Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(3): 1900–1916.
DOI: 10.3934/era.2026085
Received: 09 December 2025
Revised: 26 January 2026
Accepted: 12 February 2026
Published: 3 March 2026

Research article

Impulsive stochastic fractional integro-differential equations with delay and weakly singular kernels in Banach spaces

Fatima Mesri[1], **Abdelkrim Salim**[1,2,*] **and Mouffak Benchohra**[1]

[1] Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria

[2] Faculty of Technology, Hassiba Benbouali University of Chlef, P.O. Box 151 Chlef 02000, Algeria

* **Correspondence:** Email: salim.abdelkrim@yahoo.com, a.salim@univ-chlef.dz.

Abstract: This paper studies the existence of mild solutions for impulsive stochastic fractional integro-differential equations with finite delay and weakly singular kernels in separable Banach spaces. The model involves a Caputo derivative of order $\alpha \in (\frac{1}{2}, 1)$, a cylindrical Wiener process, instantaneous impulses, and a singular kernel $(t-s)^{-\beta}$ with $\beta \in (0, 1-\alpha)$. To the best of our knowledge, the combined presence of impulsive effects, stochastic noise, finite delay, and weakly singular kernels has not yet been analyzed in the literature within a Caputo fractional framework in Banach spaces of type 2. Using resolvent families, Itô calculus in Banach spaces, and Krasnoselskii's fixed point theorem, we establish the existence of mean-square mild solutions under natural growth and continuity assumptions. An example illustrates the applicability of the results.

Keywords: stochastic fractional differential equations; impulsive effects; weakly singular kernels; delay; mild solutions; Caputo derivative; cylindrical Wiener process; Krasnoselskii fixed point theorem

1. Introduction

Fractional differential equations provide a powerful framework for modeling processes in which past states continue to influence future dynamics. Owing to their ability to capture memory and hereditary behavior, such equations have become essential in many scientific and engineering applications [1–5]. In many realistic situations, however, the evolution of a system may be affected by abrupt external actions or sudden interventions, making it necessary to incorporate impulsive effects into the mathematical description.

At the same time, many real systems are also subject to random fluctuations [6]. When stochastic perturbations are introduced, for instance through Wiener processes [7], the resulting models are able to reflect uncertainty and noise. These stochastic versions are important in areas such as control theory, population dynamics, and financial mathematics (see, for example, [8, 9]).

During the past decade, many authors have investigated impulsive stochastic fractional differential equations involving different kinds of delays and nonlinearities; see [10, 11] and the references therein. For example, Yan and Zhang [12] studied impulsive fractional stochastic integro-differential equations with state-dependent delay and obtained existence and uniqueness results. Zhao and Song [13] analysed impulsive stochastic fractional evolution equations with infinite delay in Banach spaces and proved the existence of mild solutions. Their abstract formulation can be summarized as follows (see [13]):

$$\begin{cases} {}^{C}D_t^{\alpha}[x(t) - g(t, x_t)] = A\,x(t) + f(t, x_t) + \sigma(t, x_t)\,\dfrac{dw(t)}{dt}, \quad t \in J,\ t \neq t_i, \\ \Delta x(t_i) = I_i(x_{t_i}), \quad \Delta x'(t_i) = J_i(x_{t_i}),\ i = 1, \ldots, m, \\ x_0 = \varphi \in \mathcal{B},\ x_0' = x_1 \in H. \end{cases}$$

More recently, Dai et al. [14] investigated a class of d-dimensional nonlinear stochastic fractional integro-differential equations in Itô's sense [7]:

$$\begin{cases} D_{0+}^{\alpha} y(t) = f_0(t, y(t)) + \displaystyle\int_0^t (t-s)^{-\beta_1} f_1(t, s, y(s))\,ds \\ \qquad\qquad + \displaystyle\int_0^t (t-s)^{-\beta_2} f_2(t, s, y(s))\,\dot{d}\,W(s), \quad t \in (0, T], \\ \quad y(0) = y_0, \quad y \in \mathbb{R}^d. \end{cases}$$

However, most existing studies are limited either to finite-dimensional spaces or to kernels that are regular in time. Motivated by these works, we investigate here a new class of impulsive stochastic fractional integro-differential equations in separable Banach spaces, which in addition include weakly singular kernels of the form $(t-s)^{-\beta}$. Specifically, we consider the following problem:

$$\begin{cases} {}^{C}D_0^{\alpha} u(t) + Au(t) = \displaystyle\int_0^t (t-s)^{-\beta} f(s, u(s), u_s)\,ds + \sigma(t, u(t))\,dW(t), \\ \qquad t \in [0, T] \setminus \{t_1, \ldots, t_m\}, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), \quad k = 1, \ldots, m, \\ u(t) = \varphi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{1.1}$$

Here, $(F, \|\cdot\|_F)$ denotes a separable Banach space of type 2, and H is a separable Hilbert space. The operator ${}^{C}D_0^{\alpha}$ represents the Caputo fractional derivative of order $\alpha \in \left(\frac{1}{2}, 1\right)$, and $-A : D(A) \subset F \to F$ is a closed linear operator generating a compact analytic C_0-semigroup $\{S(t)\}_{t\geq 0}$ on F. The kernel $(t-s)^{-\beta}$, with $\beta \in (0, 1-\alpha)$, is weakly singular. The delay term is represented by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, 0]$. The noise term is modeled by a cylindrical Wiener process $W(t)$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, see [7]. The mapping $\sigma : [0, T] \times F \to \gamma(H, F)$ is the diffusion coefficient, where $\gamma(H, F)$ denotes the space of γ-radonifying operators, to be introduced in Section 2. The initial history φ is assumed to be $\mathcal{F}_0$-measurable. The nonlinear function f will be specified under appropriate assumptions later. To the best of our knowledge, the combined presence of impulsive effects, stochastic noise, finite delay, and weakly singular kernels has not yet been analyzed in the literature within a Caputo fractional framework in Banach spaces of type 2 [14–16].

The main goal of this paper is to establish the existence of mean-square mild solutions to (1.1) under natural assumptions. The proof relies on the theory of resolvent families, γ-radonifying stochastic integration in Banach spaces, and Krasnoselskii's fixed point theorem [17]. In this sense, our framework extends the previous models [13, 18] by incorporating weakly singular kernels $(t-s)^{-\beta}$. Finally, the paper is organized as follows. Section 2 introduces the functional setting and preliminary results. Section 3 presents the main existence theorem, while Section 4 provides an illustrative example. Section 5 concludes with a discussion of the main contributions and future perspectives.

