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Research article

Transitions and chaos of a business cycle model

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Abstract: This study investigates a modified Bouali model (referred to as the business cycle model) through the lens of transition dynamics theory. Under reasonable theoretical assumptions, we identify the output-capital ratio c as the key parameter that determines the transition and chaos in the business cycle model. Through rigorous mathematical analysis, we demonstrate that this model exhibits both jump (discontinuous) and continuous transitions as the output-capital ratio c increases and crosses a critical value. In particular, for the output-capital ratio large enough c, the business cycle model develops a strange attractor, which means that there exists chaos in this business cycle model.

Keywords: dynamic transition; chaos; business cycle model; bifurcation

1. Introduction

Low-dimensional dynamical systems constitute fundamental frameworks for understanding nonlinear transition phenomena in real-world systems. Lorenz's seminal 1963 work [1] established a paradigmatic 3-dimensional ordinary differential equation (ODE) model for atmospheric convection—the celebrated Lorenz system—which elegantly captures the essential dynamical complexity of atmospheric flows, including nonlinear transition dynamics and chaotic behavior. This foundational model has stimulated extensive research across nonlinear science disciplines, leading to the identification and characterization of numerous analogous systems exhibiting similar dynamical universality. Representative examples include works from Lü and Chen [2], Rössler [3], Chen and Dong [4], all of which demonstrate characteristic nonlinear phenomena particularly manifested through the emergence of strange attractors under parameter variations.

The economy functions as a complex adaptive system characterized by nonlinear interactions among heterogeneous agents at the micro level. These agents respond to macroeconomic information signals (feedback mechanisms) while simultaneously being influenced through capital and goods flows that connect disparate economic systems. Furthermore, aggregate macroeconomic variables demonstrate the cyclical fluctuations that constitute the business cycle phenomenon [5, 6]. These business cycles exhibit synchronized expansions across multiple economic sectors, invariably transitioning to recoveries, contractions, and recessions. Theoretically, such oscillatory dynamics can be rigorously modeled through nonlinear differential equations, thereby demonstrating their endogenous origin [7]. Consequently, simplified dynamical models exhibiting limit cycles or chaotic attractors prove instrumental in elucidating fundamental aspects of intereconomy interactions, particularly regarding crisis propagation mechanisms between national economies.

Motivated by the need for a parsimonious economic model to elucidate the mechanisms driving macroeconomic fluctuations, Bouali proposed an innovative 3-dimensional dynamical system in [8]. This model integrates a 2-dimensional Van der Pol oscillator with a feedback mechanism, thereby generating a 3-dimensional model that exhibits diverse oscillatory behaviors [9]. Based on recent developments in macroeconomic modeling, Amaral et al. [10] introduced a modified Bouali model that incorporates key relationships among macroeconomic variables consistent with the Keynesian IS-LM framework (investment-savings and liquidity preference-money supply). The resulting extended model demonstrates theoretical alignment with the Kaldor-Kalecki model [11,12], thus bridging short-term Keynesian dynamics with long-term growth theory.

Bifurcation theory [13, 14], as a cornerstone for analyzing complex dynamics, has demonstrated broad applicability across disciplines, including economic business cycle modeling, offering a unifying methodological framework for nonlinear systems. This theory examines the qualitative transformations of a dynamical system's behavior under parametric variations. By analyzing the points at which a small change in a parameter causes a sudden shift in the system's dynamics, bifurcation theory helps to identify critical thresholds and transitions between different states. Bifurcation analysis provides a systematic framework for understanding and predicting these transitions, making it an essential tool in fields ranging from physics to biology and economics, where systems often exhibit nonlinear behaviors and undergo significant changes in response to variations in parameters. This theory has been used to study the modified model proposed by Amaral et al. [10]. While it is well known that certain dynamical systems can exhibit transitions without accompanying bifurcations, Ma and Wang [15] developed the phase transition dynamics theory as a conceptual framework to systematically investigate the transition dynamics in dissipative systems, thereby addressing the analytical limitations inherent in traditional bifurcation analysis. This innovative theoretical framework enables researchers to systematically identify a comprehensive spectrum of transition states while providing a more holistic characterization of both stability and transition. Within the Ma and Wang [15] framework, transition states can be rigorously identified and classified through the dual perspectives of dynamic analysis and physical interpretation with the complete set of transition states being mathematically represented by a local attractor. One of the key contributions of this novel framework lies in the establishment of a systematic classification for phase transitions. According to this classification scheme, the system's fundamental state can exhibit three distinct transition regimes: 1) a continuous phase transition characterized by gradual evolution, 2) a jump phase transition, or 3) a mixed transition combining both characteristics. Key references on phase transition dynamics applications and theory include the seminal works [16–18]. This study aims to conduct a comprehensive theoretical and numerical investigation of the business cycle model proposed in [10] through the lens of phase transition dynamics [15]. Our primary objectives are twofold: 1) to identify the dominant control parameter governing the model's dynamic transitions and 2) to derive the exact analytical expression for the critical threshold of this key parameter. Building upon this foundational result, we will systematically classify the distinct types of dynamic transitions exhibited by the business cycle model, thereby establishing a rigorous framework for understanding its nonlinear dynamical behavior.

The investigation of chaos and transitions within this business cycle model holds significant importance for economic theory and policy. Understanding the onset of chaotic behavior is crucial, as it implies inherent unpredictability in economic trajectories, potentially explaining sudden market crashes or crises that defy conventional linear forecasting models. Furthermore, analyzing dynamic transitions, particularly the shift from stable equilibria to periodic oscillations or chaotic attractors, provides deep insights into how economies fundamentally restructure themselves under changing conditions, such as variations in investment efficiency or capital utilization. This study bridges rigorous dynamical systems theory with macroeconomic modeling, offering a framework to anticipate critical thresholds where economies undergo qualitative state changes. Our principal findings demonstrate that the output-capital ratio c serves as the pivotal control parameter: crossing its analytically derived critical value c_0 triggers either continuous (predictable) or discontinuous (catastrophic) transitions, with sufficiently large c inducing chaotic dynamics characterized by a strange attractor. These results not only elucidate the endogenous mechanisms of business cycles but also provide a mathematical basis for identifying early warning signals of economic instability.

In Section 2, we will present phase transition dynamics. Section 3 validates the principle of exchange stability for the business cycle model. Theoretical and numerical results are given in Section 4 and Section 5, respectively.

2. Dynamical transition

In this section, we lay the foundation for our analysis by introducing fundamental concepts and tools that will be essential to understanding the dynamic transitions in the systems we study. We begin by presenting a rigorous mathematical definition of the dynamic transition, a concept that captures the qualitative change in the behavior of a system as parameters are varied. This definition is broad enough to apply to a wide range of dissipative systems, according to Ma and Wang [15], making it a powerful framework for our subsequent investigations.

To ensure analytical clarity and conceptual simplicity, we restrict our analysis to the definition of dynamic transitions within the framework of ordinary differential equation (ODE) systems. We specifically consider ODE systems of the following form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}_\lambda(\mathbf{x}), \quad \mathbf{x}|_{t=0} = \mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

in which λ represents a control variable. We also assume the vector field $\mathbf{f}_\lambda \in C^r(\mathbb{R}^n \times \Lambda, \mathbb{R})$ with $r \geq 3$ where $\Lambda \subset \mathbb{R}$ is an open parameter interval. This measure ensures the validity of center manifold reduction and the Lyapunov coefficient per Ma and Wang's framework.

In addition, assume that $\mathbf{f}_\lambda(\mathbf{x})$ can be expanded around a equilibrium point $\mathbf{x}_0$(where $\mathbf{f}_\lambda(\mathbf{x}_0) = 0$) as

$$\mathbf{f}_\lambda(\mathbf{x}) = \frac{\partial \mathbf{f}_\lambda}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x}_0} (\mathbf{x} - \mathbf{x}_0) + O(|\mathbf{x} - \mathbf{x}_0|^2). \tag{2.2}$$

Definition 2.1 ([15]). *Let Σ_λ be an invariant set of the nonlinear system* (2.1)*, where λ is a control parameter. The system* (2.1) *undergoes a dynamic transition at $\lambda = \lambda_0$ (originating from the invariant set pair $(\Sigma_{\lambda_0}, \lambda_0)$) if the following conditions hold:*

1) *For $\lambda < \lambda_0$, Σ_λ is a local minimal attractor of system* (2.1)*;*
2) *For $\lambda > \lambda_0$, there exists an open set $U \supset \Sigma_\lambda$ (independent of λ_0) such that for any initial value $\mathbf{y} \in U \setminus (\Gamma_\lambda \cup \Sigma_\lambda)$, the solution $\mathbf{x}(t, \mathbf{y})$ of* (2.1) *satisfies*

$$\lim_{t\to+\infty} \sup dist\,(\mathbf{x}(t,\mathbf{y}), \Sigma_\lambda) \geq \delta(\lambda) > 0, \quad \lim_{\lambda\to\lambda_0} \delta(\lambda) \geq 0,$$

where Γ_λ is the stable manifold of Σ_λ with codimension ≥ 1, and dist denotes the Euclidean distance.

