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https://www.aimspress.com/journal/era

ERA, 34(3): 1885–1899.
DOI: 10.3934/era.2026084
Received: 20 January 2026
Revised: 20 January 2026
Accepted: 13 February 2026
Published: 03 March 2026

Research article

Weierstrass-type constructions, variational analysis and integral-free minimal immersions in $\mathbb{R}^n$

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Abstract: We study conformal minimal immersions into $\mathbb{R}^n$ via the classical correspondence with holomorphic null curves in $\mathbb{C}^n$. After recalling a convenient Weierstrass-type parametrization of null data on simply connected domains, we present an explicit *integral-free* construction that produces minimal immersions directly from a single holomorphic seed function, meaning that the coordinate functions are obtained through closed-form algebraic expressions involving derivatives of the seed rather than through path integration of null data. This viewpoint leads to a reconstruction identity for the seed and gives concrete formulas for the induced metric and the associated Gauss map. For polynomial seeds, we obtain explicit families in arbitrary codimension with closed-form conformal factors. As an analytic application, we derive a second-variation formula for the area under holomorphic perturbations of the seed, expressed in terms of the third derivative of the perturbation. The discussion is local and formula-driven: we do not consider global period problems, completeness, embeddedness, or topological classification, as our goal is to develop explicit analytic constructions rather than a global classification theory.

Keywords: minimal immersions; holomorphic null curves; Weierstrass-type formulas; conformal parametrizations; integral-free construction; second variation of area

1. Introduction

A conformal minimal immersion $X\colon D \to \mathbb{R}^n$ on a simply connected planar domain $D \subset \mathbb{C}$ can be described in terms of holomorphic $\mathbb{C}^n$-valued data satisfying the classical nullity condition. Equivalently, X is locally the real part of a holomorphic null curve in $\mathbb{C}^n$. This correspondence is well known.

The novelty of the present work lies not in the correspondence itself, but in the explicit closed-form parametrization of null data generated from a single holomorphic seed together with a reconstruction identity and a variational formulation written directly in terms of derivatives of the seed.

The emphasis here is local and explicit. We work on simply connected domains and focus on closed-form null data, an integral-free realization built from one holomorphic seed, and the resulting computable expressions for the metric, Gauss map, and area variation.

In contrast with classical formulations, where the immersion is reconstructed through integration of holomorphic null curves, the present framework provides algebraic coordinate expressions derived directly from derivatives of the seed function, which simplifies explicit construction and perturbation analysis.

Global questions, such as period conditions, completeness, embeddedness, and classifications are intentionally left aside.

Let $n \geq 3$ and let $D \subset \mathbb{C}$ be simply connected.

The paper is organized as follows:

i) Section 2 recalls a Weierstrass-type parametrization.
ii) Section 3 gives an explicit integral-free construction.
iii) Section 4 treats polynomial seeds.
iv) Section 5 derives a second-variation formula.

Since the correspondence between conformal minimal immersions and holomorphic null curves is classical, we avoid claims of providing a new representation in a global sense.

Instead, the contribution is the construction of a concrete parametrized class of null data with explicit coordinate realization and a direct variational interpretation.

Background research on minimal surfaces in Euclidean space was initiated by Beckenbach [1], who studied minimal surfaces in higher-dimensional Euclidean spaces. Fundamental global results for complete minimal surfaces were established by Chern and Osserman [2] and further developed in the seminal works of Osserman [3, 4]. The classical lecture notes of Nitsche [5] and the comprehensive monograph by Dierkes et al. [6] provide systematic treatments of the theory. A modern and broad perspective on submanifold theory can be found in the book Dajczer and Tojeiro [7].

The topology of complete minimal surfaces with finite total curvature was studied by Jorge and Meeks [8], while the geometry of the generalized Gauss map was investigated by Hoffman and Osserman [9]. For curvature functionals and Willmore-type energies in space forms, we refer to the works of Gruber et al. [10–12] .

In pseudo-Riemannian and δ-invariant settings, Chen's monograph [13] provides important background material, as does the work of Chen et al. [14]. For codimension-two formulas and generalized Weierstrass-Enneper representations in $\mathbb{R}^4$, see [15].

We stress again that the present paper does not pursue global classification questions.

2. A Weierstrass-type null data formula in $\mathbb{R}^n$

We record a standard holomorphic-null formulation.

Theorem 2.1 (Generalized Weierstrass representation in $\mathbb{R}^4$, [15])**.** *Let f, g, h be holomorphic on $D \subset \mathbb{C}$. Then,*

$$X(\omega, \bar{\omega}) = \Re \int \left(\frac{1}{2} f(1 - g^2 - h^2), \frac{i}{2} f(1 + g^2 + h^2), fg, fh \right) d\omega \tag{2.1}$$

defines a conformal minimal immersion into $\mathbb{R}^4$.

We include this formulation to fix notation and highlight the parametrized structure that will be exploited in the integral-free construction.

