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Research article

On the geometric behavior of Cotton solitons immersed in certain classes of GRW spacetimes

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Abstract: In the present manuscript, we investigate the geometric behavior of complete and stochastically complete Cotton solitons isometrically immersed in a generalized Robertson-Walker (GRW) spacetime whose sectional curvature of the Riemannian fiber satisfies suitable curvature constraints. In this configuration, by assuming appropriate hypotheses on the warped structure of the GRW spacetime, we jointly use a Bochner-type formula with several maximum principles, integrability conditions, and a parabolicity criterion in order to show that such a Cotton soliton must be trivial and locally conformally flat. Additionally, applications of our main results for the steady state space, the de Sitter space, the future temporal cone of the Lorentz-Minkowski space, and a specific open subset of the anti-de Sitter space are also presented.

Keywords: complete and stochastically complete Cotton solitons; GRW spacetimes; Nishikawa's maximum principle; convergence to zero at infinity; polynomial volume growth

1. Introduction

In [1], Kişisel et al. investigated several aspects of a specific geometric flow called *Cotton flow* related to a 3-dimensional Riemannian manifold (Σ^3, g), which is defined by the following evolution equation:

$$\frac{\partial}{\partial t} g(t) = \kappa C_{g(t)}, \tag{1.1}$$

where $\kappa \in \mathbb{R}$ is a constant. Here, $C = C_g$ is the *Cotton tensor*, whose components are given by the following:

$$C_{ij} = \sum_{k,l} \frac{\epsilon^{ikl}}{\sqrt{G}} \nabla_k (R_{lj} - \frac{R}{4} g_{lj}), \tag{1.2}$$

where ϵ^{ijk} denotes the Levi-Civita permutation symbol ($\epsilon^{123} = 1$), $G = |\det(g_{ij})|$, ∇ stands for the Levi-Civita conection, R_{ij} is the Ricci tensor, and R denotes the scalar curvature of (Σ^3, g). From (1.2), it follows that C is trace-free, divergence-free, and, when $C = 0$, (Σ^3, g) is locally conformally flat (see, for instance, [2]).

On the other hand, for $\kappa = 1$ in (1.1), given a smooth vector field $X \in \mathfrak{X}(\Sigma^3)$, we say that (Σ^3, g, X) is a special solution of the Cotton flow, called a *Cotton soliton*, when it is a self-similar solution of (1.1). In other words, (Σ^3, g, X) is a Cotton soliton when it satisfies the following soliton equation:

$$C + \frac{1}{2} \mathcal{L}_X g = \lambda g, \tag{1.3}$$

where $\mathcal{L}_X g_{ij}$ denotes the Lie derivative of the metric g with respect to some smooth vector field $X \in \mathfrak{X}(\Sigma^3)$, and $\lambda \in \mathbb{R}$ is called a *soliton constant*. When the smooth vector field $X \in \mathfrak{X}(\Sigma^3)$ identically vanishes, we say that (Σ^3, g, X) is a *trivial* Cotton soliton, and, from (1.2), it must be locally conformally flat. Over the last years, many authors have contributed to the theory of Cotton solitons from an intrinsic perspective (for more details, we recommend [3–8]).

Additionally, connected with this research branch, the investigation of immersed solitons related to a geometric flow has attracted several other researchers (for instance, see [9–12]). In this context, our goal here is to study the geometric behavior of complete and stochastically complete Cotton solitons immersed in certain classes of generalized Robertson-Walker (GRW) spacetimes that satisfy suitable curvature constraints in order to obtain new triviality and nonexistence results. We recall that a GRW spacetime is a warperd product space of the type $-I \times_f M^n$, where $I \subset \mathbb{R}$ is an open interval, $f : I \to \mathbb{R}$ is a positive smooth function, and (M^n, g_M) is a Riemannian manifold (for more details, see Section 2). In this setting, by employing a similar approach of [13, 14], we jointly use a Bochner-type formula with several maximum principles dealing, in particular, with the notion of convergence at infinity and polynomial volume growth, integrability conditions, and a parabolicity criterion, to show that such a Cotton soliton must be trivial and locally conformally flat (see Section 3). Furthermore, throughout Section 3, we present applications of our main results to the steady state space, the de Sitter space, the future temporal cone of the Lorentz-Minkowski space, and a specific open subset of the anti-de Sitter space.

2. Basic setup

This section is reserved to recall some basic facts that concern spacelike hypersurfaces immersed in a GRW spacetime.

Let (M^n, g_M) be an n-dimensional connected Riemmanian manifold and I an open interval of $\mathbb{R}$ (could be all $\mathbb{R}$). A GRW spacetime, denoted by $\overline{M}^{n+1} := -I \times_f M^n$, is defined as the $(n+1)$-dimensional product manifold $I \times M^n$ endowed with the following Lorentzian metric:

$$\overline{g} := -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_M^*(g_M),$$

where $\pi_I^*(dt^2)$ and $\pi_M^*(g_M)$ denote the pullback of the metrics dt^2 and g_M via canonical projections $\pi_I : \overline{M}^{n+1} \to I$ and $\pi_M : \overline{M}^{n+1} \to M^n$, respectively, and $f : I \to \mathbb{R}$ is a positive smooth function. With this structure, (M^n, g_M) is called the Riemannian fiber, (I, dt^2) is the base, and $f \in C^\infty(I)$ is the warping function.

