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Research article

Stabilization effect of the viscoelastic stress tensor on the compressible Oldroyd-B model

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Abstract: By using the delicate energy method, we prove the global existence and stability of solutions to an inviscid compressible Oldroyd-B model near an equilibrium state in a periodic domain. Especially, the result justify the stabilization effect of elasticity on the compressible flows.

Keywords: stability; compressible Oldroyd-B model; decay rate; delicate energy method

1. Introduction and main result

1.1. Model and synopsis of result

In the past few decades, non-Newtonian fluids, which deviate from a nonlinear relationship between the stress tensor and deformation tensor, have found widespread applications in engineering and industry. A specific subclass of non-Newtonian fluids, known as Oldroyd-B fluids, has proven to be effective in approximating the response of many dilute polymeric liquids. These fluids exhibit a memory effect, enabling them to describe the motion of certain viscoelastic flows. The formulations for viscoelastic flows of Oldroyd-B type were originally introduced by Oldroyd [1] and extensively discussed in [2]. The classical compressible Oldroyd-B model is expressed as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nu(\Delta \mathbf{v} + \nabla \operatorname{div} \mathbf{v}) + \nabla P = \operatorname{div} \tau, \\ \partial_t \tau + \beta \tau + \mathbf{v} \cdot \nabla \tau + g_\alpha(\tau, \nabla \mathbf{v}) = D(\mathbf{v}). \end{cases} \tag{1.1}$$

Here ρ, $\mathbf{v}$, and τ represent the density, velocity, and symmetric tensor of constraints, respectively. $P = P(\rho)$ is a smooth function of ρ representing the fluid flow pressure, satisfying $P' > 0$ and $P'(\bar{\rho}) = 1$, where $\bar{\rho} > 0$ is a constant reference density. The function $g_\alpha(\tau, \nabla \mathbf{v})$ is defined as:

$$g_\alpha(\tau, \nabla \mathbf{v}) \overset{\text{def}}{=} \Omega(\mathbf{v}) - \Omega(\mathbf{v})\tau - \alpha\left(D(\mathbf{v})\tau + \tau D(\mathbf{v})\right), \quad \text{with a parameter } \alpha \in [-1, 1]. \tag{1.2}$$

The gradient $\nabla \mathbf{v}$ is decomposed into its symmetric and skew-symmetric parts:

$$D(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \quad \text{and} \quad \Omega(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} - (\nabla \mathbf{v})^T).$$

Additionally, the model includes parameters $\nu > 0$ and $\beta > 0$.

The Oldroyd-B model, as described by the system (1.1), is of significant interest both from a physical and mathematical perspective, and it has been the subject of extensive investigation within the mathematical community. The formulation of this model was originally presented by Oldroyd in the late 1950s [1] and later expounded upon in the 1980s in the book by Bird et al. [2]. The mathematical study of the Oldroyd-B model, particularly regarding the existence of solutions over extended time periods, has been a challenging problem that has attracted attention from numerous researchers. The mathematical analysis of viscoelastic flow models was pioneered by Guillopéand Saut, who established foundational local and global existence results for differential-type fluids [3–5]. A significant advancement was later made by Molinet and Talhouk, who demonstrated that the smallness condition on the coupling parameters, required in earlier work, is not necessary for proving existence and uniqueness in both 2D and 3D [6, 7]. Lei [8, 9] and Guillopé et al. [10] investigated the incompressible limit problem of the compressible Oldroyd-B model in a torus and a bounded domain of $\mathbb{R}^3$, respectively. Fang and Zi [11] extended the study of the incompressible limit problem to $\mathbb{R}^d (d \geq 2)$, considering ill-prepared initial data in Besov spaces. Zi [12] achieved global existence of small solutions to (1.1) in the critical L^2 Besov space. Zhou et al. [13] demonstrated global well-posedness and decay rate results for (1.1) in the Sobolev space $H^2(\mathbb{R}^3)$. Furthermore, Fang and Zi [14] obtained global solutions of (1.1) with sufficiently large initial data. The collective efforts of these studies contribute to a comprehensive understanding of the Oldroyd-B model and its mathematical properties. Recently, Barrett et al. [15] and Lu and Zhang [16] delved into the exploration of the existence of global in-time weak solutions for the compressible Oldroyd-type model in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Steady-state solutions for the compressible model (1.1) (with $\beta > 0$) have also been scrutinized in several papers [17–19]. For the compressible Oldroyd-type model based on the deformation tensor, relevant results can be found in [20, 21], and references therein.

When the density ρ is constant, (1.1) is reduced to the following incompressible Oldroyd-B type model:

$$\begin{cases} \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla P = \operatorname{div} \tau, \\ \partial_t \tau + \beta \tau + \mathbf{v} \cdot \nabla \tau + g_\alpha(\tau, \nabla \mathbf{v}) = D(\mathbf{v}), \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{1.3}$$

A substantial body of well-posedness results exists for the incompressible Oldroyd-B model. Guillopé and Saut [4, 5] pioneered its mathematical analysis by establishing foundational local and global existence theorems for differential-type viscoelastic fluids. The global existence of solutions to the n-dimensional incompressible Oldroyd-B model without any damping mechanism was subsequently established in [22]. The stability properties of the generalized compressible Oldroyd-B model were investigated in [23]. Moreover, Zi et al. [24] proved global well-posedness in the critical L^p framework for the incompressible case with a non-small coupling parameter (see also the references therein).

Notably, results from [25–27] have validated the stabilizing effect of elasticity or magnetic field on incompressible or compressible flows.

The primary motivation of this present study stems from the well-established observation that solutions to the compressible Euler equations in any dimension can develop finite-time singularities-such as shocks and cusps-from smooth initial data (cf. [28, 29]). However, as shown in [30], such singularities may be prevented when the inviscid incompressible flow is coupled with a magnetic field. While classical stabilization mechanisms such as viscosity and magnetic fields are well-established, elasticity offers a distinct and often complementary pathway. Viscosity dissipates energy to suppress instabilities, acting over characteristic timescales and often smoothing perturbations. Magnetic fields, through tension and pressure, can confine and stabilize conductive fluids by altering the effective stress tensor. In contrast, elastic stabilization arises from the solid's inherent resistance to deformation, storing energy in strained bonds and providing a restoring force that acts to return the material to its reference state. This study focuses on this distinctive elastodynamic response, which can stabilize against instabilities-such as the Rayleigh-Taylor or buckling instabilities-through wave propagation and stress redistribution without significant dissipation. We elucidate the specific criteria and regimes where elasticity alone provides stabilization, highlighting its unique role distinct from viscous damping or magnetic confinement.

We expect analogous phenomena to occur in the setting of inviscid compressible flow coupled with elastic effects. In order to reveal this phenomenon, we consider the global existence and stability of solutions to an inviscid compressible Oldroyd-B model which has the following form:

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0,\\ \rho\partial_t\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla P = \operatorname{div}\tau,\\ \partial_t\tau - \mu\Delta\tau + \beta\tau + (\mathbf{v}\cdot\nabla)\tau + g_\alpha(\tau,\nabla\mathbf{v}) = D(\mathbf{v}).\end{cases} \tag{1.4}$$

We focus on the initial boundary value problem of the system (1.4) in torus $\mathbb{T}^2 = (0,1)\times(0,1)$ or $\mathbb{T}^3 = (0,1)\times(0,1)\times(0,1)$ with the initial condition

$$(\rho,\mathbf{v},\tau)(x,0) = (\rho_0,\mathbf{v}_0,\tau_0)(x), \quad x\in\mathbb{T}^d, \quad d = 2,3, \tag{1.5}$$

and the periodic boundary condition

$$(\rho,\mathbf{v},\tau)(x+\ell,t) = (\rho,\mathbf{v},\tau)(x,t), \quad t\geq 0, \quad \ell = (1,1), \text{ or } \ell = (1,1,1). \tag{1.6}$$

1.2. Main result

For notational convenience, we write

$$\bar\rho = \int_{\mathbb{T}^d}\rho_0\,dx.$$

Now we can state our main result in the following theorem.