The preceding result admits a natural extension to higher-dimensional Euclidean spaces, in which the role of the auxiliary holomorphic data is expanded to accommodate immersions into $\mathbb{R}^n$. We record this generalization below.

Theorem 2.2 (Generalized Weierstrass–Enneper representation in $\mathbb{R}^n$)**.** *Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f, g_1, \ldots, g_{n-2}$ be holomorphic functions on D. Define the holomorphic $\mathbb{C}^n$-valued differential*

$$(\phi_1, \ldots, \phi_n) = \left(\frac{1}{2} f(1 - g_1^2 - \cdots - g_{n-2}^2),\ \frac{i}{2} f(1 + g_1^2 + \cdots + g_{n-2}^2),\ f g_1, \ldots, f g_{n-2}\right).$$

Then, the map

$$X(\omega, \bar{\omega}) = \Re \int^{\omega} (\phi_1, \ldots, \phi_n)\, d\omega \tag{2.2}$$

defines a conformal minimal immersion $X : D \to \mathbb{R}^n$, provided that the induced metric does not vanish.

Proof. A direct computation shows

$$\phi_1^2 + \phi_2^2 + \cdots + \phi_n^2 = 0.$$

Thus, $(\phi_1, \ldots, \phi_n)$ is a holomorphic null curve in $\mathbb{C}^n$. Assuming that the vector $(\phi_1, \ldots, \phi_n)$ does not vanish identically, the map

$$X(\omega, \bar{\omega}) = \Re \int^{\omega} (\phi_1, \ldots, \phi_n)\, d\omega$$

is well defined up to translation on D and defines a conformal minimal immersion into $\mathbb{R}^n$ wherever the induced metric does not vanish. The conformality follows from the nullity condition, and the minimality follows from the holomorphicity of the data. □

3. Integral-free form generalization to $\mathbb{R}^n$

We now construct an integral-free realization of the same representation.

The guiding idea is to replace the classical integral reconstruction step by explicit coordinate expressions depending only on derivatives of a single holomorphic seed. This allows direct computation of geometric quantities and facilitates perturbation analysis.

Let Φ be holomorphic on a simply connected domain $D \subset \mathbb{C}$. Choose parameters $\lambda_1, \ldots, \lambda_{n-3}$ and set

$$\lambda_0 := 1, \qquad \Lambda = 1 + \lambda_1^2 + \cdots + \lambda_{n-3}^2 = \sum_{j=0}^{n-3} \lambda_j^2.$$

When we use the closed-form metric simplification in later sections, we assume $\lambda_j \in \mathbb{R}$ so that $\Lambda \in \mathbb{R}$.

Define

$$f(\omega) = \Phi'''(\omega), \qquad g_1(\omega) = \omega, \qquad g_j(\omega) = \lambda_{j-1}\omega, \quad j = 2, \ldots, n-2.$$

We consider the holomorphic map:

$$F(\omega) = (f_1(\omega), \dots, f_n(\omega)),$$

where

$$f_1(\omega) = \frac{1}{2}(1 - \Lambda\omega^2)\,\Phi''(\omega) + \Lambda\omega\,\Phi'(\omega) - \Lambda\Phi(\omega),$$

$$f_2(\omega) = \frac{i}{2}(1 + \Lambda\omega^2)\,\Phi''(\omega) - i\Lambda\omega\,\Phi'(\omega) + i\Lambda\Phi(\omega),$$

$$f_{j+3}(\omega) = \lambda_j(\omega\Phi''(\omega) - \Phi'(\omega)), \qquad j = 0, \dots, n-3.$$

Differentiating gives

$$f_1'(\omega) = \tfrac{1}{2} f(\omega)(1 - \Lambda\omega^2), \qquad f_2'(\omega) = \tfrac{i}{2} f(\omega)(1 + \Lambda\omega^2),$$

$$f_{j+3}'(\omega) = \lambda_j\,\omega\,\Phi'''(\omega) = \lambda_j\,\omega\,f(\omega), \qquad j = 0, \dots, n-3.$$

Thus, with

$$\phi_1 = f_1', \quad \phi_2 = f_2', \quad \phi_{k+2} = f\,g_k, \quad k = 1, \dots, n-2,$$

the nullity condition

$$\phi_1^2 + \cdots + \phi_n^2 = 0,$$

holds identically, so $F' = (\phi_1, \dots, \phi_n)$ is a holomorphic null curve.

Consequently, the map

$$X(\omega) = \Re\,F(\omega)$$

defines a conformal minimal immersion into $\mathbb{R}^n$ wherever the induced metric is nonzero.