Remark 2.1. *When the set Σ_λ reduces to a single equilibrium, the dynamic transition at $\lambda = \lambda_0$ is simply an exchange of stability: the system shifts from one stable equilibrium to another. This scenario is the simplest possible and can be fully analyzed with the methods developed earlier.*

In the context of equilibrium states, understanding the nature of transitions is crucial to characterizing the system's behavior as the parameters change. The lemma offers key insights into system transitions governed by (2.1), including both transition criteria and a fundamental classification scheme. This lemma serves as a key analytical tool, enabling us to identify and categorize the types of transitions that may occur at equilibrium points.

Lemma 2.1 ([15]). *Let $\mathbf{x}_0$ denote an equilibrium state of system* (2.1)*, that is, $\mathbf{f}_\lambda(\mathbf{x}_0) = 0$. Consider all eigenvalues (counting multiplicity) of matrix $\frac{\partial \mathbf{f}_\lambda}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x}_0}$, denoted by $\{\beta_i(\lambda) | i = 1, 2, \cdots, n-1, n\}$. If the following principle of exchange of stability (PES) condition holds:*

$$\begin{aligned} &\mathrm{Re}\beta_i(\lambda) \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases}, \quad 1 \leq i \leq m, \\ &\mathrm{Re}\beta_j(\lambda_0) < 0, \quad m+1 \leq j \leq n, \end{aligned} \tag{2.3}$$

then the system (2.1) *necessarily experiences a dynamic transition at the critical point $(\mathbf{x}, \lambda) = (\mathbf{x}_0, \lambda_0)$. Moreover, there exists an open set U containing $\mathbf{x} = \mathbf{x}_0$ such that the transition falls into one of the following three types:*

1) *Continuous transition: There is a dense and open set $U_\lambda \subset U$ satisfying that for any $\mathbf{y} \in U_\lambda$, the solution $\mathbf{x}(t, \mathbf{y})$ of* (2.1) *satisfies*

$$\lim_{\lambda\to\lambda_0} \lim_{t\to+\infty} \sup |\mathbf{x}(t,\mathbf{y}) - \mathbf{x}_0| = 0.$$

2) *Jump (catastrophic) transition: For each $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ for some $\epsilon > 0$, there exists a dense and open subset $U_\lambda \subset U$ such that for any $\mathbf{y} \in U_\lambda$, the solution $\mathbf{x}(t, \mathbf{y})$ of system* (2.1) *possesses the asymptotic property*

$$\lim_{t\to+\infty} \sup |\mathbf{x}(t,\mathbf{y}) - \mathbf{x}_0| \geq \delta > 0$$

where the positive constant δ does not depend on λ.

3) *Mixed (random) transition: For each $\lambda \in (\lambda_0, \lambda_0+\epsilon)$ for some $\epsilon > 0$, the open set U is decomposed into U_λ^1 and U_λ^2, which are two open sets satisfying $\overline{U} = \overline{U_\lambda^1} \cup \overline{U_\lambda^2}$, $U_\lambda^1 \cap U_\lambda^2 = \emptyset$. Moreover, the following asymptotic property holds:*

$$\lim_{\lambda\to\lambda_0} \lim_{t\to+\infty} \sup |\mathbf{x}(t,\mathbf{y}) - \mathbf{x}_0| = 0, \quad \forall\, \mathbf{y} \in U_\lambda^1,$$
$$\lim_{t\to+\infty} \sup |\mathbf{x}(t,\mathbf{y}) - \mathbf{x}_0| \geq \delta > 0, \quad \forall\, \mathbf{y} \in U_\lambda^2,$$

where U_λ^1 and U_λ^2 are referred to as the metastable domains.

Remark 2.2. *Lemma 2.1 also includes the mixed (Type-III/random) transition. In the present paper, however, the transition at $c = c_0$ is caused by a simple pair of complex conjugate eigenvalues crossing the imaginary axis (a Hopf bifurcation). The center manifold is two-dimensional, and the reduced normal form can be written as*

$$\dot{z} = (\beta(c) + i\omega_0)z + l_1 z|z|^2 + l_2 z|z|^4 + \cdots,$$

equivalently $\dot{r} = \beta(c)r + \mathrm{Re}(l_1)r^3 + \mathrm{Re}(l_2)r^5 + \cdots$. Hence, the transition type is determined by the sign of $\mathrm{Re}(l_1)$ (and $\mathrm{Re}(l_2)$ in the degenerate case), leading only to Type-I (continuous/supercritical Hopf) or Type-II (jump/subcritical Hopf) transitions in the regime considered under Assumptions (3.12) and (3.13). The mixed (Type-III/random) transition typically arises in steady-state bifurcations with real critical eigenvalues or in codimension-two multimode criticality, which is beyond the scope of the current analysis.

2.1. Methodological framework for transition type determination

This section establishes the theoretical criteria for classifying transition types at $\mathbf{x}_0$ within the nonlinear system governed by Eq (2.1). We consider two distinct cases, each offering insight into the nature of the transitions that can occur.

Case I: Consider the case where the PES condition is given by

$$\beta_1(\lambda)\begin{cases} < 0, & \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0, & \lambda > \lambda_0, \end{cases}$$
$$\mathrm{Re}\beta_j(\lambda_0) < 0, \quad 2 \leq j \leq n. \tag{2.4}$$

We use $\mathbf{e}_1$ to denote the eigenvector corresponding to the first eigenvalue $\beta_1(\lambda)$ and denote $\mathbf{e}_1^*$ as its dual eigenvector. The center-unstable space is then defined as $H_c = \{x\mathbf{e}_1 | x \in R\}$. Under the condition (2.4), let $\mathbf{h}$ be the center manifold function of system (2.1), which is constructed as $\mathbf{h}(x\mathbf{e}_1)$ on H_c. If we have

$$\begin{aligned}&(\mathbf{f}_{\lambda_0}(x\mathbf{e}_1 + \mathbf{h}(x\mathbf{e}_1)), \mathbf{e}_1^*)\\ &- \left(\frac{\partial \mathbf{f}_{\lambda_0}}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x}_0} (x\mathbf{e}_1 + \mathbf{h}(x\mathbf{e}_1) - \mathbf{x}_0), \mathbf{e}_1^* \right)\\ &= px^q + o(x^q), \quad p, q \in \mathbb{Z}^+,\end{aligned} \tag{2.5}$$

then the transition of system (2.1) at $(\mathbf{x}_0, \lambda_0)$ is classified as follows:

(i) Continuous transition occurs when $\mathbf{x}_0$ is stable at $\lambda = \lambda_0$ or when $p < 0$ with odd q.
(ii) Jump (catastrophic) transition occurs precisely when q is odd and $p > 0$.
(iii) Mixed (random) transition occurs precisely when q is an even number.

Case II: Consider the case where the PES condition is characterized by

$$\begin{aligned} &\mathrm{Re}\beta_1(\lambda)\begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases}, \quad \mathrm{Im}\beta_1(\lambda_0) \neq 0, \\ &\mathrm{Re}\beta_j(\lambda_0) < 0, \quad 2 \leq j \leq n. \end{aligned} \tag{2.6}$$

Under this condition, the transition behavior at $(\mathbf{x}_0, \lambda_0)$ is determined by the following criteria:

i) The transition emanating from $(\mathbf{x}_0, \lambda_0)$ is a continuous transition if and only if the equilibrium $\mathbf{x}_0$ remains stable at $\lambda = \lambda_0$.
ii) The transition emanating from $(\mathbf{x}_0, \lambda_0)$ is a jump transition if and only if the equilibrium $\mathbf{x}_0$ is unstable at $\lambda = \lambda_0$.

3. Economic model

3.1. Three-dimensional model

Building upon Bouali's model [8], Amaral et al. [10] developed an improved three-dimensional framework that incorporates key insights from macroeconomic theory. This extended model formally establishes the functional interrelationships among three fundamental macroeconomic variables: household savings (x), gross domestic product (GDP) (y), and foreign capital inflows (z). The temporal evolution of these variables is described by the ODE system:

$$\begin{cases} \frac{dx}{dt} = pdx + my - pxy^2, \\ \frac{dy}{dt} = wx + vy + cz, \\ \frac{dz}{dt} = sx - ry, \end{cases} \tag{3.1}$$

where eight positive parameters $\theta = (m, s, r, v, d, p, w, c)$ jointly govern the dynamics and behavior of the system. m represents the marginal propensity to save; s is the inflow-saving ratio, representing the relationship between capital inflows and savings; r stands for the indebtedness factor, measuring the system's debt accumulation tendency; v denotes the marginal propensity to consume, indicating the proportion of additional income spent on consumption; d represents the GDP potential, which is the economy's maximum sustainable output capacity; p is the profit capitalization rate, referring to the rate at which profits are reinvested; w is the saving proportion, referring to the fraction of income allocated to savings; and c is the output-capital ratio, measuring capital productivity (output per unit of capital).