Theorem 1.1. *Let $d = 2,3$. Assume that the initial data satisfies $\rho_0 - \bar\rho \in H^3(\mathbb{T}^d)$, $\mathbf{v}_0 \in H^3(\mathbb{T}^d)$, and $\tau_0 \in H^3(\mathbb{T}^d)$ with*

$$c_0 \leq \rho_0 \leq c_0^{-1}, \quad \textit{and} \quad \int_{\mathbb{T}^d}\rho_0\mathbf{v}_0\,dx = 0, \tag{1.7}$$

for some constant $0 < c_0 < 1$. *There exists a small constant* ε *such that if*

$$\|\rho_0 - \bar{\rho}\|_{H^3} + \|\mathbf{v}_0\|_{H^3} + \|\tau_0\|_{H^3} \leq \varepsilon,$$

then the system (1.4)–(1.6) *admits a unique global solution* $(\rho - \bar{\rho}, \mathbf{v}, \tau)$ *such that*

$$\begin{aligned}
&\rho - \bar{\rho} \in C([0,\infty); H^3), \quad \nabla\rho \in L^2(\mathbb{R}^+; H^2),\\
&\mathbf{v} \in C([0,\infty); H^3), \quad \nabla\mathbf{v} \in L^2(\mathbb{R}^+; H^2),\\
&\tau \in C([0,\infty); H^3) \cap L^2(\mathbb{R}^+; H^3), \quad \nabla\tau \in L^2(\mathbb{R}^+; H^3).
\end{aligned}$$

Moreover, for any $t \geq 0$, *there holds*

$$\|\rho - \bar{\rho}\|_{H^3} + \|\mathbf{v}\|_{H^3} + \|\tau\|_{H^3} \leq C_1 e^{-C_1 t}, \tag{1.8}$$

for some pure constant $C_1 > 0$ *which depends only* μ, β, *and* α.

Remark 1.2. We believe that similar results also hold in $\mathbb{R}^2$ and $\mathbb{R}^3$. This is left to be explored in further work. The fundamental difference lies in establishing the crucial dissipation estimates. On a bounded domain, our proof relies on the Poincaré inequality to control lower-order norms by higher-order dissipative terms, which directly provides a spectral gap essential for proving exponential decay. In $\mathbb{R}^d$, the Poincar inequality does not hold. The alternative is to use a spectral analysis or Fourier decomposition approach. One typically splits the solution into low-frequency and high-frequency parts. The decay of low frequencies can be analyzed using the detailed structure of the linearized system's symbol (dispersion relation), while high frequencies can be controlled by the parabolic nature of the regularized system.

Remark 1.3. It should be mentioned that if we are only interested in the global well-posedness of (1.4), then the damping term $\beta\tau$ is not necessarily required.

Remark 1.4. It is easy to check that, for sufficiently regular solutions, the averages

$$\int_{\mathbb{T}^d} \rho_0 \, dx = \bar{\rho}, \quad \int_{\mathbb{T}^d} \rho_0 \mathbf{v}_0 \, dx = 0 \tag{1.9}$$

are conserved in time,

$$\int_{\mathbb{T}^d} \rho \, dx = \bar{\rho}, \quad \int_{\mathbb{T}^d} \rho\mathbf{v} \, dx = 0. \tag{1.10}$$

Remark 1.5. In compressible inviscid flows, strong velocity gradients can lead to the formation of singularities (e.g., shock formation) or instabilities in finite time. The presence of viscoelastic stress introduces an additional, non-Newtonian restoring force that resists rapid deformation.

Remark 1.6. The stability mechanisms analyzed in this work have direct implications for several fields where elastic stresses dominate or compete with inertial and interfacial forces.

- Soft robotics and actuators: In fluid-filled soft robots or elastomeric actuators, maintaining interface stability against external pressures or during dynamic motion is crucial. The criteria derived here can inform the design of material properties and layer geometries to prevent destabilization during operation.

- Geophysical flows: In geodynamics, the analogy between viscoelastic crustal deformation and our elastic model suggests applications to the stability of sedimentary layers or salt tectonics under gravitational loading, where elasticity can delay or modify the onset of Rayleigh-Taylor-like instabilities.
- Materials processing: During the coating or layering of viscoelastic polymers (e.g., in additive manufacturing), elastic stresses can suppress interfacial fingering instabilities that are typically driven by viscosity contrasts. Our analysis provides a scaling for when elastic stabilization becomes significant.
- Biomedical engineering: In tissue engineering, the stability of layered biomaterials or cellular aggregates under centrifugation or perfusion can be influenced by their elastic modulus. The stability thresholds may offer guidelines for constructing robust, layered tissue constructs.

1.3. Difficulties and scheme of the proof

Let us now outline the core difficulty and our approach. The principal challenge arises from the absence of dissipation mechanisms in both the density and velocity equations. This lack of stabilization makes the analysis of stability and long-term behavior particularly challenging. To compensate for the lack of regularization, we consider a small perturbation of the equilibrium state $\bar{\rho}$ for the density. Within this perturbation framework, the local well-posedness of system (1.4) can be established via a now-standard procedure. The core of the proof thus lies in establishing a global bound for $(\rho - \bar{\rho}, \mathbf{v}, \tau)$.

We employ a bootstrapping argument, beginning with the ansatz that for a suitably chosen $0 < \delta < 1$,

$$\sup_{t\in[0,T]} (\|\rho - \bar{\rho}\|_{H^3} + \|\mathbf{v}\|_{H^3} + \|\tau\|_{H^3}) \leq \delta.$$

The main effort is then devoted to proving that if the initial data are sufficiently small, i.e.,

$$\|\rho_0 - \bar{\rho}\|_{H^3} + \|\mathbf{v}_0\|_{H^3} + \|\tau_0\|_{H^3} \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$, then the following improved estimate holds:

$$\sup_{t\in[0,T]} (\|\rho - \bar{\rho}\|_{H^3} + \|\mathbf{v}\|_{H^3} + \|\tau\|_{H^3}) \leq \frac{\delta}{2}. \tag{1.11}$$

Proving (1.11) is non-trivial. The central difficulty stems from the absence of dissipation or damping mechanisms in the equations for ρ and $\mathbf{v}$. Without any stabilizing effect, the norms of ρ and $\mathbf{v}$ could grow indefinitely, making (1.11) unattainable. Consequently, it is crucial to exploit any potential smoothing effects arising from the coupling and interaction among ρ, $\mathbf{v}$, and τ.

In this paper, we identify hidden dissipative structures in two key quantities. The first is a damping effect associated with the density. To make this precise, we set $\bar{\rho} = 1$ and define $a \overset{\text{def}}{=} \rho - 1$. System (1.4) is then equivalent to the following reformulated system:

$$\begin{cases} \partial_t a + \operatorname{div} \mathbf{v} = f_1, \\ \partial_t \mathbf{v} + \nabla a = \operatorname{div} \tau + f_2, \\ \partial_t \tau - \mu\Delta\tau + \beta\tau = D(\mathbf{v}) + f_3, \\ (a, \mathbf{v}, \tau)|_{t=0} = (a_0, \mathbf{v}_0, \tau_0), \end{cases} \tag{1.12}$$

where the nonlinear terms are defined by

$$\begin{aligned}
f_1 &\stackrel{\text{def}}{=} -\mathbf{v}\cdot\nabla a - a\operatorname{div}\mathbf{v},\\
f_2 &\stackrel{\text{def}}{=} -\mathbf{v}\cdot\nabla\mathbf{v} + J(a)\nabla a - I(a)\operatorname{div}\tau,\\
f_3 &\stackrel{\text{def}}{=} -\mathbf{v}\cdot\nabla\tau - g_\alpha(\tau,\nabla\mathbf{v}),
\end{aligned}$$

with

$$I(a) \stackrel{\text{def}}{=} \frac{a}{1+a}, \quad J(a) \stackrel{\text{def}}{=} \frac{P'(1+a)}{1+a} - 1.$$

In order to recover the dissipation of the density, we exploit the special symmetrize structure of the following system:

$$\begin{cases}\partial_t a + \operatorname{div}\mathbf{v} = f_1,\\ \partial_t\mathbf{v} + \nabla a = \operatorname{div}\tau + f_2.\end{cases}$$

Taking the linear term $\operatorname{div}\tau$ as a source term, we then make "cross energy estimates"

$$\sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathbf{v}\cdot\nabla^k\nabla a\,dx \tag{1.13}$$

to obtain the dissipation of the density; see Subsection 3.2 for more details.

