



Research article

Necessary and sufficient conditions of the trilinear Stein-Weiss inequality

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Abstract: In this paper, we consider a trilinear weighted Hardy-Littlewood-Sobolev inequality as follows:

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_i}}{\prod_{1 \leq j < k \leq 3} |x_j - x_k|^{\alpha_{jk}}} dx_1 dx_2 dx_3 \right| \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3},$$

which can be regarded as the natural trilinear form of the Stein-Weiss inequality. For $1 \leq p_1, p_2, p_3 \leq \infty$, we systematically establish the trilinear Stein-Weiss inequality. In particular, when $1 < p_1, p_2, p_3 < \infty$, by means of appropriate space decomposition, we give the necessary and sufficient conditions to characterize the trilinear Stein-Weiss inequality.

Keywords: Stein-Weiss inequalities; boundedness of trilinear functionals; necessary and sufficient conditions

1. Introduction

The doubly weighted Hardy-Littlewood-Sobolev inequality (often termed the Stein-Weiss inequality) was established by Stein and Weiss [1], who provided a set of sufficient conditions on the indices $\alpha, \lambda, \beta, p, q$. The necessity of these conditions was later proved by Shi et al. [2]. More precisely, for $1 < p, q < \infty$, the Stein-Weiss inequality

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x - y|^\lambda |y|^\beta} dx dy \right| \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \tag{1.1}$$

holds if and only if the following conditions are satisfied:

$$\frac{1}{p} + \frac{1}{q} + \frac{\alpha + \lambda + \beta}{n} = 2,$$

$$\alpha + \beta \geq 0, \alpha < \frac{n}{p'}, \beta < \frac{n}{q'}, \lambda < n,$$

and

$$\frac{1}{p} + \frac{1}{q} \geq 1.$$

Here, p' and q' denote the conjugate exponents of p and q , respectively, defined by $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

The Stein-Weiss inequality has been a fundamental tool in characterizing the interactions between two variables with different weights, providing a solid theoretical basis for numerous applications in various fields such as harmonic analysis, partial differential equations, and potential theory. For further results on the Stein-Weiss inequality, we refer the reader to [3–10], among others. However, in practical scenarios, the complexity of problems often demands the exploration of interactions among more than two variables with diverse weights. In the mathematical context, to address this challenge, it is natural to extend the functional $F(f, g) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dx dy$ to the multilinear setting.

Let $A = (\alpha_{ij})_{k \times k}$ ($k \geq 3$) be a real-symmetric matrix of real parameters. Consider the measurable functions f_1, \dots, f_k on \mathbb{R}^n . We define the k -linear integral functional Λ_A by

$$\Lambda_A(f_1, \dots, f_k) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \frac{\prod_{i=1}^k f_i(x_i) |x_i|^{-\alpha_{ii}}}{\prod_{1 \leq i < j \leq k} |x_i - x_j|^{\alpha_{ij}}} dx_1 \cdots dx_k.$$

This functional incorporates both pointwise weights $|x_i|^{-\alpha_{ii}}$ and pairwise interaction weights $|x_i - x_j|^{-\alpha_{ij}}$. By omitting the pointwise weights, we obtain the k -linear fractional integral functional:

$$\Lambda(f_1, \dots, f_k) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \frac{\prod_{i=1}^k f_i(x_i)}{\prod_{1 \leq i < j \leq k} |x_i - x_j|^{\alpha_{ij}}} dx_1 \cdots dx_k.$$

We also define a specialized functional that focuses on interactions with x_1 :

$$\Lambda^1(f_1, \dots, f_k) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \frac{\prod_{i=1}^k f_i(x_i) |x_i|^{-\alpha_{ii}}}{\prod_{1 < j \leq k} |x_1 - x_j|^{\alpha_{1j}}} dx_1 \cdots dx_k.$$

A fundamental question is to determine the necessary and sufficient conditions for the inequality

$$|\Lambda_A(f_1, \dots, f_k)| \leq C \prod_{i=1}^k \|f_i\|_{p_i} \tag{1.2}$$

to hold. While characterizing these conditions for general k remains challenging, some progress has been made. For the functional Λ with non-negative α_{ij} , a complete characterization was provided in [11]. Meanwhile, Zhou et al. [12] established necessary and sufficient conditions for the specialized functional Λ^1 and obtained partial results for Λ_A , including both necessary conditions and a set of sufficient conditions. However, a complete characterization for the general multilinear weighted case—that is, finding the necessary and sufficient conditions for inequality (1.2)—remains a significant challenge. Even for the simplest case beyond bilinearity, namely when $k = 3$, such a characterization has been elusive. Providing this complete characterization for the trilinear Stein-Weiss inequality is therefore a critical and natural next step, as it establishes the foundational extension of the complete Stein-Weiss theory beyond two variables.

This paper aims to close this gap by systematically establishing the trilinear Stein-Weiss inequality for $1 \leq p_1, p_2, p_3 \leq \infty$:

$$|\mathcal{F}(f_1, f_2, f_3)| \leq C \prod_{i=1}^3 \|f_i\|_{p_i}. \quad (1.3)$$

Here, the functional \mathcal{F} is the Λ_A functional in the three-variable setting, defined as

$$\mathcal{F}(f_1, f_2, f_3) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq 3} |x_i - x_j|^{-\alpha_{ij}} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3.$$

Notably, when $1 < p_1, p_2, p_3 < \infty$, we determine the complete set of necessary and sufficient conditions for the validity of inequality (1.3). This achievement mirrors the result of Shi et al. [2] for the bilinear case, thereby providing a definitive answer to a natural and open question in the multilinear theory. The necessary and sufficient conditions we establish reveal the intricate balance required between the pointwise weights (α_{ii}) and the pairwise interaction weights (α_{ij}) , a complexity not fully captured in prior partial results. Furthermore, the analytical framework developed here to manage general three-body weights is a novel approach that is expected to inform future studies on the more challenging general k -linear Stein-Weiss inequality. This result not only deepens our understanding of the trilinear Stein-Weiss inequality but also provides a more comprehensive theoretical foundation for related research.

To be more precise, based on the work of [12] (Theorem 3.1), for the range $1 \leq p_1, p_2, p_3 \leq \infty$, we deduce a necessary condition for the trilinear Stein-Weiss inequality (1.3) to hold: $\sum_{i=1}^3 1/p_i \geq 1$. In other words, when regarding $(1/p_1, 1/p_2, 1/p_3)$ as the coordinates of a point in \mathbb{R}^3 , the condition that the point $(1/p_1, 1/p_2, 1/p_3)$ lies within the heptahedron $O'C'CBAA'B'$ is necessary for the boundedness of the functional \mathcal{F} , as illustrated in Figure 1. Inspired by this geometric interpretation shown in Figure 1, we then proceed to establish the corresponding inequality (1.3) by considering different combinations of the values of p_1, p_2 , and p_3 .

Since $\sum_{i=1}^3 1/p_i \geq 1$, when $p_i > 1$ for $i = 1, 2, 3$, at most one of p_i ($i = 1, 2, 3$) can be equal to ∞ . Therefore, we can make the classification as shown in Table 1. Through Theorems 1–5 in Section 3, we establish the trilinear Stein-Weiss inequality (1.3) for the following cases: the interior of the enclosed heptahedron $O'C'CBAA'B'$ with $1 < p_1, p_2, p_3 < \infty$ and $\sum_{i=1}^3 1/p_i \geq 1$; the surface $B'BCC'$ with $p_1 = 1$ and $1 < p_2, p_3 \leq \infty$; the surface $O'AA'$ with $p_1 = \infty$, $1 < p_2, p_3 < \infty$, and $1/p_2 + 1/p_3 \geq 1$; the vertex A' with $p_1 = \infty, p_2 = p_3 = 1$; and the vertex B' with $p_1 = p_2 = p_3 = 1$. Due to the symmetry of p_1, p_2 , and p_3 , the situations where $p_2 = 1$ or $p_3 = 1$ are equivalent to the case of $p_1 = 1$, and the scenarios in which $p_2 = \infty$ or $p_3 = \infty$ are identical to the case of $p_1 = \infty$.