Examining the dynamic transition behaviors in business cycle models under varying parametric conditions presents both intellectually stimulating and analytically challenging research endeavors. It involves understanding how parameter changes can lead to shifts in the system's behavior, such as the onset of oscillations, stability changes, or other types of transitions. To address these questions, we need to delve into the specifics of the model and the parameters involved. To determine which parameter is the key determinant of transition behavior in the business cycle model, we need to analyze the sensitivity of the system's dynamics to each parameter. This can be done through a combination of analytical methods and numerical simulations. Parameters that significantly influence the stability of equilibrium points, the onset of oscillations, or the occurrence of bifurcations are likely to be key parameters.

3.2. Steady states

The steady state in the business cycle model refers to the state in which the economy is balanced, relative to variables that change over time; such states are called equilibrium points. These points are crucial for analyzing the dynamic transition behavior in the model, as they represent the stable or unstable states between which the economy can transition. To determine the equilibrium points, we need to solve the system of equations derived by setting the derivatives in the business cycle model to zero. The equilibrium points are determined by the following system:

$$\begin{cases} 0 = my + px(d - y^2), \\ 0 = wx + vy + cz, \\ 0 = sx - ry. \end{cases} \tag{3.2}$$

These equations represent the conditions under which the economy is in a steady state, with savings x, GDP y, and foreign capital inflow z remaining constant over time. The solutions to this system provide the coordinates of the equilibrium points. It is not hard to get that

$$y\left(ms + prd - pry^2\right) = 0.$$

This equation indicates that the economic model (3.1) has three equilibrium points

$$\begin{aligned} E_1 &= (0, 0, 0), \quad E_2 = -E_3, \\ E_3 &= \left(-\frac{r\alpha}{s}, -\alpha, \frac{(wr + vs)\alpha}{cs}\right), \end{aligned} \tag{3.3}$$

where

$$\alpha = \sqrt{d + \frac{ms}{pr}}.$$

3.3. Stability of equilibrium points

Taking any equilibrium point $P = (x_0, y_0, z_0)$, the linearized matrix (Jacobian matrix) $L(x_0, y_0, z_0)$ for the economic model (3.1) is as follows:

$$L(x_0, y_0, z_0) = \begin{bmatrix} p\left(d - y_0^2\right) & m - 2px_0y_0 & 0 \\ w & v & c \\ s & -r & 0 \end{bmatrix}. \tag{3.4}$$

Its characteristic polynomial is given by

$$\lambda^3 + \alpha_2(x_0, y_0, z_0)\lambda^2 + \alpha_1(x_0, y_0, z_0)\lambda + \alpha_0(x_0, y_0, z_0) = 0, \tag{3.5}$$

where

$$\begin{aligned}\alpha_2(x_0,y_0,z_0) &= -p(d-y_0^2)-v,\\ \alpha_1(x_0,y_0,z_0) &= pv(d-y_0^2)-w(m-2px_0y_0)+cr,\\ \alpha_0(x_0,y_0,z_0) &= -crp(d-y_0^2)-cs(m-2px_0y_0).\end{aligned} \tag{3.6}$$

For the equilibrium point $E_1 = (0,0,0)$, we have

$$\begin{aligned}\alpha_2(0,0,0) &= -pd-v,\\ \alpha_1(0,0,0) &= pvd--wm+cr,\\ \alpha_0(0,0,0) &= -crpd-csm,\end{aligned} \tag{3.7}$$

from which we see that when $(x_0, y_0, z_0) = (0, 0, 0)$, the characteristic polynomial (3.5) has both a root with a negative real part and a root with a positive part. We subsequently establish the following lemma:

Lemma 3.1. $E_1 = (0,0,0)$ *is linearly unstable.*

For the equilibrium point $E_2 = (r\alpha/s, \alpha, -(wr+vs)\alpha/(cs))$, we have

$$\begin{aligned}\alpha_2\left(\frac{r\alpha}{s},\alpha,-\frac{(wr+vs)\alpha}{cs}\right) &= \frac{ms-rv}{r},\\ \alpha_1\left(\frac{r\alpha}{s},\alpha,-\frac{(wr+vs)\alpha}{cs}\right) &= w\left(m+2\frac{prd}{s}\right)+cr-\frac{msv}{r},\\ \alpha_0\left(\frac{r\alpha}{s},\alpha,-\frac{(wr+vs)\alpha}{cs}\right) &= 2c(ms+prd).\end{aligned} \tag{3.8}$$

The characteristic polynomial (3.5) at the equilibrium point $E_2 = (r\alpha/s, \alpha, -(wr+vs)\alpha/(cs))$ has pure imaginary roots $\lambda = i\beta$ if and only if

$$\alpha_0|_{E_2} = \alpha_1|_{E_2}\alpha_2|_{E_2}$$

which holds if and only if

$$c = c_0, \quad c_0 := \frac{(ms-rv)\left(2dpr^2w - ms(sv-rw)\right)}{r^2 s(2dpr+ms+rv)}. \tag{3.9}$$

Let us denote the three roots of the polynomial (3.5) at the equilibrium point E_2 as follows:

$$\lambda_1 = \sigma_r + i\sigma_i, \quad \lambda_2 = \sigma_r - i\sigma_i, \quad \lambda_3 = \gamma. \tag{3.10}$$

It follows from the Routh-Hurwitz theorem that the necessary and sufficient conditions for

$$\sigma_r < 0 \quad \text{and} \quad \gamma < 0$$

hold if and only if

$$\alpha_2|_{E_2} > 0, \quad \text{and} \quad \alpha_2|_{E_2}\alpha_1|_{E_2} > \alpha_0|_{E_2} > 0. \tag{3.11}$$

Lemma 3.2. *Suppose $ms < rv$. Both the equilibrium points E_2, E_3 are linearly unstable.*

Proof. Assuming $\alpha_2|_{E_2} < 0$, it follows from the Routh–Hurwitz theorem that there exists at least one root of the polynomial (3.5) at the equilibrium point E_2, which has a positive real part. □

To ensure the existence of stable equilibrium points, based on the two preceding lemmas, for the coefficients in the economic model (3.1), we consider the following two cases:

$$\begin{cases} ms > rv, \\ sv < rw, \end{cases} \tag{3.12}$$

and

$$\begin{cases} ms > rv, \\ sv > rw, \\ ms(sv - rw) < 2dpwr^2. \end{cases} \tag{3.13}$$

Lemma 3.3. *Under the assumption* (3.12) *or* (3.13)*, if $c > c_0$, then both the equilibrium points E_2 and E_3 are linearly unstable.*

Proof. One can deduce from the assumption (3.12) or (3.13) and the condition that $c < c_0$ that

$$\alpha_2|_{E_j} > 0, \quad \text{and} \quad \alpha_2|_{E_j}\alpha_1|_{E_j} < \alpha_0|_{E_j} > 0, \quad j = 2, 3. \tag{3.14}$$

□

Lemma 3.4. *Under assumption* (3.12) *or* (3.13)*, if $c < c_0$, then both the equilibrium point E_2 and the equilibrium point E_3 are linearly stable.*

Proof. One can deduce from the assumption (3.12) and the condition that $c < c_0$ that

$$\alpha_2|_{E_j} > 0, \quad \text{and} \quad \alpha_2|_{E_j}\alpha_1|_{E_j} > \alpha_0|_{E_j} > 0, \quad j = 2, 3. \tag{3.15}$$

□

3.4. Principle of exchange stability

Consider the three eigenvalues of the system (3.4) at the equilibrium point E_2. We assume

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) \geq \lambda_3. \tag{3.16}$$

By combining Lemmas 3.3 and 3.4, we establish the following result:

Lemma 3.5. *Under the assumptions* (3.12) *or* (3.13)*, there exists $\epsilon > 0$ such that for all $c \in (c_0 - \epsilon, c_0 + \epsilon)$, λ_1 and λ_2 form a pair of complex conjugate eigenvalues of* (3.4) *at E_2, and λ_3 is a real eigenvalue. Furthermore, these eigenvalues satisfy*

$$\begin{cases} \mathrm{Re}(\lambda_i)(i = 1, 2) \begin{cases} > 0, & c + \epsilon > c > c_0, \\ = 0, & c = c_0, \\ < 0, & c < c_0, \end{cases} \\ \lambda_3 < 0, \quad c = c_0. \end{cases} \tag{3.17}$$

Based on Lemma 2.1 and under Assumption (3.12) or (3.13), the economic model (3.1) must undergo one of three types of transition at $c = c_0$. Note that c_0 is a function that depends on the parameters (m, s, r, v, d, p, w). When some parameters are given, the critical output-capital ratio c_0 can be plotted as a function of the remaining parameters.

Given parameters (r, v, p, w, s) as follows,

$$s = 1, \ r = 0.1, \ v = 0.05, \ p = 0.4, \ w = 0.6,$$

which satisfy the condition (3.12). The critical output-capital ratio c_0 is a function of d and m. For $(m, d) \in [0.02, 0.1] \times [0.2, 1]$, we plot c_0, which is shown in Figure 1.