The second crucial quantity is the dissipation of the velocity. More precisely, we first take by taking the operator div on both sides of the third equation of (1.12) so that

$$-\frac{\Delta\mathbf{v}+\nabla\operatorname{div}\mathbf{v}}{2} = -\partial_t\operatorname{div}\tau + \mu\Delta\operatorname{div}\tau - \beta\operatorname{div}\tau + \operatorname{div} f_3.$$

Then, similarly to derive the damping effect of the density, we make "cross energy estimates"

$$\sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\operatorname{div}\tau\cdot\nabla^k\mathbf{v}\,dx \tag{1.14}$$

once again to obtain the dissipation of the velocity; see Subsection 3.3 for more details.

Next, in order to use the bootstrapping argument to prove the main theorem, we need to make the following a priori nonlinear energy estimates at first:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|(a,\mathbf{v},\tau)\|_{H^3}^2 + \int_{\mathbb{T}^d}\frac{a+J(a)}{1+a}(\nabla^3 a)^2\,dx\Big) + \mu\|\nabla\tau\|_{H^3}^2 + \beta\|\tau\|_{H^3}^2\\
&\quad\le C(\|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty} + \|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty}^2 + \|(a,\tau)\|_{L^\infty}^2)\|(a,\mathbf{v},\tau)\|_{H^3}^2.
\end{aligned} \tag{1.15}$$

One can refer to Subsection 3.1 for the definition of $J(a)$. The proof proceeds as follows. First, we place these a priori nonlinear energy estimates ahead of the two types of 'cross energy estimates"; further details are provided in Subsection 3.1. Together with the preceding steps, this yields a closed inequality.

The final step is to establish estimate (1.11) and close the bootstrapping argument. Combining the energy inequality obtained in the previous step with Poincaré inequalities, we can show that

$$\|(a,\mathbf{v},\tau)(t)\|_{H^3} \le Ce^{-ct}, \tag{1.16}$$

provided $\delta > 0$ is chosen sufficiently small. In particular, the time integral of the quantity

$$\|(\nabla a, \nabla \mathbf{v}, \nabla \tau)\|_{L^\infty} + \|(\nabla a, \nabla \mathbf{v}, \nabla \tau)\|_{L^\infty}^2 + \|(a, \tau)\|_{L^\infty}^2$$

appearing in (1.15) is finite:

$$\int_0^\infty (\|(\nabla a, \nabla \mathbf{v}, \nabla \tau)\|_{L^\infty} + \|(\nabla a, \nabla \mathbf{v}, \nabla \tau)\|_{L^\infty}^2 + \|(a, \tau)\|_{L^\infty}^2)(t)\, dt \le C < \infty.$$

Applying Grönwall's inequality then gives

$$\|(a, \mathbf{v}, \tau)(t)\|_{H^3} \le C\, \|(a_0, \mathbf{v}_0, \tau_0)\|_{H^3}.$$

By making the initial norm sufficiently small, we obtain (1.11), which completes the proof of the main theorem.

Finally, we outline the notations to be used in this paper.

Notations. Let A and B be two operators. We denote by $[A, B] = AB - BA$ the commutator of A and B. Throughout the paper, $C > 0$ denotes a generic positive constant. For a Banach space X and an interval $I \subset \mathbb{R}$, if $f, g, h \in X$, we define $\|(f, g, h)\|_X \overset{\text{def}}{=} \|f\|_X + \|g\|_X + \|h\|_X$, and denote by $C(I; X)$ the set of continuous functions on I with values in X.

2. Preliminaries

We first denote by $\mathcal{F} f = (\widehat{f_{\mathbf{k}}})_{\mathbf{k}\in\widetilde{\mathbb{Z}}^d}$ the Fourier series of a distribution $f \in S'(\mathbb{T}^d)$ so that

$$f = \sum_{\mathbf{k}\in\widetilde{\mathbb{Z}}^d} \widehat{f_{\mathbf{k}}} \frac{e^{i\mathbf{k}\cdot x}}{\sqrt{|\mathbb{T}^d|}},$$

where $\widetilde{\mathbb{Z}}^d = \mathbb{Z}/r_1 \times \mathbb{Z}/r_2 \times \cdots \mathbb{Z}/r_d$ represents the set of integer lattice points, indexing the Fourier modes on the periodic domain $\mathbb{T}^d$, and $|\mathbb{T}^d|$ stands for the measure of $\mathbb{T}^d$. The inhomogeneous Sobolev space $H^s(\mathbb{T}^d)$ is defined as follows:

$$H^s(\mathbb{T}^d) \overset{\text{def}}{=} \left\{ f \in S'(\mathbb{T}^d) \middle| \|f\|_{H^s} \overset{\text{def}}{=} \Big(|\widehat{f_0}|^2 + \sum_{\mathbf{k}\neq 0} |\mathbf{k}|^{2s} |\widehat{f_{\mathbf{k}}}|^2 \Big)^{\frac{1}{2}} < \infty \right\}. \tag{2.1}$$

Next, we recall a weighted Poincaré inequality first established by Desvillettes and Villani in [31].

Lemma 2.1. *Let Ω be a bounded connected Lipschitz domain and $\bar{\varrho}$ be a positive constant. There exists a positive constant C, depending on Ω and $\bar{\varrho}$, such that for any nonnegative function ϱ satisfying*

$$\int_\Omega \varrho dx = 1, \quad \varrho \le \bar{\varrho},$$

and any $\mathbf{v} \in H^1(\Omega)$, there holds

$$\int_\Omega \varrho \left(\mathbf{v} - \int_\Omega \rho \mathbf{v} dx \right)^2 dx \le C \|\nabla \mathbf{v}\|_{L^2}^2. \tag{2.2}$$

Finally, we state some useful lemmas, which play an important role in dealing with the nonlinear terms.

Lemma 2.2. (Kato–Ponce product estimate [32,33]) *For any* $s > 0$, $1 < p < \infty$, $1 < q_1, q_2 < \infty$, *and* $1 < p_1, p_2 \le \infty$, *we have*

$$\|\nabla^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\nabla^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\nabla^s f\|_{L^{q_2}}), \tag{2.3}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

Lemma 2.3. (Commutator estimate [33]) *Let* $s > 0$. *Then, there exists a constant C such that, for any* $f \in H^s(\mathbb{T}^d) \cap W^{1,\infty}(\mathbb{T}^d)$, *and* $g \in H^{s-1}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$,

$$\|[\nabla^s, f \cdot \nabla]g\|_{L^2} \le C(\|\nabla f\|_{L^\infty}\|\nabla^s g\|_{L^2} + \|\nabla^s f\|_{L^2}\|\nabla g\|_{L^\infty}).$$

Lemma 2.4. ([34]) *Let* $s > 0$ *and* $f \in H^s(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$. *Assume that F is a smooth function on* $\mathbb{R}$ *with* $F(0) = 0$. *Then we have*

$$\|F(f)\|_{H^s} \le C(1 + \|f\|_{L^\infty})^{[s]+1}\|f\|_{H^s},$$

where the constant C depends on $\sup_{k\le[s]+2, t\le\|f\|_{L^\infty}} \|F^{(k)}(t)\|_{L^\infty}$.

3. The proof of Theorem 1.1

Consider initial data $(\rho_0 - \bar{\rho}, \mathbf{v}_0, \tau_0) \in H^3(\mathbb{T}^d)$. The local well-posedness of system (1.4) follows from methods analogous to those employed in [12]. Thus, there exists a positive time T such that a unique solution $(\rho - \bar{\rho}, \mathbf{v}, \tau) \in C([0, T]; H^3)$ exists. Furthermore, this solution obeys the following uniform pointwise bounds:

$$\frac{1}{2}c_0 \le \rho(t, x) \le 2c_0^{-1}, \qquad \forall t \in [0, T], \quad x \in \mathbb{T}^d. \tag{3.1}$$

By a standard continuity argument, global existence follows from uniform a priori bounds on the solution. Hence, the primary objective of the subsequent sections is to establish these a priori estimates.

We first introduce an auxiliary result in the form of a Poincaré-type inequality for the velocity field that will be essential for the subsequent a priori estimates.

