c}_i x^3.$$

Since for any $(x, y, t, i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times S$,

$$|f_1(x, y, t, i) - f_1(x, x, t, i)| \leq |\hat{b}_i|(e^{-0.1|x|}|x - y| + |y||e^{-0.1|x|} - e^{-0.1|y|}|)$$

$$\leq |\hat{b}_i|(e^{-0.1|x|} + 0.1|y|)|x - y|.$$

It is verified that when selecting $\epsilon_1 = 3$, $\epsilon_2 = 0$, $\epsilon_3 = 0.3$, and $r = 1$, Eq (4.2) in Assumption 5 is satisfied. To verify Assumption 6, we can show

$$\begin{aligned}
&(x - D(y,i))^T(f_1(x,x,t,i) + f_2(x,y,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \\
&\leq \begin{cases} -0.66|x - 0.1y|^2 + 0.16|y|^2 - 5.4233|x - 0.1y|^4 + 0.5303|y|^4, & i = 1, \\ -0.99|x - 0.1y|^2 + 0.255|y|^2 - 4.6485|x - 0.1y|^4 + 0.4545|y|^4, & i = 2 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&(x - D(y,i))^T(f_1(x,x,t,i) + f_2(x,y,t,i)) + \frac{q_1}{2}|g(x,y,t,i)|^2 \\
&\leq \begin{cases} -0.66|x - 0.1y|^2 + 0.2|y|^2 - 5.4233|x - 0.1y|^4 + 0.5303|y|^4, & i = 1, \\ -0.99|x - 0.1y|^2 + 0.345|y|^2 - 4.6486|x - 0.1y|^4 + 0.4545|y|^4, & i = 2. \end{cases}
\end{aligned}$$

As a result, we obtain $a_1 = -0.66, b_1 = 0.16, c_1 = 5.4233, d_1 = 0.5303, \bar{a}_1 = -0.66, \bar{b}_1 = 0.2, \bar{c}_1 = 5.4233, \bar{d}_1 = 0.5303, a_2 = -0.99$, $b_2 = 0.255, c_2 = 4.6485, d_2 = 0.4545, \bar{a}_2 = -0.99, \bar{b}_2 = 0.345, \bar{c}_2 = 4.6485, \bar{d}_2 = 0.4545$,

$$A = \begin{pmatrix} 2.32 & -1 \\ -2 & 3.98 \end{pmatrix}, A^{-1} = \begin{pmatrix} 0.5502 & 0.1382 \\ 0.2765 & 0.3207 \end{pmatrix}, \bar{A} = \begin{pmatrix} 3.64 & -1 \\ -2 & 5.96 \end{pmatrix}, \text{and } \bar{A}^{-1} = \begin{pmatrix} 0.3026 & 0.0508 \\ 0.1016 & 0.1848 \end{pmatrix}.$$

We hence show that $(\eta_1, \eta_2) = (0.6884, 0.5972)$ and $(\bar{\eta}_1, \bar{\eta}_2) = (0.3534, 0.2864)$, which show that $A, \bar{A}$ are non-singular M-matrices. We also have $N_1 = 0.4068, N_2 = 1.9975$, and $N_3 = 2.8939$. Therefore, we assume that all conditions in Assumption 7 are met. Up to this point, all the conditions of Theorem 2 are satisfied. Based on Theorem 2, it is established that the hybrid NSDDE (5.1) exhibits asymptotic stability in L^p space with $p \in [2, 4)$ when the time delay $\tau < \tau^*$. By applying the method for solving τ^*, the critical value is precisely found to be $\tau^* = 0.0014$. We now select

$$h(t) = \sum_{k=0}^{\infty}[0.2(t - 0.01k)I_{[0.01k, 0.01k+0.005)}(t) + (0.002 - 0.2(t - 0.01k))I_{[0.01k+0.005, 0.01(k+1))}(t)]. \tag{5.4}$$

It meets Assumption 1 with $h^* = 1.25$ and $\tau = 0.001$. The computer simulation results in Figure 3 intuitively verify the validity of the theoretical conclusions in this paper.