We also recall that, given an n-dimensional connected manifold Σ^n, a smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is said to be a spacelike hypersurface if Σ^n, furnished with its induced metric via ψ, is a Riemannian manifold. In this setting, let (Σ^n, g) be an n-dimensional spacelike manifold isometrically immersed in $\overline{M}^{n+1}$ endowed with a globally defined unit timelike vector field $N \in \mathfrak{X}(\Sigma^n)^\perp$. In other words, Σ^n is an oriented spacelike hypersurface of $\overline{M}^{n+1}$ with the induced metric g. In this picture, we consider the shape operator (or Weingarten operator) $A : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$, given by $AX = -\overline{\nabla}_X N$. Therefore, the mean curvature of Σ^n is defined by $H = \frac{1}{n}\mathrm{tr}A$.

We define the height function $h := (\pi_{\mathbb{R}})|_\Sigma$, where $\pi_{\mathbb{R}}$ denotes the projection of $\mathbb{R}$, and the angle function $\Theta = \overline{g}(N, \partial_t)$, which are called support functions of Σ^n and will be very important to our main results. From now on, we will consider that the unit timelike vector field N has the same time-orientation as ∂_t, that is, such that $\overline{g}(N, \partial_t) < 0$. Furthermore, from the inverse Cauchy-Schwarz inequality, we get $\Theta = \overline{g}(N, \partial_t) \leq -1$. Denoting the Levi-Civita connection of the spacelike hypersurface Σ^n by ∇ and taking into account that $\overline{\nabla}$ is the Levi-Civita connection of $-I \times_f M^n$, a direct calculation shows us that

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^\top = -\partial_t^\top = -\partial_t - \Theta N. \tag{2.1}$$

Moreover, from (2.1), we get following:

$$|\nabla h|^2 = \Theta^2 - 1,$$

where ∇h is the gradient of the height function h, and $|\cdot|$ represents the norm, both related to the metric g.

From now on, our basic geometric configuration will be as follows: let (Σ^3, g, X) be a 3-dimensional complete or stochastically complete Cotton soliton isometrically immersed in a 4-dimensional GRW spacetime $\overline{M}^4 = -I \times_f M^3$. To establish our main results in the next section, we will consider two special classes of GRW spacetimes. The first one is constituted by GRW spacetimes $\overline{M}^4 = -I \times_f M^3$, whose Riemannian fiber M^3 has a sectional curvature K_M that satisfies the following curvature constraint:

$$K_M \geq \sup_I (ff'' - (f')^2). \tag{2.2}$$

We highlight that the curvature constraint (2.2) is called a *strong null convergence condition* (SNCC), which was introduced by Alías and Colares in [15]. We observe that the SNCC is a suitable change on the so-called *null convergence condition* (NCC), which means that the Ricci curvature of $\overline{M}^4$ is nonnegative on null or lightlike directions (for more details concerning the NCC, see [16]). The second class of GRW spacetimes is formed by those $\overline{M}^4 = -I \times_f M^3$ such that the Riemannian fiber M^3 has a sectional curvature K_M that satisfies the following curvature constraint:

$$K_M \leq \inf_I (ff'' - (f')^2). \tag{2.3}$$

We point out that GRW spacetimes that obey the curvature constraint (2.3) can be regarded as special cases of spacetimes that do not satisfy the NCC, and a basic example of this kind of spacetime is given by the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^3$, which already appears in the current literature (see, for instance, [17, 18]). Furthermore, we note that both curvature constraints (2.2) and (2.3) are trivially satisfied for every spacetime having constant sectional curvature (for instance, see [16, Section 4]).

3. Main results

As previously stated, we will work with a complete or stochastically complete Cotton soliton (Σ^3, g, X) isometrically immersed in a GRW spacetime $-I \times_f M^3$ that satisfies one of the curvature constraints (2.2) or (2.3). For this section, our goal is to establish new triviality and nonexistence results that concern Cotton solitons immersed in GRW spacetimes by assuming suitable hypotheses on the warped structure of the ambient space. For this, we emphasize that a Cotton soliton (Σ^3, g, X) isometrically immersed in a GRW spacetime $-I \times_f M^3$ is a spacelike hypersurface since; by definition, a Cotton soliton is a Riemannian manifold that satisfies (1.3).

Before we present our results, we will need a key lemma from Cunha and Silva Junior [5], which corresponds to a Bochner-type formula for Cotton solitons resulting from the soliton equation (1.3).

Lemma 1. *Let* (Σ^3, g, X) *be a Cotton soliton. Then,*

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ric}(X, X),$$

where ∇X *denotes the covariant derivative of* $X \in \mathfrak{X}(\Sigma^3)$.

In addition to Lemma 1, we will use several maximum principles, integrability conditions, and a parabolicity criterion as main analytical tools to prove the next results. Therefore, we will present these results in subsections according to the analytical tool used.

3.1. Via Nishikawa's maximum principle

We start this subsection remembering the classical Nishikawa's maximum principle (see [19]).

Lemma 2. *Let* Σ^n *be a complete Riemannian manifold that has a Ricci curvature bounded from below. If* $u \in C^2(\Sigma^n)$ *is nonnegative and satisfies* $\Delta u \geq au^b$, *for some real numbers* $a > 0$ *and* $b > 1$, *then* u *identically vanishes on* Σ^n.

In order to establish our first result, we will jointly use Lemma 2 with some hypotheses on the warped structure and the Ricci curvature of Σ^3 to conclude that the Cotton soliton (Σ^3, g, X) isometrically immersed in the GRW spacetime $-I \times_f M^3$ is trivial and locally conformally flat.