2. Preliminaries

Let $(F, \|\cdot\|_F)$ be a separable Banach space type 2 and H a separable Hilbert space.

For a given $\tau > 0$, the phase space for the delay is taken as

$$\mathcal{B} := C([-\tau, 0], F), \qquad \|\phi\|_{\mathcal{B}} := \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\|_F, \quad \phi \in \mathcal{B}.$$

For a stochastic process u, the history segment is defined by $u_t(\theta) := u(t+\theta)$ for $\theta \in [-\tau, 0]$ and $t \in [0, T]$.

We consider the space

$$\mathcal{X}_T := \left\{ u : [-\tau, T] \to F \;\middle|\; \begin{array}{l} u|_{[0,T]} \text{ is } \{\mathcal{F}_t\}\text{-adapted and continuous in mean-square,} \\ u(t) = \varphi(t) \text{ for } t \in [-\tau, 0], \text{ with } \varphi \in \mathcal{B}, \\ \|u\|_{\mathcal{X}_T} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_F^2 \right] \right)^{1/2} < \infty \end{array} \right\}.$$

This space is a Banach space when endowed with the norm $\|\cdot\|_{\mathcal{X}_T}$.

Let $0 < t_1 < t_2 < \cdots < t_m < T$ be fixed impulse times. A stochastic process $u \in \mathcal{X}_T$ is said to be a solution of the impulsive system (1.1) if it satisfies the initial condition $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$, the impulsive conditions $u(t_k^+) = u(t_k^-) + I_k(u(t_k^-))$ for $k = 1, \ldots, m$, and the stochastic fractional equation on each subinterval (t_k, t_{k+1}), with $t_0 = 0$ and $t_{m+1} = T$.

Here, $\gamma(H, F)$ denotes the space of γ-radonifying operators from H to F, which is a Banach space equipped with the norm

$$\|\sigma\|_{\gamma(H,F)} := \left(\mathbb{E} \left\| \sum_{k=1}^{\infty} \gamma_k \sigma e_k \right\|_F^2 \right)^{1/2},$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of H, and $\{\gamma_k\}_{k=1}^{\infty}$ is a sequence of independent standard Gaussian random variables.

The stochastic integral $\int_0^t \sigma(s)\, dW(s)$ is understood in the sense of van Neerven, Veraar, and Weis [19], and satisfies the Itô isometry in Banach spaces.

We recall the following standard definitions.

Definition 2.1 (Cylindrical Wiener process [9])**.** *Let H be a separable Hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$. A* cylindrical Wiener motion *$W(t)$ on H is a formal series:*

$$W(t) := \sum_{k=1}^{\infty} \beta_k(t) e_k,$$

where $\{\beta_k\}_{k=1}^{\infty}$ are independent real-valued standard Brownian motions. Although this series does not converge in H, the pairing

$$\langle W(t), h\rangle_H := \sum_{k=1}^{\infty} \beta_k(t)\langle e_k, h\rangle_H, \quad \forall h \in H,$$

converges in $L^2(\Omega)$.

Definition 2.2 (γ-radonifying operators [20])**.** *Let H be a separable Hilbert space and F a separable Banach space. The space $\gamma(H,F)$ consists of all bounded linear operators $\sigma \in \mathcal{L}(H,F)$ such that, for some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H and a sequence $\{\gamma_k\}_{k=1}^{\infty}$ of independent standard Gaussian random variables,*

$$\|\sigma\|_{\gamma(H,F)} := \left(\mathbb{E}\left\|\sum_{k=1}^{\infty} \gamma_k \sigma e_k\right\|_F^2\right)^{1/2} < \infty.$$

Theorem 2.3 (Itô isometry in Banach spaces [19])**.** *Let F be a separable Banach space and H a separable Hilbert space. Let $\sigma : [0,T]\times\Omega \to \gamma(H,F)$ be a strongly measurable, $\{\mathcal{F}_t\}$-adapted process such that*

$$\mathbb{E}\left[\int_0^T \|\sigma(s)\|_{\gamma(H,F)}^2 \, ds\right] < \infty.$$

Then, the stochastic integral

$$\int_0^t \sigma(s)\, dW(s)$$

is well-defined as an F-valued random variable for every $t \in [0,T]$, belongs to $L^2(\Omega; F)$, and satisfies

$$\mathbb{E}\left[\left\|\int_0^t \sigma(s)\, dW(s)\right\|_F^2\right] = \mathbb{E}\left[\int_0^t \|\sigma(s)\|_{\gamma(H,F)}^2 \, ds\right].$$

Moreover, if the map $t \mapsto \sigma(t)$ is in $L^2(\Omega; L^2(0,T;\gamma(H,F)))$, then the process $t \mapsto \int_0^t \sigma(s)\, dW(s)$ admits a modification with continuous trajectories in F.

Remark 2.4. *Since F is a Banach space of type 2, there exists a constant $C_F > 0$, depending only on the geometry of F, such that for any $\{\mathcal{F}_t\}$-adapted process $\sigma \in L^2(\Omega; L^2(0,T;\gamma(H,F)))$, the stochastic convolution satisfies the maximal inequality:*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \sigma(s)\, dW(s)\right\|_F^2\right] \le C_F\, \mathbb{E}\left[\int_0^T \|\sigma(s)\|_{\gamma(H,F)}^2 \, ds\right].$$

This property follows from the fact that Banach spaces of type 2 have martingale type 2; see [19].

Theorem 2.5 (Krasnoselskii's theorem [17, 21])**.** *Let X be a Banach space and $\mathcal{M} \subset X$ nonempty, closed, bounded, and convex. Suppose $\mathcal{A}$, $\mathcal{B} : \mathcal{M} \to X$ satisfy the following:*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ *for all* $x, y \in \mathcal{M}$*;*
(ii) $\mathcal{A}$ *is a contraction;*
(iii) $\mathcal{B}$ *is continuous and compact on* $\mathcal{M}$*.*

Then, $\mathcal{N} = \mathcal{A} + \mathcal{B}$ *has a fixed point in* $\mathcal{M}$*.*

We now introduce the concept of a mild solution for the impulsive problem (1.1). Let $-A$ generate a compact analytic C_0-semigroup $\{S(t)\}_{t\geq 0}$ on F. Following the theory of resolvent families for the Caputo fractional derivative (see Bazhlekova [8], we define the solution operators as

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)\,d\lambda,$$
$$T_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha, A)\,d\lambda,$$

where Γ is a suitable contour in the complex plane enclosing the spectrum of A.