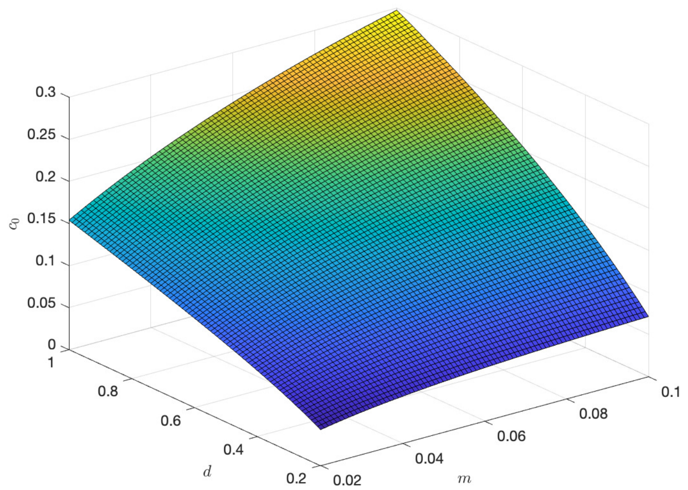

Figure 1. The critical output-capital ratio $c_0(m, d)$ as a function of (m, d), with parameters $s = 1, \ r = 0.1, \ v = 0.05, \ p = 0.4, \ w = 0.6.$

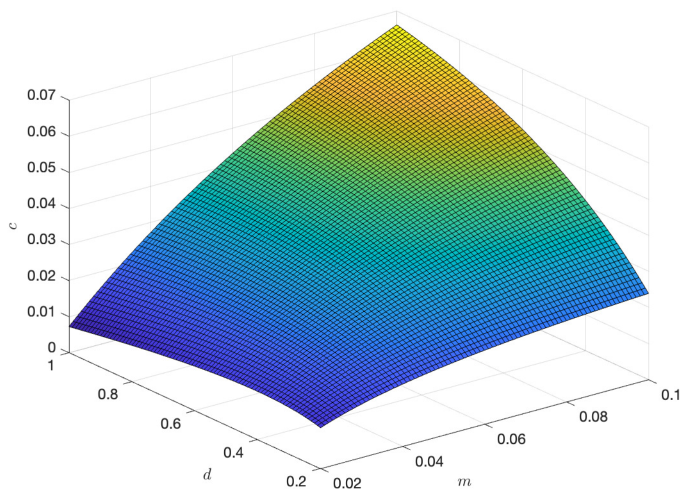

Figure 2. The critical output-capital ratio $c_0(m, d)$ as a function of (m, d), with parameters $s = 10, \ r = 0.5, \ v = 0.05, \ p = 0.8, \ w = 0.9.$

Given parameters (r, v, p, w, s) as follows,

$$s = 10,\ r = 0.5,\ v = 0.05,\ p = 0.8,\ w = 0.9,$$

that satisfy condition (3.13). The critical output-capital ratio c_0 is a function of d and m. For $(m, d) \in [0.02, 0.1] \times [0.2, 1]$, we plot c_0, which is shown in Figure 2.

4. Transition theorem

Subsequently, we systematically develop a classification methodology for transition types through Lyapunov coefficient analysis.

For analytical convenience, we formally introduce the following operators:

$$G_2(\mathbf{x}, \widetilde{\mathbf{x}}) = -\begin{pmatrix} \frac{pr\alpha}{s}\widetilde{y}^2 + 2p\alpha x\widetilde{y} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{4.1}$$

$$G_3(\mathbf{x}, \widetilde{\mathbf{x}}, \widehat{\mathbf{x}}) = -\begin{pmatrix} px\widetilde{y}^2 \\ 0 \\ 0 \end{pmatrix}, \quad \widetilde{\mathbf{x}} = \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix}, \quad \widehat{\mathbf{x}} = \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix}. \tag{4.2}$$

Based on the preceding lemmas, we have established the existence of a dynamic transition in the economic system (3.1) at the critical parameter point $\theta = (m, s, r, v, d, p, w, c) = (m, s, r, v, d, p, w, c_0)$ within the parameter space (m, s, r, v, d, p, w, c). For notational convenience, we denote the general parameter point as $\theta = (m, s, r, v, d, p, w, c)$ and the critical parameter point as $\theta_c = (m, s, r, v, d, p, w, c_0)$ within the parameter space.

Let $\{\lambda_k(\theta)\}_{k=1}^{3}$ be the eigenvalues in (3.17) corresponding to the matrix $L(r\alpha/s, \alpha, -(wr + vs)\alpha/(cs))$ given in (3.4), that is,

$$L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr + vs)\alpha}{cs}\right) = \begin{pmatrix} -\frac{ms}{r} & -\frac{2dpr}{s} - m & 0 \\ w & v & c \\ s & -r & 0 \end{pmatrix},$$

and assume $\lambda_1(\theta) = \sigma_r(\theta) + i\sigma_i(\theta) = \overline{\lambda_2(\theta)}$ and $\lambda_3(\theta) \in \mathbb{R}$, where $\sigma_i(\theta) > 0$, that is, they are roots of (3.5) with $(x_0, y_0, z_0) = (r\alpha/s, \alpha, -(wr + vs)\alpha/(cs))$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the real part and imaginary part of the eigenvector corresponding to $\lambda_1(\theta)$ and $\mathbf{e}_3(\theta)$ be the eigenvector corresponding to $\lambda_3(\theta)$.

Denote $E = (\mathbf{e}_1(\theta), \mathbf{e}_2(\theta), \mathbf{e}_3(\theta))$, and let E^* be the matrix given by

$$E^* = (\mathbf{e}_1(\theta)^*, \mathbf{e}_2(\theta)^*, \mathbf{e}_3(\theta)^*), \tag{4.3}$$

where $\mathbf{e}_1(\theta)^* + i\mathbf{e}_2(\theta)^*$ is the eigenvector of $(L(r\alpha/s, \alpha, -(wr + vs)\alpha/(cs)))^T$, and

$$(\mathbf{e}_1(\theta) + i\mathbf{e}_2(\theta), \mathbf{e}_1(\theta)^* + i\mathbf{e}_2(\theta)^*) = 1.$$

Let e_{1r} and e_{1i} be the real part and imaginary part of the eigenvector corresponding to λ_1, respectively. For $\xi = s_1\mathbf{e}_1(\theta) + s_2 i\mathbf{e}_2(\theta)$, we have

$$L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr + vs)\alpha}{cs}\right)\xi = (\sigma_r s_1 - \sigma_i s_2)\mathbf{e}_1(\theta) + (\sigma_r s_1 + \sigma_i s_2)\mathbf{e}_2(\theta).$$

We introduce the linear spaces $H_c = \text{span}\,\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$ and $H_s = \{\mathbf{e}_3(\theta)\}$ with corresponding orthogonal projections P_c and P_s. Letting $u = x\mathbf{e}_1(\theta) + y\mathbf{e}_2(\theta) + z\mathbf{e}_3(\theta)$, $u_c = P_c u$, $u_s = P_s u$, and $G = G_1 + G_2$, one can rewrite (3.1) as

$$\begin{aligned}\frac{du_c}{dt} &= L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)u_c + P_c G(u_c + u_s, u_c + u_s),\\ \frac{du_s}{dt} &= L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)u_s + P_s G(u_c + u_s, u_c + u_s).\end{aligned}$$

In order to approximate the center manifold function, we use the ansatz

$$u_s = h(u_c) = h_2(u_c) + h_3(u_c) + h_4(u_c) + O(|u_c|^5),$$

where h_k is k-linear. Then, we have

$$\begin{aligned}&L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)h + G_s(u_c, u_c) + \tilde{G}_s(u_c, h) + G_s(h, h)\\ &= \nabla h\left[L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)u_c + G_c(u_c, u_c) + \tilde{G}_c(u_c, h) + G_c(h, h)\right].\end{aligned}$$

Expanding this, we find that

$$\begin{aligned}&\nabla h_2 L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)\xi + \nabla h_3 L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)\xi + \nabla h_2 G_c(\xi, \xi)\\ &+ \nabla h_4 L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)xi + \nabla h_2 \tilde{G}_c(\xi, h_2) + \nabla h_3 G_c(\xi, \xi)\\ &= L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)h_2 + G_s(\xi, \xi) + L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)h_3 + \tilde{G}_s(\xi, h_2)\\ &+ L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)h_4 + \tilde{G}_s(\xi, h_3) + G_s(h_2, h_2) + O(|\xi|^5).\end{aligned}$$

The quadratic part of the above identity gives

$$\nabla h_2 L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)\xi - L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr+vs)\alpha}{cs}\right)h_2 = G_s(\xi, \xi).$$

The formula for h_2 is then found by simply solving a linear system. More precisely, letting $h_2(\xi) = \sum_{i=3}^{4}(x^2\phi^i_{2,0} + xy\phi^i_{1,1} + y^2\phi^i_{0,2})e_i$ and $\phi^i = (\phi^i_{2,0}, \phi^i_{1,1}, \phi^i_{0,2})^T$, one needs to solve

$$(N_2 - \beta_i)\phi^i = \begin{pmatrix}\left\langle G(e_1, e_1), e_i^*\right\rangle\\ \left\langle \tilde{G}(e_1, e_2), e_i^*\right\rangle\\ \left\langle G(e_2, e_2), e_i^*\right\rangle\end{pmatrix},$$

where

$$N_2 = \begin{pmatrix} 2\alpha & \sigma & 0\\ -2\sigma & 2\alpha & 2\sigma\\ 0 & -\sigma & 2\alpha\end{pmatrix}.$$

Similar but more complicated formulas can also be obtained for h_3 and h_4 without much work. Thus, in addition to inverting the above linear system, finding the explicit form of the eigendecomposition constitutes the core of the computational work required to reduce the system.