Theorem 1. *Let* (Σ^3, g, X) *be a complete Cotton soliton isometrically immersed in a GRW spacetime* $-I \times_f M^3$ *that satisfies the curvature constraint* (2.2). *Additionally, assume that the mean curvature* H *of* Σ^3 *is bounded and* $f''(h) \geq 0$. *If* $|X| \leq 1$ *and* $\mathrm{Ric}(X, X) \leq -\beta|X|^2$, *for some positive constant* $\beta \in \mathbb{R}$, *then* (Σ^3, g, X) *is trivial and locally conformally flat.*

Proof. We start by claiming that the Ricci curvature of such a Cotton soliton Σ^n is bounded from below. Indeed, for all $Z \in \mathfrak{X}(\Sigma^n)$, from the Gauss equation, we have the following:

$$
\begin{aligned}
\mathrm{Ric}(Z,Z) &= \overline{\mathrm{Ric}}(Z,Z) - 3H\overline{g}(AZ,Z) + |AZ|^2 \\
&= \overline{\mathrm{Ric}}(Z,Z) + \left|AZ - \frac{3}{2}HZ\right|^2 - \frac{9}{4}H^2|Z|^2 \\
&\geq \overline{\mathrm{Ric}}(Z,Z) - \frac{9}{4}H^2|Z|^2.
\end{aligned} \tag{3.1}
$$

Furthermore, defining

$$
\mathcal{Z} := |Z|^2 + |Z|^2 g(E_i,\nabla h)^2 + g(Z,\nabla h)^2 - g(Z,E_i)^2 - 2g(Z,\nabla h)g(E_i,\nabla h)g(E_i,Z),
$$

for all $Z \in \mathfrak{X}(\Sigma^3)$, we assert that $\mathcal{Z} \geq 0$.

To verify this fact, at each point of Σ^n, we take a local orthonormal frame $\{E_i\}_{i=1}^3$ on Σ^n and consider the local coordinates of $Z = \sum_{j=1}^{3} a_j E_j$ and $\nabla h = \sum_{j=1}^{3} b_j E_j$, for some real numbers $a_j, b_j \in \mathbb{R}$. In this way, for each $i \in \{1,2,3\}$, we have the following:

$$
\begin{aligned}
\mathcal{Z} &= \sum_{j=1}^{3} a_j^2 + b_i^2 \sum_{j=1}^{3} a_j^2 + \left(\sum_{j=1}^{3} a_j b_j\right)^2 - a_i^2 - 2a_i b_i \sum_{j=1}^{3} a_j b_j + a_i^2 b_i^2 - a_i^2 b_i^2 \\
&= \sum_{j=1, j\neq i}^{3} a_j^2 + b_i^2 \left(\sum_{j=1, j\neq i}^{3} a_j^2\right) + \left(\sum_{j=1, j\neq i}^{3} a_j b_j\right)^2 \geq 0.
\end{aligned} \tag{3.2}
$$

Moreover, taking into account that $K_M \geq \sup_I (ff'' - (f')^2)$, with a straightforward computation, from [20, Proposition 42 of Chapter 7], we get the following:

$$
\begin{aligned}
\overline{\mathrm{Ric}}(Z,Z) &= \sum_{i=1}^{3} \overline{g}(\overline{R}(Z,E_i)Z,E_i) \\
&= f^{-2}(h) \sum_{i=1}^{3} K_M(Z^*, E_i^*) \Big\{ |Z|^2 + |Z|^2 g(E_i,\nabla h)^2 + g(Z,\nabla h)^2 \\
&\quad - g(Z,E_i)^2 - 2g(Z,\nabla h)g(E_i,\nabla h)g(E_i,Z) \Big\} + 2((\log f)'(h))^2 |Z|^2 \\
&\quad - (\log f)''(h) g(Z,\nabla h)^2 - (\log f)''(h)|\nabla h|^2|Z|^2 \\
&\geq (\log f)''(h)\{2|Z|^2 + |Z|^2|\nabla h|^2 + g(Z,\nabla h)^2\} \\
&\quad + 2((\log f)'(h))^2|Z|^2 - (\log f)''(h)\{g(Z,\nabla h)^2 + |Z|^2|\nabla h|^2\} \\
&= 2\frac{f''(h)}{f(h)}|Z|^2 \geq 0.
\end{aligned} \tag{3.3}
$$

Thus, since we are assuming that the mean curvature H of Σ^3 is bounded, from (3.1) and (3.3), we have the following:

$$
\mathrm{Ric}(Z,Z) \geq -\frac{9}{4}H^2|Z|^2 \geq -\frac{9\alpha}{4}|Z|^2,
$$

where $\alpha \in \mathbb{R}$ is a positive constant such that $H^2 \leq \alpha$. In this setting, we conclude that the Ricci curvature of Σ^3 is bounded from below.

Furthermore, since $\mathrm{Ric}(X, X) \leq -\beta|X|^2$, for some $\beta \in \mathbb{R}$, from Lemma 1, we get that

$$\Delta|X|^2 \geq 2|\nabla X|^2 - 2\mathrm{Ric}(X, X) \geq 2\beta|X|^2.$$

Consequently, since $|X|^2 \leq 1$, we infer that

$$\Delta|X|^2 \geq 2\beta(|X|^2)^\gamma,$$

for all $1 < \gamma \in \mathbb{R}$.

Therefore, from Lemma 2, we conclude that $|X| = 0$. Returning to (1.3) and taking the trace, we have $\lambda = 0$; hence, the Cotton tensor identically vanishes.

In the following, we will deal with the *steady state space* $\mathcal{H}^4$, which is a model of the universe proposed by Bondi and Gold [21] and Hoyle [22] and is isometric to the GRW spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^3$ (for a more detailed reading, we recommend [23]). Now, we are in position to present the next corollary of Theorem 1.