Lemma 3.1. *Let* $\rho, \mathbf{v}$, *and* τ *be smooth functions to (1.4) on* $[0, \infty) \times \mathbb{T}^d$ *satisfying* (3.1), *then for any* $t \ge 0$,

$$\|\mathbf{v}(t)\|_{H^s}^2 \le C\|\nabla\mathbf{v}(t)\|_{H^s}^2, \quad \text{for any } s \in \mathbb{N}. \tag{3.2}$$

Proof. On the one hand, thanks to

$$\int_{\mathbb{T}^d} \rho\mathbf{v}\, dx = 0,$$

we can deduce from Lemma 2.1 that

$$\|(\sqrt{\rho}\mathbf{v})(t)\|_{L^2}^2 \le C\|\nabla\mathbf{v}(t)\|_{L^2}^2;$$

from this and (3.1), we can further get

$$\|\mathbf{v}(t)\|_{L^2}^2 \leq C\|\nabla\mathbf{v}(t)\|_{L^2}^2. \tag{3.3}$$

On the other hand, for any $s \in \mathbb{N}$, by using the Plancherel formula for the Fourier series, we have

$$\begin{aligned}\|\mathbf{v}\|_{H^s}^2 =& |\widehat{\mathbf{v}}(\mathbf{0})|^2 + \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|^{2s}|\widehat{\mathbf{v}}(\mathbf{k})|^2 \\ \leq& |\widehat{\mathbf{v}}(\mathbf{0})|^2 + \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|^{2s}|\mathbf{k}|^2|\widehat{\mathbf{v}}(\mathbf{k})|^2 \\ \leq& \|\mathbf{v}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{H^s}^2. \end{aligned} \tag{3.4}$$

Combined with (3.3) and (3.4), we can arrive at (3.2). Consequently, we complete the proof of the lemma. □

3.1. Nonlinear energy estimates

In this subsection, we derive the a priori energy estimates for system (1.4).

We show the first type of energy estimates in the following proposition that contains the dissipation estimate for τ.

Proposition 3.2. *Let* $(a, \mathbf{v}, \tau) \in C([0, T]; H^3)$ *be a solution to the system* (1.4)*, it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|(a, \mathbf{v}, \tau)\|_{H^3}^2 + \int_{\mathbb{T}^d} \frac{a + J(a)}{1 + a}(\nabla^3 a)^2\, dx\Big) + \mu\|\nabla\tau\|_{H^3}^2 + \beta\|\tau\|_{H^3}^2 \\ &\quad \leq C(\|(\nabla a, \nabla\mathbf{v}, \nabla\tau)\|_{L^\infty} + \|(\nabla a, \nabla\mathbf{v}, \nabla\tau)\|_{L^\infty}^2 + \|(a, \tau)\|_{L^\infty}^2)\|(a, \mathbf{v}, \tau)\|_{H^3}^2. \end{aligned} \tag{3.5}$$

Proof. We begin by establishing the L^2 energy estimate. Taking the inner product of the first equation in (1.12) with a, the second with $\mathbf{v}$, and the third with τ, respectively, and summing the results, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(a, \mathbf{v}, \tau)\|_{L^2}^2 + \mu\|\nabla\tau\|_{L^2}^2 + \beta\|\tau\|_{L^2}^2 \\ &\quad = \int_{\mathbb{T}^d} f_1 \cdot a\, dx + \int_{\mathbb{T}^d} f_2 \cdot \mathbf{v}\, dx + \int_{\mathbb{T}^d} f_3 \cdot \tau\, dx, \end{aligned} \tag{3.6}$$

where we have used the following cancellations:

$$\int_{\mathbb{T}^d} \operatorname{div}\mathbf{v} \cdot a\, dx + \int_{\mathbb{T}^d} \nabla a \cdot \mathbf{v}\, dx = 0, \quad \int_{\mathbb{T}^d} \operatorname{div}\tau \cdot \mathbf{v}\, dx + \int_{\mathbb{T}^d} D(\mathbf{v}) \cdot \tau\, dx = 0.$$

We now estimate the three terms on the right-hand side of (3.6). First, integration by parts and Hölder's inequality yield

$$\Big|\int_{\mathbb{T}^d} f_1 \cdot a\, dx\Big| \leq C\|\nabla\mathbf{v}\|_{L^\infty}\|a\|_{L^2}^2, \tag{3.7}$$

and

$$\Big|\int_{\mathbb{T}^d} \mathbf{v} \cdot \nabla\mathbf{v} \cdot \mathbf{v}\, dx\Big| \leq C\|\nabla\mathbf{v}\|_{L^\infty}\|\mathbf{v}\|_{L^2}^2. \tag{3.8}$$

Bounding the nonlinear terms involving composite functions in f_2 requires a more elaborate treatment. Throughout the analysis, we assume

$$\sup_{t\in\mathbb{R}^+,\, x\in\mathbb{T}^d} |a(t,x)| \le \frac{1}{2}, \tag{3.9}$$

which allows us to freely apply the composition estimates given in Lemma 2.4. Since $H^3(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$,, condition (3.9) will be satisfied provided the constructed solution for a has a sufficiently small norm. Consequently, Lemma 2.4 yields the following composition estimate:

$$\|(I(a), J(a))\|_{H^s} \le C\|a\|_{H^s}, \quad \text{for any } s > 0. \tag{3.10}$$

Hence, by the Hölder inequality and the Young inequality, we have

$$\begin{aligned}
\Big|\int_{\mathbb{T}^d} J(a)\nabla a\cdot \mathbf{v}\,dx\Big| &\le C\|\nabla a\|_{L^\infty}\|J(a)\|_{L^2}\|\mathbf{v}\|_{L^2}\\
&\le C\|\nabla a\|_{L^\infty}(\|a\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2),\\
\Big|\int_{\mathbb{T}^d} I(a)\mathrm{div}\,\tau\cdot \mathbf{v}\,dx\Big| &\le C\|\mathrm{div}\,\tau\|_{L^\infty}\|I(a)\|_{L^2}\|\mathbf{v}\|_{L^2}\\
&\le C\|\nabla\tau\|_{L^\infty}(\|a\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2);
\end{aligned}$$

combined with (3.8), this implies that

$$\Big|\int_{\mathbb{T}^d} f_2\cdot \mathbf{v}\,dx\Big| \le C(\|\nabla a\|_{L^\infty}^2 + \|(\nabla a, \nabla\mathbf{v}, \nabla\tau)\|_{L^\infty})\|(a,\mathbf{v},\tau)\|_{L^2}^2. \tag{3.11}$$

Due to

$$\Big|\int_{\mathbb{T}^d} \mathbf{v}\cdot\nabla\tau\cdot\tau\,dx\Big| = \Big|-\int_{\mathbb{T}^d}\frac{1}{2}\mathrm{div}\,\mathbf{v}|\tau|^2\,dx\Big| \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\tau\|_{L^2}^2,$$

and

$$\Big|\int_{\mathbb{T}^d} g_\alpha(\tau,\nabla\mathbf{v})\cdot\tau\,dx\Big| \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\tau\|_{L^2}^2,$$

we get

$$\Big|\int_{\mathbb{T}^d} f_3\cdot\tau\,dx\Big| \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\tau\|_{L^2}^2. \tag{3.12}$$

Inserting (3.7), (3.11), and (3.12) into (3.6), we arrive at a basic energy inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{v},\tau)\|_{L^2}^2 + \mu\|\nabla\tau\|_{L^2}^2 + \beta\|\tau\|_{L^2}^2\\
&\quad\le C(\|\nabla a\|_{L^\infty}^2 + \|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty})\|(a,\mathbf{v},\tau)\|_{L^2}^2.
\end{aligned} \tag{3.13}$$

Next, we derive higher-order energy estimates. By applying the differential operator ∇^k for $k = 1,2,3$ to (1.12) and then taking L^2 inner product with $(\nabla^k a, \nabla^k\mathbf{v}, \nabla^k\tau)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\nabla^k a, \nabla^k\mathbf{v}, \nabla^k\tau)\|_{L^2}^2 + \mu\|\nabla^k\nabla\tau\|_{L^2}^2 + \beta\|\nabla^k\tau\|_{L^2}^2$$

$$= \int_{\mathbb{T}^d} \nabla^k f_1 \cdot \nabla^k a \, dx + \int_{\mathbb{T}^d} \nabla^k f_2 \cdot \nabla^k \mathbf{v} \, dx + \int_{\mathbb{T}^d} \nabla^k f_3 \cdot \nabla^k \tau \, dx, \tag{3.14}$$

where we used the following cancellations:

$$\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\, \mathbf{v} \cdot a \, dx + \int_{\mathbb{T}^d} \nabla^k \nabla a \cdot \mathbf{v} \, dx = 0,$$
$$\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\, \tau \cdot \nabla^k \mathbf{v} \, dx + \int_{\mathbb{T}^d} \nabla^k D(\mathbf{v}) \cdot \nabla^k \tau \, dx = 0.$$

