Research article

Positive solutions for Hadamard fractional differential equations with sign changing nonlinearity and paremeters

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Abstract: This study investigates boundary value problems for nonlinear fractional differential equations. The differential operator is interpreted in the Hadamard sense and is coupled with a nonlinear term that involves the fractional derivative of the unknown function. The existence and multiplicity of positive solutions were established by a reducing-order technique based on the Guo–Krasnoselskii fixed-point theorem. Also, some examples are presented to illustrate the validity of our main results.

Keywords: Hadamard fractional derivative; fractional differential equation; fixed-point theorems; reducing order technique; parameter

1. Introduction

Fractional-order differential equations have gained considerable attention for their applications in a variety of fields such as engineering, biophysics, chemical physics, fluid flow, economics, and so on. For the related applications and details about fractional differential equations, see [1–5]. As a powerful tool of modeling the above many abnormal phenomena, in the last few decades, the fractional calculus theory has been enriched, and several different fractional derivatives and integrals such as the Riemann-Liouville [6], Atangana [7], Caputo [8–10], Caputo-Fabrizio [11], Hilfer [12], Hadamard derivatives [13–17] and so on. In comparison, the Hadamard derivative is a nonlocal, fractional derivative with a singular logarithmic kernel. Therefore, compared with others, the study of Hadamard fractional differential equations is relatively difficult [18–29].

In fractional differential equation studies, boundary value problems stand out as an essential research area, particularly regarding the existence of positive solutions, and lots of excellent results have been obtained by means of fixed-point theorems, topological degree methods, monotone iterative techniques,

upper and lower solutions techniques, and so forth [30–38]. When the nonlinear term changes sign, these problems are called as semipositone; for the relevant results, see [39–45].

This study addresses the existence and multiplicity of positive solutions to the following semipositone boundary value problem associated with the Hadamard-type fractional differential equation

$$
\begin{cases}
{}^{H}D_{1^+}^{\alpha}x(t) + \lambda f(t, x(t), {}^{H}D_{1^+}^{\beta}x(t)) = 0, & 1 < t < e, \\
x(1) = 0, \quad {}^{H}D_{1^+}^{\beta}x(1) = {}^{H}D_{1^+}^{\beta+1}x(1) = 0, \\
{}^{H}D_{1^+}^{\beta+1}x(e) = \mu {}^{H}D_{1^+}^{\beta+1}x(\zeta),
\end{cases}
\tag{1.1}
$$

where ${}^{H}D_{1^+}^{\alpha}, {}^{H}D_{1^+}^{\beta}$ are the Hadamard fractional derivatives, $3 < \alpha \le 4, 0 < \beta < 1, \alpha - \beta \le 3, \lambda > 0$ is a parameter, and $1 < \zeta < e, 0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1, f : (1, e) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a given continuous function, and may be singular at $t = 1$ and/or $t = e$. Futher, it depends on the unknown functions as well as their lower order fractional derivative, in which $e = \exp(1), \mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)$.

In [8], the authors studied the following Caputo nonlinear fractional differential equations with integral boundary value problems:

$$
\begin{cases}
{}^{C}D_{0^+}^{\alpha}u(t) + a(t)f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) = u'(0) = 0, \quad u(1) = \int_0^1 u(t)dA(t),
\end{cases}
\tag{1.2}
$$

where $2 < \alpha \le 3, {}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative. $f : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $a \in L^\infty[0, 1]$ is nonnegative, $\int_0^1 u(t)dA(t)$ denotes the Riemann-Stieltjes integral of u with respect to A. By using the Guo–Krasnoselskii fixed-point theorem and the Leggett-Williams fixed-point theorem, the authors have successfully established results regarding the existence and multiplicity of positive solutions of the given problem, (1.2).

In [31], the authors explored the existence and uniqueness of solutions for nonlocal boundary value problems associated with nonlinear fractional differential equations:

$$
\begin{cases}
D_{0^+}^{\alpha}u(t) + f(t, v(t), D^{\gamma}v(t)) = 0, & 0 < t < 1, 2 < \alpha, \beta \le 3, \\
D_{0^+}^{\beta}v(t) + g(t, u(t), D^{\delta}u(t)) = 0, & 0 < t < 1, 1 < \gamma, \delta \le 2, \\
u(0) = u'(0) = 0, \quad u'(1) = \mu u'(\xi), \\
v(0) = v'(0) = 0, \quad v'(1) = \mu v'(\xi),
\end{cases}
\tag{1.3}
$$

where $\alpha - \delta \ge 1, \beta - \gamma \ge 1, 0 < \xi < 1, 0 < \mu\xi^{\alpha-2} < 1, 0 < \mu\xi^{\beta-2} < 1, D_{0^+}^{\alpha}$, and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives. $f, g : [0, 1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions and depend on the unknown functions and their lower-order fractional derivatives. The authors in [31] established both the uniqueness and presence of solutions for problem (1.3) through the application of the Banach contraction mapping principle, and the Schauder fixed-point theorem.

In [9], the focus was on investigating the existence and nonexistence of positive solutions for boundary value problems related to Caputo nonlinear fractional differential equations such as

$$
\begin{cases}
{}^{c}D_{0^+}^{\theta}p(y) + \mu f(y, p(y)) = 0, & y \in [0, 1], \\
p(0) = p''(0) = 0, p(1) = \int_0^1 p(y)dA(y),
\end{cases}
\tag{1.4}
$$

where $\theta \in (2, 3)$, $^cD_{0^+}^{\theta}$ is the Caputo fractional derivative, μ is a parameter, and A is a bounded variation function with $B = \int_0^1 y dA(y) < 1$, $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}_0^+)$, $\mathbb{R}_0^+ = (0, \infty)$. By using the Guo–Krasnoselskii fixed-point theorem, some results on the existence and nonexistence of positive solutions to problem (1.4) were established.

In [46], the authors considered the following nonlinear fractional differential equation,

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, & n \geq 3, \end{cases} \tag{1.5}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $n - 1 < \alpha < n$, $\lambda > 0$, f : $[0, 1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, and $f \geq -\omega(t)$ for a given nonnegative function ω. By using the Guo–Krasnoselskii fixed-point theorem, the existence of positive solutions was established at different parameter intervals.

Motivated by the excellent work above, in this paper, based on the Guo–Krasnoselskii fixed-point theorem and the Leggett-Williams fixed-point theorem, by the reducing order technique, the existence and multiplicity of positive solutions for the problem (1.1) are obtained. Compared with [8, 9, 46], the boundary value problem (1.1) we study contains a nonlinearity term that depends on lower-order fractional derivatives. We overcome the difficulty of f depending on derivatives by using the reducing-order technique. Compared with [8, 9, 46], the nonlinearity we study may be singular at $t = 1$ and/or $t = e$, and we obtain the existence and multiplicity of positive solutions; our results are therefore more abundant than those in [9, 46]. Compared with [8, 9, 31], the nonlinearity f from (1.1) is not a nonnegative function, but it may change sign and may be singular at some points; that is to say, (1.1) is a semipositone problem. At present, there are relatively few results on triple solutions when the Leggett-Williams fixed-point theorem is used under this circumstance. In [8, 9, 31, 46], emphasis was placed on investigating the Riemann-Liouville or Caputo fractional derivative. In contrast, our study in this paper revolves around the examination of the Hadamard fractional derivatives.

The methodology employed in this study unfolds in the following manner: Section 2 presents a review of essential definitions and preliminary lemmas which are key tools for our main results. In Section 3, we delve into existing findings to the problem mentioned above and give complete proofs. Finally, Section 4 offers some illustrative examples to demonstrate our findings.

2. Preliminaries and lemmas

In this section, for the convenience of reader, we introduce some notations and definitions that will be used in the proof of our main results.

Definition 2.1. (See [47].) Let $g : [1, \infty) \to \mathbb{R}$. The Hadamard fractional integral of order $q > 0$ for a function g is defined as

$$^HI_{1^+}^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Definition 2.2. (See [47].) Let $g : [1, \infty) \to \mathbb{R}$. The Hadamard-type fractional q-order derivative is

defined as

$$^{H}D_{1^{+}}^{q}u(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds, \quad n-1 < q < n, \quad n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q, and $\log(\cdot) = \log_{e}(\cdot)$.

Lemma 2.1. *(See [47].) If $u, v : \mathbb{R}_{0}^{+} \to \mathbb{R}$ with order $\gamma > 0$, then*

$$^{H}D_{t}^{\gamma}(u(t) + v(t)) =^{H} D_{t}^{\gamma}u(t) +^{H} D_{t}^{\gamma}v(t).$$

Lemma 2.2. *(See [47].)*

1) *If $u \in L^{1}(1, e), v > \sigma > 0$, then*

$$^{H}I^{v\,H}I^{\sigma}u(t) =^{H} I^{v+\sigma}u(t), \quad ^{H}D_{t}^{\sigma\,H}I^{v}u(t) =^{H} I^{v-\sigma}u(t), \quad ^{H}D_{t}^{\sigma\,H}I^{\sigma}u(t) = u(t).$$

2) *If $v > 0, \sigma > 0, a > 0$, then* $^{H}D_{a}^{v}\left(\log\left(\frac{t}{a}\right)^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-v)}\left(\log\frac{x}{a}\right)^{\sigma-v-1}$.