Corollary 1. *Let (Σ^3, g, X) be a complete Cotton soliton isometrically immersed in the steady state space $\mathcal{H}^4$. Additionally, assume that the mean curvature H of Σ^3 is bounded. If $|X| \leq 1$ and $\mathrm{Ric}(X, X) \leq -\beta|X|^2$, for some positive constant $\beta \in \mathbb{R}$, then (Σ^3, g, X) is trivial and locally conformally flat.*

According to [16, Example 4.2], the 4-dimensional de Sitter space $\mathbb{S}^4_1$ is isometric to the GRW spacetime $-\mathbb{R}\times_{\cosh t}\mathbb{S}^3$, where $\mathbb{S}^3$ stands for the 3-dimensional unit Euclidean sphere. With this previous discussion, we finish this subsection by presenting one more corollary of Theorem 1.

Corollary 2. *Let (Σ^3, g, X) be a complete Cotton soliton isometrically immersed in the de Sitter space $\mathbb{S}^4_1$. Additionally, assume that the mean curvature H of Σ^3 is bounded. If $|X| \leq 1$ and $\mathrm{Ric}(X, X) \leq -\beta|X|^2$, for some positive constant $\beta \in \mathbb{R}$, then (Σ^3, g, X) is trivial and locally conformally flat.*

3.2. Via stochastic completeness

We recall that a Riemannian manifold (Σ^n, g) is said to be *stochastically complete* if, for some (and, hence, for any) $(x, \tau) \in \Sigma^n \times (0, +\infty)$, the heat kernel $p(x, y, \tau)$ of the Laplace-Beltrami operator Δ satisfies the following conservation property:

$$\int_\Sigma p(x, y, \tau)d\mu(y) = 1. \tag{3.4}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have an infinite life time. Furthermore, for the Brownian motion on a manifold, the conservation property (3.4) means that the total probability of the particle to be found in the state space is constant and equal to one (cf. [24–27]).

Following the terminology introduced by Pigola et al. in [28], the Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold (Σ^n, g) if, for any smooth function $u \in C^2(\Sigma^n)$ with $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ that satisfies the following:

$$\lim_k u(p_k) = \sup_\Sigma u, \quad \lim_k |\nabla u(p_k)| = 0 \quad \text{and} \quad \limsup_k \Delta u(p_k) \leq 0.$$

In this point of view, the classical result given by Omori and Yau in [29, 30] states that Omori-Yau's maximum principle holds on every complete Riemannian manifold with a Ricci curvature bounded from below.

On the other hand, as it was also observed by Pigola, Rigoli, and Setti in [28], the validity of Omori-Yau's maximum principle on Σ^n does not depend on the curvature bounds as would be expected. For instance, Omori-Yau's maximum principle holds on every Riemannian manifold which is properly immersed into a Riemannian space form with a controlled mean curvature (see [28, Example 1.14]). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, following again the terminology introduced in [28], the weak Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold (Σ^n, g) if, for any smooth function $u \in C^2(\Sigma^n)$ with $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ with the following properties:

$$\lim_k u(p_k) = \sup_\Sigma u \quad \text{and} \quad \limsup_k \Delta u(p_k) \leq 0.$$

As was proven by Pigola et al. [28, 31], the fact that the weak Omori-Yau's maximum principle holds on Σ^n is equivalent to its stochastic completeness. More precisely, the following Lemma holds.

Lemma 3. *A Riemannian manifold* (Σ^n, g) *is stochastically complete if and only if the weak Omori-Yau's maximum principle holds on* (Σ^n, g).

Next, we will deal with the *first Newton transformation* $P_1 : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ of a hypersurface Σ^n, which is defined by the following:

$$P_1 X = nHX - AX.$$

After this previous discussion, using Lemma 3, we can prove the following result.

Theorem 2. *Let* (Σ^3, g, X) *be a stochastically complete Cotton soliton isometrically immersed in a GRW spacetime* $-I \times_f M^3$ *that satisfies the curvature constraint* (2.3). *Additionally, assume that* $A \circ P_1$ *is positive semi-definite in the direction of* X *and* $\dfrac{f''(h)}{f(h)} \leq -\alpha$, *for some positive constant* $\alpha \in \mathbb{R}$. *If* $|X| \in L^\infty(\Sigma^3)$, *then* (Σ^3, g, X) *is trivial and locally conformally flat.*

Proof. Let us start by proving that $\mathrm{Ric}(X, X) \leq -2\alpha|X|^2$. Indeed, taking into account that $A \circ P_1$ is positive semi-definite in the direction of X, from the Gauss equation, we have the following:

$$\begin{aligned}
\mathrm{Ric}(X, X) &= \sum_{i=1}^{3} \overline{g}(\overline{R}(X, E_i)X, E_i) - 3H\overline{g}(AX, X) + \overline{g}(AX, AX) \\
&= \sum_{i=1}^{3} \overline{g}(\overline{R}(X, E_i)X, E_i) - \overline{g}(AX, P_1 X) \\
&= \sum_{i=1}^{3} \overline{g}(\overline{R}(X, E_i)X, E_i) - \overline{g}(X, A \circ P_1 X) \\
&\leq \sum_{i=1}^{3} \overline{g}(\overline{R}(X, E_i)X, E_i),
\end{aligned}$$

where $\{E_i\}_{i=1}^3$ is an (local) orthonormal frame on Σ^3. On the other hand, from (3.2), we have that $\mathcal{X} \geq 0$, where

$$\mathcal{X} := |X|^2 + |X|^2 g(E_i, \nabla h)^2 + g(X, \nabla h)^2 - g(X, E_i)^2 - 2g(X, \nabla h)g(E_i, \nabla h)g(E_i, X).$$