Proposition 3.1 (Reconstruction of the seed function)**.** *With notation as above, the holomorphic seed function Φ can be recovered from the data $f_1, \dots, f_n$ by*

$$\Phi(\omega) = \frac{\Lambda\omega^2 - 1}{2\Lambda}\,f_1(\omega) - i\,\frac{\Lambda\omega^2 + 1}{2\Lambda}\,f_2(\omega) - \frac{\omega}{\Lambda}\sum_{j=0}^{n-3}\lambda_j\,f_{j+3}(\omega).$$

Proof. From the definition of the last $(n-2)$ components, we have

$$\sum_{j=0}^{n-3}\lambda_j\,f_{j+3} = \sum_{j=0}^{n-3}\lambda_j^2(\omega\Phi'' - \Phi') = \Lambda(\omega\Phi'' - \Phi').$$

Using the explicit expressions for f_1 and f_2, one verifies by direct substitution that the coefficients of Φ'' and Φ' cancel each other out, while the coefficient of Φ is equal to 1, yielding $\Phi(\omega)$. □

Remark 3.2. The reconstruction identity shows that the seed function is uniquely encoded in the coordinate functions of the immersion. This fact is crucial for the perturbation analysis developed later, since variations of the seed correspond directly to geometric deformations of the immersion.

4. Polynomial seed functions and explicit minimal surfaces

We now focus on polynomial seed functions and obtain explicit families of minimal surfaces in all dimensions.

Theorem 4.1. *Let $n \geq 3$ and let $\Phi(\omega) = \omega^m$ with $m \geq 3$. Fixing real parameters $\lambda_1, \ldots, \lambda_{n-3}$ and setting*

$$\Lambda = 1 + \lambda_1^2 + \cdots + \lambda_{n-3}^2,$$

let us consider the immersion

$$X(\omega) = \Re(f_1(\omega), \ldots, f_n(\omega))$$

determined by the integral-free formulas above.

Then:

1) *Each coordinate function $f_k(\omega)$ is a polynomial of degree $m - 1$ or m.*
2) *The induced metric is*
$$ds^2 = C_m^2 \, |\omega|^{2m-6} \, (1 + \Lambda|\omega|^2)^2 \, |d\omega|^2,$$
where $C_m = m(m-1)(m-2)$.
3) *The generalized Gauss map takes values in the complex quadric*
$$Q^{n-2} = \{[z_1 : \cdots : z_n] \in \mathbb{C}P^{n-1} \, ; \, z_1^2 + \cdots + z_n^2 = 0\}$$
and is a rational map of ω.

This theorem demonstrates that polynomial seeds generate families of minimal immersions whose metric and Gauss map admit closed-form expressions, providing explicit test cases for analytic and numerical investigations in higher codimension.

Proof. Let $\Phi(\omega) = \omega^m$ with $m \geq 3$. Then

$$\Phi'(\omega) = m\omega^{m-1}, \qquad \Phi''(\omega) = m(m-1)\omega^{m-2}, \qquad \Phi'''(\omega) = C_m \, \omega^{m-3}.$$

Substituting these into the integral-free formulas yields the stated polynomial structure and metric expression. □

Example. Let $n = 5$, $\Phi(\omega) = \omega^4$, and choose $\lambda_1 = \lambda_2 = 1$. Then $C_4 = 24$ and the induced metric becomes

$$ds^2 = 24^2|\omega|^2(1 + 3|\omega|^2)^2 \, |d\omega|^2,$$

yielding an explicit polynomial minimal surface in $\mathbb{R}^5$.

Remark 4.2. Polynomial seed functions yield infinite families of explicit minimal surfaces in all dimensions. The parameters λ_j distribute curvature among normal directions, while the exponent m controls growth and complexity of the Gauss map.

5. Stability under holomorphic null deformations in $\mathbb{R}^n$

In this section, we study how the minimal immersions constructed from polynomial seed functions $\Phi(\omega) = \omega^m$ behave under holomorphic perturbations of the seed. The key point is that in our integral-free representation, all geometry is encoded in the third derivative Φ''', so that varying the seed by $\Phi \mapsto \Phi + \varepsilon h$ produces a very transparent expression for the induced metric and hence for the area functional.

We fix once and for all the Weierstrass parameters

$$\lambda_1, \ldots, \lambda_{n-3} \in \mathbb{R}, \qquad \Lambda = 1 + \lambda_1^2 + \cdots + \lambda_{n-3}^2,$$

and work with the corresponding integral-free representation of the previous section. For the polynomial seed $\Phi(\omega) = \omega^m$, $m \geq 3$, we obtain the minimal immersion $X : D \to \mathbb{R}^n$ of Theorem 4.1 with conformal factor

$$\lambda_0(\omega) = C_m^2 |\omega|^{2m-6}(1 + \Lambda|\omega|^2)^2, \qquad C_m = m(m-1)(m-2).$$

We now describe how the area of X changes when the seed Φ is perturbed holomorphically.

Theorem 5.1 (Holomorphic stability)**.** *Let $n \geq 3$ and let $\Phi(\omega) = \omega^m$ with $m \geq 3$. Fix real parameters $\lambda_1, \ldots, \lambda_{n-3}$ and set*

$$\Lambda = 1 + \lambda_1^2 + \cdots + \lambda_{n-3}^2.$$

Let $X : D \to \mathbb{R}^n$ be the minimal immersion associated with Φ and these parameters via the integral-free representation. Consider a holomorphic variation of the seed

$$\Phi_\varepsilon(\omega) = \Phi(\omega) + \varepsilon\, h(\omega),$$

where h is holomorphic on D, and let X_ε be the family of immersions obtained from Φ_ε by the same construction.