Lemma 2.3. *(See [47].) Given $\gamma > 0$ and $x \in C[1, \infty) \cap L^{1}[1, \infty)$, then the solution of Hadamard fractional equation $^{H}D_{1^{+}}^{\gamma}x(t) = 0$ is*

$$x(t) = \sum_{i=1}^{n} c_{i}(\log t)^{\gamma-i},$$

and the following formula holds:

$$^{H}I_{1^{+}}^{\gamma\,H}D_{1^{+}}^{\gamma}x(t) = x(t) + \sum_{i=1}^{n} c_{i}(\log t)^{\gamma-i},$$

where $c_{i} \in \mathbb{R}, i = 1, 2, ..., n$, and $n = [\gamma] + 1$.

Lemma 2.4. *Let $h \in C(1, e) \cap L^{1}(1, e)$ with $0 < \int_{1}^{e} h(s)\frac{ds}{s} < \infty$. If $2 < \alpha - \beta \le 3, 0 < \Omega = 1 - \mu(\log\zeta)^{\alpha-\beta-2} < 1$, then the boundary value problem*

$$\begin{cases} ^{H}D_{1^{+}}^{\alpha-\beta}y(t) + h(t) = 0, & 1 < t < e, \\ y(1) = \delta y(1) = 0, & \delta y(e) = \mu\delta y(\zeta) \end{cases} \tag{2.1}$$

can be represented as

$$y(t) = \int_{1}^{e} G(t, s)h(s)\frac{ds}{s}, \tag{2.2}$$

where $\delta = t(d/dt)$, and

$$G(t, s) = \begin{cases} \dfrac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \mu\left(\log\frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \Omega\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}, \\ \hspace{8cm} 1 \le s \le t \le e, s \le \zeta, \\[4pt] \dfrac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \Omega\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}, \hspace{2cm} 1 \le \zeta \le s \le t \le e, \\[4pt] \dfrac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \mu\left(\log\frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}, \hspace{0.3cm} 1 \le t \le s \le \zeta < e, \\[4pt] \dfrac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}, \hspace{3cm} 1 \le t \le s \le e, \zeta \le s. \end{cases} \tag{2.3}$$

Proof. Utilizing Definitions 2.1 and 2.2 and Lemma 2.3, we can express the solution to (2.1) as

$$y(t) = -^H I_{1^+}^{\alpha-\beta} h(t) + c_1 (\log t)^{\alpha-\beta-1} + c_2 (\log t)^{\alpha-\beta-2} + c_3 (\log t)^{\alpha-\beta-3}$$

$$= -\frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} h(s) \frac{ds}{s} + c_1 (\log t)^{\alpha-\beta-1} + c_2 (\log t)^{\alpha-\beta-2} + c_3 (\log t)^{\alpha-\beta-3},$$

where c_1, c_2, and c_3 represent real constants.

Notice that with $y(1) = \delta y(1) = 0$, we have $c_2 = c_3 = 0$. Therefore,

$$y(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} h(s) \frac{ds}{s} + c_1 (\log t)^{\alpha-\beta-1}. \tag{2.4}$$

Based on Lemma 2.2, it follows that

$$\delta y(e) = -\frac{1}{\Gamma(\alpha-\beta-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} h(s) \frac{ds}{s} + (\alpha-\beta-1)c_1,$$

$$\delta y(\zeta) = -\frac{1}{\Gamma(\alpha-\beta-1)} \int_1^\zeta \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} h(s) \frac{ds}{s} + (\alpha-\beta-1)c_1 (\log \zeta)^{\alpha-\beta-2}.$$

By employing the boundary condition $\delta y(e) = \mu \delta y(\zeta)$, we can obtain

$$c_1 = \frac{\int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} h(s) \frac{ds}{s} - \mu \int_1^\zeta \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} h(s) \frac{ds}{s}}{\Omega \Gamma(\alpha-\beta)}. \tag{2.5}$$

Substituting (2.5) into (2.4), we have

$$y(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} h(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-\beta-1}}{\Omega \Gamma(\alpha-\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} h(s) \frac{ds}{s}$$

$$- \frac{\mu (\log t)^{\alpha-\beta-1}}{\Omega \Gamma(\alpha-\beta)} \int_1^\zeta \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} h(s) \frac{ds}{s}.$$

For $1 \le t \le \zeta < e$, we have

$$y(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} h(s) \frac{ds}{s} + \frac{\left[\int_1^t + \int_t^\zeta + \int_\zeta^e\right] \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} h(s) \frac{ds}{s}}{\Omega \Gamma(\alpha-\beta)}$$

$$- \frac{\mu \left[\int_1^t + \int_t^\zeta\right] \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} h(s) \frac{ds}{s}}{\Omega \Gamma(\alpha-\beta)}$$

$$= \int_1^t \frac{\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \mu \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \Omega \left(\log \frac{t}{s}\right)^{\alpha-\beta-1}}{\Omega \Gamma(\alpha-\beta)} h(s) \frac{ds}{s}$$

$$+ \int_t^\zeta \frac{\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \mu \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1}}{\Omega \Gamma(\alpha-\beta)} h(s) \frac{ds}{s}$$

$$+ \int_\zeta^e \frac{\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1}}{\Omega \Gamma(\alpha-\beta)} h(s) \frac{ds}{s} = \int_1^e G(t,s) h(s) \frac{ds}{s}.$$

For $1 \leq \zeta \leq t \leq e$, we have

$$
y(t) = -\frac{\left[\int_1^\zeta + \int_\zeta^t\right]\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}h(s)\frac{ds}{s}}{\Gamma(\alpha-\beta)} + \frac{\left[\int_1^\zeta + \int_\zeta^t + \int_t^e\right]\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1}h(s)\frac{ds}{s}}{\Omega\Gamma(\alpha-\beta)}
$$

$$
- \frac{\mu\int_1^\zeta\left(\log\frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1}h(s)\frac{ds}{s}}{\Omega\Gamma(\alpha-\beta)}
$$

$$
= \int_1^\zeta \frac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \mu\left(\log\frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \Omega\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}h(s)\frac{ds}{s}
$$

$$
+ \int_\zeta^t \frac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \Omega\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}h(s)\frac{ds}{s}
$$

$$
+ \int_t^e \frac{\left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}h(s)\frac{ds}{s} = \int_1^e G(t,s)h(s)\frac{ds}{s}.
$$

On the other hand, if y satisfies (2.2), we can easily show that y satisfies (2.1).

Lemma 2.5. *The function $G(t,s)$ given by (2.3) possesses the following properties:*

1) $G(t,s) \geq 0$, *for* $(t,s) \in [1,e] \times [1,e]$;
2) $(\log t)^{\alpha-\beta-1}G(e,s) \leq G(t,s) \leq G(e,s)$ *for* $(t,s) \in [1,e] \times [1,e]$.

Proof. Let

$$
\phi_1(t,s) = \left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \mu\left(\log\frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \Omega\left(\log\frac{t}{s}\right)^{\alpha-\beta-1},
$$

$$
\phi_2(t,s) = \left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \Omega\left(\log\frac{t}{s}\right)^{\alpha-\beta-1},
$$

$$
\phi_3(t,s) = \left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - \mu\left(\log\frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1},
$$

$$
\phi_4(t,s) = \left(\log\frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1}.
$$

1) For $s \leq t$, $s \leq \zeta$, let $\Lambda = (1 - \log s/\log\zeta)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-1} - (\log(t/s))^{\alpha-\beta-1}$.