6. Conclusions

This paper investigates the delay-dependent stability of highly nonlinear hybrid NSDDEs, overcoming two major limitations of classical theory and advancing the study of such systems into a regime of pronounced nonlinearity. By innovatively decomposing the drift coefficient into a two-component form, the Lipschitz coefficient of the delayed component is extended from a fixed constant to a polynomial function, significantly broadening the scope of applicability of the theoretical models. Meanwhile, for time-varying delay functions, this work removes the traditional constraints of differentiability and

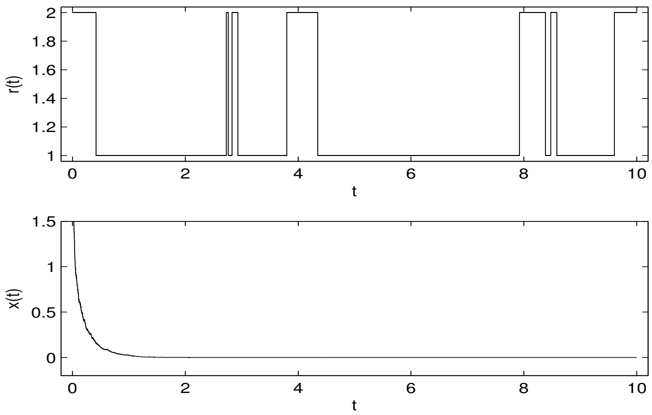

Figure 3. The computer simulation of the sample paths of the Markov chain and NSDDE (5.1) with the time delay $h(t)$ given in (5.4), $\xi(t) = 3 + 0.1sin(t)$ for $t \in [-0.001, 0]$, and $r_0 = 2$, using the truncated Euler-Maruyama method with step size of 10^{-4}.

a strictly positive lower bound, requiring only more inclusive measure theoretic conditions [22, 23]. Methodologically, by refining the construction technique of Lyapunov functionals and introducing an adjustable free-parameter mechanism, the interference of delay effects on stability analysis is systematically mitigated. Building on these advancements, a generalized criterion system encompassing H_∞ stability and moment asymptotic stability is established. The simulation example fully demonstrates the effectiveness and flexibility of this theoretical framework in analyzing the stability of complex dynamical systems, offering a more universal research paradigm for the field of highly nonlinear NSDDEs. Although the paper has improved existing results, the delay upper bound τ^* still is conservative. In the future, we will refine our methods to obtain a less conservative upper bound for the time delay.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. L. C. Feng, Z. H. Wu, J. D. Cao, S. Q. Zheng, F. E. Alsaadi, Exponential stability for nonlinear hybrid stochastic systems with time varying delays of neutral type, *Appl. Math. Lett.*, **107** (2020), 106468. https://doi.org/10.1016/j.aml.2020.106468
2. L. C. Feng, L. Liu, J. D. Cao, L. Rutkowski, G. P. Lu, General decay stability for nonautonomous neutral stochastic systems with time-varying delays and Markovian switching, *IEEE T. Cybern.*, **52** (2022), 5441–5453. https://doi.org/10.1109/TCYB.2020.3031992
3. Y. G. Kao, J. Xie, C. H. Wang, H. R. Karimi, A sliding mode approach to non-fragile observer-based control design for uncertain Markovian neutral-type stochastic systems, *Automatica*, **52** (2015), 218–226. https://doi.org/10.1016/j.automatica.2014.10.095
4. M. X. Shen, C. Fei, W. Y. Fei, X. R. Mao, Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations, *Syst. Control Lett.*, **137** (2020), 104645. https://doi.org/10.1016/j.sysconle.2020.104645
5. P. P. Zhang, Y. G. Kao, J. Hu, B. Niu, Robust observer-based sliding mode H_∞ control for stochastic Markovian jump systems subject to packet losses, *Automatica*, **130** (2021), 109665. https://doi.org/10.1016/j.automatica.2021.109665
6. J. Xie, J. Meng, Y. G. Kao, Z. Liu, Quantized feedback control for Markovian jumping singular systems driven by fractional Brownian motions, *Int. J. Robust Nonlinear Control*, **31** (2021), 7498–7512. https://doi.org/10.1002/rnc.5707
7. W. H. Chen, W. X. Zheng, Y. J. Shen, Delay-dependent stochastic stability and H_∞-control of uncertain neutral stochastic systems with time delay, *IEEE Trans. Autom. Control*, **54** (2009), 1660–1667. https://doi.org/10.1109/TAC.2009.2017981
8. J. W. Xia, J. Y. Yu, Y. M. Li, H. X. Zheng, New delay-interval-dependent exponential stability for stochastic neural networks with interval time-varying delay and distributed delay, *Circuits Syst. Signal Process.*, **31** (2012) ,1535–1557. https://doi.org/10.1007/s00034-011-9383-9
9. L. V. Hien, H. Trinh, Delay-dependent stability and stabilisation of two-dimensional positive Markov jump systems with delays, *IET Control Theory Appl.*, **11** (2017), 1603–1610. https://doi.org/10.1049/iet-cta.2016.1358
10. W. Y. Fei, L. J. Hu, X. R. Mao, M. X. Shen, Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica*, **82** (2017), 165–170. https://doi.org/10.1016/j.automatica.2017.04.050
11. M. X. Shen, W. Y. Fei, X. R. Mao, Y. Liang, Stability of highly nonlinear neutral stochastic differential delay equations, *Syst. Control Lett.*, **115** (2018), 1–8. https://doi.org/10.1016/j.sysconle.2018.02.013
12. W. Y. Fei, L. J. Hu, X. R. Mao, M. X. Shen, Generalized criteria on delay-dependent stability of highly nonlinear hybrid stochastic systems, *Int. J. Robust Nonlinear Control*, **29** (2019), 1201–1215. https://doi.org/10.1002/rnc.4402
13. M. X. Shen, C. Fei, W. Y. Fei, X. R. Mao, C. H. Mei, Delay-dependent stability of highly nonlinear neutral stochastic functional differential equations, *Int. J. Robust Nonlinear Control*, **32** (2022), 9957–9976. https://doi.org/10.1002/rnc.6384