Definition 2.6 (Mean-square mild solution)**.** *A stochastic process* $u : [-\tau, T] \times \Omega \to F$ *is called a* mean-square mild solution *of* (1.1) *if for each* $t \in [0, T]$, $\mathbb{P}$*-a.s.,*

$$\begin{aligned} u(t) = S_\alpha(t)\varphi(0) &+ \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k^-)) \\ &+ \int_0^t T_\alpha(t-s)\left(\int_0^s (s-\xi)^{-\beta} f(\xi, u(\xi), u_\xi)\,d\xi\right)ds \\ &+ \int_0^t T_\alpha(t-s)\,\sigma(s, u(s))\,dW(s). \end{aligned}$$

This formulation follows the standard construction for Caputo fractional stochastic evolution equations with impulses; see [3, 8, 15].

3. Existence of mild solution

We impose the following assumptions:

(H$_1$) The operator $-A$ generates a compact analytic C_0-semigroup on F.
(H$_2$) The solution operators $S_\alpha(t)$ and $T_\alpha(t)$ associated with the linear part of the system satisfy the following estimates: There exist positive constants M, C_T, C_H and an exponent $\lambda_0 \in \left(0, \alpha - \frac{1}{2}\right)$ such that

$$\|S_\alpha(t)\| \leq M, \quad \|T_\alpha(t)\| \leq C_T t^{\alpha-1} \quad (t > 0),$$

and, moreover, the operator family $T_\alpha(t)$ is assumed to be Hölder continuous with respect to t in the operator norm; that is,

$$\|T_\alpha(t) - T_\alpha(s)\| \leq C_H|t-s|^{\lambda_0}, \quad 0 < s < t \leq T.$$

(H$_3$) The function $f : [0,T] \times F \times \mathcal{B} \to F$ is measurable in t, continuous in (x,ϕ), and there exist $b > 0$ and $a \in L^{1/\alpha}([0,T],\mathbb{R}_+)$ such that

$$\|f(t,x,\phi)\|_F \le a(t) + b(\|x\|_F + \|\phi\|_{\mathcal{B}}), \quad t \in [0,T],\ x \in F,\ \phi \in \mathcal{B}.$$

(H$_4$) The diffusion coefficient $\sigma : [0,T]\times F \to \gamma(H,F)$ satisfies Lipschitz and linear growth conditions: There exist constants $L_\sigma, C_\sigma > 0$ such that for all $t \in [0,T]$ and $x, y \in F$,

$$\|\sigma(t,x) - \sigma(t,y)\|_{\gamma(H,F)} \le L_\sigma \|x - y\|_F, \qquad \|\sigma(t,x)\|_{\gamma(H,F)} \le C_\sigma(1 + \|x\|_F).$$

(H$_5$) For each $k = 1,\dots,m$, the impulsive function $I_k : F \to F$ is continuous and bounded; that is, there exists $d_k \ge 0$ such that

$$\|I_k(x)\|_F \le d_k, \quad \forall x \in F.$$

Fix $R > 0$, and consider the following closed convex set:

$$B_R := \Big\{u \in \mathcal{X}_T : \|u\|_{\mathcal{X}_T} \le R\Big\}.$$

Define the constants:

$$A := M\|\varphi\|_{\mathcal{B}} + M\sum_{k=1}^{m} d_k + \sqrt{2}\,C_T\,\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} T^{1-\beta}B(\alpha, 2-\alpha-\beta),$$
$$L_f := 2\sqrt{2}\,b\,C_T\,\frac{T^{\alpha+1-\beta}}{1-\beta}B(\alpha, 2-\beta),$$
$$K := \sqrt{2C_F}\,C_T\,C_\sigma\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}.$$

We assume that R satisfies

$$R \ge \frac{A+K}{1-L_f-K}. \tag{3.1}$$

Theorem 3.1. *Assume that hypotheses* (H$_1$)–(H$_5$) *hold. Suppose further that*

$$L_f + K < 1, \qquad C_T L_\sigma \sqrt{\frac{C_F T^{2\alpha-1}}{2\alpha-1}} < 1.$$

Then, the impulsive system (1.1) *admits at least one mean-square mild solution $u \in \mathcal{X}_T$.*

Proof. Let the operator $\mathcal{N} : B_R \to \mathcal{X}_T$ be defined for $t \in [0,T]$ by

$$\begin{aligned}(\mathcal{N}u)(t) &= S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k^-)) \\ &\quad + \int_0^t T_\alpha(t-s)\left(\int_0^s (s-\xi)^{-\beta} f(\xi, u(\xi), u_\xi)\,d\xi\right)ds \\ &\quad + \int_0^t T_\alpha(t-s)\sigma(s,u(s))\,dW(s).\end{aligned}$$

We split $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ with

$$(\mathcal{N}_1 u)(t) := \int_0^t T_\alpha(t-s)\sigma(s,u(s))\,dW(s),$$
$$(\mathcal{N}_2 u)(t) := S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k^-)) + \int_0^t T_\alpha(t-s)\left(\int_0^s (s-\xi)^{-\beta} f(\xi,u(\xi),u_\xi)\,d\xi\right)ds.$$

Set $\mathcal{F}(s) := \int_0^s (s-\xi)^{-\beta} f(\xi,u(\xi),u_\xi)\,d\xi.$

We shall show that the operator $\mathcal{N}$ has a fixed point in $B_R \subset \mathcal{X}_T$, which is a mean-square mild solution of (1.1).

Step 1: proving $\mathcal{N}$ is well-defined and $\mathcal{N}(B_R) \subset B_R$. Let $u \in B_R$ and $t \in [0,T]$.