We now estimate the nonlinear terms on the right-hand side of (3.14). For the first term in f_1, we write

$$\int_{\mathbb{T}^d} \nabla^k(\mathbf{v} \cdot \nabla a) \cdot \nabla^k a \, dx = \int_{\mathbb{T}^d} (\nabla^k(\mathbf{v} \cdot \nabla a) - \mathbf{v} \cdot \nabla \nabla^k a) \cdot \nabla^k a \, dx + \int_{\mathbb{T}^d} \mathbf{v} \cdot \nabla \nabla^k a \cdot \nabla^k a \, dx.$$

Using Lemma 2.3 and integrating by parts, we obtain

$$\begin{aligned} \Big| \int_{\mathbb{T}^d} \nabla^k(\mathbf{v} \cdot \nabla a) \cdot \nabla^k a \, dx \Big| &\le C \left\| [\nabla^k, \mathbf{v} \cdot \nabla] a \right\|_{L^2} \|\nabla^k a\|_{L^2} + C \|\mathrm{div}\, \mathbf{v}\|_{L^\infty} \|\nabla^k a\|_{L^2}^2 \\ &\le C(\|\nabla \mathbf{v}\|_{L^\infty} \|\nabla^k a\|_{L^2} + \|\nabla^k \mathbf{v}\|_{L^2} \|\nabla a\|_{L^\infty}) \|\nabla^k a\|_{L^2} \\ &\le C(\|\nabla \mathbf{v}\|_{L^\infty} + \|\nabla a\|_{L^\infty})(\|\nabla^k a\|_{L^2}^2 + \|\nabla^k \mathbf{v}\|_{L^2}^2). \end{aligned} \tag{3.15}$$

To bound the second term in f_1, we decompose it as

$$\int_{\mathbb{T}^d} \nabla^k(a \mathrm{div}\, \mathbf{v}) \cdot \nabla^k a \, dx = \sum_{k=1,2} \int_{\mathbb{T}^d} \nabla^k(a \mathrm{div}\, \mathbf{v}) \cdot \nabla^k a \, dx + \int_{\mathbb{T}^d} \nabla^3(a \mathrm{div}\, \mathbf{v}) \cdot \nabla^3 a \, dx. \tag{3.16}$$

The first part on the right-hand side of (3.16) can be estimated directly by

$$\begin{aligned} \Big| \sum_{k=1,2} \int_{\mathbb{T}^d} \nabla^k(a \mathrm{div}\, \mathbf{v}) \cdot \nabla^k a \, dx \Big| &\le C(\|a\|_{L^\infty} \|\mathrm{div}\, \mathbf{v}\|_{H^k} + \|\mathrm{div}\, \mathbf{v}\|_{L^\infty} \|a\|_{H^k}) \|a\|_{H^k} \\ &\le C(\|a\|_{L^\infty} \|(a, \mathbf{v})\|_{H^3}^2 + \|\nabla \mathbf{v}\|_{L^\infty} \|a\|_{H^3}^2). \end{aligned} \tag{3.17}$$

Estimating the second term is not straightforward because it involves four derivatives acting on $\mathbf{v}$. To handle this, we reduce the order of differentiation to three. Using the Leibniz rule, we decompose the integral as

$$\int_{\mathbb{T}^d} \nabla^3(a \mathrm{div}\, \mathbf{v}) \cdot \nabla^3 a \, dx = \sum_{\ell=0}^{2} \int_{\mathbb{T}^d} \nabla^{3-\ell} a \nabla^\ell \mathrm{div}\, \mathbf{v} \cdot \nabla^3 a \, dx + \int_{\mathbb{T}^d} a \nabla^3 \mathrm{div}\, \mathbf{v} \cdot \nabla^3 a \, dx. \tag{3.18}$$

It then follows from interpolation inequalities that

$$\begin{aligned} \Big| \sum_{\ell=0}^{2} \int_{\mathbb{T}^d} \nabla^{3-\ell} a \nabla^\ell \mathrm{div}\, \mathbf{v} \cdot \nabla^3 a \, dx \Big| &\le C \|\mathrm{div}\, \mathbf{v}\|_{L^\infty} \|\nabla^3 a\|_{L^2}^2 + C \|\nabla a\|_{L^\infty} \|\nabla^2 \mathrm{div}\, \mathbf{v}\|_{L^2} \|\nabla^3 a\|_{L^2} \\ &\le C \|\nabla \mathbf{v}\|_{L^\infty} \|a\|_{H^3}^2 + C \|\nabla a\|_{L^\infty} \|(a, \mathbf{v})\|_{H^3}^2. \end{aligned} \tag{3.19}$$

To bound the second term on the right-hand side of (3.18), we make use of the continuity equation

$$\mathrm{div}\,\mathbf{v} = -\frac{\partial_t a + \mathbf{v}\cdot\nabla a}{1+a}$$

to show that

$$\begin{aligned}\int_{\mathbb{T}^d} a\nabla^3 \mathrm{div}\,\mathbf{v}\cdot\nabla^3 a\,dx &= -\int_{\mathbb{T}^d} a\nabla^3\left(\frac{\partial_t a + \mathbf{v}\cdot\nabla a}{1+a}\right)\cdot\nabla^3 a\,dx\\ &= -\int_{\mathbb{T}^d} a\nabla^3\left(\frac{\partial_t a}{1+a}\right)\cdot\nabla^3 a\,dx - \int_{\mathbb{T}^d} a\nabla^3\left(\frac{\mathbf{v}\cdot\nabla a}{1+a}\right)\cdot\nabla^3 a\,dx. \end{aligned}\tag{3.20}$$

Now, the remaining terms in (3.20) can be estimated in a manner similar to (4.19); for brevity, we omit the details here. Consequently, we obtain

$$\begin{aligned}\int_{\mathbb{T}^d} a\nabla^3 \mathrm{div}\,\mathbf{v}\cdot\nabla^3 a\,dx \le &-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{a}{1+a}(\nabla^3 a)^2\,dx\\ &+C(\|\nabla\mathbf{v}\|_{L^\infty}+\|\nabla a\|_{L^\infty}+\|\mathbf{v}\|_{L^\infty}\|\nabla a\|_{L^\infty})\|(a,\mathbf{v})\|_{H^3}^2. \end{aligned}\tag{3.21}$$

Inserting (3.19) and (3.21) into (3.18), and combining with (3.17), we can infer from (3.16) that

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla^k(a\mathrm{div}\,\mathbf{v})\cdot\nabla^k a\,dx \le &-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{a}{1+a}(\nabla^3 a)^2\,dx\\ &+C(\|\nabla\mathbf{v}\|_{L^\infty}+\|\nabla a\|_{L^\infty}+\|\mathbf{v}\|_{L^\infty}\|\nabla a\|_{L^\infty})\|(a,\mathbf{v})\|_{H^3}^2. \end{aligned}\tag{3.22}$$

Combining with (3.15) and (3.22) yields

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla^k f_1\cdot\nabla^k a\,dx \le &-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{a}{1+a}(\nabla^k a)^2\,dx\\ &+C(\|\nabla\mathbf{v}\|_{L^\infty}+\|\nabla a\|_{L^\infty}+\|\mathbf{v}\|_{L^\infty}\|\nabla a\|_{L^\infty})\|(a,\mathbf{v})\|_{H^3}^2. \end{aligned}\tag{3.23}$$

We now deal with the terms in f_3. For the first term in f_3, a similar process as in (3.15) yields

$$\int_{\mathbb{T}^d}\nabla^k(\mathbf{v}\cdot\nabla\tau)\cdot\nabla^k\tau\,dx \le C(\|\nabla\mathbf{v}\|_{L^\infty}+\|\nabla\tau\|_{L^\infty})\|(\mathbf{v},\tau)\|_{H^3}^2.$$

For the last two terms in f_3, we get by a similar derivation of (3.17) that

$$\int_{\mathbb{T}^d}\nabla^k g_\alpha(\tau,\nabla\mathbf{v})\cdot\nabla^k\tau\,dx \le \frac{\mu}{16}\|\nabla^{k+1}\tau\|_{L^2}^2 + C(\|\nabla\mathbf{v}\|_{L^\infty}^2+\|\tau\|_{L^\infty}^2)\|\nabla^k\tau\|_{L^2}^2.$$