The structure of the present paper is as follows: In Section 2, we introduce the necessary notations and lemmas. In Section 3, we present the main theorems along with their proofs. Section 4 provides a discussion on the implications and applicability of our results. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Preliminaries

Throughout this work, all function spaces are defined on the Euclidean space \mathbb{R}^n equipped with the Lebesgue measure. For any measurable function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and for $1 \leq p \leq \infty$, the L^p -norm of f is

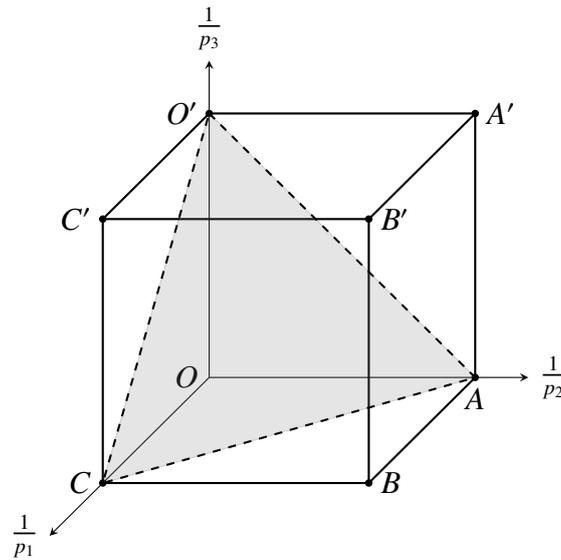


Figure 1. Bounded region of the functional \mathcal{F} : The heptahedron $O'C'CBAA'B'$.

Table 1. Classification of cases for the validity of the trilinear Stein-Weiss inequality.

Necessary condition $\sum_{i=1}^3 1/p_i \geq 1$	Region in the bounded domain of the functional \mathcal{F}	Remarks
1. $1 < p_1, p_2, p_3 < \infty$ (Theorem 1)	Interior region of the heptahedron $O'C'CBAA'B'$	Includes the interior of the bottom-face triangle $O'CA$
2. $p_1 = 1$ (Theorem 2)	Region of the side-face $B'BCC'$	Excludes points B, B', C' .
3. $p_1 = \infty$ (Theorems 3 and 4)	Interior of the triangle $O'AA'$ ($1 < p_2, p_3 < \infty$) The point A' ($p_2 = p_3 = 1$)	Excludes points O', A .
4. $p_1 = p_2 = p_3 = 1$ (Theorem 5)	Corresponding to point B'	$\sum_{i=1}^3 1/p_i = 3$

defined in the standard manner:

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|, & \text{if } p = \infty. \end{cases}$$

The Lebesgue space $L^p(\mathbb{R}^n)$ consists of all measurable functions f for which $\|f\|_p < \infty$. To facilitate the later proofs of the theorems, we introduce the following notations:

$$G(x_1, x_2, x_3) := \prod_{1 \leq i < j \leq 3} |x_i - x_j|^{-\alpha_{ij}}, \quad H(x_1, x_2, x_3) := G(x_1, x_2, x_3) \prod_{i=1}^3 |x_i|^{-\alpha_{ii}}.$$

Furthermore, we denote the product space $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ by \mathbb{R}^{3n} . Similarly, the product space $\mathbb{R}^n \times \mathbb{R}^n$ will be written as \mathbb{R}^{2n} .

Let J be a subset of $\{1, 2, \dots, k\}$. We denote the cardinality of J , namely the number of elements it contains, by $|J|$. We use the symbol $f \lesssim g$ to signify that there exists a positive constant C such that

$f \leq Cg$. When $0 < \alpha_1, \alpha_2 < n$ and $\alpha_1 + \alpha_2 > n$, we introduce the following notation:

$$\varphi(\alpha_1, \alpha_2, n) = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n-\alpha_1}{2})\Gamma(\frac{n-\alpha_2}{2})\Gamma(\frac{\alpha_1+\alpha_2-n}{2})}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})\Gamma(n - \frac{\alpha_1+\alpha_2}{2})},$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ is the gamma function. For $0 < d_1, d_2, d_3 < n$ and $d_1 + d_2 + d_3 = 2n$, we define

$$\phi(d_1, d_2, d_3, n) = \pi^{\frac{n}{2}} \prod_{j=1}^3 \frac{\Gamma(\frac{n-d_j}{2})}{\Gamma(\frac{d_j}{2})}. \quad (2.1)$$

Lemma 1 ([13, 14]). (1) When $\max\{d_1, d_2\} < n < d_1 + d_2$, the following equation is valid:

$$\int_{\mathbb{R}^n} |x_1 - t|^{-d_1} |x_2 - t|^{-d_2} dt = \varphi(d_1, d_2, n) |x_1 - x_2|^{n-d_1-d_2}.$$

(2) Provided that $\max\{d_1, d_2, d_3\} < n$ and $d_1 + d_2 + d_3 = 2n$, the following equation holds:

$$\int_{\mathbb{R}^n} \prod_{i=1}^3 |x_i - t|^{-d_i} dt = \phi(d_1, d_2, d_3, n) |x_1 - x_2|^{d_3-n} |x_1 - x_3|^{d_2-n} |x_2 - x_3|^{d_1-n}.$$

Lemma 2 ([12], Lemma 2.4). If $a_i \leq b_i$ for $i = 1, \dots, n$, and

$$C \in \left[\sum_{i=1}^n a_i, \sum_{i=1}^n b_i \right],$$

then there exists c_i such that $C = \sum_{i=1}^n c_i$ and $a_i \leq c_i \leq b_i$.

Consider the operator $S_{\alpha, \lambda, \beta}$ defined by

$$S_{\alpha, \lambda, \beta} f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|y|^\alpha |x - y|^\lambda |x|^\beta} dy.$$

The following two results are fundamental.

Lemma 3 ([2], Theorem 1.3). Let $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For $1 < q < \infty$, there exists a constant C such that

$$\|S_{\alpha, \lambda, \beta} f\|_{q'} \leq C \|f\|_p,$$

if and only if the following conditions

$$\frac{1}{p} + \frac{1}{q} \geq 1, \quad \alpha + \beta \geq 0, \quad \alpha < \frac{n}{p'}, \quad \beta < \frac{n}{q'}, \quad \lambda < n,$$

and

$$\frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \lambda}{n} = 2$$

hold simultaneously.

Lemma 4 ([2], Theorem 1.7). *Let $f \in L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. The inequality*

$$\|S_{\alpha,\lambda,\beta}f\|_{\infty} \leq C\|f\|_p$$

holds if and only if $\beta = n/p' - \alpha - \lambda$, $\alpha < n/p'$, $\lambda < n/p'$, and $\alpha + \lambda > n/p'$.

Remark 1. *Lemma 4 has the following equivalent form: Let $\varphi \in L^p(\mathbb{R}^n)$ with $1 < p \leq \infty$. Then there exists a constant C such that the inequality*

$$\left| \int_{\mathbb{R}^n} \frac{\varphi(x)}{|s-x|^{\beta}|t-x|^{\lambda}} dx \right| \leq C\|\varphi\|_{L^p} |s-t|^{\frac{n}{p'}-\beta-\lambda}$$

holds if and only if $\beta < n/p'$, $\lambda < n/p'$, and $\beta + \lambda > n/p'$.

The subsequent lemma pertains to the necessary conditions for the validity of the generalized Hardy-Littlewood-Sobolev inequality with power weights.