1) *The first variation of the area vanishes:*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Area}(X_\varepsilon) = 0.$$

In particular, every holomorphic perturbation of the seed produces a Jacobi field along X.

2) *On any relatively compact subdomain $\Omega \subset D$ where the immersion X is regular, the second variation of area is given by the weighted L^2-norm of the third derivative of h:*

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \mathrm{Area}(X_\varepsilon) = 2 \int_\Omega (1 + \Lambda|\omega|^2)^2\, |h'''(\omega)|^2\, dA,$$

where dA denotes the Euclidean area element in the parameter ω.

3) *For every holomorphic h, the second variation is nonnegative. Moreover, it vanishes identically if and only if $h'''(\omega) \equiv 0$ on D, that is, precisely when*

$$h(\omega) = a + b\omega + c\omega^2$$

for complex constants a, b, c. In particular, modulo these obvious degeneracies (which do not change the Weierstrass data), the immersion X is strictly stable under holomorphic deformations of the seed.

Proof. We divide the argument into several steps.

Step 1: By construction, for each fixed ε, the map X_ε is a conformal minimal immersion on the regular set of Φ_ε. Indeed, its Weierstrass data are obtained from the same null-holomorphic recipe as in Section 3, applied to the seed Φ_ε, and the nullity condition is preserved for all ε.

It is a standard fact in the theory of minimal surfaces that if X_ε is a smooth 1-parameter family of immersions with X_0 minimal, then the first variation of area at $\varepsilon = 0$ is given by the L^2-pairing of the variation field

$$V = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X_\varepsilon$$

with the mean curvature vector of X_0. Since $X_0 = X$ is minimal, its mean curvature vanishes identically, and hence

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Area}(X_\varepsilon) = 0.$$

This proves item (1). In particular, the variation field V is a Jacobi field along X.

Step 2: We now use the explicit form of the Weierstrass representation. Set

$$f(\omega) = \Phi'''(\omega),$$

so that for the unperturbed seed $\Phi(\omega) = \omega^m$, we have $f(\omega) = C_m\omega^{m-3}$. As in the integral-free construction, the components of the holomorphic null curve are

$$\phi_1 = \tfrac{1}{2} f(1 - \Lambda\omega^2), \qquad \phi_2 = \tfrac{i}{2} f(1 + \Lambda\omega^2), \qquad \phi_{k+2} = f\, g_k, \quad k = 1, \ldots, n-2,$$

with $g_1(\omega) = \omega$ and $g_j(\omega) = \lambda_{j-1}\omega$ for $j \geq 2$. It is convenient to write these in the factored form

$$\phi_i(\omega) = a_i(\omega)\, f(\omega), \qquad i = 1, \ldots, n,$$

where the coefficients $a_i(\omega)$ depend only on ω and the fixed parameters λ_j, but not on the particular choice of seed. In particular, the same functions $a_i(\omega)$ appear for every member of the family X_ε.

For the perturbed seed

$$\Phi_\varepsilon(\omega) = \Phi(\omega) + \varepsilon h(\omega),$$

we have

$$\Phi_\varepsilon'''(\omega) = f(\omega) + \varepsilon h'''(\omega),$$

so that the Weierstrass data of X_ε are

$$\phi_i(\varepsilon, \omega) = a_i(\omega)\big(f(\omega) + \varepsilon h'''(\omega)\big), \qquad i = 1, \ldots, n.$$

Step 3: For a conformal immersion in complex coordinates ω, the induced metric has the form

$$ds_\varepsilon^2 = \lambda(\varepsilon, \omega)\, |d\omega|^2, \qquad \lambda(\varepsilon, \omega) = 2 \sum_{i=1}^{n} |\phi_i(\varepsilon, \omega)|^2.$$

Using the factorization from Step 2, we obtain

$$\lambda(\varepsilon, \omega) = 2\Big|f(\omega) + \varepsilon h'''(\omega)\Big|^2 \sum_{i=1}^{n} |a_i(\omega)|^2.$$

At $\varepsilon = 0$, this reduces to

$$\lambda(0, \omega) = 2|f(\omega)|^2 \sum_{i=1}^{n} |a_i(\omega)|^2.$$

On the other hand, by Theorem 4.1, we already know that for $\Phi(\omega) = \omega^m$, the conformal factor is

$$\lambda(0, \omega) = C_m^2 |\omega|^{2m-6} (1 + \Lambda|\omega|^2)^2.$$

Recalling that $f(\omega) = C_m \omega^{m-3}$, we may cancel $C_m^2 |\omega|^{2m-6}$ from both expressions and obtain the identity

$$2 \sum_{i=1}^{n} |a_i(\omega)|^2 = (1 + \Lambda|\omega|^2)^2.$$

Substituting back into the general expression for $\lambda(\varepsilon, \omega)$ gives the clean formula

$$\lambda(\varepsilon, \omega) = (1 + \Lambda|\omega|^2)^2 \left| f(\omega) + \varepsilon h'''(\omega) \right|^2.$$