After having performed all these calculations, we arrive at a set of reduced equations of the form

$$\frac{dx}{dt} = \sigma_r x - \sigma_i y + \sum_{2\leq p+q\leq 5} a^1_{pq} x^p y^q + O(|(x,y)|^6), \tag{4.4}$$

$$\frac{dy}{dt} = \sigma_r y + \sigma_i x + \sum_{2\leq p+q\leq 5} a^2_{pq} x^p y^q + O(|(x,y)|^6), \tag{4.5}$$

where the coefficients a^i_{pq}, $i = 1, 2$, $2 \leq p + q \leq 5$, can be determined numerically by using the procedure outlined above.

From [13], for c in the vicinity of c_0, it is known that the system (4.4) and (4.5) can be reduced to the following system:

$$\begin{aligned}\frac{dw}{dt} &= (\alpha(\theta) + \sigma(\theta)i)\, w + \sigma(\theta) l_1(\theta)\, |w|^2\, w \\ &\quad + \sigma(\theta) l_2(\theta)\, |w|^4\, w + O(|w|^6),\end{aligned} \tag{4.6}$$

where $l_1(\theta)$ and $l_2(\theta)$ are called the first and second Lyapunov coefficients, respectively, which are determined by E, G_2, and G_3 completely.

Hence, the transition type in the economical system (3.1) at the critical point θ is determined by that of the reduced system (4.6). Based on the preceding reduced system, we have the following results:

Theorem 4.1. *Consider the economic system* (3.1) *under the assumptions of Lemma 3.5 (transversality condition), and assume the vector field is at least C^3-smooth near the equilibria. Let $\theta_c = (m, s, r, v, d, p, w, c_0)$ be the critical parameter point where c_0 is as defined in* (3.9)*. Then, the following assertions hold:*

1) If $l_1(\theta_c) < 0$, or if $l_1(\theta_c) = 0$ and $l_2(\theta_c) < 0$, the system undergoes a continuous transition (i.e., a supercritical Hopf bifurcation) at θ_c. Specifically, for $c > c_0$, the equilibrium E_2 bifurcates into a stable periodic orbit Γ.

2) If $l_1(\theta_c) < 0$, or if $l_1(\theta_c) = 0$ and $l_2(\theta_c) > 0$, the system undergoes a jump transition (i.e., a subcritical Hopf bifurcation) at θ_c. In this case, an unstable periodic orbit emerges, and the system may transition discontinuously to another attractor. Furthermore, there exists a subcritical point

$$\theta_{cc} = (m, s, r, v, d, p, w, c^*(m, s, r, v, d, p, w))$$

at which there is a separation of periodic orbits such that the nonzero attractor Γ_1 *bifurcates from E_2. In particular, there is no stable periodic bifurcation solution bifurcating on $c > c_0$.*

3) If $l_1(\theta_c) = 0$ and $l_2(\theta_c) = 0$, the bifurcation is degenerate, and higher-order terms are needed (not analyzed here).

Proof. It suffices to demonstrate that the trivial solution to the reduced system (4.6) exhibits stability (instability) at the critical point $\theta = \theta_c$ when $l_1(\theta_c) < 0$ ($l_1(\theta_c) > 0$). Specifically, note that the dynamics of the system are governed by

$$\frac{d\,|\tilde{w}|^2}{dt} = 2\sigma(\theta) l_1(\theta)\, |\tilde{w}|^4 + o(|\tilde{w}|^4), \quad \sigma(\theta) > 0. \tag{4.7}$$

This differential relationship readily implies the stability (instability) of the zero solution for the reduced system (4.6) at $\theta = \theta_c$, under the condition that $l_1(\theta_c) < 0$ ($l_1(\theta_c) > 0$).

When $l_1(\theta_c) = 0$, the stability of the trivial solution is determined by the second Lyapunov coefficient $l_2(\theta_c)$. Specifically, the dynamics of the reduced system (4.6) are governed by the higher-order expansion:

$$\frac{d\,|\tilde{w}|^2}{dt} = 2\sigma(\theta)l_2(\theta)\,|\tilde{w}|^6 + o(|\tilde{w}|^6), \quad \sigma(\theta) > 0.$$

Thus, if $l_2(\theta_c) < 0$, the zero solution is stable (continuous transition); if $l_2(\theta_c) > 0$, it is unstable (jump transition). The conclusions in this case can also be derived from Theorems 2.3.7 and 2.5.7 in [15]. □

The topological structures for continuous transition (depicted in Figure 3) and jump transition (illustrated in Figure 4) described in Theorem 4.1 are presented comparatively.

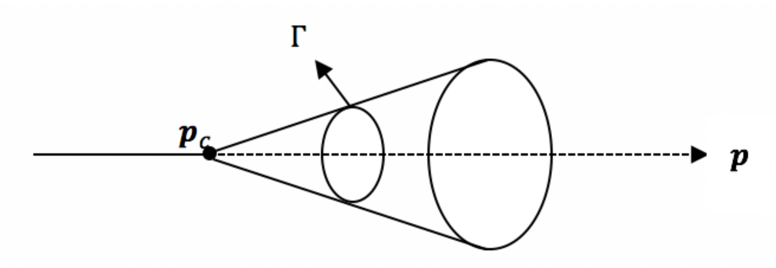

Figure 3. The continuous transition at θ_c.

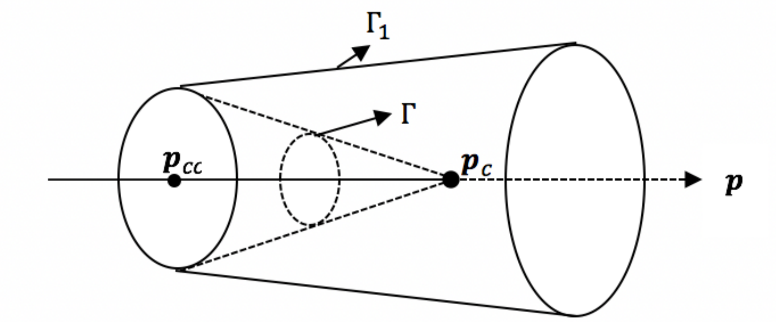

Figure 4. The jump transition at θ_c.

Remark 4.1. *The aforementioned bifurcation and transition phenomena are inherently associated with the critical equilibrium E_2. Due to the symmetry $(x, y, z) \longrightarrow (-x, -y, -z)$ of system* (3.1)*, the results apply equally to E_3. However, E_3 is economically infeasible due to negative capital/output under parameter constraints.*

Remark 4.2. *The qualitative distinction between the "jump" and "continuous" transitions fundamentally arises from the vanishing condition of the first Lyapunov coefficient l_1.*

5. Numerical investigations

5.1. Types of dynamic transition

In the preceding analysis, we formally demonstrated that the economic system (3.1) exhibits either continuous or catastrophic transitions at $c = c_0$. In this subsection, we provide numerical verification of these transition types at the corresponding critical control parameter values. By employing the theoretical framework established in the preceding sections, we recast the verification task as the computational determination of the first Lyapunov coefficient $l_1(m, p, d, w, v, c_c, s, r)$, where c_c denotes the critical value of the control parameter c. Furthermore, utilizing the invariant formula for the first Lyapunov coefficient derived in [13] (cf. formula (5.39)), the explicit expression for $l_1(m, p, d, w, v, c_c, s, r)$ corresponding to the economic system (3.1) is obtained as

$$\begin{aligned} l_1(\theta_c) = \frac{1}{2\sigma(\theta_c)} \mathrm{Re}\Big[& (\mathbf{e}^*, G_3(\mathbf{e}, \mathbf{e}, \bar{\mathbf{e}})) \\ & + 2\,(\mathbf{e}^*, G_2\,(\mathbf{e}, \mathbf{h}_{11})) + (\mathbf{e}^*, G_2\,(\bar{\mathbf{e}}, \mathbf{h}_{20}))\Big], \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \mathbf{e} &= \mathbf{e}_1(\theta_c) + i\mathbf{e}_2(\theta_c), \\ \mathbf{e}^* &= \mathbf{e}_1^*(\theta_c) + i\mathbf{e}_2^*(\theta_c), \\ \mathbf{h}_{11} &= -\left(L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr + vs)\alpha}{cs}\right)\right)^{-1} G_2(\mathbf{e}, \bar{\mathbf{e}}), \\ \mathbf{h}_{20} &= (2i\sigma(\theta_c)I_3 - \left(L\left(\frac{r\alpha}{s}, \alpha, -\frac{(wr + vs)\alpha}{cs}\right)\right)^{-1} G_2(\mathbf{e}, \mathbf{e}). \end{aligned}$$

To evaluate the first Lyapunov coefficient $l_1(\theta_c)$ in (5.1) numerically, we proceed as follows.