Lemma 5 ([12], Theorem 3.1). *If the multilinear fractional integral functional Λ_A defined by*

$$\Lambda_A(f_1, \dots, f_k) := \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \frac{\prod_{i=1}^k f_i(x_i)}{\prod_{1 \leq i < j \leq k} |x_i - x_j|^{\alpha_{ij}} \prod_{i=1}^k |x_i|^{\alpha_{ii}}} dx_1 \dots dx_k \quad (2.2)$$

is bounded on $\bigotimes_{i=1}^k L^{p_i}(\mathbb{R}^n)$, where $1 \leq p_i \leq \infty$, then the indices p_i , α_{ii} and α_{ij} satisfy the following conditions simultaneously:

(i)

$$\sum_{i=1}^k \frac{1}{p_i} + \sum_{1 \leq i < j \leq k} \frac{\alpha_{ij}}{n} + \sum_{i=1}^k \frac{\alpha_{ii}}{n} = k; \quad (2.3)$$

(ii)

$$\sum_{i=1}^k \frac{1}{p_i} \geq 1; \quad (2.4)$$

(iii)

$$\sum_{i=1}^k \alpha_{ii} \geq 0; \quad (2.5)$$

(iv) *For any subset of $\{1, 2, \dots, k\}$ denoted as I , which at least contains two elements,*

$$\sum_{i,j \in I, i < j} \frac{\alpha_{ij}}{n} < |I| - 1 \quad (2.6)$$

holds;

(v) *For any proper subset of $\{1, 2, \dots, k\}$ denoted as J , which at least contains one element,*

$$\sum_{i,j \in J, i < j} \frac{\alpha_{ij}}{n} + \sum_{i \in J} \left(\frac{1}{p_i} + \frac{\alpha_{ii}}{n} \right) < |J| \quad (2.7)$$

holds, or

$$\sum_{i,j \in J, i < j} \frac{\alpha_{ij}}{n} + \sum_{i \in J} \left(\frac{1}{p_i} + \frac{\alpha_{ii}}{n} \right) = |J|,$$

and

$$\sum_{i \in \{1, \dots, k\} \setminus J} \frac{1}{p_i} \geq 1, \quad \sum_{i \in J} \frac{1}{p_i} \geq 1$$

hold simultaneously.

Remark 2. If J has less than two elements, we naturally appoint $\sum_{i,j \in J, i < j} \frac{\alpha_{ij}}{n} = 0$.

For special trilinear functional \mathcal{F}^1 defined by

$$\mathcal{F}^1(f_1, f_2, f_3) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}}}{|x_1 - x_2|^{\alpha_{12}} |x_1 - x_3|^{\alpha_{13}}} dx_1 dx_2 dx_3,$$

we have the following results.

Lemma 6 ([12], Theorem 1.3 for $k = 3$). Let $f_i \in L^{p_i}(\mathbb{R}^n)$ for $1 < p_i < \infty$ with $i = 1, 2, 3$. Then the inequality

$$|\mathcal{F}^1(f_1, f_2, f_3)| \leq C \prod_{i=1}^3 \|f_i\|_{p_i}$$

holds if and only if

$$\sum_{i=1}^3 \frac{1}{p_i} + \sum_{1 < j < 3} \frac{\alpha_{1j}}{n} + \sum_{k=1}^3 \frac{\alpha_{kk}}{n} = 3, \quad (2.8)$$

$$\sum_{i=1}^3 \frac{1}{p_i} \geq 1, \quad (2.9)$$

$$0 < \frac{n}{p_i'} - \alpha_{ii} < \alpha_{1i} < n \quad (i = 2, 3), \quad (2.10)$$

and

$$\alpha_{11} > \sum_{i=2}^3 \max \left\{ \frac{n}{p_i'} - \alpha_{ii} - \alpha_{1i}, -\alpha_{ii} \right\}$$

hold simultaneously, or the conditions (2.8)–(2.10) and

$$\alpha_{11} = \sum_{i=2}^3 \max \left\{ \frac{n}{p_i'} - \alpha_{ii} - \alpha_{1i}, -\alpha_{ii} \right\}$$

with $\alpha_{1i} \neq n/p_i'$ ($i = 2, 3$) hold simultaneously.

Remark 3. Specifically, Lemma 6 can be regarded as a special instance of Theorem 1. It corresponds to the scenario in which the index $\alpha_{23} = 0$ within the functional \mathcal{F} . In general, Lemma 6 pertains to the case where one of the three indices α_{12} , α_{13} , and α_{23} in the functional \mathcal{F} equals zero.

Lemma 7. Let $f_i \in L^{p_i}(\mathbb{R}^n)$ for $p_1 = \infty$, $1 < p_2$, $p_3 < \infty$. Then the inequality

$$|\mathcal{F}^1(f_1, f_2, f_3)| \leq C \|f_1\|_{\infty} \|f_2\|_{p_2} \|f_3\|_{p_3} \quad (2.11)$$

holds if and only if

$$\frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{12}}{n} + \frac{\alpha_{13}}{n} + \sum_{i=1}^3 \frac{\alpha_{ii}}{n} = 3, \quad (2.12)$$

$$\frac{1}{p_2} + \frac{1}{p_3} \geq 1, \quad (2.13)$$

$$0 < \frac{n}{p_i'} - \alpha_{ii} < \alpha_{1i} < n \quad (i = 2, 3), \quad (2.14)$$

and

$$\sum_{i=1}^3 \alpha_{ii} \geq 0 \quad (2.15)$$

hold simultaneously.

Proof. Let $\alpha_{11} = \beta_{12} + \beta_{13}$ with $\beta_{12}, \beta_{13} \geq 0$. Reformulating the functional $\mathcal{F}^1(f_1, f_2, f_3)$ and applying Hölder's inequality, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} f_1(x_1) \int_{\mathbb{R}^n} \frac{f_2(x_2)}{|x_1|^{\beta_{12}} |x_1 - x_2|^{\alpha_{12}} |x_2|^{\alpha_{22}}} dx_2 \int_{\mathbb{R}^n} \frac{f_3(x_3)}{|x_1|^{\beta_{13}} |x_1 - x_3|^{\alpha_{13}} |x_3|^{\alpha_{33}}} dx_3 dx_1 \\ &= \int_{\mathbb{R}^n} f_1(x_1) S_{\alpha_{22}, \alpha_{12}, \beta_{12}} f_2(x_1) S_{\alpha_{33}, \alpha_{13}, \beta_{13}} f_3(x_1) dx_1 \\ &\leq \|f_1\|_{L^{\infty}} \|S_{\alpha_{22}, \alpha_{12}, \beta_{12}} f_2\|_{L^{q_2'}} \|S_{\alpha_{33}, \alpha_{13}, \beta_{13}} f_3\|_{L^{q_3'}}, \end{aligned}$$

where $\alpha_{11} = \beta_{12} + \beta_{13}$, $1 < q_2', q_3' < \infty$ and $1/q_2' + 1/q_3' = 1$. In order to apply Lemma 3 to obtain inequality

$$\|S_{\alpha_{ii}, \alpha_{1i}, \beta_{1i}} f_i\|_{L^{q_i'}} \leq C \|f_i\|_{L^{p_i}} \quad (i = 2, 3),$$

the parameters $\alpha_{ii}, \alpha_{1i}, \beta_{1i}, p_i, q_i$ must satisfy

$$\frac{1}{p_i} + \frac{1}{q_i} + \frac{\alpha_{ii} + \alpha_{1i} + \beta_{1i}}{n} = 2,$$

$$\alpha_{ii} + \beta_{1i} \geq 0, \quad 0 < \frac{n}{p_i'} - \alpha_{ii} < \alpha_{1i} < n, \quad \alpha_{ii} + \alpha_{1i} + \beta_{1i} \leq n.$$

Since equality (2.12) holds, we can appropriately choose the parameters q_i, β_{1i} such that equality

$$\frac{1}{p_i} + \frac{1}{q_i} + \frac{\alpha_{ii} + \alpha_{1i} + \beta_{1i}}{n} = 2 \quad (i = 2, 3)$$

is satisfied. We also need to show the existence of β_{1i} ($i = 2, 3$) satisfying

$$-\alpha_{ii} \leq \beta_{1i} \leq n - \alpha_{ii} - \alpha_{1i} \quad (i = 2, 3).$$

By Lemma 2, such β_{1i} exists provided that

$$-\alpha_{22} - \alpha_{33} \leq \alpha_{11} \leq 2n - \alpha_{22} - \alpha_{33} - \alpha_{12} - \alpha_{13}.$$

Notice that the inequality $-\alpha_{22} - \alpha_{33} \leq \alpha_{11}$ is equivalent to (2.15). Moreover, given that equality (2.12) holds, the inequality

$$\alpha_{11} \leq 2n - \alpha_{22} - \alpha_{33} - \alpha_{12} - \alpha_{13}$$

is equivalent to

$$\frac{1}{p_2} + \frac{1}{p_3} \geq 1,$$

which coincides with (2.13). This completes the proof of the sufficiency of the conditions.