Step 4: Let $\Omega \subset D$ be a relatively compact domain on which X is regular. The area of X_ε over Ω is

$$\mathrm{Area}(X_\varepsilon) = \int_\Omega \lambda(\varepsilon, \omega)\, dA = \int_\Omega (1 + \Lambda|\omega|^2)^2 \left| f(\omega) + \varepsilon h'''(\omega) \right|^2 dA.$$

Differentiating twice with respect to ε and evaluating at $\varepsilon = 0$ yields

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \mathrm{Area}(X_\varepsilon) = \int_\Omega (1 + \Lambda|\omega|^2)^2 \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \left| f(\omega) + \varepsilon h'''(\omega) \right|^2 dA.$$

Since for any complex numbers u, v we have

$$|u + \varepsilon v|^2 = |u|^2 + 2\varepsilon \Re(u\bar{v}) + \varepsilon^2 |v|^2,$$

it follows that

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} |u + \varepsilon v|^2 = 2|v|^2.$$

Applying this with $u = f(\omega)$ and $v = h'''(\omega)$ gives

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \left| f(\omega) + \varepsilon h'''(\omega) \right|^2 = 2|h'''(\omega)|^2,$$

and hence

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \mathrm{Area}(X_\varepsilon) = 2 \int_\Omega (1 + \Lambda|\omega|^2)^2 |h'''(\omega)|^2\, dA.$$

This proves the formula in item 2).

Step 5: The integrand in the expression of the second variation is nonnegative at every point of Ω, so the second variation is always ≥ 0. It vanishes over Ω if and only if

$$(1 + \Lambda|\omega|^2)^2 |h'''(\omega)|^2 \equiv 0 \quad \text{on } \Omega,$$

which is equivalent to $h'''(\omega) \equiv 0$ on Ω. By the identity principle for holomorphic functions, this implies $h'''(\omega) \equiv 0$ on the whole domain D.

Conversely, if $h'''(\omega) \equiv 0$ on D, then h must be a polynomial of degree at most 2, say

$$h(\omega) = a + b\omega + c\omega^2,$$

with complex constants a, b, c. For such h, we clearly have $h'''(\omega) = 0$, and the second variation vanishes identically.

Thus, the null-space of the quadratic form, given by the second variation, consists precisely of quadratic polynomials, and for any holomorphic h with $h'''(\omega) \not\equiv 0$, the integrand is strictly positive on a nonempty open subset of D, which implies that the second variation is strictly positive. This establishes item 3) and completes the proof. □

Remark 5.2. Theorem 5.1 shows that the family of minimal immersions arising from polynomial seeds $\Phi(\omega) = \omega^m$ is remarkably rigid with respect to holomorphic deformations of the seed: apart from the obvious degeneracies coming from adding a quadratic polynomial

$$h(\omega) = a + b\omega + c\omega^2,$$

every holomorphic perturbation increases the area to second order. At the level of the Weierstrass data, these quadratic perturbations are precisely the ones that do not change Φ''' and therefore leave the null curve and its associated immersion unchanged up to translation.

6. Examples and projection visualization

Although the representations developed in this work apply to all dimensions $n \geq 3$, it is instructive to restrict to $n = 4$ in order to obtain concrete two-dimensional projections that can be plotted. Let

$$X = (X_1, X_2, X_3, X_4) : D \subset \mathbb{C} \to \mathbb{R}^4$$

be a conformal minimal immersion obtained from the integral-free Weierstrass representation of the previous section. Then, each ordered pair of coordinate functions

$$(X_i, X_j), \qquad 1 \leq i < j \leq 4,$$

defines a map from D into $\mathbb{R}^2$. Thus, there are six canonical orthogonal projections:

$$(X_1, X_2),\ (X_1, X_3),\ (X_1, X_4),\ (X_2, X_3),\ (X_2, X_4),\ (X_3, X_4).$$

Choice of Weierstrass data in $\mathbb{R}^4$: When $n = 4$, the general Weierstrass data specialize to

$$f,\ g_1,\ g_2 \qquad \text{holomorphic on } D,$$

and the null curve is

$$\phi_1 = \tfrac{1}{2} f\,(1 - g_1^2 - g_2^2), \qquad \phi_2 = \tfrac{i}{2} f\,(1 + g_1^2 + g_2^2), \qquad \phi_3 = f\,g_1, \qquad \phi_4 = f\,g_2.$$