With the above discussion, from [20, Proposition 42 of Chapter 7] and taking into account that the sectional curvature of the Riemannian fiber M^3 satisfies $K_M \leq \inf_I(ff'' - (f')^2) \leq f^2(h)(\log f)''(h)$, we obtain the following:

$$\begin{aligned}\sum_{i=1}^3 \overline{g}(\overline{R}(X, E_i)X, E_i) = f^{-2}(h) \sum_{i=1}^3 & K_M(X^*, E_i^*) \Big\{|X|^2 + |X|^2 g(E_i, \nabla h)^2 + g(X, \nabla h)^2 \\ & -g(X, E_i)^2 - 2g(X, \nabla h)g(E_i, \nabla h)g(E_i, X)\Big\} + 2((\log f)'(h))^2|X|^2 \\ & - (\log f)''(h)g(X, \nabla h)^2 - (\log f)''(h)|\nabla h|^2|X|^2 \\ \leq\ & (\log f)''(h)\{2|X|^2 + |X|^2|\nabla h|^2 + g(X, \nabla h)^2\} \\ & + 2((\log f)'(h))^2|X|^2 - (\log f)''(h)\{g(X, \nabla h)^2 + |X|^2|\nabla h|^2\} \\ =\ & 2\frac{f''(h)}{f(h)}|X|^2 \\ \leq\ & -2\alpha|X|^2,\end{aligned}$$

where we use the hypothesis that $\dfrac{f''(h)}{f(h)} \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$, in the last inequality. Furthermore, we conclude that $\mathrm{Ric}(X, X) \leq -2\alpha|X|^2$.

Now, from Lemma 1, we get the following:

$$\Delta|X|^2 = 2|\nabla X|^2 - 2\mathrm{Ric}(X, X) \geq 4\alpha|X|^2.$$

However, since we are assuming that $|X| \in L^\infty(\Sigma^3)$ and (Σ^3, g, X) is stochastically complete, from Lemma 3, we get a sequence $(q_k) \in \Sigma^3$ such that

$$\lim_k |X|^2(q_k) = \sup_\Sigma |X|^2 \quad \text{and} \quad \limsup_k \Delta|X|^2(q_k) \leq 0.$$

In this picture, we have the following:

$$0 \geq \limsup_k \Delta|X|^2(q_k) \geq 4\alpha \limsup_k |X|^2(q_k) = 4\alpha \sup_\Sigma |X|^2 \geq 0.$$

Therefore, we conclude that $\sup_\Sigma |X|^2 \equiv 0$, which implies that $|X|^2 = 0$. Moreover, taking into account that the Cotton tensor is trace-free, taking the trace of (1.3), we verify that $\lambda = 0$ and, consequently, $C \equiv 0$. In this way, we have that (Σ^3, g, X) is trivial and locally conformally flat.

Motivated by [16, Example 4.3], we will consider the open subset Ω of the 4-dimensional anti-de Sitter space $\mathbb{H}_1^4$ that is isometric to the GRW spacetime $-\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \times_{\cos t} \mathbb{H}^3$, where $\mathbb{H}^3$ stands for the 3-dimensional hyperbolic space. In this context, it is not difficult to verify that we obtain the following corollary of Theorem 2.

Corollary 3. *Let (Σ^3, g, X) be a stochastically complete Cotton soliton isometrically immersed in $\Omega = -\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} \mathbb{H}^3$. Additionally, assume that $A \circ P_1$ is positive semi-definite in the direction of X. If $|X| \in L^\infty(\Sigma^3)$, then (Σ^3, g, X) is trivial and locally conformally flat.*

3.3. Via polynomial volume growth

Let us consider a (connected oriented) complete Riemannian manifold (Σ^n, g) and denote a geodesic ball with radius r by B_r. We say that Σ^n has a *polynomial volume growth* when there exists a polynomial function $\sigma : (0, +\infty) \to (0, +\infty)$ such that

$$\mathrm{vol}(B_r) = O(\sigma(r))$$

as $t \to \infty$. The following lemma corresponds to a particular case of a more general maximum principle from Alías, Caminha, and Nascimento (see [32, Theorem 2.1]).

Lemma 4. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $Y \in \mathfrak{X}(\Sigma^n)$ be a bounded smooth vector field on Σ^n, with $|Y| \le c$ for some positive constant $c \in \mathbb{R}$. Let $v \in C^\infty(\Sigma^n)$ be a smooth function such that $g(\nabla v, Y) \ge 0$ and $\mathrm{div}(Y) \ge av$ on Σ^n, for some positive constant $a \in \mathbb{R}$. If (Σ^n, g) has a polynomial volume growth, then $v \le 0$ on Σ^n.*

Our next triviality result deals with a complete Cotton soliton with a polynomial volume growth isometrically immersed in a GRW spacetime. In this setting, assuming suitable hypotheses on the warped structure, we will show that the Cotton soliton must be trivial and locally conformally flat.