The necessity of (2.12), (2.13), (2.15) follows easily from Lemma 5. It remains to prove the necessity of (2.14). Taking $I_j = \{1, j\}$ for $j = 2, 3$, condition (iv) of Lemma 5 gives

$$\frac{\alpha_{1j}}{n} < |I_j| - 1 = 1,$$

i.e., $\alpha_{1j} < n$. For $I_i = \{i\}$ ($i = 2, 3$), inequality (2.7) of Lemma 5 together with Remark 2 yields

$$\frac{1}{p_i} + \frac{\alpha_{ii}}{n} < 1,$$

which is exactly $\alpha_{ii} < n/p'_i$. Finally, for $I^i = \{1, 2, 3\} \setminus \{i\}$ ($i = 2, 3$), inequality (2.7) gives

$$\sum_{\substack{k < l \\ k, l \in I^i}} \frac{\alpha_{kl}}{n} + \sum_{k \in I^i} \left(\frac{1}{p_k} + \frac{\alpha_{kk}}{n} \right) < |I^i| = 2.$$

Comparing this with condition (2.12), we obtain

$$\frac{\alpha_{1i}}{n} + \frac{1}{p_i} + \frac{\alpha_{ii}}{n} > 1,$$

i.e., $\frac{n}{p'_i} - \alpha_{ii} < \alpha_{1i}$ for $i = 2, 3$. The three conditions that have been proved are precisely (2.14).

3. Main results and proof

Theorem 1. *Let $f_i \in L^{p_i}(\mathbb{R}^n)$ for $1 < p_i < \infty$ with $i = 1, 2, 3$. Then the trilinear Stein-Weiss inequality (1.3) holds if and only if the conditions*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{11} + \alpha_{22} + \alpha_{33}}{n} + \frac{\alpha_{12} + \alpha_{13} + \alpha_{23}}{n} = 3, \quad (3.1)$$

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1, \quad (3.2)$$

$$\alpha_{11} + \alpha_{22} + \alpha_{33} \geq 0, \quad (3.3)$$

$$\alpha_{ij} < n, \quad i \neq j, \quad (3.4)$$

$$\alpha_{ii} < \frac{n}{p'_i} \quad (i = 1, 2, 3), \quad (3.5)$$

$$\alpha_{ii} + \sum_{j \neq i} \alpha_{ij} > \frac{n}{p'_i} \quad (i = 1, 2, 3), \quad (3.6)$$

and

$$\alpha_{12} + \alpha_{13} + \alpha_{23} < 2n \quad (3.7)$$

hold simultaneously.

Remark 4. If the condition $1/p_1 + 1/p_2 + 1/p_3 = 1$ is satisfied, then, in view of $\alpha_{12} + \alpha_{13} + \alpha_{23} < 2n$ and Eq (3.1), it is inevitable that $\alpha_{11} + \alpha_{22} + \alpha_{33} > 0$. It is particularly significant to observe that when exactly one of the real-valued indices α_{12} , α_{13} , α_{23} is equal to 0, Theorem 1 reduces to the $k = 3$ case of Theorem 1.3 in [12]. In other words, Theorem 1 generalizes the $k = 3$ situation of Theorem 1.3 in [12], offering a more comprehensive and inclusive framework.

Proof. The necessity of the conditions stated in Theorem 1 can be derived from Lemma 5. More precisely, when $k = 3$, Eq (2.3) in the condition (i) is consistent with the condition (3.1) in Theorem 1; the inequality (2.4) in the condition (ii) is consistent with the condition (3.2) in Theorem 1; and the inequality (2.5) in the condition (iii) is consistent with the condition (3.3) in Theorem 1. When $I = \{i, j\}$, where $i, j \in \{1, 2, 3\}$, $i < j$, by the inequality (2.6) in condition (iv), we have

$$\frac{\alpha_{ij}}{n} < |I| - 1 = 1,$$

which is (3.4). When $I_i = \{i\}$, $i = 1, 2, 3$, according to Remark 2 and the inequality (2.7) in the condition (v), we have

$$\frac{1}{p_i} + \frac{\alpha_{ii}}{n} < |I_i| = 1,$$

which is (3.5). When $I^i = \{1, 2, 3\} \setminus \{i\}$, $i = 1, 2, 3$, by the inequality (2.7) in the condition (v), we have

$$\sum_{k,l \in I^i, k < l} \frac{\alpha_{kl}}{n} + \sum_{j \in I^i} \left(\frac{1}{p_j} + \frac{\alpha_{jj}}{n} \right) < |I^i| = 2. \quad (3.8)$$

Comparing the inequality (3.8) with Eq (2.3) in the condition (i) when $k = 3$, we obtain

$$\frac{1}{p_i} + \frac{\alpha_{ii}}{n} + \sum_{j \neq i} \frac{\alpha_{ij}}{n} > 1,$$

which is (3.6). When $I = \{1, 2, 3\}$, by the inequality (2.6) in the condition (iv), we have

$$\frac{\alpha_{12} + \alpha_{13} + \alpha_{23}}{n} < |I| - 1 = 2,$$

which is (3.7).

Next, we shall prove the sufficiency of these conditions. Without loss of generality, we assume that $f_i \geq 0$ for $i = 1, 2, 3$. Based on (3.5) and (3.6), we obtain

$$\sum_{j \neq i} \alpha_{ij} > 0, i = 1, 2, 3.$$

From the inequality stated above, it can be deduced that at most one of α_{12} , α_{13} , and α_{23} is less than or equal to 0. Consequently, we will consider the following two cases.

Case I: $\min\{\alpha_{12}, \alpha_{13}, \alpha_{23}\} > 0$. We first define the following sets:

$$\begin{aligned} U_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : |x_2 - x_3| \geq \max\{|x_1 - x_2|, |x_1 - x_3|\}\}, \\ U_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : |x_1 - x_3| \geq \max\{|x_1 - x_2|, |x_2 - x_3|\}\}, \\ U_3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : |x_1 - x_2| \geq \max\{|x_1 - x_3|, |x_2 - x_3|\}\}. \end{aligned}$$

It then follows that

$$\mathbb{R}^{3n} \subseteq U_1 \cup U_2 \cup U_3.$$

As a consequence, we deduce the following inequality:

$$\int_{\mathbb{R}^{3n}} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3 \leq \sum_{i=1}^3 \int_{U_i} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3.$$

Since the regions U_1 , U_2 , and U_3 share similar properties, for the sake of simplicity, we will only estimate the integral over U_3 . A direct calculation reveals that

$$\begin{aligned} & \int_{U_3} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3 \\ &= \int_{U_3} |x_1 - x_2|^{-\beta_1} |x_1 - x_3|^{-\alpha_{13}} |x_1 - x_2|^{-\beta_2} |x_2 - x_3|^{-\alpha_{23}} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\ &\leq \int_{U_3} |x_1 - x_3|^{-\alpha_{13} - \beta_1} |x_2 - x_3|^{-\alpha_{23} - \beta_2} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\ &\leq \int_{\mathbb{R}^{3n}} \frac{\prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}}}{|x_1 - x_3|^{\alpha_{13} + \beta_1} |x_2 - x_3|^{\alpha_{23} + \beta_2}} dx_1 dx_2 dx_3 =: \Gamma_{\beta_1, \beta_2}^3(f_1, f_2, f_3), \end{aligned} \quad (3.9)$$

where β_1 and β_2 satisfy $\beta_1 + \beta_2 = \alpha_{12}$ and $\beta_1, \beta_2 \geq 0$.