In the integral-free setting of the previous section, the Weierstrass data are determined from a holomorphic seed Φ by setting

$$f(\omega) = \Phi'''(\omega), \qquad g_1(\omega) = \omega, \qquad g_2(\omega) = \lambda\,\omega,$$

where $\lambda \in \mathbb{R}$ is a fixed parameter and

$$\Lambda = 1 + \lambda^2.$$

The minimal immersion is given by

$$X(\omega) = \Re \int^{\omega} (\phi_1, \phi_2, \phi_3, \phi_4)\, d\omega.$$

Let

$$\Phi(\omega) = \omega^4,$$

so that

$$\Phi'(\omega) = 4\omega^3, \qquad \Phi''(\omega) = 12\omega^2, \qquad \Phi'''(\omega) = 24\omega.$$

We now take

$$g_1(\omega) = \omega, \qquad g_2(\omega) = \lambda\,\omega,$$

with $\lambda \in \mathbb{R}$. The Weierstrass data become

$$\phi_1 = 12\omega(1 - \omega^2 - \lambda^2\omega^2), \quad \phi_2 = 12i\omega(1 + \omega^2 + \lambda^2\omega^2), \quad \phi_3 = 24\omega^2, \quad \phi_4 = 24\lambda\omega^2.$$

The induced metric is

$$ds^2 = 24^2\,|\omega|^2\,(1 + \Lambda|\omega|^2)^2\,|d\omega|^2, \qquad \Lambda = 1 + \lambda^2.$$

By introducing polar coordinates $\omega = re^{i\theta}$, with $u = r\cos\theta$ and $v = r\sin\theta$, the immersion can be expressed in the trigonometric form

$$X(r,\theta) = \begin{pmatrix} 6r^2\cos(2\theta) \;-\; 3(1+\lambda^2)r^4\cos(4\theta) \\ -6r^2\sin(2\theta) \;-\; 3(1+\lambda^2)r^4\sin(4\theta) \\ 8r^3\cos(3\theta) \\ 8\lambda\, r^3\cos(3\theta) \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

The projections of the minimal surface X in the (u, v) coordinates are shown in Figures 1–2, while those in the (r, θ) coordinates are given in Figures 3–4.

The immersion may be sampled numerically on a bounded domain $\Omega \subset \mathbb{C}$. For each ordered pair $1 \le i < j \le 4$, we consider the projection

$$(X_i, X_j) : \Omega \to \mathbb{R}^2.$$

Similarly, one may consider three-dimensional projection surfaces obtained from triples of coordinate functions:

$$(X_i, X_j, X_k) : \Omega \to \mathbb{R}^3, \qquad 1 \le i < j < k \le 4.$$

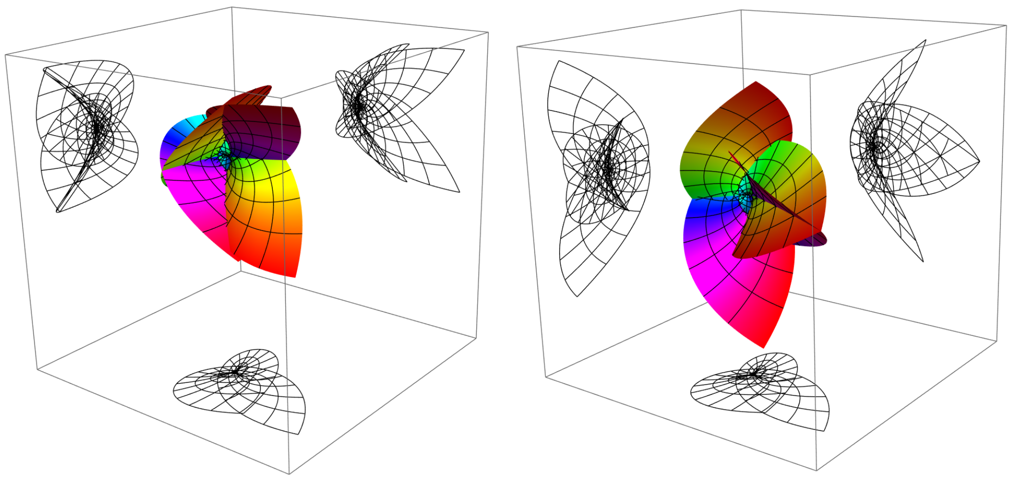

Figure 1. Projections of the minimal surface $X(u, v)$ into the $X_1X_2X_3$-space (left) and $X_1X_2X_4$-space (right).