14. H. L. Xu, X. R. Mao, Improved delay-dependent stability of superlinear hybrid stochastic systems with general time-varying delays, *Nonlinear Anal. Hybrid Syst*, **50** (2023), 101413. https://doi.org/10.1016/j.nahs.2023.101413

15. H. F. Min, S. Y. Xu, B. Y. Zhang, Q. Ma, Output-feedback control for stochastic nonlinear systems subject to input saturation and time-varying delay, *IEEE Trans. Autom. Control*, **64** (2018), 359–364. https://doi.org/10.1109/TAC.2018.2828084

16. C. Fei, W. Y. Fei, X. R. Mao, D. F. Xia, L. T. Yan, Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, *IEEE Trans. Autom. Control*, **65** (2020), 2899–2912. https://doi.org/10.1109/tac.2019.2933604

17. H. L. Dong, X. R. Mao, Advances in stabilization of highly nonlinear hybrid delay systems, *Automatica*, **136** (2022), 110086. https://doi.org/10.1016/j.automatica.2021.110086

18. M. L. Li, F. Q. Deng, Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise, *Nonlinear Anal. Hybrid Syst.*, **24** (2017), 171–185. https://doi.org/10.1016/j.nahs.2017.01.001

19. Y. Xie, C. J. Zhang, Asymptotical boundedness and moment exponential stability for stochastic neutral differential equations with time-variable delay and Markovian switching, *Appl. Math. Lett.*, **70** (2017), 46–51. https://doi.org/10.1016/j.aml.2017.03.003

20. X. R. Mao, C. G. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

21. X. Y. Li, W. Liu, X. R. Mao, J. S. Zhao, Stabilization and destabilization of hybrid systems by periodic stochastic controls, *Syst. Control Lett.*, **152** (2021), 104929. https://doi.org/10.1016/j.sysconle.2021.104929

22. W. R. Li, C. Fei, M. X. Shen, W. Y. Fei, X. R. Mao, A stabilization analysis for highly nonlinear neutral stochastic delay hybrid systems with superlinearly growing jump coefficients by variable-delay feedback control, *J. Franklin Inst.*, **360** (2023), 11932–11964. https://doi.org/10.1016/j.jfranklin.2023.08.028

23. H. Yu, A. L. Wu, Stabilization by discrete-time feedback control for highly nonlinear hybrid neutral stochastic functional differential equations with infinite delay, *Math. Methods Appl. Sci.*, **47** (2024), 8187–8207. https://doi.org/10.1002/mma.10010

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