If $\Lambda > 0$, by $2 < \alpha - \beta \le 3$, $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, $1 < \zeta < e$, we have

$$\phi_1(t, s) = \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \mu \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1}$$

$$- \left[1 - \mu(\log \zeta)^{\alpha-\beta-2}\right] \left(\log \frac{t}{s}\right)^{\alpha-\beta-1}$$

$$= \left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \left(\log \frac{t}{s}\right)^{\alpha-\beta-1}\right]$$

$$- \mu(\log \zeta)^{\alpha-\beta-2} \left[\left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \left(\log \frac{t}{s}\right)^{\alpha-\beta-1}\right]$$

$$> \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1}$$

$$= \left[(1 - \log s)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}\right] (\log t)^{\alpha-\beta-1} \ge 0.$$

If $\Lambda \le 0$, by $2 < \alpha - \beta \le 3$, $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, $1 < \zeta < e$, we have

$$\phi_1(t, s) = \left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \left(\log \frac{t}{s}\right)^{\alpha-\beta-1}\right] - \mu(\log \zeta)^{\alpha-\beta-2}\Lambda$$

$$\ge (\log t)^{\alpha-\beta-1} \left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-1}\right] \ge 0.$$

For $\zeta \le s \le t$, by $2 < \alpha - \beta \le 3$, $0 < \Omega < 1$, $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, we have

$$\phi_2(t, s) = \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \Omega \left(\log \frac{t}{s}\right)^{\alpha-\beta-1}$$

$$> \left(\log \frac{e}{s}\right)^{\alpha-\beta-1} (\log t)^{\alpha-\beta-1} - (\log t - \log s)^{\alpha-\beta-1}$$

$$= (\log t)^{\alpha-\beta-1} \left[(1 - \log s)^{\alpha-\beta-1} - \left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-1}\right] \ge 0.$$

For $t \le s \le \zeta$, by $2 < \alpha - \beta \le 3$, $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, $1 < \zeta < e$, we have

$$\phi_3(t, s) = \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \mu \left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1}$$

$$= \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1} - \mu(\log \zeta)^{\alpha-\beta-2} \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2} (\log t)^{\alpha-\beta-1}$$

$$> (\log t)^{\alpha-\beta-1} \left[(1 - \log s)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}\right] \ge 0.$$

For $t \le s$, $\zeta \le s$, $\phi_4(t, s) \ge 0$ holds clearly. Therefore, $G(t, s) \ge 0$ for $t, s \in [1, e]$.

2) Next, we prove $\phi_i(t, s)$ $(i = 1, 2, 3, 4)$ are monotone increasing functions for t.

For $s \le t$, $s \le \zeta$, if $t < \zeta$, by $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, $1 < \zeta < e$, we have

$$\frac{\partial \phi_1(t, s)}{\partial t} = (\alpha - \beta - 1)\left(\log \frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - \mu(\alpha - \beta - 1)\left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2}$$

$$- (\alpha - \beta - 1)\left[1 - \mu(\log \zeta)^{\alpha-\beta-2}\right]\left(\log \frac{t}{s}\right)^{\alpha-\beta-2}$$

$$= (\alpha - \beta - 1)\left[(1 - \log s)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - \left(\log \frac{t}{s}\right)^{\alpha-\beta-2}\right]$$

$$- (\alpha - \beta - 1)\mu(\log \zeta)^{\alpha-\beta-2}\left[\left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - \left(\log \frac{t}{s}\right)^{\alpha-\beta-2}\right]$$

$$\ge (\alpha - \beta - 1)\left[(1 - \log s)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}\right](\log t)^{\alpha-\beta-2} \ge 0.$$

If $t \ge \zeta$, by $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, $1 < \zeta < e$, we have

$$\frac{\partial \phi_1(t, s)}{\partial t} = (\alpha - \beta - 1)\left(\log \frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - \mu(\alpha - \beta - 1)\left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2}$$

$$- (\alpha - \beta - 1)\left[1 - \mu(\log \zeta)^{\alpha-\beta-2}\right]\left(\log \frac{t}{s}\right)^{\alpha-\beta-2}$$

$$= (\alpha - \beta - 1)\left[(1 - \log s)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - (\log t - \log s)^{\alpha-\beta-2}\right]$$

$$- (\alpha - \beta - 1)\mu(\log \zeta)^{\alpha-\beta-2}\left[\left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - (\log t - \log s)^{\alpha-\beta-2}\right]$$

$$= (\alpha - \beta - 1)(\log t)^{\alpha-\beta-2}\left[(1 - \log s)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-2}\right]$$

$$+ (\alpha - \beta - 1)(\log t)^{\alpha-\beta-2}\mu(\log \zeta)^{\alpha-\beta-2}\left[\left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}\right] \ge 0.$$

For $\zeta \le s \le t$, by $2 < \alpha - \beta \le 3$, $0 < \Omega < 1$, $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, we have

$$\frac{\partial \phi_2(t, s)}{\partial t} = (\alpha - \beta - 1)\left(\log \frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - (\alpha - \beta - 1)\Omega\left(\log \frac{t}{s}\right)^{\alpha-\beta-2}$$

$$> (\alpha - \beta - 1)\left[(1 - \log s)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - (\log t - \log s)^{\alpha-\beta-2}\right]$$

$$= (\alpha - \beta - 1)(\log t)^{\alpha-\beta-2}\left[(1 - \log s)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-2}\right] \ge 0.$$

For $t \leq s \leq \zeta$, by $2 < \alpha - \beta \leq 3$, $0 < \mu(\log \zeta)^{\alpha-\beta-2} < 1$, $1 < \zeta < e$, we have

$$
\begin{aligned}
\frac{\partial \phi_3(t,s)}{\partial t} &= (\alpha - \beta - 1)\left(\log \frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - \mu(\alpha - \beta - 1)\left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} \\
&= (\alpha - \beta - 1)\left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} - \mu(\log \zeta)^{\alpha-\beta-2}\left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2}\right] \\
&> (\alpha - \beta - 1)\left[(1 - \log s)^{\alpha-\beta-2} - \left(1 - \frac{\log s}{\log \zeta}\right)^{\alpha-\beta-2}\right](\log t)^{\alpha-\beta-2} \\
&\geq 0.
\end{aligned}
$$

For $t \leq s$, $\zeta \leq s$, $\partial\phi_4(t,s)/\partial t = (\alpha - \beta - 1)\left(\log(e/s)\right)^{\alpha-\beta-2}(\log t)^{\alpha-\beta-2} \geq 0$, so $G(t,s)$ is a monotone-increasing function for t. Therefore, $G(t,s) \leq G(e,s)$ for $t, s \in [1, e]$.

On the other hand, for $1 \leq s \leq t \leq e$, $s \leq \zeta$, we have

$$
\begin{aligned}
G(t,s) &= \frac{(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}\left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} - \mu\left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} - \Omega\left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-1}\right] \\
&\geq \frac{(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}\left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} - \mu\left(\log \frac{\zeta}{s}\right)^{\alpha-\beta-2} - \Omega(1 - \log s)^{\alpha-\beta-1}\right] \\
&= (\log t)^{\alpha-\beta-1}G(e,s).
\end{aligned}
$$

For $1 \leq \zeta \leq s \leq t \leq e$, we have

$$
\begin{aligned}
G(t,s) &= \frac{(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}\left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} - \Omega\left(1 - \frac{\log s}{\log t}\right)^{\alpha-\beta-1}\right] \\
&\geq \frac{(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)}\left[\left(\log \frac{e}{s}\right)^{\alpha-\beta-2} - \Omega(1 - \log s)^{\alpha-\beta-1}\right] \\
&= (\log t)^{\alpha-\beta-1}G(e,s).
\end{aligned}
$$

By a similar calculation, we can obtain $G(t,s) \geq (\log t)^{\alpha-\beta-1}G(e,s)$. This completes the proof.

In order to overcome difficulty of f depending on derivatives, we consider the following modified version of Eq (1.1):

$$
\begin{cases}
{}^H D_{1^+}^{\alpha-\beta}y(t) + \lambda f(t, {}^H I_{1^+}^{\beta}y(t), y(t)) = 0, & 1 < t < e, \\
y(1) = \delta y(1) = 0, & \delta y(e) = \mu \delta y(\zeta).
\end{cases}
\tag{2.6}
$$

Lemma 2.6. *If $y \in C[1, e]$ is a positive solution of the problem* (2.6)*, then* ${}^H I_{1^+}^{\beta}y$ *is a positive solution of the problem* (1.1)*.*

Proof. Suppose $y \in C[1, e]$ is a positive solution of the problem (2.6) and let $x(t) = {}^H I_{1^+}^{\beta}y(t)$. From Lemma 2.2, we can obtain

$$
{}^H D_{1^+}^{\beta}x(t) = y(t), \quad {}^H D_{1^+}^{\alpha}x(t) = \frac{d^n}{dt^n}I_{1^+}^{n-\alpha}I_{1^+}^{\beta}y(t) = \frac{d^n}{dt^n}I_{1^+}^{n-\alpha+\beta}y(t) = {}^H D_{1^+}^{\alpha-\beta}y(t).
\tag{2.7}
$$

From (2.7), we have

$$^H D_{1^+}^\alpha x(t) + \lambda f(t, x(t), {}^H D_{1^+}^\beta x(t)) = {}^H D_{1^+}^{\alpha-\beta} y(t) + \lambda f(t, {}^H I_{1^+}^\beta y(t), y(t)) = 0,$$

and

$$\begin{aligned}
{}^H D_{1^+}^\beta x(1) &= {}^H D_{1^+}^\beta I_{1^+}^\beta y(1) = y(1) = 0, \\
{}^H D_{1^+}^{\beta+1} x(1) &= {}^H D_{1^+}^{\beta+1} I_{1^+}^\beta y(1) = \delta y(1) = 0, \\
{}^H D_{1^+}^{\beta+1} x(e) &= {}^H D_{1^+}^{\beta+1} I_{1^+}^\beta y(e) = \delta y(e), \\
{}^H D_{1^+}^{\beta+1} x(\zeta) &= {}^H D_{1^+}^{\beta+1} I_{1^+}^\beta y(\zeta) = \delta y(\zeta), \\
x(1) &= 0.
\end{aligned}$$

Thus, $^H I_{1^+}^\beta y(t)$ is a positive solution of the problem (1.1). This completes the proof.