Therefore,

$$\int_{\mathbb{T}^d}\nabla^k f_3\cdot\nabla^k\tau\,dx \le \frac{\mu}{16}\|\nabla^{k+1}\tau\|_{L^2}^2 + C(\|\nabla\mathbf{v}\|_{L^\infty}+\|\nabla\tau\|_{L^\infty}+\|\tau\|_{L^\infty}^2)\|(\mathbf{v},\tau)\|_{H^3}^2.\tag{3.24}$$

Finally, we have to bound the terms in f_2. To do this, we write

$$\int_{\mathbb{T}^d}\nabla^k f_2\cdot\nabla^k\mathbf{v}\,dx = \Pi_1+\Pi_2+\Pi_3\tag{3.25}$$

with

$$\Pi_1 \stackrel{\text{def}}{=} -\int_{\mathbb{T}^d} \nabla^k(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\nabla^k\mathbf{v}\,dx,$$
$$\Pi_2 \stackrel{\text{def}}{=} \int_{\mathbb{T}^d} \nabla^k(J(a)\nabla a)\cdot\nabla^k\mathbf{v}\,dx,$$
$$\Pi_3 \stackrel{\text{def}}{=} -\int_{\mathbb{T}^d} \nabla^k(I(a)\mathrm{div}\,\tau)\cdot\nabla^k\mathbf{v}\,dx.$$

The term Π_1 can be bounded as in (3.15):

$$\left|\Pi_1\right| \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla^k\mathbf{v}\|_{L^2}^2.$$

Similarly,

$$\begin{aligned}\left|\Pi_3\right| \le& C(\|I(a)\|_{L^\infty}\|\nabla^k\mathrm{div}\,\tau\|_{L^2} + \|\nabla^k I(a)\|_{L^2}\|\mathrm{div}\,\tau\|_{L^\infty})\|\nabla^k\mathbf{v}\|_{L^2}\\ \le& \frac{\mu}{16}\|\nabla^{k+1}\tau\|_{L^2}^2 + C(\|a\|_{L^\infty}^2 + \|\nabla\tau\|_{L^\infty}^2)(\|\nabla^k a\|_{L^2}^2 + \|\nabla^k\mathbf{v}\|_{L^2}^2).\end{aligned} \tag{3.26}$$

The term Π_2 can be estimated similarly to A_5 in (4.21); for brevity we omit the details. Consequently, we obtain the following bound for Π_2:

$$\begin{aligned}\Pi_2 \le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{J(a)}{1+a}(\nabla^3 a)^2\,dx\\ &+ C(\|a\|_{L^\infty}\|(\nabla a,\nabla\mathbf{v})\|_{L^\infty} + \|\nabla\mathbf{v}\|_{L^\infty} + \|\mathbf{v}\|_{L^\infty}\|\nabla a\|_{L^\infty})\|(a,\mathbf{v})\|_{H^3}^2.\end{aligned}$$

Inserting the bounds for Π_1 through Π_3 into (3.25), we get

$$\begin{aligned}\int_{\mathbb{T}^d}\nabla^k f_2\cdot\nabla^k\mathbf{v}\,dx \le& \frac{\mu}{16}\|\nabla^{k+1}\tau\|_{L^2}^2 - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{J(a)}{1+a}(\nabla^3 a)^2\,dx\\ &+ C(\|a\|_{L^\infty}\|(\nabla a,\nabla\mathbf{v})\|_{L^\infty} + \|\nabla\mathbf{v}\|_{L^\infty} + \|\mathbf{v}\|_{L^\infty}\|\nabla a\|_{L^\infty})\|(a,\mathbf{v})\|_{H^3}^2.\end{aligned} \tag{3.27}$$

Plugging (3.23), (3.24), and (3.27) into (3.14) and combining with (3.13), we arrive at the desired estimate (3.5). This completes the proof of Proposition 3.2. □

3.2. The dissipation of the density

In this subsection, we aim to recover the dissipation estimate for the density.

Proposition 3.3. *Assume the solution* $(a(t),\mathbf{v}(t),\tau(t))$ *to* (1.4) *satisfies*

$$\sup_{t\in[0,T]}\|(a(t),\mathbf{v}(t),\tau(t))\|_{H^3} \le \delta \tag{3.28}$$

for some $0<\delta<1$. *Then,*

$$\|\nabla a\|_{H^2}^2 + \sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathbf{v}\cdot\nabla^k\nabla a\,dx \le C\delta^2\|\nabla a\|_{H^2}^2 + C\|\nabla\mathbf{v}\|_{H^2}^2 + C\|\tau\|_{H^3}^2. \tag{3.29}$$

Proof. Let us first recall the second equation in (1.12):

$$\nabla a = -\partial_t \mathbf{v} + \operatorname{div}\tau + f_2.$$

Applying ∇^k to this identity and then taking the inner product with $\nabla^k\nabla a$, we obtain

$$\begin{aligned}\|\nabla^k\nabla a\|_{L^2}^2 &= -\int_{\mathbb{T}^d}\nabla^k\partial_t\mathbf{v}\cdot\nabla^k\nabla a\,dx + \int_{\mathbb{T}^d}\nabla^k\operatorname{div}\tau\cdot\nabla^k\nabla a\,dx + \int_{\mathbb{T}^d}\nabla^k f_2\cdot\nabla^k\nabla a\,dx\\ &\overset{\text{def}}{=} M_1 + M_2 + M_3.\end{aligned} \tag{3.30}$$

The first term M_1 involves the time derivative $\nabla^k\partial_t\mathbf{v}$. To handle it, we integrate by parts in both the time and space variables and use the continuity equation to get

$$M_1 = -\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathbf{v}\cdot\nabla^k\nabla a\,dx + \|\nabla^k\operatorname{div}\mathbf{v}\|_{L^2}^2 - \int_{\mathbb{T}^d}\nabla^k\operatorname{div}\mathbf{v}\cdot\nabla^k f_1\,dx. \tag{3.31}$$

Remember that

$$f_1 = -\,\mathbf{v}\cdot\nabla a - a\operatorname{div}\mathbf{v}.$$

The last term of M_1 in (3.31) can be bounded as

$$\begin{aligned}-\int_{\mathbb{T}^d}\nabla^k\operatorname{div}\mathbf{v}\cdot\nabla^k f_1\,dx &\le C(1+\|a\|_{L^\infty})\,\|\operatorname{div}\mathbf{v}\|_{H^2}^2 + C\,\|\mathbf{v}\|_{H^2}^2\|a\|_{H^3}^2\\ &\le C\|\nabla\mathbf{v}\|_{H^2}^2 + C\delta^2\|a\|_{H^3}^2,\end{aligned} \tag{3.32}$$

where we have used (3.28).

Thanks to Hölder's inequality and Cauchy's inequality, we directly get

$$M_2 \le \frac{1}{4}\|\nabla a\|_{H^2}^2 + C\|\operatorname{div}\tau\|_{H^2}^2. \tag{3.33}$$

Due to $H^2(\mathbb{T}^d)\hookrightarrow L^\infty(\mathbb{T}^d)$, it holds that

$$\begin{aligned}\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{H^2}^2 &\le C\|\mathbf{v}\|_{H^2}^2\|\nabla\mathbf{v}\|_{H^2}^2 \le C\delta^2\|\nabla\mathbf{v}\|_{H^2}^2,\\ \|J(a)\nabla a\|_{H^2}^2 &\le C\|J(a)\|_{H^2}^2\|\nabla a\|_{H^2}^2 \le C\delta^2\|\nabla a\|_{H^2}^2,\\ \|I(a)\operatorname{div}\tau\|_{H^2}^2 &\le C\|I(a)\|_{H^2}^2\|\operatorname{div}\tau\|_{H^2}^2 \le C\delta^2\|\operatorname{div}\tau\|_{H^2}^2,\end{aligned}$$

which implies that

$$\|f_2\|_{H^2}^2 \le C\delta^2\|\nabla\mathbf{v}\|_{H^2}^2 + C\delta^2\|\nabla a\|_{H^2}^2 + C\delta^2\|\operatorname{div}\tau\|_{H^2}^2. \tag{3.34}$$

Hence, we can obtain

$$\begin{aligned}M_3 &= \int_{\mathbb{T}^d}\nabla^k f_2\cdot\nabla^k\nabla a\,dx\\ &\le \frac{1}{16}\|\nabla^k\nabla a\|_{L^2}^2 + C\|f_2\|_{H^2}^2\\ &\le \frac{1}{4}\|\nabla a\|_{H^2}^2 + C\delta^2\|\nabla\mathbf{v}\|_{H^2}^2 + C\delta^2\|\nabla a\|_{H^2}^2 + C\delta^2\|\operatorname{div}\tau\|_{H^2}^2.\end{aligned} \tag{3.35}$$

Inserting (3.32) into M_1 and combining with (3.33) and (3.35), we can arrive at (3.29) by summing up $k = 0, 1, 2$. This completes the proof of Proposition 3.3. □

3.3. The dissipation of the velocity

In this subsection, we are concerned with the dissipation estimate for the velocity.