First, for the stochastic part of $\mathcal{N}_1$, using Remark 2.4, (H_2), and (H_4), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t T_\alpha(t-s)\,\sigma(s,u(s))\,dW(s)\right\|_F^2\right]\\
&\le C_F\,\mathbb{E}\left[\int_0^T \sup_{t\ge s}\|T_\alpha(t-s)\|^2\,\|\sigma(s,u(s))\|^2_{\gamma(H,F)}\;ds\right]\\
&\le C_F\,\mathbb{E}\left[\int_0^T \left(C_T(T-s)^{\alpha-1}\right)^2\|\sigma(s,u(s))\|^2_{\gamma(H,F)}\;ds\right]\\
&= C_F C_T^2\int_0^T (T-s)^{2\alpha-2}\,\mathbb{E}\left[\|\sigma(s,u(s))\|^2_{\gamma(H,F)}\right]ds\\
&\le C_F C_T^2 C_\sigma^2\int_0^T (T-s)^{2\alpha-2}\,\mathbb{E}\left[(1+\|u(s)\|_F)^2\right]ds\\
&\le C_F C_T^2 C_\sigma^2\int_0^T (T-s)^{2\alpha-2}\,2\,(1+R^2)\,ds\\
&= 2C_F C_T^2 C_\sigma^2(1+R^2)\int_0^T (T-s)^{2\alpha-2}\,ds\\
&= 2C_F C_T^2 C_\sigma^2(1+R^2)\frac{T^{2\alpha-1}}{2\alpha-1}.
\end{aligned}$$

Taking square roots and the supremum over $t \in [0,T]$, we obtain

$$\begin{aligned}
\|\mathcal{N}_1 u\|_{\mathcal{X}_T} &\le \sqrt{2\,C_F}C_T C_\sigma\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\sqrt{1+R^2}\\
&= K\sqrt{1+R^2}.
\end{aligned}$$

Second, for the deterministic part $\mathcal{N}_2$, we estimate each component. Since φ is deterministic, by (H_2),

$$\|S_\alpha(\cdot)\varphi(0)\|_{\mathcal{X}_T} = \left(\mathbb{E}\left[\sup_{t\in[0,T]}\|S_\alpha(t)\varphi(0)\|_F^2\right]\right)^{1/2} = \sup_{t\in[0,T]}\|S_\alpha(t)\varphi(0)\|_F \le M\|\varphi\|_{\mathcal{B}}.$$

By (H_2) and (H_5), and since $\|I_k(u(t_k^-))\|_F \le d_k$,

$$\left\|\sum_{0<t_k<\cdot} S_\alpha(\cdot - t_k)I_k(u(t_k^-))\right\|_{\mathcal{X}_T} = \left(\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k^-))\right\|_F^2\right]\right)^{1/2} \le M\sum_{k=1}^m d_k.$$

For the integral term, by (H_3) and the definition of $\mathcal{F}(s)$,

$$\begin{aligned}
\|\mathcal{F}(s)\|_F &= \left\|\int_0^s (s-\xi)^{-\beta} f(\xi, u(\xi), u_\xi)\,d\xi\right\|_F \\
&\le \int_0^s (s-\xi)^{-\beta}\big\|f(\xi, u(\xi), u_\xi)\big\|_F\,d\xi \\
&\le \int_0^s (s-\xi)^{-\beta}\big(a(\xi) + b(\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}})\big)\,d\xi \\
&= \int_0^s (s-\xi)^{-\beta}a(\xi) + b\int_0^s (s-\xi)^{-\beta}\big((\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}})\big)\,d\xi \\
&\le \|a\|_{L^{1/\alpha}}\left(\int_0^s (s-\xi)^{-\frac{\beta}{1-\alpha}}d\xi\right)^{1-\alpha} + b\int_0^s (s-\xi)^{-\beta}\big((\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}})\big)\,d\xi \\
&= \|a\|_{L^{1/\alpha}}\left(\frac{s^{1-\frac{\beta}{1-\alpha}}}{1-\frac{\beta}{1-\alpha}}\right)^{1-\alpha} + b\int_0^s (s-\xi)^{-\beta}\big((\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}})\big)\,d\xi \\
&= \|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} s^{1-\alpha-\beta} + b\int_0^s (s-\xi)^{-\beta}\big((\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}})\big)\,d\xi. \qquad (3.2)
\end{aligned}$$

Hence, using (H_2),

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t T_\alpha(t-s)\mathcal{F}(s)\,ds\right\|_F^2\right] \\
&\le C_T^2\,\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{\alpha-1}\|\mathcal{F}(s)\|_F\,ds\right)^2\right] \\
&\le 2C_T^2\,\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{\alpha-1}\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} s^{1-\alpha-\beta}\,ds\right)^2\right] \\
&\quad + 2C_T^2 b^2\,\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\xi)^{-\beta}(\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}})\,d\xi\,ds\right)^2\right] \\
&=: 2C_T^2\,\mathbb{E}\left[\sup_{t\in[0,T]} I_1(t)^2\right] + 2C_T^2 b^2\,\mathbb{E}\left[\sup_{t\in[0,T]} I_2(t)^2\right].
\end{aligned}$$

We now estimate $I_1(t)$ and $I_2(t)$ separately.

Estimate of $I_1(t)$: We have

$$I_1(t) = \int_0^t (t-s)^{\alpha-1} \|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} s^{1-\alpha-\beta}\, ds$$
$$= \|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} t^{1-\beta} \int_0^1 (1-r)^{\alpha-1} r^{1-\alpha-\beta}\, dr \quad (\text{via } s = tr)$$
$$= \|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} t^{1-\beta} B(\alpha, 2-\alpha-\beta).$$

Therefore,

$$\sup_{t\in[0,T]} I_1(t) \le \|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} T^{1-\beta} B(\alpha, 2-\alpha-\beta),$$

and consequently,

$$\mathbb{E}\left[\sup_{t\in[0,T]} I_1(t)^2\right] \le \left(\|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha} T^{1-\beta} B(\alpha, 2-\alpha-\beta)\right)^2.$$

Estimate of $I_2(t)$: For any $\xi \in [0,t]$, we have

$$\|u(\xi)\|_F + \|u_\xi\|_{\mathcal{B}} \le \|u(\xi)\|_F + \sup_{\theta\in[-\tau,0]} \|u(\xi+\theta)\|_F \le 2 \sup_{r\in[-\tau,T]} \|u(r)\|_F =: 2 \sup_{r\in[-\tau,T]} \|u(r)\|_F.$$

Therefore,

$$I_2(t) \le 2 \sup_{r\in[-\tau,T]} \|u(r)\|_F \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\xi)^{-\beta}\, d\xi\, ds$$
$$= 2 \sup_{r\in[-\tau,T]} \|u(r)\|_F \int_0^t (t-s)^{\alpha-1} \frac{s^{1-\beta}}{1-\beta}\, ds$$
$$\le \frac{2\sup_{r\in[-\tau,T]} \|u(r)\|_F}{1-\beta} \int_0^t (t-s)^{\alpha-1} s^{1-\beta}\, ds$$
$$= \frac{2\sup_{r\in[-\tau,T]} \|u(r)\|_F}{1-\beta} t^{\alpha+1-\beta} B(\alpha, 2-\beta).$$