In this paper, we make the following assumptions:

(H_1) The function $f \in C((1, e) \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ may be singular at $t = 1$ and/or $t = e$, and there exist the functions $\psi, Z \in C(1, e) \cap L^1(1, e)$ with $\psi, Z > 0$, $g \in C([1, e] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ such that $-\psi(t) \le f(t, u_1, u_2) \le Z(t)g(t, u_1, u_2)$ for all $(t, u_1, u_2) \in (1, e) \times \mathbb{R}^+ \times \mathbb{R}^+$.

(H_2) There exists $1 < \tau < e$ such that

$$\liminf_{u_1+u_2 \to +\infty} \frac{f(t, u_1, u_2)}{u_1 + u_2} = +\infty \text{ uniformly on } t \in [\tau, e].$$

(H_3) There exists $1 < \tau < e$ such that

$$\liminf_{u_1+u_2 \to +\infty} f(t, u_1, u_2) > \xi \text{ uniformly on } t \in [\tau, e],$$

where $\xi = \frac{2\Delta}{\int_\tau^e G(e,s)\frac{ds}{s}}$, $\Delta = \frac{1}{\Omega\Gamma(\alpha-\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} \psi(s)\frac{ds}{s}$.

(H_4)

$$\limsup_{u_1+u_2 \to +\infty} \frac{g(t, u_1, u_2)}{u_1 + u_2} = 0 \text{ uniformly on } t \in [1, e].$$

Lemma 2.7. *Assume that (H_1) holds. Let $\lambda > 0$, $2 < \alpha - \beta \le 3$. Then, the following boundary value problem of fractional differential equation*

$$\begin{cases} {}^H D_{1^+}^{\alpha-\beta} \omega(t) + \lambda\psi(t) = 0, & 1 < t < e, \\ \omega(1) = \delta\omega(1) = 0, & \delta\omega(e) = \mu\delta\omega(\zeta) \end{cases} \tag{2.8}$$

has a unique solution

$$\omega(t) = \lambda \int_1^e G(t, s)\psi(s)\frac{ds}{s}, \tag{2.9}$$

and

$$0 \le \omega(t) \le \lambda\Delta(\log t)^{\alpha-\beta-1}, \tag{2.10}$$

where

$$\Delta = \frac{1}{\Omega\Gamma(\alpha-\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\beta-2} \psi(s)\frac{ds}{s}.$$

Proof. From Lemma 2.4, let $h(t) = \lambda\psi(t)$. We have (2.9) immediately.

By (2.3), we can obtain

$$
\begin{aligned}
0 \leq \omega(t) &= \lambda \int_1^e G(t,s)\psi(s)\frac{ds}{s} \\
&\leq \frac{\lambda(\log t)^{\alpha-\beta-1}}{\Omega\Gamma(\alpha-\beta)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\beta-2} \psi(s)\frac{ds}{s} \\
&= \lambda\Delta(\log t)^{\alpha-\beta-1}.
\end{aligned}
$$

Therefore, (2.10) holds. This completes the proof.

Lemma 2.8. *Suppose that $\omega = \omega(t)$ is the solution of the problem of* (2.8). *$y(t)$ is the positive solution of the problem* (2.6) *if and only if $u(t) = y(t) + \omega(t)$ with $u(t) > \omega(t)$ as the solution of the following problem:*

$$
\begin{cases}
{}^H D_{1^+}^{\alpha-\beta} u(t) + \lambda\left[f(t, {}^H I_{1^+}^{\beta}\,[u(t) - \omega(t)]^+, [u(t) - \omega(t)]^+) + \psi(t)\right] = 0, & 1 < t < e, \\
u(1) = \delta u(1) = 0, \qquad \delta u(e) = \mu\delta u(\zeta),
\end{cases}
\tag{2.11}
$$

where $[\rho(t)]^+ = \max\{\rho(t), 0\}$.

Proof. From Lemma 2.1, if $y(t)$ is the positive solution of the problem (2.6), we have

$$
\begin{aligned}
{}^H D_{1^+}^{\alpha-\beta}(y(t) + \omega(t)) &= {}^H D_{1^+}^{\alpha-\beta} y(t) + {}^H D_{1^+}^{\alpha-\beta}\omega(t) \\
&= -\lambda f(t, {}^H I_{1^+}^{\beta}\,[y(t)]^+, [y(t)]^+) - \lambda\psi(t) \\
&= -\lambda\left[f(t, {}^H I_{1^+}^{\beta}\,[y(t)]^+, [y(t)]^+) + \psi(t)\right]
\end{aligned}
$$

such that

$$
{}^H D_{1^+}^{\alpha-\beta}(y(t) + \omega(t)) + \lambda\left[f(t, {}^H I_{1^+}^{\beta}\,[y(t)]^+, [y(t)]^+) + \psi(t)\right] = 0.
$$

Let $u(t) = y(t) + \omega(t)$. We can obtain $y(t) = u(t) - \omega(t)$ and

$$
{}^H D_{1^+}^{\alpha-\beta} u(t) + \lambda\left[f(t, {}^H I_{1^+}^{\beta}\,[u(t) - \omega(t)]^+, [u(t) - \omega(t)]^+) + \psi(t)\right] = 0.
$$

By the boundary conditions of the problems (2.6) and (2.8), we have $u(1) = \delta u(1) = 0, \delta u(e) = \mu\delta u(\zeta)$. Therefore, $u(t)$ is the solution of the problem (2.11).

On the other hand, if $u(t)$ and $\omega(t)$ are the solutions of the problems (2.11) and (2.8), respectively, and in a similar way, when $u(t) > \omega(t)$, $y(t) = u(t) - \omega(t) > 0$ is a positive solution of the problem (2.6), also from Lemma 2.6, $x(t) = {}^H I_{1^+}^{\beta} y(t) = {}^H I_{1^+}^{\beta}(u(t) - \omega(t))$ is a positive solution of the problem (1.1).

In this paper, the space $E = C[1, e]$ will be used, and $(E, \|\cdot\|)$ is a Banach space if it is endowed with the norm $\|u\| = \max_{t\in[1,e]} |u(t)|$.

The cone $K \subset E$ is defined by $K = \left\{u \in E : u(t) \geq (\log t)^{\alpha-\beta-1}\|u\|, t \in [1, e]\right\}$. Similar to [41], for $\lambda > 0$, we introduce the operator $\mathscr{F} : K \to E$:

$$
\mathscr{F}u(t) = \lambda \int_1^e G(t,s)\left[f(s, {}^H I_{1^+}^{\beta}\,[u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s)\right]\frac{ds}{s}.
\tag{2.12}
$$

It is clear that u is a solution of the problem (2.11) if and only if u is a fixed point of operator $\mathscr{F}$.

Lemma 2.9. *Assume that if (H_1) holds, then $\mathscr{F} : K \to K$ is a completely continuous operator.*

Proof. If for any $u \in K$, we can obtain that there exists a constant $R > 0$ such that $\|u\| \le R$, then for $t \in [1, e]$, we have

$$0 \le [u(t) - \omega(t)]^+ \le u(t) \le \|u\| \le R,$$

$$0 \le^H I_{1^+}^\beta [u(t) - \omega(t)]^+ = \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} [u(s) - \omega(s)]^+ \frac{ds}{s} \le \frac{R}{\Gamma(\beta+1)}.$$

So,

$$\mathscr{F}u(t) = \lambda \int_1^e G(t, s) \left[f(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$\le \lambda \int_1^e G(e, s) \left[Z(s) g(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$\le \lambda M \int_1^e G(e, s) \left(Z(s) + \psi(s) \right) \frac{ds}{s} < +\infty,$$

where

$$M = \max \left\{ \max_{[1,e] \times [0, \frac{R}{\Gamma(\beta+1)}] \times [0,R]} g(t, u_1, u_2), 1 \right\}.$$

Thus, $\mathscr{F} : K \to E$ is well defined.

From Lemma 2.6, for any $u \in K$, we have

$$\mathscr{F}u(t) = \lambda \int_1^e G(t, s) \left[f(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$\ge \lambda (\log t)^{\alpha-\beta-1} \int_1^e G(e, s) \left[f(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$= (\log t)^{\alpha-\beta-1} \|\mathscr{F}u\|, \quad t \in [1, e].$$

Therefore, $\mathscr{F}u \in K$, which implies that $\mathscr{F} : K \to K$.