1) For a fixed set of parameters (m, s, r, v, d, p, w) satisfying Assumption (3.12) (or (3.13)), we compute the equilibrium E_i $(i = 2, 3)$ at $c = c_c$ and evaluate the Jacobian matrix $A = Df(E_i; \theta_c)$, where f denotes the vector field of (3.1). The critical value c_c is determined by enforcing that A has a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_c$ and that all other eigenvalues have negative real parts.
2) Let $\mathbf{e} \in \mathbb{C}^3$ be the right eigenvector of A associated with $i\omega_c$, and let $\mathbf{e}^* \in \mathbb{C}^3$ be the left (adjoint) eigenvector associated with $-i\omega_c$, that is, $Ae = i\omega_c\mathbf{e}$ and $A^\top e^* = -i\omega_c\mathbf{e}^*$. We normalize them by the standard condition $\langle \mathbf{e}^*, \mathbf{e} \rangle = 1$ (complex inner product), which fixes the scaling in (5.1).
3) We construct the symmetric multilinear forms $G_2(\cdot, \cdot)$ and $G_3(\cdot, \cdot, \cdot)$ from the second and third derivatives of f at the equilibrium,

$$G_2(u, v) = \frac{1}{2}D^2 f(E_i; \theta_c)[u, v], \qquad G_3(u, v, w) = \frac{1}{6}D^3 f(E_i; \theta_c)[u, v, w].$$

   In implementation, $D^2 f$ and $D^3 f$ are obtained either by symbolic differentiation (recommended for accuracy) or by centered finite differences with a prescribed step size.

4) Following the standard center-manifold/normal-form reduction, we compute

$$h_{20} = (2i\omega_c I - A)^{-1} G_2(\mathbf{e}, \mathbf{e}), \qquad h_{11} = (-A)^{-1} G_2(\mathbf{e}, \bar{\mathbf{e}}),$$

where $\bar{e}$ is the complex conjugate of e. The inverses are computed by solving the corresponding linear systems.

5) Finally, we substitute $(e, e^*, G_2, G_3, h_{20}, h_{11})$ into (5.1) and take the real part. The Hopf bifurcation classified by the sign of $\mathrm{Re}(l_1)$: $\mathrm{Re}(l_1) < 0$ implies a supercritical Hopf (continuous transition), and $\mathrm{Re}(l_1) > 0$ implies a subcritical Hopf (jump transition).

To assess the sign of l_1 reliably, we verify that $\mathrm{Re}(l_1)$ is insensitive to reasonable changes in numerical settings (e.g., differentiation step size and floating-point precision). When $|\,\mathrm{Re}(l_1)|$ is close to zero (within a prescribed tolerance), we additionally confirm the classification by direct numerical integration for $c = c_c \pm \varepsilon$ and checking whether the bifurcating periodic orbit is stable or unstable.

From a numerical perspective, the transition type can be rigorously characterized through the exact analytical expression presented in (5.1). Specifically, as established in Theorem 4.1, the initial step to determine the transition type at the critical parameter point $\theta = \theta_c$ requires estimating the first Lyapunov coefficient $l_1(m, p, d, w, v, c_0, s, r)$. In Section 4.1, we have seen that for the given parameters (m, p, d, w, v, s, r), there exists a transition from the equilibrium point E_2 at $(m, p, d, w, v, c_0(m, p, d, w, v, s, r), s, r)$ if one of assumptions (3.12) and (3.13) is satisfied, where the critical output-capital ratio c_0 is a function of (m, p, d, w, v, s, r) with the exact expression

$$c_0(m, p, d, w, v, s, r) = \frac{(ms - rv)\left(2dpr^2 w + ms(rw - sv)\right)}{r^2 s(2dpr + ms + rv)}.$$

If we take $s = 1,\ r = 0.1,\ v = 0.05,\ p = 0.4,\ w = 0.6$, and $(m, d) \in [0.01, 0.1] \times [0.1, 1]$, we can verify that Assumption (3.12) is valid. If we take $s = 5,\ r = 0.5,\ v = 0.05,\ p = 0.8,\ w = 0.9$, and $(m, d) \in [0.01, 0.1] \times [0.1, 1]$, Assumption (3.13) holds.

When the parameters (p, w, v, s, r) are specified, we can regard the first Lyapunov coefficient l_1 as a function with two variables, m and d. In this case, the specific transition type in the economic system (3.1) can be determined by numerically estimating the sign of the first Lyapunov coefficient l_1. Preliminary numerical explorations of l_1 are presented in Figure 5 with $s = 1,\ r = 0.1,\ v = 0.05,\ p = 0.4,\ w = 0.6$, $(m, d) \in [0.01, 0.1] \times [0.1, 1]$, and in Figure 6 with $s = 5,\ r = 0.5,\ v = 0.05,\ p = 0.8,\ w = 0.9$, $(m, d) \in [0.01, 0.1] \times [0.1, 1]$.

The results in Figures 5 and 6 show that the economic system (3.1) captures both the continuous and jump transitions at these critical points $(m, p, d, w, v, c_0(m, p, d, w, v, s, r), s, r)$ for appropriate values of m and d when (p, w, v, s, r) are specified. More precisely, from Figure 5, we can infer that for $s = 1,\ r = 0.1,\ v = 0.05,\ p = 0.4,\ w = 0.6$, when (m, d) takes its values in the vicinity of $(0.02, 0.8)$, l_1 on the parameter plane (m, d) is negative, whereas l_1 is positive when (m, d) takes its values near $(0.07, 0.5)$ on the parameter plane (m, d). For $s = 5,\ r = 0.5,\ v = 0.05,\ p = 0.8,\ w = 0.9$, we can infer from Figure 6 that when (m, d) takes its values in the vicinity of $(0.09, 0.2)$ on the parameter plane (m, d). l_1 is negative, whereas l_1 is positive when (m, d) takes its values near $(0.05, 0.7)$ on the parameter plane (m, d).

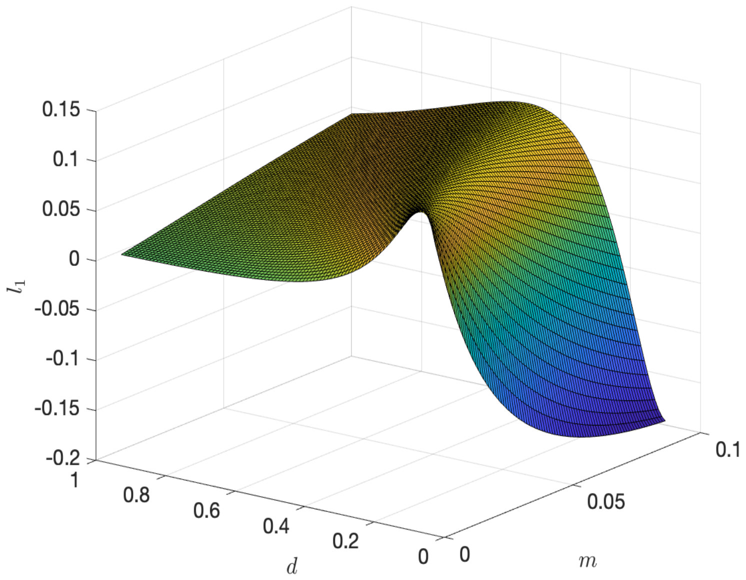

Figure 5. The Lyapunov coefficient $l_1(m, d)$ as a function of (m, d), with parameters $s = 1,\ r = 0.1,\ v = 0.05,\ p = 0.4,\ w = 0.6$.

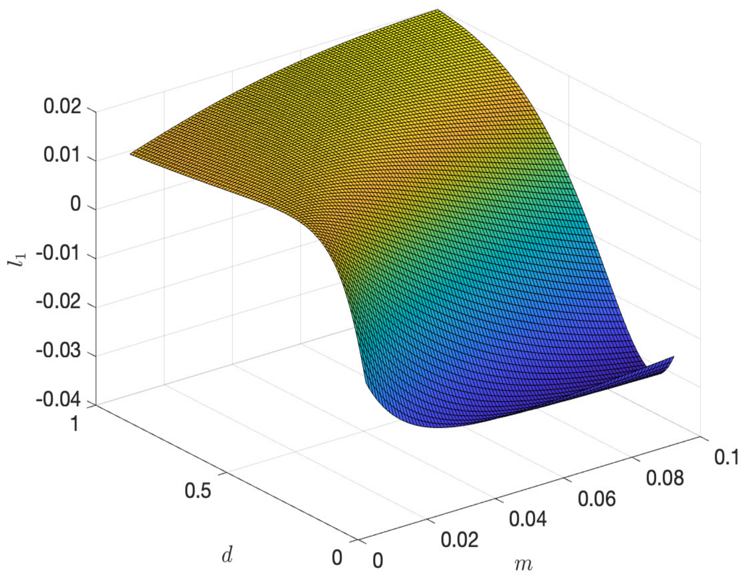

Figure 6. The Lyapunov coefficient $l_1(m, d)$ as a function of (m, d), where parameters $s = 5,\ r = 0.5,\ v = 0.05,\ p = 0.8,\ w = 0.9$.

Because the system exhibits both types of dynamical transitions (continuous and discontinuous), a fundamental research question concerns the identification of parameter space regions that give rise to distinct transition types. From a numerical perspective, once the parameters (p, w, v, s, r) are fixed, evaluating the mapping $(m, d) \to l_1$ becomes a computationally intensive yet tractable task.