Theorem 3. *Let $\left(\Sigma^3, g, X\right)$ be a complete Cotton soliton with a polynomial volume growth isometrically immersed in a GRW spacetime $-I \times_f M^3$ that satisfies the curvature constraint (2.3). Additionally, assume that $A \circ P_1$ is positive semi-definite in the direction of X and $\dfrac{f''(h)}{f(h)} \le -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. If $|X|, |\nabla X| \in L^\infty(\Sigma^3)$, then $\left(\Sigma^3, g, X\right)$ is trivial and locally conformally flat.*

Proof. In a similar way of the proof of Theorem 2, we get $\mathrm{Ric}(X, X) \le -2\alpha|X|^2$. From Lemma 1, we get the following:

$$\Delta|X|^2 = 2|\nabla X|^2 - 2\mathrm{Ric}(X, X) \ge 4\alpha|X|^2.$$

Now, let us take the smooth vector field $Y = \nabla|X|^2 \in \mathfrak{X}(\Sigma^3)$. We observe that

$$\mathrm{div}(Y) \ge 4\alpha|X|^2 \quad \text{and} \quad g(Y, \nabla|X|^2) = |\nabla|X|^2|^2 \ge 0. \tag{3.5}$$

Furthermore, taking into account that $|X|, |\nabla X| \in L^\infty(\Sigma^3)$, from Kato's inequality, we have the following:

$$|Y| = |\nabla|X|^2| = 2|X||\nabla|X|| \le 2|X||\nabla X| \in L^\infty(\Sigma^3).$$

If (Σ^3, g, X) is complete, noncompact, and has a polynomial volume growth, we can apply Lemma 4 to get that $|X|^2 = 0$. If (Σ^3, g, X) is compact (without boundary), by integrating both sides of the first inequality of (3.5), then we obtain the following:

$$0 = \int_\Sigma \mathrm{div}(Y) \ge 4\alpha \int_\Sigma |X|^2 \ge 0.$$

Therefore, $|X|^2 = 0$.

Considering both cases, returning to (1.3) and taking the trace, we have $\lambda = 0$ and, consequently, the Cotton tensor identically vanishes. This finishes the proof.

Inspired by Corollary 3, we state the following consequence of Theorem 3.

Corollary 4. *There is no complete Cotton soliton* $\left(\Sigma^3, g, X\right)$ *with a polynomial volume growth isometrically immersed in* $\Omega = -\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} \mathbb{H}^3$ *such that* $A \circ P_1$ *is positive semi-definite in the direction of* X *and* $|X|, |\nabla X| \in L^\infty(\Sigma^3)$.

Proof. By contradiction, suppose that there is such a complete Cotton soliton (Σ^3, g, X). In this setting, it is not difficult to see that all hypotheses of Theorem 3 are satisfied and, therefore, (Σ^3, g, X) is trivial. In other words, (Σ^3, g, X) is a slice of $\Pi = -\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} \mathbb{H}^3$, which implies that (Σ^3, g, X) has and exponential volume growth because $\mathbb{H}^3$ has an exponential volume growth. However, this contradicts the hypothesis that (Σ^3, g, X) has a polynomial volume growth.

3.4. Via Integrability

Here, we will explore some integrability conditions to get our next results. In this setting, we will assume suitable hypotheses on the warped structure, jointly with (3.6), to present our first integrability result as follows.

Theorem 4. *Let* $\left(\Sigma^3, g, X\right)$ *be a complete noncompact Cotton soliton isometrically immersed in a GRW spacetime* $-I \times_f M^3$ *that satisfies the curvature constraint* (2.3)*. Additionally, assume that* $A \circ P_1$ *is positive semi-definite in the direction of* X *and* $\dfrac{f''(h)}{f(h)} \leq -\alpha$ *for some positive constant* $\alpha \in \mathbb{R}$*. If* $|X| \in L^\infty(\Sigma^3)$ *and*

$$\int_\Sigma |\nabla X| = o(r), \tag{3.6}$$

then (Σ^3, g, X) *is trivial and locally conformally flat.*

Proof. Similar to the proof of Theorem 2, we get $\mathrm{Ric}(X, X) \leq -2\alpha|X|^2$. From Lemma 1, we obtain the following:

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ric}(X, X) \geq 2\alpha|X|^2. \tag{3.7}$$

In this picture, integrating (3.7) on a ball and using Stokes' Theorem, from Kato's inequality, we obtain the following:

$$\begin{aligned} 2\alpha \int_{B_r} |X|^2 \leq \frac{1}{2}\int_{B_r} \Delta|X|^2 = \frac{1}{2}\int_{\partial B_r} g(\nabla|X|^2, \nu) \\ \leq \frac{1}{2}\int_{\partial B_r} |\nabla|X|^2| \\ \leq \int_{\partial B_r} |X||\nabla X|. \end{aligned} \tag{3.8}$$

Since $|X| \in L^\infty(\Sigma^3)$, there is a positive constant $C \in \mathbb{R}$ such that $|X| \leq C$. Thus, from (3.8), we have the following:

$$2\alpha \int_{B_r} |X|^2 \leq \int_{\partial B_r} |X||\nabla X| \leq C \int_{\partial B_r} |\nabla X|.$$

Thus, from Fubini's Theorem and (3.6), we have the following:

$$\int_{B_r} |X|^2 \to 0$$

as $r \to +\infty$. Therefore, we verify the following:

$$\int_{\Sigma} |X|^2 = 0.$$

In this setting, we conclude that $|X|$ identically vanishes and taking the trace of (1.3), we get $\lambda = 0$. In other words, (Σ^3, g, X) is trivial and locally conformally flat.

One more time, we will reason as in Corollary 3 to establish our next result, which is a corollary of Theorem 4.

Corollary 5. *Let* $\left(\Sigma^3, g, X\right)$ *be a complete noncompact Cotton soliton isometrically immersed in* $\Omega = -\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} \mathbb{H}^3$. *Additionally, assume that* $A \circ P_1$ *is positive semi-definite in the direction of* X. *If* $|X| \in L^\infty(\Sigma^3)$ *and*

$$\int_{\Sigma} |\nabla X| = o(r),$$

then (Σ^3, g, X) *is trivial and locally conformally flat.*

In the following, let us consider the space of Lebesgue q-integrable functions on a Riemannian manifold Σ^n, denoted by $L^q(\Sigma^n)$ and given by the following:

$$L^q(\Sigma^n) := \left\{ u \in C^\infty(\Sigma^n); \int_{\Sigma} |u|^q < +\infty, \quad 1 \leq q < +\infty \right\}.$$

At this point, we recall that a smooth function $u \in C^\infty(\Sigma^n)$ on a Riemannian manifold Σ^n is called a harmonic function when $\Delta u = 0$. When $\Delta u \geq 0$ on Σ^n, u is called a subharmonic function.