Hence,

$$\sup_{t\in[0,T]} I_2(t) \le \sup_{r\in[-\tau,T]} \|u(r)\|_F \frac{2B(\alpha,2-\beta)}{1-\beta} T^{\alpha+1-\beta}.$$

We obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} I_2(t)^2\right] \le \left(\frac{2B(\alpha,2-\beta)}{1-\beta} T^{\alpha+1-\beta}\right)^2 \mathbb{E}\left[\left(\sup_{r\in[-\tau,T]} \|u(r)\|_F\right)^2\right]$$
$$= \left(\frac{2B(\alpha,2-\beta)}{1-\beta} T^{\alpha+1-\beta}\right)^2 \|u\|_{\mathcal{X}_T}^2$$
$$\le \left(\frac{2B(\alpha,2-\beta)}{1-\beta} T^{\alpha+1-\beta}\right)^2 R^2.$$

Adding the two components, we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t T_\alpha(t-s)\mathcal{F}(s)\,ds\right\|_F^2\right]\\
&\le 2C_T^2\left(\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha}T^{1-\beta}B(\alpha,2-\alpha-\beta)\right)^2\\
&+2C_T^2b^2\left(\frac{2B(\alpha,2-\beta)}{1-\beta}T^{\alpha+1-\beta}\right)^2R^2\\
&=2C_T^2\left[\left(\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha}T^{1-\beta}B(\alpha,2-\alpha-\beta)\right)^2\right.\\
&\quad\left.+\left(\frac{2bB(\alpha,2-\beta)}{1-\beta}T^{\alpha+1-\beta}\right)^2R^2\right].
\end{aligned}$$

Combining these three estimates, we obtain for every $t\in[0,T]$,

$$\begin{aligned}
\|\mathcal{N}_2u\|_{\mathcal{X}_T} &\le \|S_\alpha(\cdot)\varphi(0)\|_{\mathcal{X}_T}+\left\|\sum_{0<t_k<\cdot}S_\alpha(\cdot-t_k)I_k(u(t_k^-))\right\|_{\mathcal{X}_T}\\
&\quad+\left\|\int_0^{\cdot}T_\alpha(\cdot-s)\mathcal{F}(s)\,ds\right\|_{\mathcal{X}_T}\\
&\le M\|\varphi\|_{\mathcal{B}}+M\sum_{k=1}^m d_k\\
&\quad+\sqrt{2}C_T\left(\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha}T^{1-\beta}B(\alpha,2-\alpha-\beta)\right)\\
&\quad+\sqrt{2}C_Tb\left(\frac{2B(\alpha,2-\beta)}{1-\beta}T^{\alpha+1-\beta}\right)R\\
&=A+L_fR.
\end{aligned}$$

Therefore, since $\sqrt{1+R^2}\le 1+R$ for all $R\ge 0$, we obtain

$$\begin{aligned}
\|\mathcal{N}u\|_{\mathcal{X}_T} &\le \|\mathcal{N}_1u\|_{\mathcal{X}_T}+\|\mathcal{N}_2u\|_{\mathcal{X}_T}\\
&\le K\sqrt{1+R^2}+A+L_fR\\
&\le K(1+R)+A+L_fR.
\end{aligned}$$

By the choice of R in (3.1), we have $K(1+R)+A+L_fR\le R$, and hence $\|\mathcal{N}u\|_{\mathcal{X}_T}\le R$. Thus, $\mathcal{N}(B_R)\subset B_R$.

Step 2: $\mathcal{N}_1$ is a contraction. Let $u, v \in B_R$. Then,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathcal{N}_1 u(t) - \mathcal{N}_1 v(t)\|_F^2\right] \\
&= \mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t T_\alpha(t-s)[\sigma(s,u(s)) - \sigma(s,v(s))]\, dW(s) \right\|_F^2\right] \\
&\le C_F\, \mathbb{E}\left[\int_0^T \sup_{t\ge s} \|T_\alpha(t-s)\|^2 \, \|\sigma(s,u(s)) - \sigma(s,v(s))\|_{\gamma(H,F)}^2 \, ds\right] \\
&\le C_F\, \mathbb{E}\left[\int_0^T \left(C_T (T-s)^{\alpha-1}\right)^2 \|\sigma(s,u(s)) - \sigma(s,v(s))\|_{\gamma(H,F)}^2 \, ds\right] \\
&\le C_F C_T^2 \int_0^T (T-s)^{2\alpha-2}\, \mathbb{E}\left[L_\sigma^2 \|u(s) - v(s)\|_F^2\right] ds \\
&\le C_F C_T^2 L_\sigma^2 \|u - v\|_{\mathcal{X}}^2 \int_0^T (T-s)^{2\alpha-2}\, ds \\
&= C_F C_T^2 L_\sigma^2 \|u - v\|_{\mathcal{X}}^2 \frac{T^{2\alpha-1}}{2\alpha - 1}.
\end{aligned}
$$

Taking the supremum over $t \in [0, T]$ and the square root,

$$
\|\mathcal{N}_1(u) - \mathcal{N}_1(v)\|_{\mathcal{X}_T} \le C_T L_\sigma \sqrt{\frac{C_F T^{2\alpha-1}}{2\alpha - 1}}\, \|u - v\|_{\mathcal{X}_T}.
$$

By assumption, the constant on the right-hand side is strictly less than 1, so $\mathcal{N}_1$ is a contraction.

Step 3: Equicontinuity of $\mathcal{N}_2(B_R)$. Let $\vartheta_1, \vartheta_2 \in [0, T]$ with $\vartheta_1 < \vartheta_2$, and assume that no impulse time lies in $(\vartheta_1, \vartheta_2]$ (possible since the set of impulses is finite). For any $u \in B_R$, we have

$$
\begin{aligned}
&\|\mathcal{N}_2 u(\vartheta_2) - \mathcal{N}_2 u(\vartheta_1)\|_F \\
&= \Bigg\| \left[S_\alpha(\vartheta_2)\varphi(0) - S_\alpha(\vartheta_1)\varphi(0)\right] \\
&\qquad + \left[\sum_{0<t_k<\vartheta_2} S_\alpha(\vartheta_2 - t_k) I_k(u(t_k^-)) - \sum_{0<t_k<\vartheta_1} S_\alpha(\vartheta_1 - t_k) I_k(u(t_k^-))\right] \\
&\qquad + \left[\int_0^{\vartheta_2} T_\alpha(\vartheta_2 - s)\mathcal{F}(s)\, ds - \int_0^{\vartheta_1} T_\alpha(\vartheta_1 - s)\mathcal{F}(s)\, ds\right] \Bigg\|_F \\
&\le \|S_\alpha(\vartheta_2)\varphi(0) - S_\alpha(\vartheta_1)\varphi(0)\|_F \\
&\quad + \left\| \sum_{0<t_k<\vartheta_1} [S_\alpha(\vartheta_2 - t_k) - S_\alpha(\vartheta_1 - t_k)] I_k(u(t_k^-)) \right\|_F \\
&\quad + \left\| \int_0^{\vartheta_2} T_\alpha(\vartheta_2 - s)\mathcal{F}(s)\, ds - \int_0^{\vartheta_1} T_\alpha(\vartheta_1 - s)\mathcal{F}(s)\, ds \right\|_F .
\end{aligned}
$$