From (H_1), according to the Lebesgue dominated convergence theorem and the Arzelà–Ascoli theorem, we can easily obtain that $\mathscr{F} : K \to K$ is completely continuous. This completes the proof. $\square$

Lemma 2.10. *(See [48].) (Guo–Krasnoselskii fixed-point theorem) Let E be a real Banach space, $P \subset E$ be a cone. Assume Ω_1, Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

1) $\|Tx\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$, or
2) $\|Tx\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$, $x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2.3. (See [49].) A continuous mapping $\theta : P \to [1, \infty)$ is said to be a nonnegative continuous concave functional on the cone P of a real Banach space E provided that

$$\theta(\lambda u + (1 - \lambda)v) \ge \lambda\theta(u) + (1 - \lambda)\theta(v)$$

for all $u, v \in P$ and $\lambda \in [0, 1]$.

Let $a, b, d > 0$ be constants. We define $P_d = \{u \in P : \|u\| < d\}$, $\overline{P}_d = \{u \in P : \|u\| \leq d\}$ and $P(\theta, a, b) = \{u \in P : \theta(u) \geq a, \|u\| \leq b\}$.

Lemma 2.11. *(See [49].) (Leggett-Williams fixed point theorem) Let P be a cone in the real Banach space E and $c > 0$ be a constant. Assume that there exists a concave, nonnegative, continuous functional θ on P with $\theta(u) \leq \|u\|$ for all $u \in \overline{P}_c$. Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator. Suppose that there exist constants $0 < a < b < d \leq c$ such that the following conditions hold:*

(i) *$\{u \in P(\theta, b, d) : \theta(u) > b\} \neq \varnothing$ and $\theta(Tu) > b$ for $u \in P(\theta, b, d)$;*
(ii) *$\|Tu\| < a$ for $\|u\| \leq a$;*
(iii) *$\theta(Tu) > b$ for $u \in P(\theta, b, c)$ with $\|Tu\| > d$.*

Then, T has at least three fixed points, u_1, u_2, and u_3, in $\overline{P}_c$. Furthermore, $\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.

3. Results

Now, we state our main theorems and give their proofs.

Theorem 3.1. *Suppose (H_1) and (H_2) hold. Then there exists $\lambda^* > 0$ such that, for any $0 < \lambda < \lambda^*$, boundary value problem (1.1) has at least one positive solution.*

Proof. Let $\Omega_1 = \{u \in K : \|u\| < R_1\}$, where $R_1 > 0$. Choose $\lambda^* = \min\left\{R_1/\Delta, R_1/(A \int_1^e G(e, s)(Z(s) + \psi(s))\frac{ds}{s})\right\}$ with $A = \max\left\{\max_{[1,e] \times [0, R_1/\Gamma(\beta+1)] \times [0, R_1]} g(t, u_1, u_2), 1\right\}$.

Let $\lambda \in (0, \lambda^*)$ for $\forall u \in K \cap \partial\Omega_1$, $t \in [1, e]$. We have

$$0 \leq [u(t) - \omega(t)]^+ \leq u(t) \leq \|u\| \leq R_1,$$

$$0 \leq^H I_{1^+}^\beta [u(t) - \omega(t)]^+ = \frac{1}{\Gamma(\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} [u(s) - \omega(s)]^+ \frac{ds}{s}$$

$$\leq \frac{R_1}{\Gamma(\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} \frac{ds}{s} \leq \frac{R_1}{\Gamma(\beta+1)}.$$

By (H_1), for any $u \in K \cap \partial\Omega_1$, $t \in [1, e]$, we have

$$|\mathscr{F}u(t)| = \left|\lambda \int_1^e G(t, s)\left[f(s, ^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s)\right]\frac{ds}{s}\right|$$

$$\leq \lambda \int_1^e G(e, s)\left[Z(s)g(s, ^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s)\right]\frac{ds}{s}$$

$$\leq \lambda^* A \int_1^e G(e, s)(Z(s) + \psi(s))\frac{ds}{s}$$

$$\leq R_1 = \|u\|.$$

Therefore, $\|\mathscr{F}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$.

On the other hand, we choose a real number $B > 0$ such that $\lambda B(\log \tau)^{2(\alpha-\beta-1)} \int_\tau^e G(e, s)\frac{ds}{s}/(\Delta + 1) \geq 1$.

By (H_2), for any $t \in [\tau, e]$, there exists a constant $\gamma_1 > 0$ such that

$$\frac{f(t, u_1, u_2)}{u_1 + u_2} \geq B, \qquad u_1 + u_2 \geq \gamma_1. \tag{3.1}$$

Take $R_2 = \max\{\lambda(\Delta + 1), 2R_1, \gamma_1(\Delta + 1)/(\log \tau)^{\alpha-\beta-1}\}$, let $\Omega_2 = \{u \in K : \|u\| < R_2\}$. By $\omega(t) \leq \lambda\Delta(\log t)^{\alpha-\beta-1}$ for all $t \in [1, e]$, we can obtain that for any $u \in K \cap \partial\Omega_2$,

$$
\begin{aligned}
u(t) - \omega(t) &\geq u(t) - \lambda\Delta(\log t)^{\alpha-\beta-1} \\
&\geq u(t) - \frac{\lambda\Delta}{R_2}u(t) = (1 - \frac{\lambda\Delta}{R_2})u(t) \\
&\geq (1 - \frac{\lambda\Delta}{\lambda(\Delta + 1)})u(t) = \frac{1}{\Delta + 1}u(t) \\
&\geq \frac{(\log \tau)^{\alpha-\beta-1}R_2}{\Delta + 1} \geq \gamma_1 > 0, \quad t \in [\tau, e],
\end{aligned}
\tag{3.2}
$$

and then

$$
\begin{aligned}
{}^H I_{1^+}^{\beta} [u(t) - \omega(t)]^+ + [u(t) - \omega(t)]^+ &= {}^H I_{1^+}^{\beta} [u(t) - \omega(t)] + [u(t) - \omega(t)] \\
&= \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} [u(s) - \omega(s)]\frac{ds}{s} + [u(t) - \omega(t)] \\
&\geq u(t) - \omega(t) \geq \gamma_1 > 0, \quad t \in [\tau, e].
\end{aligned}
\tag{3.3}
$$

Thus, (3.1)–(3.3) imply that, for any $u \in K \cap \partial\Omega_2, t \in [\tau, e]$, we have

$$
\begin{aligned}
|\mathscr{F}u(t)| &= \left|\lambda \int_1^e G(t, s)\left[f(s, {}^H I_{1^+}^{\beta} [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s)\right]\frac{ds}{s}\right| \\
&\geq \lambda \int_\tau^e G(t, s)\left[f(s, {}^H I_{1^+}^{\beta} [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+)\right]\frac{ds}{s} \\
&\geq \lambda B(\log \tau)^{\alpha-\beta-1} \int_\tau^e G(e, s)\left({}^H I_{1^+}^{\beta} [u(s) - \omega(s)]^+ + [u(s) - \omega(s)]^+\right)\frac{ds}{s} \\
&\geq \frac{\lambda B(\log \tau)^{2(\alpha-\beta-1)}R_2 \int_\tau^e G(e, s)\frac{ds}{s}}{\Delta + 1} \\
&\geq R_2 = \|u\|.
\end{aligned}
$$

Therefore, $\|\mathscr{F}u\| \geq \|u\|, u \in K \cap \partial\Omega_2$.

By Lemma 2.10, $\mathscr{F}$ has a fixed point u_1 in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $R_1 \leq \|u_1\| \leq R_2$. By $R_1 \geq \lambda^*\Delta > \lambda\Delta$, we can obtain that for any $\|u_1\| \geq R_1$,

$$
\begin{aligned}
u_1(t) - \omega(t) &\geq (\log t)^{\alpha-\beta-1}\|u_1\| - \lambda\Delta(\log t)^{\alpha-\beta-1} \\
&\geq (\log t)^{\alpha-\beta-1}(R_1 - \lambda\Delta).
\end{aligned}
$$

therefore, $u_1(t) \geq \omega(t) + \Theta_1(\log t)^{\alpha-\beta-1} > 0, t \in (1, e)$, where $\Theta_1 = R_1 - \lambda\Delta$. If $y_1(t) = u_1(t) - \omega(t)$, then $y_1(t)$ is a positive solution of the problem (2.6). Moreover, from Lemma 2.6, $x_1(t) = {}^H I_{1^+}^{\beta} y_1(t) = {}^H I_{1^+}^{\beta} (u_1(t) - \omega(t))$ is a positive solution of boundary value problem (1.1).