Consequently, the aforementioned computational procedure can be reliably implemented using the bisection method without significant technical challenges. This approach yields a boundary curve in the (m, d) parameter space, approximately corresponding to l_1, that defines an effective boundary separating the regions of continuous and jump transition, as illustrated in Figures 7 and 8.

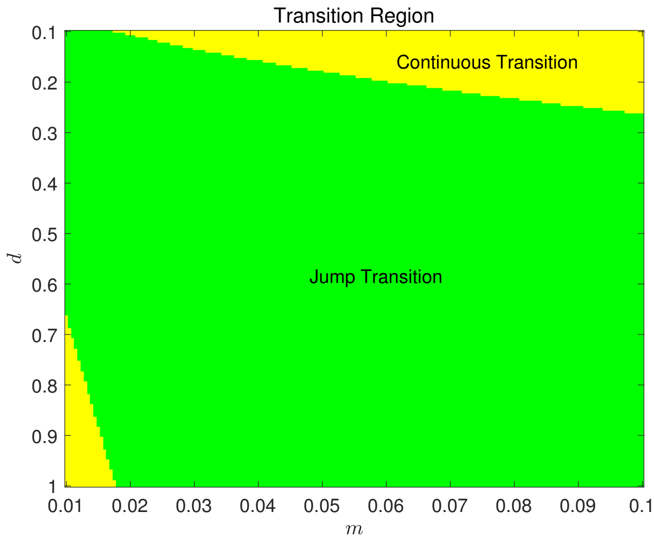

Figure 7. The continuous and jump transition region: $s = 1,\ r = 0.1,\ v = 0.05,\ p = 0.4,\ w = 0.6$.

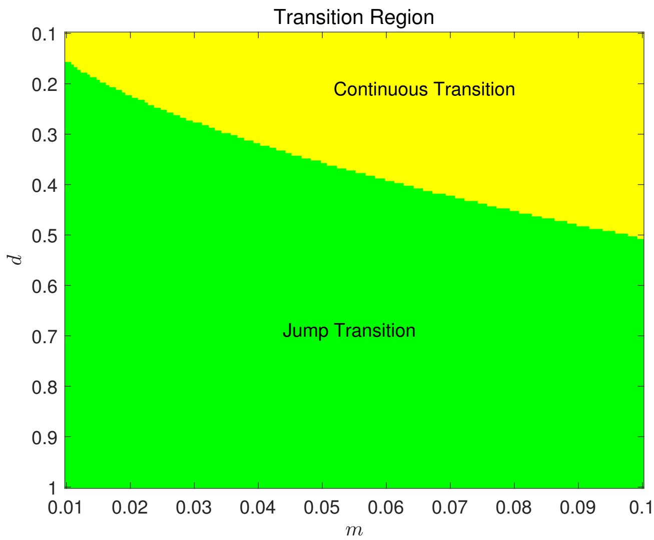

Figure 8. The continuous and jump transition region: $s = 5,\ r = 0.5,\ v = 0.05,\ p = 0.8,\ w = 0.9$.

5.2. Numerical integration

All simulations of (3.1) are performed in double precision using an ode15s time integrator. Unless otherwise stated, we set the relative and absolute error tolerances to RelTol $= 10^{-10}$ and AbsTol $=$

10^{-12}. For each parameter set, trajectories are integrated on $[0, T]$ with T chosen sufficiently large to capture the asymptotic regime (e.g., $T = 1.6 \times 10^4$ in Figures 9 and 10). To remove transients, we discard the initial interval $[0, T_{\text{tr}}]$ (e.g., $T_{\text{tr}} = 8 \times 10^3$) and generate phase portraits and statistics on $[T_{\text{tr}}, T]$. For visualization and spectral diagnostics, solutions are sampled uniformly with step Δt (e.g., $\Delta t = 0.1$) after transient removal.

Initial conditions are chosen as small perturbations around the relevant equilibrium (typically E_2 or E_3),

$$(x(0), y(0), z(0)) = E_i + \varepsilon\xi, \qquad i \in \{2, 3\},$$

where ξ is a unit vector, and ε is a small amplitude (e.g., $\varepsilon = 10^{-3}$). We also repeat simulations for multiple perturbation directions and amplitudes to verify that the reported long-time behavior is robust.

In Figures 9 and 10, the scalar quantity

$$l(t) = \left\|(x(t), y(t), z(t)) - E_2\right\|_2$$

is used to measure the distance to the equilibrium E_2.

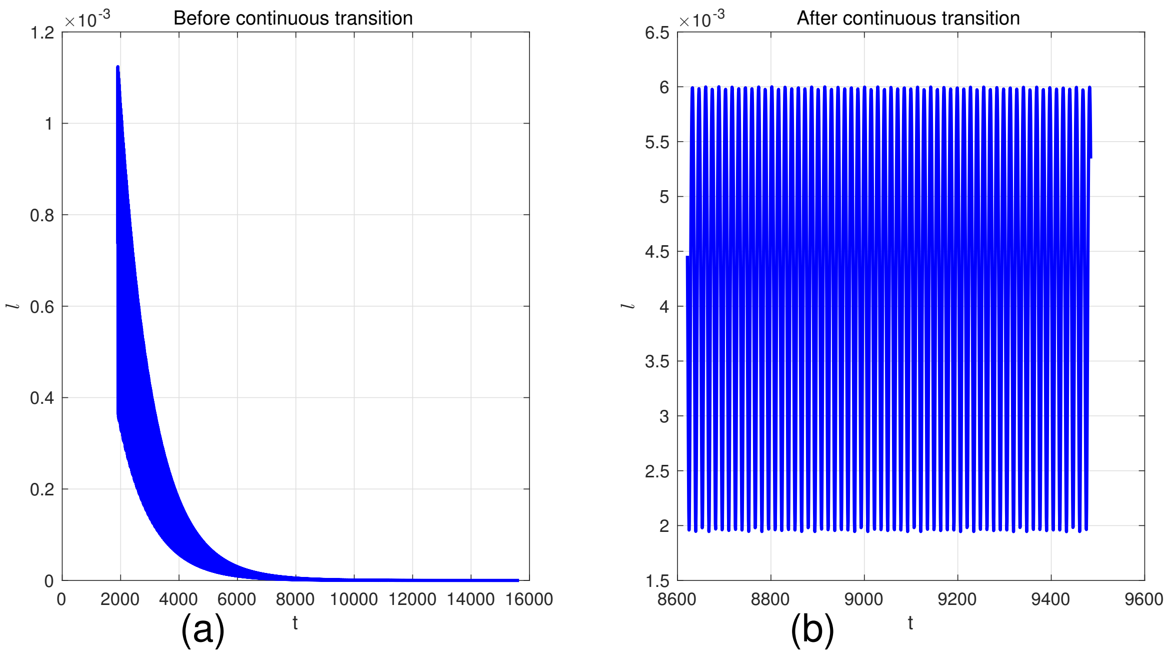

Figure 9. $l = |E_2 - (x, y, z)|, (s, r, v, p, w, m, d) = (1, 0.1, 0.05, 0.4, 0.6, 0.01, 0.95), c_0 = 0.0256$, (a) $c = 0.0246$; (b) $c = 0.0266$.

For given (m, p, d, w, v, s, r) satisfying one of the assumptions (3.12) and (3.13), we have theoretically established that the economic system (3.1) exhibits two distinct transition regimes when the output-capital ratio parameter c exceeds a critical threshold c_0, which is determined by the system parameters (m, p, d, w, v, s, r). Specifically, the asymptotic equilibrium state of the system (3.1) transitions from either E_2 or E_3 to alternative equilibrium states under this condition. In this section, we conduct numerical simulations of the economic system (3.1) to empirically validate our theoretical findings through phase space trajectory analysis.