We now estimate the first two terms together as the initial and impulsive part, and the last term as the deterministic integral part.

(i) Initial and impulsive terms:

$$\begin{aligned}
&\|S_\alpha(\vartheta_2)\varphi(0) - S_\alpha(\vartheta_1)\varphi(0)\|_F + \left\| \sum_{0<t_k<\vartheta_1} [S_\alpha(\vartheta_2 - t_k) - S_\alpha(\vartheta_1 - t_k)] I_k(u(t_k^-)) \right\|_F \\
&\le \|S_\alpha(\vartheta_2) - S_\alpha(\vartheta_1)\| \, \|\varphi\|_{\mathcal{B}} + \sum_{k=1}^{m} \|S_\alpha(\vartheta_2 - t_k) - S_\alpha(\vartheta_1 - t_k)\| \, \|I_k(u(t_k^-))\|_F \\
&\le \|S_\alpha(\vartheta_2) - S_\alpha(\vartheta_1)\| \, \|\varphi\|_{\mathcal{B}} + \sum_{k=1}^{m} \|S_\alpha(\vartheta_2 - t_k) - S_\alpha(\vartheta_1 - t_k)\| \, d_k.
\end{aligned}$$

By (H_1), the operator family $S_\alpha(t)$ is strongly continuous on $[0, T]$. Since the set $\{I_k(u(t_k^-)) : u \in B_R\}$ is bounded in F (by (H_5)) and $S_\alpha(t)$ is compact for $t > 0$, the mapping

$$t \mapsto S_\alpha(t - t_k) I_k(u(t_k^-))$$

is continuous uniformly with respect to $u \in B_R$. Therefore,

$$\left\| [S_\alpha(\vartheta_2 - t_k) - S_\alpha(\vartheta_1 - t_k)] I_k(u(t_k^-)) \right\|_F \to 0 \quad \text{as } |\vartheta_2 - \vartheta_1| \to 0,$$

uniformly for all $u \in B_R$ and for each $k = 1, \ldots, m$.

(ii) Deterministic integral terms: We estimate the mean-square difference. For any $u \in B_R$,

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^{\vartheta_2} T_\alpha(\vartheta_2 - s)\mathcal{F}(s)\, ds - \int_0^{\vartheta_1} T_\alpha(\vartheta_1 - s)\mathcal{F}(s)\, ds \right\|_F^2 \right] \\
&\le 2\,\mathbb{E}\left[\left\| \int_{\vartheta_1}^{\vartheta_2} T_\alpha(\vartheta_2 - s)\mathcal{F}(s)\, ds \right\|_F^2 \right] \\
&\quad + 2\,\mathbb{E}\left[\left\| \int_0^{\vartheta_1} [T_\alpha(\vartheta_2 - s) - T_\alpha(\vartheta_1 - s)]\mathcal{F}(s)\, ds \right\|_F^2 \right] \\
&\le 2C_T^2\,\mathbb{E}\left[\left(\int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{\alpha-1} \|\mathcal{F}(s)\|_F\, ds \right)^2 \right] \\
&\quad + 2C_H^2 |\vartheta_2 - \vartheta_1|^{2\lambda_0}\,\mathbb{E}\left[\left(\int_0^{\vartheta_1} \|\mathcal{F}(s)\|_F\, ds \right)^2 \right].
\end{aligned}$$

By the estimate (3.2) established in Step 1, for all $s \in [0, T]$, we have

$$\|\mathcal{F}(s)\|_F \le \|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta} \right)^{1-\alpha} s^{1-\alpha-\beta} + 2bM_u \frac{s^{1-\beta}}{1-\beta},$$

where $M_u = \sup_{r \in [-\tau, T]} \|u(r)\|_F$. Then $\mathbb{E}[M_u^2] = \|u\|_{\mathcal{X}_T}^2 \le R^2$.

Therefore,

$$\|\mathcal{F}(s)\|_F \le \|a\|_{L^{1/\alpha}} \left(\frac{1-\alpha}{1-\alpha-\beta} \right)^{1-\alpha} s^{1-\alpha-\beta} + 2bM_u \frac{s^{1-\beta}}{1-\beta}.$$

Thus,

$$\mathbb{E}\left[\left(\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-s)^{\alpha-1}\|\mathcal{F}(s)\|_F ds\right)^2\right] \le 2\left(\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-s)^{\alpha-1}\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha}s^{1-\alpha-\beta}ds\right)^2$$
$$+ 8b^2\mathbb{E}[M_u^2]\left(\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-s)^{\alpha-1}\frac{s^{1-\beta}}{1-\beta}ds\right)^2$$
$$\le 2\left(\|a\|_{L^{1/\alpha}}\left(\frac{1-\alpha}{1-\alpha-\beta}\right)^{1-\alpha}\frac{(\vartheta_2-\vartheta_1)^\alpha}{\alpha}T^{1-\alpha-\beta}\right)^2$$
$$+ 8b^2R^2\left(\frac{T^{1-\beta}}{1-\beta}\frac{(\vartheta_2-\vartheta_1)^\alpha}{\alpha}\right)^2$$
$$\longrightarrow 0 \quad \text{as } \vartheta_2 \to \vartheta_1.$$

Therefore,

$$\lim_{|\vartheta_2-\vartheta_1|\to 0}\sup_{u\in B_R}\mathbb{E}\left[\|\mathcal{N}_2u(\vartheta_2)-\mathcal{N}_2u(\vartheta_1)\|_F^2\right]=0,$$

so the family $\mathcal{N}_2(B_R)$ is equicontinuous on each interval $[t_k, t_{k+1}]$.