Theorem 3.2. *Suppose (H_1), (H_3), and (H_4) hold. Then, there exists $\lambda^* > 0$ such that, for any $\lambda > \lambda^*$, boundary value problem (1.1) has at least one positive solution.*

Proof. By (H_3), for any $t \in [\tau, e]$, there exists a constant $\gamma_2 > 0$ such that

$$
f(t, u_1, u_2) \geq \xi, \quad u_1 + u_2 \geq \gamma_2.
\tag{3.4}
$$

Define $\lambda^* = \gamma_2/(\Delta(\log \tau)^{\alpha-\beta-1})$ and let $\lambda > \lambda^*$, $\Omega_3 = \{u \in K : \|u\| < R_3\}$, where $R_3 = 2\lambda\Delta > 0$. For all $u \in K \cap \partial\Omega_3$, $t \in [1, e]$, we have

$$
\begin{aligned}
u(t) - \omega(t) &\geq (\log t)^{\alpha-\beta-1} R_3 - \lambda\Delta(\log t)^{\alpha-\beta-1} \\
&= (\log t)^{\alpha-\beta-1}(2\lambda\Delta - \lambda\Delta) \\
&= (\log t)^{\alpha-\beta-1}\lambda\Delta \\
&> (\log t)^{\alpha-\beta-1}\lambda^*\Delta \\
&= \frac{\gamma_2(\log t)^{\alpha-\beta-1}}{(\log \tau)^{\alpha-\beta-1}} \\
&\geq 0.
\end{aligned}
\tag{3.5}
$$

Then, for all $u \in K \cap \partial\Omega_3$, $t \in [\tau, e]$, we have

$$
{}^H I_{1^+}^{\beta} [u(t) - \omega(t)]^+ + [u(t) - \omega(t)]^+ \geq [u(t) - \omega(t)]^+ \geq \frac{\gamma_2(\log t)^{\alpha-\beta-1}}{(\log \tau)^{\alpha-\beta-1}} \geq \gamma_2.
\tag{3.6}
$$

Thus, (3.4) and (3.6) imply that, for any $u \in K \cap \partial\Omega_3$,

$$
\begin{aligned}
\|\mathscr{F}u\| = \max_{t\in[1,e]} |\mathscr{F}u(t)| &= \left| \lambda \int_1^e G(e, s) \left[f(s, {}^H I_{1^+}^{\beta} [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s} \right| \\
&\geq \lambda \int_\tau^e G(e, s) \left[f(s, {}^H I_{1^+}^{\beta} [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s} \\
&\geq \lambda\xi \int_\tau^e G(e, s) \frac{ds}{s} \\
&= \lambda \frac{2\Delta}{\int_\tau^e G(e, s)\frac{ds}{s}} \int_\tau^e G(e, s) \frac{ds}{s} \\
&= R_3 = \|u\|.
\end{aligned}
$$

Therefore, $\|\mathscr{F}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_3$.

On the other hand, if we choose $\eta = 1/(2\lambda(1 + \frac{1}{\Gamma(\beta+1)}) \int_1^e G(e, s)Z(s)\frac{ds}{s}) > 0$, then by (H_4), for any $t \in [1, e]$, there exists a constant $\gamma_3 > R_3$ such that

$$
g(t, u_1, u_2) \leq \eta(u_1 + u_2), \qquad u_1 + u_2 \geq \gamma_3.
$$

Also, because g is continuous, let $\overline{M} = \max_{t\in[1,e], u_1+u_2\leq\gamma_3} g(t, u_1, u_2)$.

Take $R_4 = \max\{R_3 + \gamma_3, 2\lambda \int_1^e G(e, s)(\overline{M}Z(s) + \psi(s))\frac{ds}{s}\}$ and let $\Omega_4 = \{u \in K : \|u\| < R_4\}$. For all $u \in K \cap \partial\Omega_4$, $t \in [1, e]$, we have

$$
\begin{aligned}
u(t) - \omega(t) &\geq (\log t)^{\alpha-\beta-1}\|u\| - \lambda\Delta(\log t)^{\alpha-\beta-1} \\
&= (\log t)^{\alpha-\beta-1}(R_4 - \lambda\Delta) > (\log t)^{\alpha-\beta-1}(R_3 - \lambda\Delta) > 0,
\end{aligned}
\tag{3.7}
$$

and if ${}^H I_{1^+}^{\beta} [u(t) - \omega(t)]^+ + [u(t) - \omega(t)]^+ = {}^H I_{1^+}^{\beta} [u(t) - \omega(t)] + [u(t) - \omega(t)] \geq \gamma_3$, we have

$$
\begin{aligned}
g(t, {}^H I_{1^+}^{\beta} [u(t) - \omega(t)]^+, [u(t) - \omega(t)]^+) &\leq \eta \left({}^H I_{1^+}^{\beta} [u(t) - \omega(t)] + [u(t) - \omega(t)] \right) \\
&\leq \eta \left(1 + \frac{1}{\Gamma(\beta + 1)} \right) R_4.
\end{aligned}
\tag{3.8}
$$

Thus, for any $u \in K \cap \partial\Omega_4$, $t \in [1, e]$, (3.7) and (3.8) imply that

$$\|\mathscr{F}u\| = \max_{t\in[1,e]} |\mathscr{F}u(t)|$$

$$= \lambda \int_1^e G(e, s) \left[f(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$\leq \lambda \int_1^e G(e, s) \left[Z(s) g(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$\leq \lambda \int_{{}^H I_{1^+}^\beta [u(s) - \omega(s)]^+ + [u(s) - \omega(s)]^+ \geq \gamma_3} G(e, s) Z(s) g(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) \frac{ds}{s}$$

$$+ \lambda \int_{{}^H I_{1^+}^\beta [u(s) - \omega(s)]^+ + [u(s) - \omega(s)]^+ \leq \gamma_3} G(e, s) Z(s) g(s, {}^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) \frac{ds}{s}$$

$$+ \lambda \int_1^e G(e, s) \psi(s) \frac{ds}{s}$$

$$\leq \lambda \eta \left(1 + \frac{1}{\Gamma(\beta + 1)} \right) R_4 \int_1^e G(e, s) Z(s) \frac{ds}{s} + \lambda \int_1^e G(e, s) \left(\overline{M} Z(s) + \psi(s) \right) \frac{ds}{s}$$

$$\leq \frac{R_4}{2} + \frac{R_4}{2}$$

$$= R_4 = \|u\|.$$

Therefore, $\|\mathscr{F}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_4$.

By Lemma 2.10, $\mathscr{F}$ has a fixed point u_2 in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$ such that $R_3 \leq \|u_2\| \leq R_4$. From $\|u_2\| \geq R_3$, we have

$$u_2(t) - \omega(t) \geq (\log t)^{\alpha-\beta-1} \|u_2\| - \lambda\Delta(\log t)^{\alpha-\beta-1}$$
$$> (\log t)^{\alpha-\beta-1}(R_3 - \lambda\Delta)$$
$$= 2\lambda\Delta(\log t)^{\alpha-\beta-1}$$

and then $u_2(t) \geq \omega(t) + \Theta_2(\log t)^{\alpha-\beta-1} > 0$, $t \in (1, e)$, where $\Theta_2 = 2\lambda\Delta$. Let $y_2(t) = u_2(t) - \omega(t)$. Then $y_2(t)$ is a positive solution of the problem (2.6). Moreover, from Lemma 2.6, $x_2(t) = {}^H I_{1^+}^\beta y_2(t) = {}^H I_{1^+}^\beta (u_2(t) - \omega(t))$ is a positive solution of the boundary value problem (1.1).

Before proving the Theorem 3.3, we set

$$N_1 = \frac{1}{\lambda \int_1^e G(e, s) \left[(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s}}, \quad N_2 = \frac{1}{\lambda(\log \tau)^{\alpha-\beta-1} \int_\tau^e G(e, s) \left[(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s}},$$

and define a concave, nonnegative, continuous functional on K by $\theta(u) = \min_{t\in[\tau,e]} u(t)$.

Theorem 3.3. *Suppose the function $f \in C((1, e) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R})$ may be singular at $t = 1$ and/or $t = e$, and there exists a constant $0 < \sigma < 1$ such that $(\log t)^\sigma f(t, u_1, u_2)$ is continuous on $[1, e] \times \mathbb{R}^+ \times \mathbb{R}^+$. Let $0 < \lambda\Delta < r < a < b < d(\log \tau)^{\alpha-\beta-1} < d < c$ with $aN_1 > 1$, $rN_2 > 1$, $\tau \in (1, e)$ and suppose that f satisfies the following conditions:*

  (C_1) $(\log t)^\sigma f(t, u_1, u_2) < aN_1 - 1$ *for all* $(t, u_1, u_2) \in [1, e] \times [0, \frac{a}{\Gamma(\beta+1)}] \times [0, a]$;

  (C_2) $(\log t)^\sigma f(t, u_1, u_2) > bN_2 - 1$ *for all* $(t, u_1, u_2) \in [\tau, e] \times [\frac{(b-\lambda\Delta)(\log \tau)^\beta}{\Gamma(\beta+1)}, \frac{d}{\Gamma(\beta+1)}] \times [b - \lambda\Delta, d]$;

(C_3) $(\log t)^\sigma f(t, u_1, u_2) < cN_1 - 1$ *for all* $(t, u_1, u_2) \in [1, e] \times [0, \frac{c}{\Gamma(\beta+1)}] \times [0, c]$;

(C_4) $(\log t)^\sigma f(t, u_1, u_2) > rN_2 - 1$ *for all* $(t, u_1, u_2) \in [\tau, e] \times [0, \frac{a}{\Gamma(\beta+1)}] \times [0, a]$.