Proposition 3.4. *Assume the solution* $(a(t), \mathbf{v}(t), \tau(t))$ *to* (1.4) *satisfies*

$$\sup_{t\in[0,T]} \|(a(t), \mathbf{v}(t), \tau(t))\|_{H^3} \leq \delta \tag{3.36}$$

for some $0 < \delta < 1$. *Then,*

$$\begin{aligned} &\|\nabla \mathbf{v}\|_{H^2}^2 + \|\mathrm{div}\,\mathbf{v}\|_{H^2}^2 + \sum_{k=0}^{2} \frac{d}{dt}\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \mathbf{v}\,dx \\ &\leq (\frac{1}{8} + C\delta^2)\|\nabla a\|_{H^2}^2 + C\|\tau\|_{H^3}^2 + C\|\nabla \tau\|_{H^3}^2 + C\delta^2\|\nabla \mathbf{v}\|_{H^2}^2, \end{aligned} \tag{3.37}$$

where $\varepsilon > 0$ *is a fixed small number.*

Proof. We first take the operator div on both sides of the third equation of (1.12) to get

$$-\frac{\Delta \mathbf{v} + \nabla \mathrm{div}\,\mathbf{v}}{2} = -\partial_t \mathrm{div}\,\tau + \mu\Delta \mathrm{div}\,\tau - \beta \mathrm{div}\,\tau + \mathrm{div}\, f_3.$$

Applying ∇^k to the above equation and then multiplying it by $\nabla^k \mathbf{v}$, we obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla^k \nabla \mathbf{v}\|_{L^2}^2 + \frac{1}{2}\|\nabla^k \mathrm{div}\,\mathbf{v}\|_{L^2}^2 \\ &= -\int_{\mathbb{T}^d} \nabla^k \partial_t \mathrm{div}\,\tau \cdot \nabla^k \mathbf{v}\,dx + \mu\int_{\mathbb{T}^d} \nabla^k \Delta \mathrm{div}\,\tau \cdot \nabla^k \mathbf{v}\,dx \\ &\quad -\beta\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \mathbf{v}\,dx + \int_{\mathbb{T}^d} \nabla^k \mathrm{div}\, f_3 \cdot \nabla^k \mathbf{v}\,dx \\ &\overset{\text{def}}{=} N_1 + N_2 + N_3 + N_4. \end{aligned} \tag{3.38}$$

To estimate N_1, we write it as

$$N_1 = -\frac{d}{dt}\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \mathbf{v}\,dx + \int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \partial_t \mathbf{v}\,dx. \tag{3.39}$$

According to the velocity equation

$$\partial_t \mathbf{v} + \nabla a = \mathrm{div}\,\tau + f_2,$$

we can further divide the last term in (3.39) into three terms:

$$\begin{aligned} \int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \partial_t \mathbf{v}\,dx = &\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \mathrm{div}\,\tau\,dx \\ &- \int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \nabla a\,dx + \int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k f_2\,dx \end{aligned}$$

$$\stackrel{\text{def}}{=} N_{1,1} + N_{1,2} + N_{1,3}.$$

Thanks to the Hölder inequality and the Young inequality, it holds that

$$\begin{aligned} N_{1,1} + N_{1,2} &\leq \frac{1}{8}\|\nabla a\|_{H^2}^2 + C\|\mathrm{div}\,\tau\|_{H^2}^2, \\ N_{1,3} &\leq C\|\mathrm{div}\,\tau\|_{H^2}^2 + C\|f_2\|_{H^2}^2. \end{aligned}$$

From the above inequality and (3.34), we can infer from (3.39) that

$$N_1 = -\frac{d}{dt}\int_{\mathbb{T}^d} \nabla^k \mathrm{div}\,\tau \cdot \nabla^k \mathbf{v}\,dx + (\varepsilon + C\delta^2)\|\nabla a\|_{H^2}^2 + C\|\nabla\tau\|_{H^3}^2 + C\delta^2\|\nabla\mathbf{v}\|_{H^2}^2. \tag{3.40}$$

By the integration by parts, Hölder's inequality, and Cauchy's inequality, we have

$$N_2 \leq C\|\nabla^k \Delta\tau\|_{L^2}\|\nabla^k\nabla\mathbf{v}\|_{L^2} \leq \frac{1}{8}\|\nabla\mathbf{v}\|_{H^2}^2 + C\|\nabla\tau\|_{H^3}^2. \tag{3.41}$$

Similarly,

$$N_3 \leq \frac{1}{8}\|\nabla\mathbf{v}\|_{H^2}^2 + C\|\tau\|_{H^3}^2. \tag{3.42}$$

Finally, we have to deal with the last term in (3.38). By using the integration by parts and Hölder's inequality, it holds that

$$N_4 \leq C\|\nabla^k f_3\|_{L^2}\|\nabla^k\nabla\mathbf{v}\|_{L^2} \leq C\|\nabla\mathbf{v}\|_{H^2}\|f_3\|_{H^2}. \tag{3.43}$$

Owing to $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ with $d = 2, 3$, we have

$$\begin{aligned} \|f_3\|_{H^2} &\leq C\|(\mathbf{v}\cdot\nabla)\tau\|_{H^2} + C\|g_\alpha(\tau, \nabla\mathbf{v})\|_{H^2} \\ &\leq C\|\mathbf{v}\|_{H^3}\|\tau\|_{H^3} + C\|\tau\|_{H^3}\|\nabla\mathbf{v}\|_{H^2}. \end{aligned}$$

From the above inequality and (3.43), we have

$$\begin{aligned} N_4 &\leq \frac{1}{8}\|\nabla\mathbf{v}\|_{H^2}^2 + C\|\mathbf{v}\|_{H^3}^2\|\tau\|_{H^3}^2 + C\|\tau\|_{H^3}^2\|\nabla\mathbf{v}\|_{H^2}^2 \\ &\leq \frac{1}{8}\|\nabla\mathbf{v}\|_{H^2}^2 + C\|\mathbf{v}\|_{H^3}^2\|\tau\|_{H^3}^2 \\ &\leq \frac{1}{8}\|\nabla\mathbf{v}\|_{H^2}^2 + C\delta^2\|\tau\|_{H^3}^2. \end{aligned} \tag{3.44}$$

Plugging (3.40)–(3.42), and (3.44) into (3.38), we deduce (3.37) for any $0 \leq k \leq 2$. This completes the proof of Proposition 3.4. □

3.4. Complete the proof of Theorem 1.1

In this subsection, we prove Theorem 1.1. Given initial data $(a_0, \mathbf{v}_0, \tau_0) \in H^3(\mathbb{T}^d)$, the local well-posedness of system (1.4) follows directly from the energy method. Hence, we may assume there exists a time $T > 0$ and a unique solution

$$(a, \mathbf{v}, \tau) \in C([0, T]; H^3(\mathbb{T}^d))$$

satisfying (1.4). Furthermore, we may impose the smallness condition

$$\sup_{t\in[0,T]} (\|a\|_{H^3} + \|\mathbf{v}\|_{H^3} + \|\tau\|_{H^3}) \le \delta, \tag{1}$$

for some constant $0 < \delta < 1$, to be chosen later.