Step 4: $\mathcal{N}_2$ is compact on B_R. To apply the Arzelà–Ascoli theorem, we verify the two required properties: equicontinuity and pointwise relative compactness. Equicontinuity of $\mathcal{N}_2(B_R)$ on each interval $[t_k, t_{k+1}]$ was established in Step 3. Moreover, since the resolvent family $\{T_\alpha(t)\}_{t>0}$ is compact (by (H_1))) and $\mathcal{F}(s)$ is uniformly bounded for $u \in B_R$ (as shown in Step 1), the set $\{(\mathcal{N}_2u)(t) : u \in B_R\}$ is relatively compact in F for each $t \in (0, T]$. At $t = 0$, $(\mathcal{N}_2u)(0) = S_\alpha(0)\varphi(0) = \varphi(0)$ is constant, hence trivially compact. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{N}_2(B_R)$ is relatively compact in $\mathcal{X}_T$. The continuity of f and I_k ensures that $\mathcal{N}_2$ is continuous, hence completely continuous (i.e., compact).

From Steps 1–4, we have that $\mathcal{N}(B_R) \subset B_R$, $\mathcal{N}_1$ is a contraction on B_R, and $\mathcal{N}_2$ is continuous and compact on B_R. Since B_R is a nonempty, closed, bounded, and convex subset of the Banach space $\mathcal{X}$, all the conditions of Krasnoselskii's fixed point theorem (Theorem 2.5) are satisfied. Therefore, the operator $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ admits at least one fixed point $u \in B_R \subset \mathcal{X}_T$. This fixed point is a mean-square mild solution of the system (1.1).

4. Example

Consider the following impulsive stochastic fractional partial integro-differential equation with delay on the interval $(0, \pi)$:

$$\begin{cases} {}^C D_0^\alpha u(t,\xi) = \dfrac{\partial^2 u}{\partial \xi^2}(t,\xi) + \displaystyle\int_0^t (t-s)^{-\beta}\frac{\sin s}{1+\|u_s(0)\|_{L^2}^2}(u(s,\xi)+u(s-\tau,\xi))\,ds \\ \qquad\qquad + \dfrac{\cos t}{2+\|u(t)\|_{L^2}}\,u(t,\xi)\,dW(t,\xi), \quad t\in[0,0.2]\setminus\{t_k\}_{k=1}^m,\ \xi\in(0,\pi), \\ u(t_k^+,\xi) = u(t_k^-,\xi) + \dfrac{1}{k^2(1+\|u(t_k^-)\|_{L^2}^2)}, \quad k=1,\ldots,m, \\ u(t,\xi) = \varphi(t,\xi), \quad t\in[-\tau,0],\ \xi\in(0,\pi). \end{cases} \tag{4.1}$$

Subject to the Dirichlet boundary conditions,

$$u(t,0) = u(t,\pi) = 0, \quad t \in [0, 0.2],$$

where the following are true:

$\alpha \in \left(\frac{1}{2}, 1\right), \beta \in (0, 1-\alpha)$, $\tau > 0$, and $0 < t_1 < \cdots < t_m < 0.2$ are fixed;
$W(t)$ is a cylindrical Wiener process on $H = L^2(0,\pi)$;
the delay segment is $u_t(\theta)(\xi) = u(t+\theta, \xi)$ for $\theta \in [-\tau, 0]$;
$\varphi \in C([-\tau, 0], L^2(0,\pi))$ is a given initial history function.

We show that this problem fits into the abstract framework of Section 3. Let $F = L^2(0,\pi)$, and define the operator $A : D(A) \subset F \to F$ by

$$Ay = \frac{d^2y}{dx^2}, \qquad D(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

Then, $-A$ generates a compact analytic C_0-semigroup on F, and the associated solution operators $S_\alpha(t)$ and $T_\alpha(t)$ satisfy the estimates in (H_2); see [9].
where $f : [0, 0.2] \times F \times \mathcal{B} \to F$, and $I_k : F \to F$ are defined by

$$f(t,x,\phi)(\xi) := \frac{\sin(t)}{1 + \|\phi(0)\|_{L^2}^2}(x(\xi) + \phi(-\tau)(\xi)),$$
$$I_k(x)(\xi) := \frac{1}{k^2(1 + \|x\|_{L^2}^2)}, \quad \text{for a.e. } \xi \in (0,\pi).$$

Let $(\varphi_k)_{k\geq 1}$ be the orthonormal basis of $F = L^2(0,\pi)$ given by eigenfunctions:

$$\varphi_k(\xi) := \sqrt{\frac{2}{\pi}}\sin(k\xi), \qquad k \geq 1.$$

Choose a sequence $(c_k)_{k\geq 1}$ such that $\sum_{k=1}^{\infty} c_k^2 < \infty$ (for instance $c_k = 1/k^2$). Let $(e_k)_{k\geq 1}$ be an orthonormal basis of H (one may take $e_k = \varphi_k$).

For $(t,x) \in [0, 0.2] \times F$, define the operator as

$$\sigma(t,x)h := \frac{\cos t}{2 + \|x\|_{L^2}} \sum_{k=1}^{\infty} c_k \langle h, e_k\rangle_H \, \varphi_k, \qquad h \in H.$$

Then, $\sigma(t,x) \in \gamma(H,F)$ and

$$\|\sigma(t,x)\|_{\gamma(H,F)} = \frac{|\cos t|}{2 + \|x\|_{L^2}}\Big(\sum_{k=1}^{\infty} c_k^2\Big)^{1/2}.$$

These functions satisfy the required assumptions:

(i) For all t, x, ϕ, we have

$$\|f(t,x,\phi)\|_{L^2} \leq \|x\|_{L^2} + \|\phi\|_{\mathcal{B}},$$

so (H_3) holds with $a \equiv 0$, $b = 1$;

(ii) The function σ satisfies the Lipschitz and growth conditions: there exist constants $L_\sigma, C_\sigma > 0$ such that

$$\|\sigma(t,x) - \sigma(t,y)\|_{\gamma(H,F)} \le L_\sigma \|x - y\|_{L^2},$$

$$\|\sigma(t,x)\|_{\gamma(H,F)} \le C_\sigma(1 + \|x\|_{L^2}),$$

because the map $x \mapsto (2 + \|x\|_{L^2})^{-1}$ is Lipschitz on $[0, +\infty)$ and $\sum_k c_k^2 < \infty$. Thus (H_4) holds.