Then boundary value problem (1.1) *has at least three positive solutions,* $x_1, x_2,$ *and* $x_3,$ *in* $\Omega^*,$ *where* $\Omega^* = \left\{ I_{1^+}^\beta y : y = u - \omega, u \in \overline{K}_c \right\}.$

Proof. To complete the proof, we will show that all the conditions of Lemma 2.11 are satisfied for $\mathscr{F}$ defined by (2.12) and divide it into the following four steps.

Step 1. We will show that $\mathscr{F} : \overline{K}_c \to \overline{K}_c$ is completely continuous. For $u \in \overline{K}_c$, we have $\|u\| \le c$; thus,

$$0 \le [u(t) - \omega(t)]^+ \le u(t) \le \|u\| \le c,$$

and

$$0 \le^H I_{1^+}^\beta [u(t) - \omega(t)]^+ \le \frac{c}{\Gamma(\beta + 1)}.$$

By (C_3), we can get $(\log t)^\sigma f(t,^H I_{1^+}^\beta [u(t) - \omega(t)]^+, [u(t) - \omega(t)]^+) < cN_1 - 1, t \in [1, e]$. For all $u \in \overline{K}_c$, we have

$$
\begin{aligned}
\|\mathscr{F}u\| &= \max_{t\in[1,e]} |\mathscr{F}u(t)| \\
&= \max_{t\in[1,e]} \lambda \int_1^e G(t, s) \left[(\log s)^{-\sigma}(\log s)^\sigma f(s,^H I_{1^+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s} \\
&\le \lambda \int_1^e G(e, s) \left[(cN_1 - 1)(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s} \\
&\le \lambda cN_1 \int_1^e G(e, s) \left[(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s} \\
&= c.
\end{aligned}
$$

Therefore, $\mathscr{F} : \overline{K}_c \to \overline{K}_c$. By Lemma 2.9, we have that $\mathscr{F} : \overline{K}_c \to \overline{K}_c$ is completely continuous.

Step 2. We will show that $\|\mathscr{F}u\| < a$ for $\|u\| \le a$. By assumption (C_1), we can similarly obtain that for $\|u\| \le a$, we have $\|\mathscr{F}u\| < a$. Thus, condition (ii) of Lemma 2.11 is satisfied.

Step 3. We will show that $\{u \in K(\theta, b, d) : \theta(u) > b\} \ne \emptyset$ and $\theta(\mathscr{F}u) > b$ for $u \in K(\theta, b, d)$.

Let $\tilde{u}(t) = (b + d)/2$. Obviously, $\tilde{u} \in K(\theta, b, d)$. From the definition of $\theta(u)$, we have $\theta(u) = \theta(b + d)/2 > b$; thus, $\{u \in K(\theta, b, d) : \theta(u) > b\} \ne \emptyset$.

On the other hand, for $u \in K(\theta, b, d)$, we can obtain $\theta(u) = \min_{t\in[\tau,e]} u(t) \ge b, \|u\| \le d$, and then for $t \in [\tau, e]$,

$$u(t) - \omega(t) \ge b - \lambda\Delta(\log t)^{\alpha-\beta-1} \ge b - \lambda\Delta > 0, \quad {}^H I_{1^+}^\beta [u(t) - \omega(t)] \ge \frac{(b - \lambda\Delta)(\log \tau)^\beta}{\Gamma(\beta + 1)} > 0,$$

and

$$u(t) - \omega(t) \le u(t) \le \|u\| \le d \le c, \quad {}^H I_{1^+}^\beta [u(t) - \omega(t)] \le \frac{d}{\Gamma(\beta + 1)} \le \frac{c}{\Gamma(\beta + 1)}.$$

By (C_2), we can get $(\log t)^\sigma f(t, {}^H I_{1+}^\beta [u(t) - \omega(t)]^+, [u(t) - \omega(t)]^+) > bN_2 - 1, t \in [\tau, e]$. For all $u \in K(\theta, b, d)$,

$$\theta(\mathscr{F}u) = \min_{t\in[\tau,e]} \mathscr{F}u(t) \geq \min_{t\in[\tau,e]} \lambda(\log t)^{\alpha-\beta-1} \int_1^e G(e,s) \left[f(s, {}^H I_{1+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$\geq \lambda(\log \tau)^{\alpha-\beta-1} \int_\tau^e G(e,s) \left[(\log s)^{-\sigma}(\log s)^\sigma f(s, {}^H I_{1+}^\beta [u(s) - \omega(s)]^+, [u(s) - \omega(s)]^+) + \psi(s) \right] \frac{ds}{s}$$

$$> \lambda bN_2(\log \tau)^{\alpha-\beta-1} \int_\tau^e G(e,s) \left[(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s}$$

$$= b.$$

This shows that for all $u \in K(\theta, b, d)$, we have $\theta(\mathscr{F}u) > b$. Thus, condition (i) of Lemma 2.11 is satisfied.

Step 4. We will show that $\theta(\mathscr{F}u) > b$ for $u \in K(\theta, b, c)$ with $\|\mathscr{F}u\| > d$.

We assume that if $u \in K(\theta, b, c)$, then

$$\theta(\mathscr{F}u) = \min_{t\in[\tau,e]} |\mathscr{F}u(t)|$$

$$\geq \min_{t\in[\tau,e]} (\log t)^{\alpha-\beta-1} \|\mathscr{F}u\|$$

$$= (\log \tau)^{\alpha-\beta-1} \|\mathscr{F}u\|$$

$$> d(\log \tau)^{\alpha-\beta-1} > b.$$

Thus, condition (iii) of Lemma 2.11 is satisfied.

To summarize, all hypotheses of Lemma 2.11 are satisfied. So, the integral operator $\mathscr{F}$ has at least three fixed points, u_1, u_2, u_3, in $\overline{K}_c$ such that $\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.

For $\|u_1\| < a$, $[u(t) - \omega(t)]^+ \leq a$, ${}^H I_{1+}^\beta [u_1(t) - \omega(t)]^+ \leq a/\Gamma(\beta+1)$, by (C_4), we have $\theta(u_1) > r$. Then, we have

$$u_i(t) \geq (\log t)^{\alpha-\beta-1} \|u_i\| > r(\log t)^{\alpha-\beta-1} > \lambda\Delta(\log t)^{\alpha-\beta-1} \geq \omega(t), \quad i = 1, 2, 3.$$

Let $y_i(t) = u_i(t) - \omega(t)$. Then, $y_i(t)$ $(i = 1, 2, 3)$ are positive solutions for the problem (2.6). Furthermore, from Lemma 2.6, $x_i(t) = {}^H I_{1+}^\beta y_i(t)$ $(i = 1, 2, 3)$ are positive solutions for boundary value problem (1.1), and $x_i \in \Omega^*$ $(i = 1, 2, 3)$. This completes the proof.

4. Applications

Example 4.1. Consider the following fractional differential equation:

$$(4.1) \quad \begin{cases} {}^H D_{1+}^{\frac{25}{8}} x(t) + \lambda \left(\dfrac{(x(t) + {}^H D_{1+}^{\frac{7}{8}} x(t))^2}{\sqrt[3]{\log t(1 - \log t)^2}} - \dfrac{2}{\sqrt{\log t}} \right) = 0, \quad 1 < t < e, \\[4mm] x(1) = 0, \quad {}^H D_{1+}^{\frac{7}{8}} x(1) = {}^H D_{1+}^{\frac{15}{8}} x(1) = 0, \quad {}^H D_{1+}^{\frac{15}{8}} x(e) = \dfrac{15}{8} {}^H D_{1+}^{\frac{15}{8}} x(\dfrac{5}{4}), \end{cases}$$

where $\alpha = \frac{25}{8}$, $\beta = \frac{7}{8}$, $\mu = \frac{3}{4}$, $\zeta = \frac{5}{4}$, $f(t, u_1, u_2) = \frac{(u_1+u_2)^2}{\sqrt[3]{\log t(1-\log t)^2}} - \frac{2}{\sqrt{\log t}}$, and

$$\psi(t) = \frac{2}{\sqrt{\log t}}, \quad g(t, u_1, u_2) = (u_1 + u_2)^2, \quad Z(t) = \frac{1}{\sqrt[3]{\log t(1 - \log t)^2}}.$$

It easy to see that (H_1) is satisfied. Letting $\tau = 1.5$, we have

$$\liminf_{u_1+u_2\to+\infty} \frac{\frac{(u_1+u_2)^2}{\sqrt[3]{\log t(1-\log t)^2}} - \frac{2}{\sqrt{\log t}}}{u_1 + u_2} = +\infty, \text{ uniformly on } t \in [1.5, e].$$

Hence, (H_2) is satisfied. Let $R_1 = 1$. Because $\Omega \approx 0.4845$, $\Delta \approx 6.3684$, we have $\lambda^* \approx \min\{0.1570, 0.0306\} = 0.0306$. It follows from Theorem 3.1 that the fractional differential equation (4.1) has at least one positive solution, provided $\lambda \in (0, 0.0306)$.