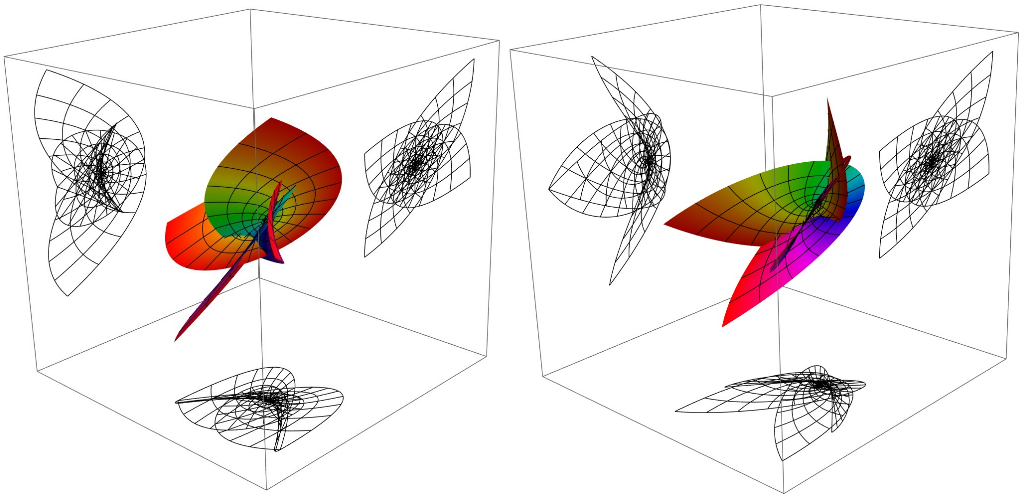

Figure 2. Projections of the minimal surface $X(u, v)$ into the $X_1X_3X_4$-space (left) and $X_2X_3X_4$-space (right).

Remark 6.1 (Orthogonal projections in $\mathbb{R}^4$). Let $X = (X_1, X_2, X_3, X_4) : D \to \mathbb{R}^4$ be a minimal immersion obtained from the integral-free representation. Each ordered pair of coordinate functions

$$(X_i, X_j), \qquad 1 \le i < j \le 4,$$

defines a canonical orthogonal projection $D \to \mathbb{R}^2$. Since there are $\binom{4}{2} = 6$ such coordinate pairs, one obtains six distinct planar projections:

$$(X_1, X_2),\ (X_1, X_3),\ (X_1, X_4),\ (X_2, X_3),\ (X_2, X_4),\ (X_3, X_4).$$

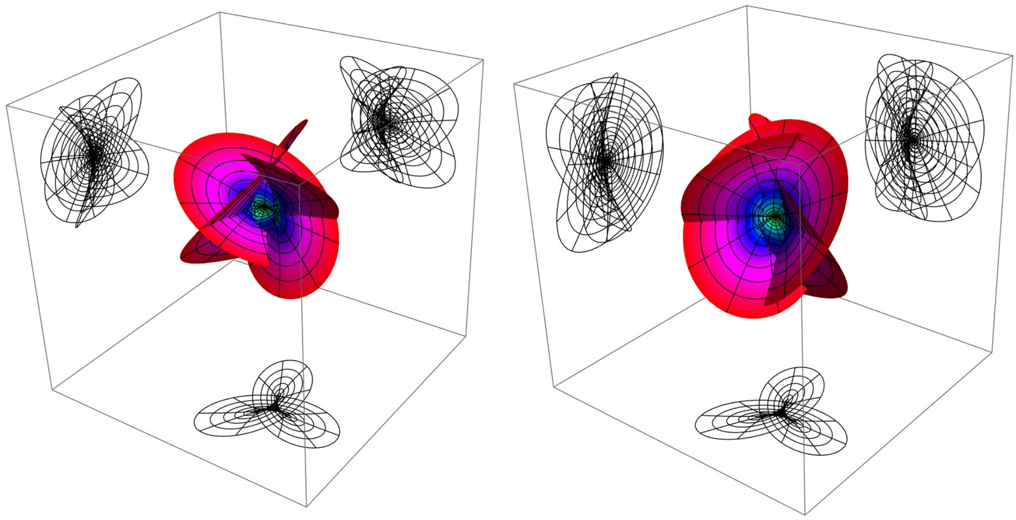

Figure 3. Projections of the minimal surface $X(r, \theta)$ into the $X_1X_2X_3$-space (left) and $X_1X_2X_4$-space (right).