When the conditions

$$\sum_{i=1}^3 \frac{1}{p_i} > 1, \quad (3.10)$$

$$\sum_{i=1}^3 \frac{1}{p_i} + \frac{\alpha_{ii}}{n} + \frac{\alpha_{13} + \beta_1}{n} + \frac{\alpha_{23} + \beta_2}{n} = 3, \quad (3.11)$$

$$\alpha_{ii} < \frac{n}{p_i'} \quad (i = 1, 2), \quad (3.12)$$

$$\frac{n}{p_i'} - \alpha_{ii} < \alpha_{i3} + \beta_i < n \quad (i = 1, 2), \quad (3.13)$$

and

$$\alpha_{33} > \sum_{i=1}^2 \max \left\{ \frac{n}{p_i'} - \alpha_{ii} - \alpha_{i3} - \beta_i, -\alpha_{ii} \right\}$$

hold, or when the conditions (3.10)–(3.13) and

$$\alpha_{33} = \sum_{i=1}^2 \max \left\{ \frac{n}{p_i'} - \alpha_{ii} - \alpha_{i3} - \beta_i, -\alpha_{ii} \right\}$$

with $\alpha_{i3} + \beta_i \neq n/p_i'$ for $i = 1, 2$ hold, it follows from Remark 3 on Lemma 6 that the inequality

$$\Gamma_{\beta_1, \beta_2}^3(f_1, f_2, f_3) \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3} \quad (3.14)$$

holds for $1 < p_1, p_2, p_3 < \infty$.

We first prove that β_1 and β_2 satisfying (3.13) must exist. In fact, when (3.6) and $\alpha_{i3} < n$ hold with $i = 1, 2$, we can choose β_i so that

$$\frac{n}{p_i'} - \alpha_{ii} < \alpha_{i3} + \beta_i \leq \sum_{j \neq i} \alpha_{ij}$$

and

$$\alpha_{i3} \leq \alpha_{i3} + \beta_i < n$$

hold, respectively, for $i = 1, 2$. Further, we note that (3.13) is equivalent to

$$\frac{n}{p_i'} - \alpha_{ii} - \alpha_{i3} < \beta_i < n - \alpha_{i3} \quad (i = 1, 2). \quad (3.15)$$

Since

$$\frac{n}{p_1'} + \frac{n}{p_2'} - \alpha_{11} - \alpha_{22} - \alpha_{13} - \alpha_{23} < \alpha_{12} < 2n - \alpha_{13} - \alpha_{23}$$

is equivalent to $\alpha_{33} < n/p_3'$ and $\alpha_{12} + \alpha_{13} + \alpha_{23} < 2n$. Therefore, by Lemma 2, it is clear that such β_1 and β_2 exist. Then, we can choose β_i satisfying (3.13) and $\alpha_{i3} + \beta_i \neq n/p_i'$ for $i = 1, 2$ so that the inequality (3.14) holds for $1 < p_1, p_2, p_3 < \infty$.

The estimations for the remaining two regions follow a similar pattern. Therefore, we have completed the proof for this case.

Case II: One of α_{12} , α_{13} , and α_{23} is non-positive. Without loss of generality, we assume $\alpha_{12} \leq 0$. Given that the case $\alpha_{12} = 0$ has already been covered in Remark 3 on Lemma 6, we further assume $\alpha_{12} < 0$. We then obtain the following inequality:

$$|x_1 - x_2|^{-\alpha_{12}} \leq (|x_1 - x_3| + |x_2 - x_3|)^{-\alpha_{12}} \lesssim |x_1 - x_3|^{-\alpha_{12}} + |x_2 - x_3|^{-\alpha_{12}}.$$

Consequently, we have

$$\begin{aligned} |\mathcal{F}(f_1, f_2, f_3)| &\lesssim \iiint \frac{f_1(x_1)f_2(x_2)f_3(x_3)}{|x_1|^{\alpha_{11}}|x_2|^{\alpha_{22}}|x_3|^{\alpha_{33}}|x_1-x_3|^{\alpha_{13}+\alpha_{12}}|x_2-x_3|^{\alpha_{23}}} dx_1 dx_2 dx_3 \\ &+ \iiint \frac{f_1(x_1)f_2(x_2)f_3(x_3)}{|x_1|^{\alpha_{11}}|x_2|^{\alpha_{22}}|x_3|^{\alpha_{33}}|x_1-x_3|^{\alpha_{13}}|x_2-x_3|^{\alpha_{12}+\alpha_{23}}} dx_1 dx_2 dx_3 \\ &=: \Gamma_{\alpha_{12}, 0}^3(f_1, f_2, f_3) + \Gamma_{0, \alpha_{12}}^3(f_1, f_2, f_3). \end{aligned}$$

Since each term on the right-hand side of the above inequality can be estimated using the same approach as that for the functional $\Gamma_{\beta_1, \beta_2}^3$ presented in (3.9) to achieve the desired result, we hereby complete the proof of Theorem 1.

Theorem 2. Let $f_i \in L^{p_i}(\mathbb{R}^n)$ for $p_1 = 1$ and $1 < p_2, p_3 \leq \infty$. If the conditions

$$\frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{12} + \alpha_{13} + \alpha_{23}}{n} + \frac{\alpha_{11} + \alpha_{22} + \alpha_{33}}{n} = 2, \quad (3.16)$$

$$\alpha_{11} + \alpha_{22} + \alpha_{33} > 0, \quad (3.17)$$

$$\alpha_{11} < 0 < \alpha_{11} + \alpha_{12} + \alpha_{13},$$

$$\alpha_{ii} < \frac{n}{p_i'} \quad (i = 2, 3),$$

$$\alpha_{23} < \frac{n}{p_2'} - \alpha_{22} < \alpha_{12} < \frac{n}{p_2'} < \alpha_{12} + \alpha_{22} + \alpha_{23},$$

and

$$\alpha_{23} < \frac{n}{p_3'} - \alpha_{33} < \alpha_{13} < \frac{n}{p_3'} < \alpha_{13} + \alpha_{23} + \alpha_{33}$$

hold simultaneously, then the inequality

$$|\mathcal{F}(f_1, f_2, f_3)| \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}$$

holds.

Proof. Without loss of generality, we assume $f_i \geq 0$ for $i = 1, 2, 3$. In order to prove

$$|\mathcal{F}(f_1, f_2, f_3)| \leq E \|f_1\|_1 \|f_2\|_{p_2} \|f_3\|_{p_3},$$

it is sufficient to prove

$$\left| \int_{\mathbb{R}^{2n}} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3 \right| \leq E \|f_2\|_{p_2} \|f_3\|_{p_3}. \quad (3.18)$$

Case I: $\alpha_{23} > 0$. We define the following subsets of \mathbb{R}^{2n} :

$$\begin{aligned}
 U_1 &= \left\{ (x_2, x_3) \in \mathbb{R}^{2n} : \frac{|x_3|}{2} \leq |x_2| \leq 2|x_3| \right\}, \\
 U_2 &= \left\{ (x_2, x_3) \in \mathbb{R}^{2n} : |x_2| < \frac{|x_3|}{2} \right\}, \\
 U_3 &= \left\{ (x_2, x_3) \in \mathbb{R}^{2n} : |x_3| < \frac{|x_2|}{2} \right\}.
 \end{aligned}$$

It then follows that $\mathbb{R}^{2n} \subseteq U_1 \cup U_2 \cup U_3$. To establish the inequality (3.18), we are required to estimate the following integral inequalities:

$$\int_{U_i} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3 \leq E \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}$$

for $i = 1, 2, 3$, respectively.