Multiplying by a suitably large constant A_1 on both sides of (3.37) and then summing up (3.29), we get

$$\begin{aligned} & A_1(\|\nabla\mathbf{v}\|_{H^2}^2 + \|\mathrm{div}\,\mathbf{v}\|_{H^2}^2) + \frac{7}{8}\|\nabla a\|_{H^2}^2 \\ & \quad + A_1\sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathrm{div}\,\tau\cdot\nabla^k\mathbf{v}\,dx + \sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathbf{v}\cdot\nabla^k\nabla a\,dx \\ & \le C\delta^2 A_1\|\nabla a\|_{H^2}^2 + C(A_1+1)\|\tau\|_{H^3}^2 + CA_1\|\nabla\tau\|_{H^3}^2 + CA_1\delta^2\|\nabla\mathbf{v}\|_{H^2}^2. \end{aligned} \tag{3.45}$$

After choosing a small enough δ in (3.45), we have

$$\begin{aligned} & A_1(\|\nabla\mathbf{v}\|_{H^2}^2 + \|\mathrm{div}\,\mathbf{v}\|_{H^2}^2) + \|\nabla a\|_{H^2}^2 \\ & \quad + A_1\sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathrm{div}\,\tau\cdot\nabla^k\mathbf{v}\,dx + \sum_{k=0}^{2}\frac{d}{dt}\int_{\mathbb{T}^d}\nabla^k\mathbf{v}\cdot\nabla^k\nabla a\,dx \\ & \le C(A_1+1)\|\tau\|_{H^3}^2 + CA_1\|\nabla\tau\|_{H^3}^2. \end{aligned} \tag{3.46}$$

Multiplying by a suitably large constant $A_2 \ge 2C(A_1 + 1)$ on both sides of (3.5) and then summing up (3.46), we finally get

$$\begin{aligned} & \frac{d}{dt}\Big\{\frac{A_2}{2}\|(a,\mathbf{v},\tau)\|_{H^3}^2 + \frac{A_2}{2}\int_{\mathbb{T}^d}\frac{a+J(a)}{1+a}(\nabla^3 a)^2\,dx \\ & \quad + \sum_{k=0}^{2}\int_{\mathbb{T}^d}\Big(A_1\nabla^k\mathrm{div}\,\tau\cdot\nabla^k\mathbf{v} + \nabla^k\mathbf{v}\cdot\nabla^k\nabla a\Big)\,dx\Big\} \\ & \quad + A_2\|\nabla\tau\|_{H^3}^2 + A_2\|\tau\|_{H^3}^2 + A_1(\|\nabla\mathbf{v}\|_{H^2}^2 + \|\mathrm{div}\,\mathbf{v}\|_{H^2}^2) + \|\nabla a\|_{H^2}^2 \\ & \le CA_2(\|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty} + \|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty}^2 + \|(a,\tau)\|_{L^\infty}^2)\|(a,\mathbf{v},\tau)\|_{H^3}^2. \end{aligned} \tag{3.47}$$

By using the embedding relation $H^2(\mathbb{T}^d) \hookrightarrow L^\infty((\mathbb{T}^d))$ once again, we get

$$\begin{aligned} & \|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty} + \|(\nabla a,\nabla\mathbf{v},\nabla\tau)\|_{L^\infty}^2 + \|(a,\tau)\|_{L^\infty}^2 \\ & \quad \le C\|(a,\mathbf{v},\tau)\|_{H^3} + C\|(a,\mathbf{v},\tau)\|_{H^3}^2 \end{aligned}$$

$$\leq C\delta(1+\delta). \tag{3.48}$$

Denote

$$\begin{aligned}\mathcal{E}(t) =&\frac{A_2}{2}\|(a,\mathbf{v},\tau)\|_{H^3}^2 + \frac{A_2}{2}\int_{\mathbb{T}^d}\frac{a+J(a)}{1+a}(\nabla^3 a)^2\,dx\\&+\sum_{k=0}^{2}\int_{\mathbb{T}^d}\Big(A_1\nabla^k \mathrm{div}\,\tau\cdot\nabla^k\mathbf{v}+\nabla^k\mathbf{v}\cdot\nabla^k\nabla a\Big)\,dx,\end{aligned}$$

and

$$\mathcal{D}(t) =A_2\|\nabla\tau\|_{H^3}^2 + A_2\|\tau\|_{H^3}^2 + A_1(\|\nabla\mathbf{v}\|_{H^2}^2 + \|\mathrm{div}\,\mathbf{v}\|_{H^2}^2) + \|\nabla a\|_{H^2}^2.$$

Then, we deduce from (3.47) and (3.48) that

$$\frac{d}{dt}\mathcal{E}(t) + \frac{1}{2}\mathcal{D}(t) \leq CA_2\delta(1+\delta)\|(a,\mathbf{v},\tau)\|_{H^3}^2. \tag{3.49}$$

On the one hand, due to $\|a\|_{L^\infty} \ll 1$, by elementary computation, there exists an A_2 large enough such that

$$\mathcal{E}(t) \geq \|(a,\mathbf{v},\tau)\|_{H^3}^2. \tag{3.50}$$

On the other hand, from the fact that $\int_{\mathbb{T}^d} a\,dx = 0$, we can infer that

$$\|a\|_{H^3} \leq C\|\nabla a\|_{H^2}. \tag{3.51}$$

Thus, by using the Lemma 3.1, (3.51) and making $\delta > 0$ suitably small in (3.49), we can conclude that

$$\frac{d}{dt}\mathcal{E}(t) + \frac{1}{2}\mathcal{D}(t) \leq 0.$$

Furthermore, it is straightforward to verify that

$$\mathcal{E}(t) \leq C\|(a,\mathbf{v},\tau)\|_{H^3}^2.$$

Combining this with (3.51) and Lemma 3.1 yields

$$\mathcal{E}(t) \leq \frac{1}{c}\mathcal{D}(t).$$

Hence, we obtain the differential inequality

$$\frac{d}{dt}\mathcal{E}(t) + c\mathcal{E}(t) \leq 0.$$

Solving it gives

$$\mathcal{E}(t) \leq Ce^{-ct},$$

which in particular implies

$$\int_0^t \|(a, \mathbf{v}, \tau)(t')\|_{H^3}\, dt' \le C. \tag{3.52}$$

Applying (3.48) together with Proposition 3.2 once more, we have

$$\begin{aligned}\|(a(t), \mathbf{v}(t), \tau(t))\|_{H^3}^2 \le& \|(a_0, \mathbf{v}_0, \tau_0)\|_{H^3}^2 \\ &+ C\int_{\mathbb{T}^d} (\|(a, \mathbf{v}, \tau)\|_{H^3} + \|(a, \mathbf{v}, \tau)\|_{H^3}^2)\|(a, \mathbf{v}, \tau)\|_{H^3}^2\, dt'.\end{aligned} \tag{3.53}$$

Now, applying Gronwall's inequality to (3.53) and using the bound (3.52), we obtain

$$\|(a, \mathbf{v}, \tau)\|_{H^3}^2 \le C\|(a_0, \mathbf{v}_0, \tau_0)\|_{H^3}^2 \le C\varepsilon^2.$$

Choosing ε sufficiently small so that $C\varepsilon \le \delta/2$, a standard continuity argument shows that the local solution can be extended globally in time. This completes the proof of the main theorem.

4. Conclusions

In this paper, we have investigated the initial-boundary value problem for the inviscid compressible Oldroyd-B model on the periodic domain $\mathbb{T}^d$ for dimensions $d = 2, 3$. We have proved that, provided the initial data is sufficiently close to a constant equilibrium state, the system admits a unique global classical solution that decays to equilibrium at an exponential rate.

Our main result, Theorem 1.1, establishes the global existence, uniqueness, and nonlinear stability of solutions near equilibrium. This result rigorously justifies the stabilizing effect of elastic stress on compressible fluid motion. The primary challenge stems from the complete absence of viscosity or damping mechanisms in the density and momentum equations, which typically leads to finite-time singularity formation in inviscid compressible flows (e.g., shocks in the Euler equations). Our analysis demonstrates that the coupling with the viscoelastic stress tensor τ introduces sufficient dissipation to prevent this blowup, effectively regularizing the flow.

The proof relies on a delicate energy method that uncovers hidden dissipative structures. The core innovation lies in constructing two key "cross-energy" functionals (see (1.13) and (1.14)) that extract damping effects for the density $a = \rho - \bar{\rho}$ and the velocity $\mathbf{v}$ from the coupled system structure. These estimates are then combined with the explicit dissipation from the stress equation $\partial_t \tau - \mu\Delta\tau + \beta\tau$ to close a global energy inequality via a bootstrapping argument.

This work provides a mathematical foundation for understanding how elasticity can suppress hydrodynamic instability and singularity formation in compressible media. A natural and important question for future research is whether an analogous global stability result holds in the whole space $\mathbb{R}^d$ with critical regularity instead of a periodic torus.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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