Example 4.2.

$$\begin{cases} {}^H D_{1^+}^{\frac{13}{4}} x(t) + \lambda \left(\frac{1}{\sqrt{\log t(1 - \log t)}} \cdot \log(1 + x(t) + {}^H D_{1^+}^{\frac{3}{4}} x(t)) - \frac{1}{\sqrt[3]{(\log t)^2}} \right) = 0, \quad 1 < t < e, \\ x(1) = 0, \quad {}^H D_{1^+}^{\frac{3}{4}} x(1) = {}^H D_{1^+}^{\frac{7}{4}} x(1) = 0, \quad {}^H D_{1^+}^{\frac{7}{4}} x(e) = \frac{3}{4} {}^H D_{1^+}^{\frac{7}{4}} x(\frac{5}{4}), \end{cases} \tag{4.2}$$

where $\alpha = \frac{13}{4}, \beta = \frac{3}{4}, \mu = \frac{3}{4}, \zeta = \frac{5}{4}$, $f(t, u_1, u_2) = \frac{1}{\sqrt{\log t(1-\log t)}} \cdot \log(1 + u_1 + u_2) - \frac{1}{\sqrt[3]{(\log t)^2}}$, and

$$\psi(t) = \frac{1}{\sqrt[3]{(\log t)^2}}, \quad g(t, u_1, u_2) = \log(1 + u_1 + u_2), \quad Z(t) = \frac{1}{\sqrt{\log t(1 - \log t)}}.$$

Clearly, (H_1) holds.

Let $\tau = 1.3$. We have

$$\liminf_{u_1+u_2\to+\infty} f(t, u_1, u_2) > \xi \approx 16.7341, \text{ uniformly on } t \in [1.3, e].$$

So, (H_3) is satisfied. Letting $\gamma_2 = 280720$, then $\lambda^* \approx 710411$. On the other hand, we have

$$\limsup_{u_1+u_2\to+\infty} \frac{g(t, u_1, u_2)}{u_1 + u_2} = 0 \text{ uniformly on } t \in [1, e].$$

Consequently, it follows from Theorem 3.2 that the fractional differential equation (4.2) has at least one positive solution provided that $\lambda > 710411$.

Example 4.3.

$$\begin{cases} {}^H D_{1^+}^{\frac{22}{7}} x(t) + f(t, x(t), {}^H D_{1^+}^{\frac{2}{7}} x(t)) = 0, \quad 1 < t < e, \\ x(1) = 0, \quad {}^H D_{1^+}^{\frac{2}{7}} x(1) = {}^H D_{1^+}^{\frac{9}{7}} x(1) = 0, \quad {}^H D_{1^+}^{\frac{9}{7}} x(e) = \frac{3}{4} {}^H D_{1^+}^{\frac{9}{7}} x(\frac{5}{4}), \end{cases} \tag{4.3}$$

where $\alpha = 22/7, \beta = 2/7, \mu = 3/4, \zeta = 5/4, \lambda = 1, \tau = 1.6, \psi(t) = 10/\sqrt{\log t}$, and

$$f(t, u_1, u_2) = \begin{cases} \dfrac{34.5 - 10^{-4}(u_2 - 3000)}{\sqrt{\log t}}, & (t, u_1, u_2) \in (1, e] \times [0, 125] \times [0, +\infty), \\ \dfrac{34.5 + 0.6(u_1 - 125)^2 - 10^{-4}(u_2 - 3000)}{\sqrt{\log t}}, & (t, u_1, u_2) \in (1, e] \times (125, 150] \times [0, +\infty), \\ \dfrac{409.5 - 10^{-4}(u_2 - 3000)}{\sqrt{\log t}}, & (t, u_1, u_2) \in (1, e] \times (150, +\infty) \times [0, +\infty). \end{cases}$$

For illustrating Theorem 3.3, we choose $\sigma = 1/2$, with the given data, it is found that $\Omega = 1 - \mu(\log \zeta)^{\alpha - \beta - 2} = 1 - 3/4(\log 5/4)^{6/7}$, and

$$N_1 = \frac{1}{\lambda \int_1^e G(e, s) \left[(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s}} = \frac{1}{\int_1^e G(e, s) \frac{11}{\sqrt{\log s}} \frac{ds}{s}}$$

$$= \left(\int_1^{\frac{5}{4}} \frac{(1 - \log s)^{\frac{6}{7}} - \Omega(1 - \log s)^{\frac{13}{7}}}{\Omega\Gamma(\frac{20}{7})} \frac{11}{\sqrt{\log s}} \frac{ds}{s} \right.$$

$$\left. + \int_{\frac{5}{4}}^e \frac{(1 - \log s)^{\frac{6}{7}} - \frac{3}{4}(\log \frac{5}{4} - \log s)^{\frac{6}{7}} - \Omega(1 - \log s)^{\frac{13}{7}}}{\Omega\Gamma(\frac{20}{7})} \frac{11}{\sqrt{\log s}} \frac{ds}{s} \right)^{-1}$$

$$\approx 0.3288,$$

$$N_2 = \frac{1}{\lambda(\log \tau)^{\alpha - \beta - 1} \int_\tau^e G(e, s) \left[(\log s)^{-\sigma} + \psi(s) \right] \frac{ds}{s}}$$

$$= \frac{1}{(\log 1.6)^{\frac{13}{7}} \int_{1.6}^e \frac{(1 - \log s)^{\frac{6}{7}} - \frac{3}{4}(\log \frac{5}{4} - \log s)^{\frac{6}{7}} - \Omega(1 - \log s)^{\frac{13}{7}}}{\Omega\Gamma(\frac{20}{7})} \frac{11}{\sqrt{\log s}} \frac{ds}{s}}$$

$$\approx 3.4644,$$

$\lambda\Delta = \Delta = 1/(\Omega\Gamma(\alpha - \beta)) \int_1^e (\log(e/s))^{\alpha - \beta - 2} \psi(s) \frac{ds}{s} = 1/(\Omega\Gamma(20/7)) \int_1^e (1 - \log s)^{6/7} 10/\sqrt{\log s} \frac{ds}{s} \approx 9.9635$, $\sqrt{\log t} f(t, u_1, u_2)$ is continuous on $[1, e] \times \mathbb{R}^+ \times \mathbb{R}^+$.

Taking $r = 10$, $a = 110$, $b = 115$, $d = 2900$, $c = 3000$, we have $d(\log \tau)^{\alpha - \beta - 1} > b$, and

$$\sqrt{\log t} f(t, u_1, u_2) < 35, \quad (t, u_1, u_2) \in [1, e] \times [0, 122] \times [0, 110];$$

$$\sqrt{\log t} f(t, u_1, u_2) > 397, \quad (t, u_1, u_2) \in [1.6, e] \times [160, 3223] \times [114, 2900];$$

$$\sqrt{\log t} f(t, u_1, u_2) < 985, \quad (t, u_1, u_2) \in [1, e] \times [0, 3334] \times [0, 3000];$$

$$\sqrt{\log t} f(t, u_1, u_2) > 34, \quad (t, u_1, u_2) \in [1.6, e] \times [0, 122] \times [0, 110].$$

Consequently, it follows from Theorem 3.3 that the fractional differential equation (4.3) has at least three positive solutions.

5. Conclusions

This work studies the existence and multiplicity of positive solutions for a class of singular fractional differential equations with the lower-order fractional derivative. By applying a reducing-order technique and the fixed-point theorem, some new results of the existence and multiplicity of positive solutions for the equation are established. The interesting points are that the nonlinearity f from (1.1) contains the lower-order fractional derivative and that the nonlinearity is not a nonnegative function, but it may change sign and may be singular at some points. However, in this study, as we considered it as a semipositone problem, the conditions $f(t, u_1, u_2) \geq -\psi(t)$ for all $(t, u_1, u_2) \in (1, e) \times \mathbb{R}^+ \times \mathbb{R}^+$ are required. Furthermore, if $f(t, u_1, u_2) \geq -\psi(t, u_1)$ for all $(t, u_1, u_2) \in (1, e) \times \mathbb{R}^+ \times \mathbb{R}^+$, that is, the nonlinearity f is not a nonnegative function, it may change sign; its lower bound is not only related to the time variable but also to the space variable. Therefore, the existence of positive solutions for the equation are still valued. These interesting problems are worth future study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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