On the region U_1 , given that $|x_3|/2 \leq |x_2| \leq 2|x_3|$, we can deduce that $|x_2 - x_3| \leq 3|x_2|$. Let $\alpha_{33} = u_1 + u_2$, where $\alpha_{22} + u_1 \geq 0$. Then, we have

$$|x_2 - x_3|^{\alpha_{22} + u_1} \lesssim |x_2|^{\alpha_{22} + u_1} \lesssim |x_2|^{\alpha_{22}} |x_3|^{u_1}.$$

Consequently, we get the following chain of inequalities:

$$\begin{aligned}
 & \int_{U_1} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3 \\
 &= \int_{U_1} |x_1|^{-\alpha_{11}} G(x_1, x_2, x_3) \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3 \\
 &\lesssim |x_1|^{-\alpha_{11}} \int_{U_1} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} |x_2 - x_3|^{-\alpha_{23} - \alpha_{22} - u_1} |x_3|^{-u_2} \prod_{i=2}^3 f_i(x_i) dx_2 dx_3 \\
 &\leq |x_1|^{-\alpha_{11}} \int_{\mathbb{R}^{2n}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} |x_2 - x_3|^{-\alpha_{23} - \alpha_{22} - u_1} |x_3|^{-u_2} \prod_{i=2}^3 f_i(x_i) dx_2 dx_3 \\
 &= |x_1|^{-\alpha_{11}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_2(x_2)}{|x_1 - x_2|^{\alpha_{12}} |x_3 - x_2|^{\alpha_{23} + \alpha_{22} + u_1}} dx_2 |x_1 - x_3|^{-\alpha_{13}} |x_3|^{-u_2} f_3(x_3) dx_3. \tag{3.19}
 \end{aligned}$$

Drawing on Remark 1, we can deduce the following inequality:

$$\int_{\mathbb{R}^n} \frac{f_2(x_2)}{|x_1 - x_2|^{\alpha_{12}} |x_3 - x_2|^{\alpha_{23} + \alpha_{22} + u_1}} dx_2 \leq C |x_1 - x_3|^{n/p_2' - \alpha_{12} - \alpha_{23} - \alpha_{22} - u_1} \|f_2\|_{p_2},$$

where the indices must satisfy the conditions: $\alpha_{12} < n/p_2'$, $\alpha_{23} + \alpha_{22} + u_1 < n/p_2'$, and $\alpha_{12} + \alpha_{23} + \alpha_{22} + u_1 > n/p_2'$. Subsequently, substituting the above-derived formula back into (3.19) and once again applying Remark 1, we obtain that

$$\begin{aligned}
& \int_{U_1} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3 \\
& \leq C |x_1|^{-\alpha_{11}} \|f_2\|_{p_2} \int_{\mathbb{R}^n} |x_1 - x_3|^{n/p_2' - \alpha_{12} - \alpha_{23} - \alpha_{22} - u_1 - \alpha_{13}} |x_3|^{-u_2} f_3(x_3) dx_3 \\
& \leq C |x_1|^{-\alpha_{11}} \|f_2\|_{p_2} |x_1|^{n/p_3' + n/p_2' - \alpha_{12} - \alpha_{23} - \alpha_{22} - u_1 - \alpha_{13} - u_2} \|f_3\|_{p_3} \\
& = C \|f_2\|_{p_2} \|f_3\|_{p_3}.
\end{aligned}$$

It is crucial to note that the indices in this step must satisfy the following conditions: First, $u_2 < n/p_3'$. Second, $\alpha_{12} + \alpha_{23} + \alpha_{22} + u_1 + \alpha_{13} - n/p_2' < n/p_3'$. Based on Eq (3.16), this condition is equivalent to $\alpha_{11} + u_2 > 0$. Third, $\alpha_{12} + \alpha_{23} + \alpha_{22} + u_1 + \alpha_{13} - n/p_2' + u_2 > n/p_3'$. According to Eq (3.16), this condition is equivalent to $\alpha_{11} < 0$.

We remain tasked with proving the existence of u_1 and u_2 that fulfill the subsequent conditions: $\alpha_{23} + \alpha_{22} + u_1 < n/p_2'$, $\alpha_{12} + \alpha_{23} + \alpha_{22} + u_1 > n/p_2'$, $u_2 < n/p_3'$, $\alpha_{11} + u_2 > 0$. This problem is equivalent to demonstrating the existence of u_1 and u_2 , where u_1 adheres to the condition

$$\frac{n}{p_2'} - \alpha_{12} - \alpha_{23} - \alpha_{22} < u_1 < \frac{n}{p_2'} - \alpha_{23} - \alpha_{22},$$

and u_2 satisfies the condition $-\alpha_{11} < u_2 < n/p_3'$. In reality, given that $u_1 + u_2 = \alpha_{33}$, according to Lemma 2, our focus can be narrowed down to showing that α_{33} meets the following double-sided inequality:

$$\frac{n}{p_2'} - \alpha_{12} - \alpha_{23} - \alpha_{22} - \alpha_{11} < \alpha_{33} < \frac{n}{p_2'} + \frac{n}{p_3'} - \alpha_{23} - \alpha_{22}.$$

Upon recalling Eq (3.16), it becomes readily apparent that the first inequality in the above double-sided inequality is equivalent to the condition $\alpha_{13} < n/p_3'$, whereas the second inequality is equivalent to $0 < \alpha_{11} + \alpha_{12} + \alpha_{13}$.

Moreover, considering that u_1 and u_2 must satisfy $\alpha_{22} + u_1 \geq 0$ and $\alpha_{11} + u_2 > 0$ concurrently, it follows that the inequality (3.17) holds.

Next, we proceed to estimate the integral over the region U_2 . On U_2 , given that $|x_2| < |x_3|/2$, we can infer that $|x_2 - x_3| \geq |x_3| - |x_2| > |x_3|/2$. Since $\alpha_{23} > 0$, it follows that $|x_2 - x_3|^{-\alpha_{23}} \lesssim |x_3|^{-\alpha_{23}}$. Let $\alpha_{11} = \eta_{12} + \eta_{13}$, where $\eta_{12} = n/p_2' - \alpha_{22} - \alpha_{12}$ and $\eta_{13} = n/p_3' - \alpha_{23} - \alpha_{33} - \alpha_{13}$. Then, we obtain

$$\begin{aligned}
& \int_{U_2} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3 \\
& = \int_{U_2} |x_1|^{-\eta_{12} - \eta_{13}} G(x_1, x_2, x_3) \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3 \\
& \lesssim \int_{U_2} |x_1|^{-\eta_{12} - \eta_{13}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} |x_2|^{-\alpha_{22}} |x_3|^{-\alpha_{23} - \alpha_{33}} \prod_{i=2}^3 f_i(x_i) dx_2 dx_3 \\
& \leq \int_{\mathbb{R}^{2n}} |x_1|^{-\eta_{12} - \eta_{13}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} |x_2|^{-\alpha_{22}} |x_3|^{-\alpha_{23} - \alpha_{33}} \prod_{i=2}^3 f_i(x_i) dx_2 dx_3
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \frac{f_2(x_2)}{|x_1|^{\eta_{12}}|x_1 - x_2|^{\alpha_{12}}|x_2|^{\alpha_{22}}} dx_2 \times \int_{\mathbb{R}^n} \frac{f_3(x_3)}{|x_1|^{\eta_{13}}|x_1 - x_3|^{\alpha_{13}}|x_3|^{\alpha_{23} + \alpha_{33}}} dx_3 \\
&= S_{\alpha_{22}, \alpha_{12}, \eta_{12}} f_2(x_1) \times S_{\alpha_{23} + \alpha_{33}, \alpha_{13}, \eta_{13}} f_3(x_1).
\end{aligned}$$

Given that the following conditions hold:

$$\begin{aligned}
\alpha_{22} < \frac{n}{p_2'}, & \quad \alpha_{12} < \frac{n}{p_2'}, & \quad \alpha_{12} + \alpha_{22} > \frac{n}{p_2'}, \\
\alpha_{23} + \alpha_{33} < \frac{n}{p_3'}, & \quad \alpha_{13} < \frac{n}{p_3'}, & \quad \alpha_{13} + \alpha_{23} + \alpha_{33} > \frac{n}{p_3'},
\end{aligned}$$

by virtue of Lemma 4, we have

$$S_{\alpha_{22}, \alpha_{12}, \eta_{12}} f_2(x_1) \times S_{\alpha_{23} + \alpha_{33}, \alpha_{13}, \eta_{13}} f_3(x_1) \leq E \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}.$$

Consequently, we achieve the desired estimation.