In [33], by generalizing a previous result due to Gaffney [34], Yau established the following version of Stokes' Theorem on an n-dimensional complete noncompact Riemannian manifold Σ^n: *if* ω *is an integrable* $(n-1)$*-differential form on* Σ^n*, then there exists a sequence* B_i *of domains on* Σ^n *such that*

$$B_i \subset B_{i+1}, \quad \Sigma^n = \bigcup_{i \geq 1} B_i \quad \text{and} \quad \lim_{i \to +\infty} \int_{B_i} d\omega = 0.$$

As a consequence of this result, in [33], Yau obtained the following lemma.

Lemma 5. *If* u *is a subharmonic function on a complete Riemannian manifold* Σ^n *with* $|\nabla u| \in L^1(\Sigma^n)$*, then* u *is harmonic.*

In this setting, as a consequence of Lemma 5, we present the next lemma, which is a Liouville type result also due to Yau [33] (additionally, see [35]).

Lemma 6. *Let* u *be a nonnegative smooth subharmonic function on a complete Riemannian manifold* Σ^n. *If* $u \in L^q(\Sigma^n)$*, for some* $q > 1$*, then* u *is constant.*

In this picture, using Lemmas 5 and 6, we can prove our next triviality result that concerns Cotton solitons isometrically immersed in a GRW spacetime.

Theorem 5. *Let $\left(\Sigma^3, g, X\right)$ be a complete Cotton soliton isometrically immersed in a GRW spacetime $-I \times_f M^3$ that satisfies the curvature constraint* (2.3). *Additionally, assume that $f''(h) \leq 0$ and $A \circ P_1$ is positive definite in the direction of X. If either $|X| \in L^\infty(\Sigma^3)$ and $|\nabla X| \in L^1(\Sigma^3)$ or $|X|^2 \in L^q(\Sigma^3)$, for some $q > 1$, then (Σ^3, g, X) is trivial and locally conformally flat.*

Proof. Similar to the proof of Theorem 2, we can verify the following:

$$\mathrm{Ric}(X, X) \leq 2\frac{f''(h)}{f(h)}|X|^2 \leq 0,$$

where, in the last inequality, we use the hypothesis that $f''(h) \leq 0$. Furthermore, from Lemma 1, we have the following:

$$\Delta|X|^2 = 2|\nabla X|^2 - 2\mathrm{Ric}(X, X) \geq 0. \tag{3.9}$$

If $|X| \in L^\infty(\Sigma^3)$ and $|\nabla X| \in L^1(\Sigma^3)$, from Kato's inequality, we have the following:

$$|\nabla|X|^2| = 2|X||\nabla|X|| \leq 2|X||\nabla X|.$$

Therefore, $|\nabla|X|^2| \in L^1(\Sigma^3)$ and, from Lemma 5, we obtain $\Delta|X|^2 = 0$. On the other hand, if $|X|^2 \in L^q(\Sigma^3)$, for some $q > 1$, then we get that $|X|^2$ is constant and, consequently, $\Delta|X|^2 = 0$.

In both cases, from (3.9), we conclude that $\mathrm{Ric}(X, X) = 0$. Since $A \circ P_1$ is positive definite in the direction of X and $\overline{g}(A \circ P_1 X, X) = 0$, we have that $|X| = 0$. This conclude the proof.

As in [36, Section 4], the future temporal cone Υ^+ of the Lorentz-Minkowski space $\mathbb{R}^4_1$ is defined by the following:

$$\Upsilon^+ : \{x \in \mathbb{R}^4_1; g_1(x, x) < 0 \quad \text{and} \quad g_1(x, e_1) < 0\},$$

where $e_1 = (1, 0, 0, 0)$, and g_1 denotes the canonical metric of the Lorentz-Minkowski space $\mathbb{R}^4_1$.

We observe that Υ^+ can be viewed as the following GRW spacetime:

$$-(0, +\infty) \times_t \mathbb{H}^3,$$

where $\mathbb{H}^3$ denotes the 3-dimensional hyperbolic space. In this setting, it is not difficult to verify that we have the following application of Theorem 5.

Corollary 6. *Let $\left(\Sigma^3, g, X\right)$ be a complete Cotton soliton isometrically immersed in $\Upsilon^+ = -(0, +\infty) \times_t \mathbb{H}^3$. Additionally, assume that $A \circ P_1$ is positive definite in the direction of X. If either $|X| \in L^\infty(\Sigma^3)$ and $|\nabla X| \in L^1(\Sigma^3)$ or $|X|^2 \in L^q(\Sigma^3)$, for some $q > 1$, then (Σ^3, g, X) is trivial and locally conformally flat.*

3.5. *Via a parabolicity criterion*

We recall that a Riemannian manifold Σ^n is said to be parabolic when the only subharmonic function bounded from above are the constant functions. In other words, Σ^n is parabolic when given a $u \in C^\infty(\Sigma^n)$ such that

$$\Delta u \geq 0 \quad \text{and} \quad \sup_\Sigma u < +\infty;$$

then, u is a constant function.

In this setting, to prove our next results, we need to quote the following lemma from [37, Theorem 4.4].