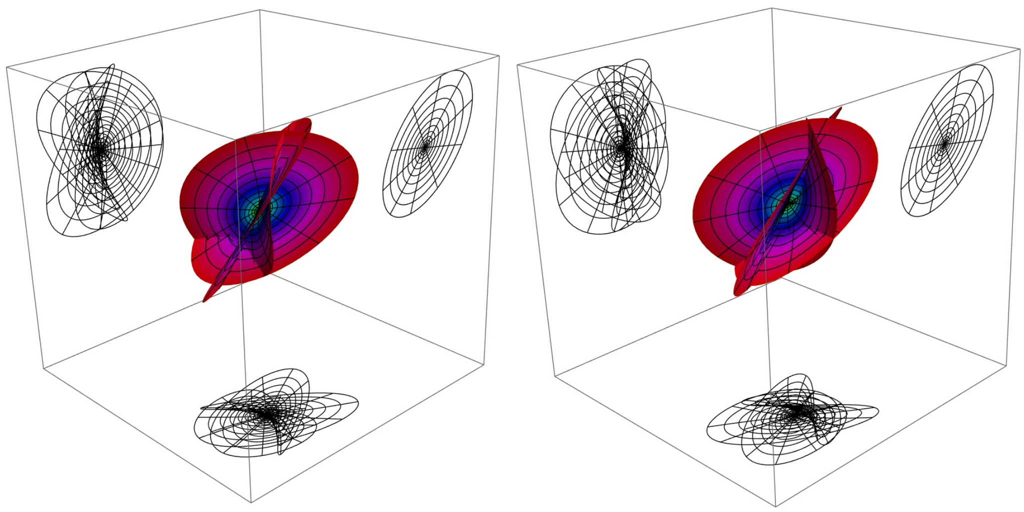

Figure 4. Projections of the minimal surface $X(r, \theta)$ into the $X_1X_3X_4$-space (left) and $X_2X_3X_4$-space (right).

In addition, the corresponding three-dimensional projection surfaces arise from triples of coordinate functions. Since $\binom{4}{3} = 4$, these surfaces are given by

$$(X_1, X_2, X_3), \qquad (X_1, X_2, X_4), \qquad (X_1, X_3, X_4), \qquad (X_2, X_3, X_4).$$

Remark 6.2 (Complex visualization parameter). In the theoretical discussion of Sections 2–5, we restricted to real values of the Weierstrass parameter $\lambda \in \mathbb{R}$, so that $\Lambda = 1 + \lambda^2 \in \mathbb{R}$; and hence, the induced metric takes the closed form stated in Theorem 4.1. The figures appearing in this paper have been generated from the same Weierstrass data, but with the complex choice $\lambda = 1 + i$ (see Figures 1–4).

This choice is made purely for visualization: the holomorphic nullity condition

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 0$$

continues to hold, so that $X(\omega) = \Re \int(\phi_1, \phi_2, \phi_3, \phi_4)\, d\omega$ remains a conformal minimal immersion in $\mathbb{R}^4$. However, since

$$\Lambda = 1 + \lambda^2 = 1 + 2i$$

is now complex, the compact real metric formula from Theorem 4.1 does not simplify in the same way, and the metric must be computed directly from

$$ds^2 = 2 \sum_{i=1}^{4} |\phi_i|^2 \, |d\omega|^2.$$

7. Applications to physics and related sciences

In string theory and M-theory, extended objects such as membranes (2-branes) and higher-dimensional branes minimize the Nambu-Goto action, which is the area functional in ambient spacetime. The null-holomorphic representation offers an explicit analytic construction of such extremal embeddings. Polynomial seed functions correspond to branes of polynomial growth, while the deformation theory from Theorem 5.1 relates to the stability of small fluctuations around these configurations.

In nonlinear sigma models and harmonic map theory, minimal immersions represent energy-minimizing field configurations. The higher-dimensional Gauss map into the quadric Q^{n-2} captures orientational degrees of freedom associated with these fields, and the explicit stability formula can be interpreted as a quadratic energy functional for perturbations of the corresponding fields.

7.1. Applications to materials science

Materials exhibiting triply periodic minimal surface (TPMS) geometry appear in nanofabrication, photonic crystals, and self-assembled block copolymers. Our representation allows systematic construction of higher-codimension generalizations of TPMS patterns, offering templates for multi-phase or anisotropic materials, where several independent normal directions encode different physical interactions. The parameters λ_j play the role of anisotropy parameters in the normal bundle.

Minimal surfaces describe the equilibrium geometry of amphiphilic interfaces, with the isotropic nullity condition corresponding to vanishing mean curvature. By varying the parameters λ_j, one can distribute the curvature among multiple normal directions, thus modeling elastic anisotropy in complex fluids and liquid crystalline phases.

7.2. Applications to biology and biophysics

Cellular membranes and protein assemblies often approximate minimal geometries to reduce bending energy. The polynomial seed families $X(\omega)$ derived above provide explicit models for membrane folds, branching, and geometric transitions. The stability framework of Theorem 5.1 gives a way to analyze which perturbations increase or decrease bending energy at second order.

During biological growth, surfaces evolve through curvature-driven flows. Our representation gives closed-form examples of candidate equilibrium states and stable perturbations around them, as quantified in Theorem 5.1. Such explicit models may be useful for testing numerical schemes and for understanding qualitative features of morphogenetic patterns.

Remark 7.1. The ability to model minimal surfaces in all dimensions $n \geq 3$ with explicit analytic formulas, stability criteria, and curvature control makes the present framework suitable for applications where geometry, energy minimization, and multidimensional interaction structures play central roles.

Acknowledgements

This work is dedicated to Sophie Germain, whose foundational contributions to symmetry theory, elasticity, and the conceptual structure of mathematical truth, as well as her influential writings on geometric analysis, continue to be a source of intellectual inspiration.

Use of AI tools declaration

The authors used AI-assisted tools solely for grammar correction and language editing. All mathematical ideas, results, proofs, and scientific content are entirely the authors' original work.

Conflict of interest

The authors declare there is no conflict of interest.

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