One can estimate the integral $\int_{U_3} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3$ by the same method, so we omit the details.

Case II: $\alpha_{23} \leq 0$. Given that $\alpha_{23} \leq 0$, we obtain the following inequality:

$$|x_2 - x_3|^{-\alpha_{23}} \leq (|x_2 - x_1| + |x_1 - x_3|)^{-\alpha_{23}} \lesssim |x_2 - x_1|^{-\alpha_{23}} + |x_1 - x_3|^{-\alpha_{23}}. \quad (3.20)$$

Based on the inequality (3.20), we can derive the following:

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_2 dx_3 \\
&\lesssim \int_{\mathbb{R}^{2n}} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-(\alpha_{12} + \alpha_{23})} |x_1 - x_3|^{-\alpha_{13}} \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3 \\
&+ \int_{\mathbb{R}^{2n}} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-(\alpha_{13} + \alpha_{23})} \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3.
\end{aligned} \quad (3.21)$$

For the first term on the right-hand side of the inequality (3.21), let $\alpha_{11} = \sigma_{12} + \sigma_{13}$, where $\sigma_{12} = \frac{n}{p_2'} - \alpha_{22} - \alpha_{12} - \alpha_{23}$ and $\sigma_{13} = \frac{n}{p_3'} - \alpha_{33} - \alpha_{13}$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-(\alpha_{12} + \alpha_{23})} |x_1 - x_3|^{-\alpha_{13}} \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3 \\
&= \int_{\mathbb{R}^n} \frac{f_2(x_2)}{|x_1|^{\sigma_{12}} |x_1 - x_2|^{\alpha_{12} + \alpha_{23}} |x_2|^{\alpha_{22}}} dx_2 \times \int_{\mathbb{R}^n} \frac{f_3(x_3)}{|x_1|^{\sigma_{13}} |x_1 - x_3|^{\alpha_{13}} |x_3|^{\alpha_{33}}} dx_3 \\
&= S_{\alpha_{22}, \alpha_{12} + \alpha_{23}, \sigma_{12}} f_2(x_1) \times S_{\alpha_{33}, \alpha_{13}, \sigma_{13}} f_3(x_1) \\
&\leq E \|f_2\|_{p_2} \|f_3\|_{p_3}.
\end{aligned}$$

The validity of the last inequality above is ensured by Lemma 4 because the following conditions hold:

$$\begin{aligned}
\alpha_{22} < \frac{n}{p_2'}, & \quad \alpha_{12} + \alpha_{23} < \frac{n}{p_2'}, & \quad \alpha_{12} + \alpha_{22} + \alpha_{23} > \frac{n}{p_2'}, \\
\alpha_{33} < \frac{n}{p_3'}, & \quad \alpha_{13} < \frac{n}{p_3'}, & \quad \alpha_{13} + \alpha_{33} > \frac{n}{p_3'}.
\end{aligned}$$

The second term on the right-hand side of the inequality (3.21) can be estimated using the same approach. Thus, the proof of Theorem 2 is completed.

Theorem 3. Let $f_i \in L^{p_i}(\mathbb{R}^n)$ for $p_1 = \infty$, $1 < p_2, p_3 < \infty$ with $1/p_2 + 1/p_3 \geq 1$. If the conditions

$$\frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_{12} + \alpha_{13} + \alpha_{23}}{n} = 3,$$

$$\alpha_{11} + \alpha_{22} + \alpha_{33} > 0,$$

$$\alpha_{ij} < n, \quad i \neq j,$$

$$\alpha_{11} > 0,$$

$$0 < \frac{n}{p_i'} - \alpha_{ii} < \sum_{j \neq i} \alpha_{ij} \quad (i = 1, 2, 3), \quad (3.22)$$

and

$$\alpha_{12} + \alpha_{13} + \alpha_{23} < 2n$$

hold simultaneously, then the inequality

$$|\mathcal{F}(f_1, f_2, f_3)| \leq C \|f_1\|_{\infty} \|f_2\|_{p_2} \|f_3\|_{p_3}$$

holds.

Proof. The proof of Theorem 3 follows the same line of reasoning as that of Theorem 1. In the following proof, Lemma 7 takes the place of Lemma 6 and plays a crucial role.

Without loss of generality, we can assume that $f_i \geq 0$ for $i = 1, 2, 3$. From (3.22), we know that at most one of α_{12} , α_{13} , and α_{23} is non-positive.

Let us first consider the case where α_{12} , α_{13} , and α_{23} are all positive. We define the following sets:

$$\begin{aligned} U_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : |x_2 - x_3| \geq \max\{|x_1 - x_2|, |x_1 - x_3|\}\}, \\ U_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : |x_1 - x_3| \geq \max\{|x_1 - x_2|, |x_2 - x_3|\}\}, \\ U_3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : |x_1 - x_2| \geq \max\{|x_1 - x_3|, |x_2 - x_3|\}\}. \end{aligned}$$

It then follows that $\mathbb{R}^{3n} \subseteq U_1 \cup U_2 \cup U_3$. Consequently, we have

$$\begin{aligned} |\mathcal{F}(f_1, f_2, f_3)| &\leq \int_{\mathbb{R}^{3n}} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3 \\ &\leq \sum_{i=1}^3 \int_{U_i} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3. \end{aligned}$$

To complete the proof, we need to estimate the following inequality separately for $i = 1, 2, 3$:

$$\int_{U_i} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3 \leq E \|f_1\|_{\infty} \|f_2\|_{p_2} \|f_3\|_{p_3}. \quad (3.23)$$

We begin by estimating the integral over the region U_1 . Given that $\alpha_{23} > 0$, we let $\alpha_{23} = \theta_1 + \theta_2$, where $\theta_1, \theta_2 \geq 0$. Then, we obtain the following chain of inequalities:

$$\begin{aligned}
 & \int_{U_1} H(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) dx_1 dx_2 dx_3 \\
 &= \int_{U_1} |x_1 - x_2|^{-\alpha_{12}} |x_2 - x_3|^{-\theta_1 - \theta_2} |x_1 - x_3|^{-\alpha_{13}} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\
 &\leq \int_{U_1} |x_1 - x_2|^{-\alpha_{12} - \theta_1} |x_1 - x_3|^{-\alpha_{13} - \theta_2} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\
 &\leq \int_{\mathbb{R}^{3n}} \frac{\prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}}}{|x_1 - x_2|^{\alpha_{12} + \theta_1} |x_1 - x_3|^{\alpha_{13} + \theta_2}} dx_1 dx_2 dx_3 =: \mathcal{F}_{\theta_1, \theta_2}^1(f_1, f_2, f_3). \tag{3.24}
 \end{aligned}$$

When the following conditions hold simultaneously:

$$\begin{aligned}
 \frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{12} + \theta_1}{n} + \frac{\alpha_{13} + \theta_2}{n} + \sum_{i=1}^3 \frac{\alpha_{ii}}{n} &= 3, \\
 \frac{1}{p_2} + \frac{1}{p_3} &\geq 1, \quad \sum_{i=1}^3 \alpha_{ii} \geq 0, \\
 0 < \frac{n}{p_2'} - \alpha_{22} < \alpha_{12} + \theta_1 < n, \\
 0 < \frac{n}{p_3'} - \alpha_{33} < \alpha_{13} + \theta_2 < n,
 \end{aligned}$$

by virtue of Lemma 7, we have

$$|\mathcal{F}_{\theta_1, \theta_2}^1(f_1, f_2, f_3)| \leq E \|f_1\|_{\infty} \|f_2\|_{p_2} \|f_3\|_{p_3}$$

for $1 < p_2, p_3 < \infty$. This yields the desired estimate.