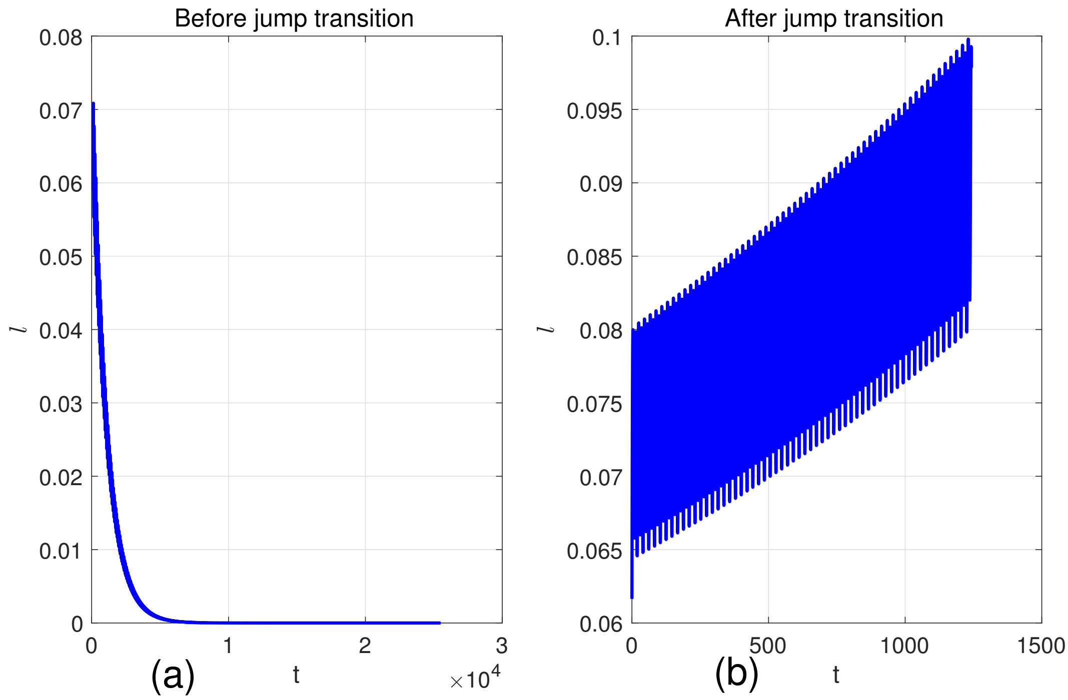

Figure 10. $l = |E_2 - (x, y, z)|$, $(s, r, v, p, w, m, d) = (1, 0.1, 0.05, 0.4, 0.6, 0.1, 1)$, $c_0 = 0.2978$, (a): $c = 0.2878$; (b): $c = 0.2988$.

Theorem 4.1 directly implies that for a continuous transition at the critical parameter value c_0:

1) The equilibrium state E_2 (or equivalently E_3) remains locally stable for all parameters $c < c_0$;
2) It undergoes a supercritical bifurcation to a stable periodic solution when $c > c_0$.

For a jump (subcritical) transition at the critical parameter value c_0 :

1) The equilibrium state E_2 (or equivalently E_3) remains locally stable for all $c < c_0$;
2) The system exhibits a discontinuous transition to an alternative state when $c > c_0$.

To empirically validate the theoretical findings presented in Theorem 4.1, we present two illustrative numerical examples.

Example 5.1. *Taking* $(s, r, v, p, w) = (1, 0.1, 0.05, 0.4, 0.6)$ *from Figure 7, we see that the point* $(m, d) = (0.01, 0.95)$ *resides within a continuous transition region, that is,* $l_1 < 0$. *In that case, it is theoretically demonstrated that a stable periodic solution bifurcates from the point*

$$(s, r, v, p, w, m, d, c_0) = (1, 0.1, 0.05, 0.4, 0.6, 0.01, 0.95, 0.0256).$$

Take

$$(s, r, v, p, w, m, d, c) = (1, 0.1, 0.05, 0.4, 0.6, 0.01, 0.95, 0.0266),$$

which is a point close to the critical point $(s, r, v, p, w, m, d, c_0)$ *and is on the right side of* $c = c_0$, *the numerical simulation presented in Figure 9. This confirms that equilibrium* E_2 *remains stable prior to the continuous transition, serving as the asymptotic state of system* (3.1). *Following this continuous transition, the asymptotic state of the system evolves from equilibrium* E_2 *to a stable periodic solution. This transition behavior matches precisely Statement 1) of Theorem 4.1.*

Example 5.2. *Taking* $(s, r, v, p, w) = (1, 0.1, 0.05, 0.4, 0.6)$ *from Figure 7, we see that the point* $(m, d) = (0.1, 1)$ *resides within a continuous transition region, that is,* $l_1 < 0$*. In that case, it is theoretically demonstrated that a stable periodic solution bifurcates from the point*

$$(s, r, v, p, w, m, d, c_0) = (1, 0.1, 0.05, 0.4, 0.6, 0.1, 1, 0.2978).$$

Take

$$(s, r, v, p, w, m, d, c) = (1, 0.1, 0.05, 0.4, 0.6, 0.1, 1, 0.2988),$$

which is a point close to the critical point $(s, r, v, p, w, m, d, c_0)$ *and is on the right side of* $c = c_0$*. The numerical simulation presented in Figure 10 demonstrates that the equilibrium state* E_2 *maintains stability prior to the occurrence of a jump transition, which is an asymptotically stable state of the system* (3.1)*. Following a jump transition, the asymptotic state of the economic system* (3.1) *transitions from equilibrium* E_2 *to an alternative stable state. This outcome is in accordance with Statement 1) of Theorem 4.1.*

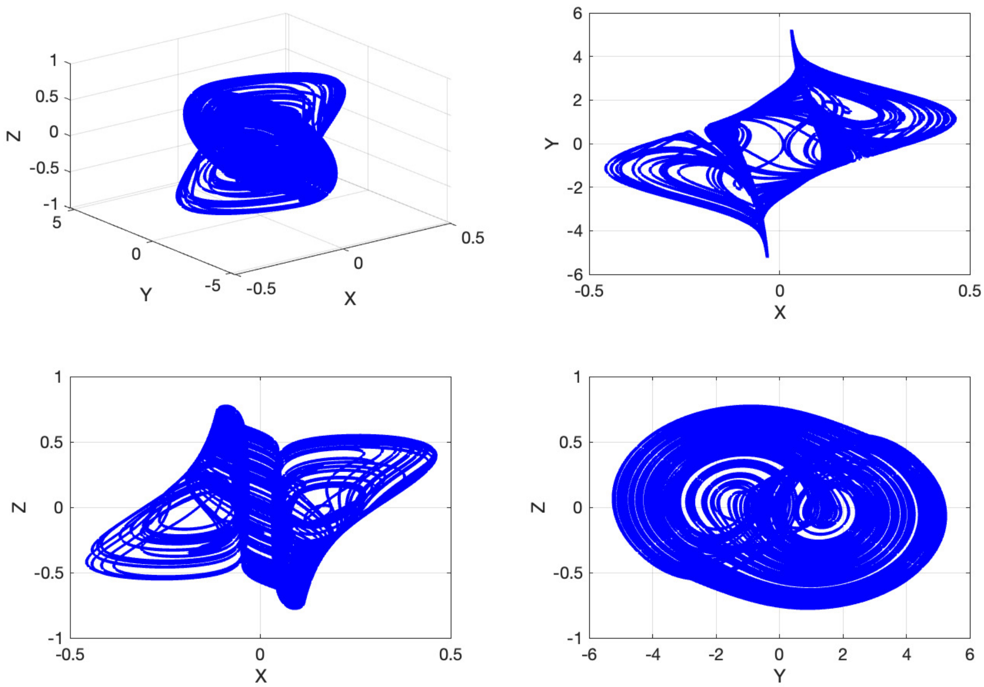

Figure 11. $s = 1,\ r = 0.1,\ v = 0.05,\ p = 0.4,\ w = 0.6, m = 0.08\ , d = 0.9,\ c = 4.2446$

Because the system (3.1) only has three equilibrium points, they are all unstable as $c > c_0$. After a continuous transition, the new stable state is periodic, whereas after a jump transition, we cannot explicitly determine the new stable state. Intuitively, there are three possible situations for $c > c_0$. First, the system (3.1) will go to infinity. Second, the system (3.1) may have other stable periodic solutions. Third, the system (3.1) has a bounded global attractor, which is a strange attractor in which any orbit of the system (3.1) is dense. Performing further numerical simulations, we find that for $c > c_0$ but c is very close to c_0, after a jump transition, the system (3.1) exhibits the first dynamic behavior. In contrast, for $c > c_0$ but c is not close to c_0, after a jump transition, any orbit of the system (3.1) is dense in a bounded set; see Figure 11. The result in Figure 11 shows that for a large output–capital ratio, the economic system (3.1) exhibits complex, chaotic behavior.

6. Conclusions

The ODE system (3.1) represents a modified Bouali model [8], formulated as an idealized macroeconomic framework for business-cycle generation. The dimensionless variables x, y, and z represent savings, gross domestic product (GDP), and foreign capital inflows, respectively, and the model reflects LM-IS-style macroeconomic interactions. The system (3.1) depends on parameters s, r, v, p, w, m, d, and the output-capital ratio c. We analyze how qualitative changes in dynamics arise as parameters vary, using the phase transition dynamic theory in [15].

Under assumption (3.12) or (3.13), we identify c as the primary control parameter for the dynamical transition of (3.1). Specifically, we establish an explicit critical threshold

$$c_0 := \frac{(ms - rv)\left(2dpr^2w - ms(sv - rw)\right)}{r^2 s(2dpr + ms + rv)}.$$

When $c < c_0$, the equilibria are locally asymptotically stable and govern the long-time behavior of trajectories. At $c = c_0$, a Hopf bifurcation occurs, and the transition type is determined by the first Lyapunov coefficient $l_1(s, r, v, p, w, m, d, c_0)$. If $l_1 < 0$, the Hopf bifurcation is supercritical, yielding a stable periodic orbit (continuous transition). If $l_1 > 0$, the Hopf bifurcation is subcritical (catastrophic transition), and solutions may exhibit multistability and more complex postcritical dynamics. Our numerical results in Section 5 further suggest that, for sufficiently large c, trajectories can become aperiodic and display features consistent with chaotic dynamics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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