(iii) Each I_k is continuous and bounded, since $\|I_k(x)\|_{L^2} \le \dfrac{\sqrt{\pi}}{k^2}$ for all $x \in F$, so (H_5) holds with $d_k = \dfrac{\sqrt{\pi}}{k^2}$.

All assumptions (H_1)–(H_5) are satisfied. To ensure the smallness conditions in Theorem 3.1, choose $\alpha = 0.6$, $\beta = 0.2$, $T = 0.2$, $C_T = C_F = 1$, $b = 1$, and $C_\sigma = 0.2$. Then,

$$L_f := 2\sqrt{2}\, b\, C_T \frac{T^{\alpha+1-\beta}}{1-\beta} B(\alpha, 2-\beta) \approx 0.25, \quad K := \sqrt{2C_F}\, C_T\, C_\sigma \sqrt{\frac{T^{2\alpha-1}}{2\alpha - 1}} \approx 0.65,$$

so that $L_f + K \approx 0.90 < 1$, and

$$C_T L_\sigma \sqrt{\frac{C_F T^{2\alpha-1}}{2\alpha - 1}} \approx 0.10 < 1.$$

Therefore, the existence of a mean-square mild solution of problem (4.1) is guaranteed on $[0, T]$ with $T = 0.2$.

5. Conclusions

This paper addresses the existence of mean-square mild solutions for a class of impulsive stochastic fractional integro-differential equations with finite delay and weakly singular memory kernels of the form $(t - s)^{-\beta}$, where $\beta \in (0, 1 - \alpha)$. The model combines Caputo fractional dynamics of order $\alpha \in (\frac{1}{2}, 1)$, cylindrical Wiener noise, instantaneous impulses, and nonlocal integral terms, all set within a separable Banach space F of type 2. This functional setting ensures the well-posedness of the stochastic convolution via γ-radonifying operators and supports the compactness arguments required for fixed point methods.

The proof hinges on the properties of the resolvent families $S_\alpha(t)$ and $T_\alpha(t)$ generated by the linear operator $-A$, assumed to induce a compact analytic C_0-semigroup. Under natural growth conditions on the nonlinearity f and Lipschitz continuity of the diffusion coefficient σ, the solution operator is decomposed into a contraction and a completely continuous part. Krasnoselskii's fixed point theorem then yields the desired existence result without requiring global Lipschitz assumptions on f.

This framework extends existing models by incorporating weakly singular kernels into a stochastic impulsive fractional setting. Future work will explore generalizations to alternative fractional derivatives such as Hilfer or Hadamard types, investigate stability and controllability properties of mild solutions, and develop numerical approximation schemes for stochastic fractional systems with memory and impulses. These directions aim to bridge theoretical analysis with applications in viscoelasticity, anomalous transport, and mathematical finance, where memory effects, randomness, and abrupt changes coexist.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. S. Abbas, B. Ahmad, M. Benchohra, A. Salim, *Fractional Difference, Differential Equations and Inclusions: Analysis and Stability*, Morgan Kaufmann, Cambridge, 2024. https://doi.org/10.1016/C2023-0-00030-9
2. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012.
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.
4. A. Salim, S. Bouriah, M. Benchohra, J. E. Lazreg, E. Karapınar, A study on k-generalized ψ-Hilfer fractional differential equations with periodic integral conditions, *Math. Methods Appl. Sci.*, **47** (2023), 1–18. https://doi.org/10.1002/mma.9056
5. A. Salim, M. Benchohra, J. E. Lazreg, Nonlocal k-generalized ψ-Hilfer impulsive initial value problem with retarded and advanced arguments, *Appl. Anal. Optim.*, **6** (2022), 21–47.
6. A. Salim, F. Mesri, M. Benchohra, C. Tunç, Controllability of second order semilinear random differential equations in Fréchet spaces, *Mediterr. J. Math.*, **20** (2023), 1–12. https://doi.org/10.1007/s00009-023-02299-0
7. B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 6th edition, Springer, Berlin, 2013.
8. E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces*, Ph.D thesis, Eindhoven University of Technology, 2001.
9. G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd edition, Cambridge University Press, Cambridge, 2014. https://doi.org/10.1017/CBO9781107295513
10. M. Abouagwa, F. Cheng, J. Li, Impulsive stochastic fractional differential equations driven by fractional Brownian motion, *Adv. Differ. Equations*, **2020** (2020), 57. https://doi.org/10.1186/s13662-020-2533-2
11. S. Selvarasu, P. Kalamani, M. Arjunan, Approximate controllability of nonlocal impulsive fractional neutral stochastic integro-differential equations with state-dependent delay in Hilbert spaces, *Malaya J. Math.*, **4** (2016), 354–362.
12. Z. Yan, H. Zhang, Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay, *Electron. J. Differ. Equations*, **2013** (2013), 1–15.

13. S. Zhao, M. Song, Stochastic impulsive fractional differential evolution equations with infinite delay, *Filomat*, **31** (2017), 4261–4274. https://doi.org/10.2298/FIL1713261Z
14. X. Dai, A. Xiao, W. Bu, Stochastic fractional integro-differential equations with weakly singular kernels: Well-posedness and Euler–Maruyama approximation, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 4231–4253. https://arxiv.org/abs/1901.10333
15. A. Ahmadova, I. T. Huseynov, N. I. Mahmudov, Controllability of fractional stochastic delay dynamical systems, preprint, arXiv:2009.10654.
16. Y. Zang, J. Li, Approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions, *Boundary Value Probl.*, **2013** (2013), 1–15. https://doi.org/10.1186/1687-2770-2013-193
17. J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure Applied Mathematics, Marcel Dekker, New York, **60** (1980).
18. P. Kalamani, D. Baleanu, S. Selvarasu, M. M. Arjunan, On existence results for impulsive fractional neutral stochastic integro-differential equations with nonlocal and state-dependent delay conditions, *Adv. Contin. Discrete Models*, **2016** (2016), 88. https://doi.org/10.1186/s13662-016-0885-4
19. J. van Neerven, M. Veraar, L. Weis, Stochastic integration in UMD Banach spaces, *Ann. Probab.*, **35** (2007), 1438–1478. https://doi.org/10.1214/009117906000001006
20. J. van Neerven, M. C. Veraar, L. Weis, Maximal L^p-regularity for stochastic evolution equations, *J. Funct. Anal.*, **255** (2011), 940–993. https://doi.org/10.1137/110832525
21. M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964.