Lemma 7. *Let $-I\times_f M^n$ be a GRW spacetime, whose Riemannian fiber M^n is complete with a parabolic universal Riemannian covering, and let $\psi:\Sigma^n\to -I\times_f M^n$ be a complete spacelike hypersurface such that its hyperbolic angle function θ is bounded. If the restriction $f(h)$ on Σ^n of the warping function f of $\overline{M}^{n+1}$ satisfies*

$$0<\inf_{\Sigma^n} f(h)\leq \sup_{\Sigma^n} f(h)<+\infty,$$

then, Σ^n is parabolic.

Keeping in mind the concept of parabolicity and Lemma 7, we are in position to get the following triviality result.

Theorem 6. *Let $\left(\Sigma^3,g,X\right)$ be a complete noncompact Cotton soliton isometrically immersed in a GRW spacetime $-I\times_f M^3$ that satisfies the curvature constraint* (2.3)*, whose fiber M^3 has a parabolic universal Riemannian covering. Additionally, assume that $A\circ P_1$ is positive definite and $f''(h)\leq 0$. If Σ^3 is contained in a slab, the hyperbolic angle function θ is bounded, and $|X|\in L^\infty(\Sigma^3)$, then (Σ^3,g,X) is trivial and locally conformally flat.*

Proof. Similar to the proof of Theorem 5, we can verify that $\mathrm{Ric}(X,X)\leq 0$. Moreover, from Lemma 1, we get the following:

$$\Delta|X|^2=2|\nabla X|^2-2\mathrm{Ric}(X,X)\geq 0. \tag{3.10}$$

On the other hand, since Σ^3 is contained in a slab, we get $0<\inf f(h)\leq\sup f(h)<+\infty$. Thus, taking into account that the hyperbolic angle function θ is bounded, from Lemma 7, we have that Σ^3 is parabolic. Furthermore, since $|X|\in L^\infty(\Sigma^3)$, from (3.10), we conclude that $|X|^2$ is a bounded subharmonic smooth function on Σ^3 and, consequently, $|X|^2$ is constant.

Thereby, returning to (3.10), we get $\Delta|X|^2=0$. In particular, we also get $\mathrm{Ric}(X,X)=0$. Since $A\circ P_1$ is positive definite and $\overline{g}(A\circ P_1X,X)=0$, we have that $|X|=0$. In other words, (Σ^3,g,X) is trivial and locally conformally flat.

3.6. *Via convergence at infinity*

To state the next result, we need the concept to convergence to zero at infinity. Given a (connected) complete noncompact Riemannian manifold (Σ^n,g) and denoting the Riemannian distance of Σ^n measured from a fixed point $o\in\Sigma^n$ by $d(\cdot,o):\Sigma^n\to[0,+\infty)$, a function $h\in C^\infty(\Sigma^n)$ converges to zero at infinity when

$$\lim_{d(x,o)\to\infty} h(x)=0.$$

The following lemma is due to Alías et al. [38].

Lemma 8. *Let (Σ^n,g) be a complete noncompact Riemannian manifold, and let $Y\in\mathfrak{X}(\Sigma^n)$ be a smooth vector field on Σ^n. Assume that there exists a nonnegative, non-identically vanishing function $v\in C^\infty(\Sigma^n)$ which converges to zero at infinity and such that $g(\nabla v,Y)\geq 0$. If $\mathrm{div}(Y)\geq 0$ on Σ^n, then $g(\nabla v,Y)\equiv 0$ on Σ^n.*

After this discussion, we will use the concept of convergence to zero at infinity to obtain the last results of this manuscript. The first one is the next theorem.

Theorem 7. *Let $\left(\Sigma^3, g, X\right)$ be a complete noncompact Cotton soliton isometrically immersed in a GRW spacetime $-I \times_f M^3$ that satisfies the curvature constraint* (2.3). *Additionally, assume that $f''(h) \leq 0$ and $A \circ P_1$ is positive semi-definite in the direction of X. If $|X|$ converges to zero at infinity, then $\left(\Sigma^3, g, X\right)$ is trivial and locally conformally flat.*

Proof. Following the same steps of the proof of Theorem 2, we can verify the following:

$$\mathrm{Ric}(X, X) \leq 2\frac{f''(h)}{f(h)}|X|^2 \leq 0,$$

where, in the last inequality, we use the hypothesis that $f''(h) \leq 0$. Thus, from Lemma 1, we get the following:

$$\Delta|X|^2 = 2|\nabla X|^2 - 2\mathrm{Ric}(X, X) \geq 0.$$

Taking the smooth vector field $Y = \nabla|X|^2 \in \mathfrak{X}(\Sigma^3)$, we also get the following:

$$\mathrm{div}(Y) = \Delta|X|^2 \geq 0 \quad \text{and} \quad g(Y\nabla|X|^2) = |\nabla|X|^2|^2 \geq 0.$$

Since we suppose that $|X|$ converges to zero at infinity, from Lemma 8, we have $|\nabla|X|^2|^2 \equiv 0$ and, consequently, $|X|^2$ is a constant function. However, since $|X|^2$ is a constant function and converges to zero at infinity, we conclude that $|X| \equiv 0$. Finally, taking the trace of (1.3), we conclude that $\lambda = 0$. Thus, (Σ^3, g, X) is trivial and locally conformally flat.

We finish our paper by presenting the following corollary of Theorem 7.

Corollary 7. *Let $\left(\Sigma^3, g, X\right)$ be a complete noncompact Cotton soliton isometrically immersed in $\Upsilon^+ = -(0, +\infty) \times_t \mathbb{H}^3$. Additionally, assume that $A \circ P_1$ is positive semi-definite in the direction of X. If $|X|$ converges to zero at infinity, then $\left(\Sigma^3, g, X\right)$ is trivial and locally conformally flat.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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