Note that when $p_1 = \infty$, the inequality

$$\int_{U_i} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_1 dx_2 dx_3 \leq E \|f_2\|_{p_2} \|f_3\|_{p_3} \tag{3.25}$$

implies the inequality (3.23). Consequently, it suffices to prove inequality (3.25) for $i = 2, 3$.

We now turn our attention to the integral over the region U_2 . Given that $\alpha_{13} > 0$, we can select non-negative real numbers γ_1 and γ_2 such that $\alpha_{13} = \gamma_1 + \gamma_2$. Then, we have

$$\begin{aligned}
& \int_{U_2} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_1 dx_2 dx_3 \\
& \leq \int_{\mathbb{R}^{3n}} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-(\alpha_{12} + \gamma_1)} |x_2 - x_3|^{-(\alpha_{23} + \gamma_2)} \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\
& = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-(\alpha_{12} + \gamma_1)} dx_1 |x_2 - x_3|^{-(\alpha_{23} + \gamma_2)} \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3.
\end{aligned}$$

Given that the following conditions hold:

$$\alpha_{11} < n, \quad \alpha_{12} + \gamma_1 < n, \quad \alpha_{11} + \alpha_{12} + \gamma_1 > n$$

along with

$$\begin{aligned}
\frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{11} + \alpha_{12} + \gamma_1 + \alpha_{22} - n + \alpha_{23} + \gamma_2 + \alpha_{33}}{n} &= 2, \quad \frac{1}{p_2} + \frac{1}{p_3} \geq 1, \\
\alpha_{23} + \gamma_2 < n, \quad \alpha_{33} < \frac{n}{p_3'} < \alpha_{23} + \gamma_2 + \alpha_{33}, \quad \alpha_{11} + \alpha_{12} + \gamma_1 + \alpha_{22} - n + \alpha_{33} &\geq 0.
\end{aligned}$$

By applying Lemmas 1 (1) and 3, we can infer the following chain of equalities and inequality:

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-(\alpha_{12} + \gamma_1)} dx_1 \right) |x_2 - x_3|^{-(\alpha_{23} + \gamma_2)} \prod_{i=2}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_2 dx_3 \\
& = C \int_{\mathbb{R}^{2n}} |x_2|^{n - \alpha_{11} - \alpha_{12} - \gamma_1 - \alpha_{22}} |x_2 - x_3|^{-(\alpha_{23} + \gamma_2)} |x_3|^{-\alpha_{33}} \prod_{i=2}^3 f_i(x_i) dx_2 dx_3 \\
& = C \int_{\mathbb{R}^{2n}} \frac{f_2(x_2) f_3(x_3)}{|x_2|^{\alpha_{11} + \alpha_{12} + \gamma_1 + \alpha_{22} - n} |x_2 - x_3|^{\alpha_{23} + \gamma_2} |x_3|^{\alpha_{33}}} dx_2 dx_3 \\
& \leq C \|f_2\|_{p_2} \|f_3\|_{p_3}.
\end{aligned}$$

One can establish the inequality

$$\int_{U_3} H(x_1, x_2, x_3) f_2(x_2) f_3(x_3) dx_1 dx_2 dx_3 \leq E \|f_2\|_{p_2} \|f_3\|_{p_3}$$

in the same way, so we omit the details.

Next, we examine the case where one of α_{12} , α_{13} , and α_{23} is non-positive. Without loss of generality, we assume $\alpha_{23} \leq 0$. Given that the case $\alpha_{23} = 0$ has already been addressed in Lemma 7, we further assume $\alpha_{23} < 0$. In this scenario, we have the following inequality:

$$|x_2 - x_3|^{-\alpha_{23}} \leq (|x_2 - x_1| + |x_1 - x_3|)^{-\alpha_{23}} \lesssim |x_2 - x_1|^{-\alpha_{23}} + |x_1 - x_3|^{-\alpha_{23}}.$$

From this, we can deduce that

$$\begin{aligned}
|\mathcal{F}(f_1, f_2, f_3)| &\lesssim \int_{\mathbb{R}^{3n}} |x_1 - x_2|^{-(\alpha_{12} + \alpha_{23})} |x_1 - x_3|^{-\alpha_{13}} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\
&+ \int_{\mathbb{R}^{3n}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-(\alpha_{13} + \alpha_{23})} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\
&=: \mathcal{F}_{\alpha_{23}, 0}^1(f_1, f_2, f_3) + \mathcal{F}_{0, \alpha_{23}}^1(f_1, f_2, f_3).
\end{aligned}$$

Since each term on the right-hand side of the above inequality can be estimated in the same manner as the functional $\mathcal{F}_{\theta_1, \theta_2}^1$ presented in (3.24) to obtain the desired result, we hereby conclude the proof of Theorem 3.

Theorem 4. Let $f_i \in L^{p_i}(\mathbb{R}^n)$ for $p_1 = \infty$, $p_2 = p_3 = 1$. If the conditions

$$\frac{\alpha_{11} + \alpha_{22} + \alpha_{33}}{n} + \frac{\alpha_{12} + \alpha_{13} + \alpha_{23}}{n} = 1,$$

$$\alpha_{11}, \alpha_{12}, \alpha_{13} < n,$$

$$\alpha_{11} + \alpha_{12} + \alpha_{13} = 2n,$$

$$\alpha_{22} = \alpha_{13} - n,$$

and

$$\alpha_{33} = \alpha_{12} - n$$

hold, there exists a constant $C = \phi(\alpha_{11}, \alpha_{12}, \alpha_{13}, n)$ defined in (2.1), such that

$$|\mathcal{F}(f_1, f_2, f_3)| \leq \phi(\alpha_{11}, \alpha_{12}, \alpha_{13}, n) \|f_1\|_{\infty} \|f_2\|_1 \|f_3\|_1,$$

furthermore, the constant $\phi(\alpha_{11}, \alpha_{12}, \alpha_{13}, n)$ is sharp.

We have derived two results corresponding to the endpoints C' and B of the heptahedron $O'C'CBAA'B'$ that are similar to Theorem 4. However, for the sake of brevity, we will not elaborate on them in this paper.

Proof. Let $tr(A_3) = \alpha_{11} + \alpha_{22} + \alpha_{33}$ and $s_{i < j}(A_3) = \alpha_{12} + \alpha_{13} + \alpha_{23}$, and denote $\phi(\alpha_{11}, \alpha_{12}, \alpha_{13}, n)$ as $\phi(\cdot)$. Given the conditions $\alpha_{11}, \alpha_{12}, \alpha_{13} < n$ and $\alpha_{11} + \alpha_{12} + \alpha_{13} = 2n$, according to Lemma 1 (2), we obtain

$$\int_{\mathbb{R}^n} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} dx_1 = \phi(\cdot) |x_2|^{\alpha_{13} - n} |x_2 - x_3|^{\alpha_{11} - n} |x_3|^{\alpha_{12} - n}.$$

Since $\alpha_{22} = \alpha_{13} - n$, $\alpha_{33} = \alpha_{12} - n$, and $\alpha_{11} + \alpha_{12} + \alpha_{13} = 2n$, we can easily derive that $tr(A_3) = 0$. Combining $tr(A_3) = 0$ with $tr(A_3) + s_{i < j}(A_3) = n$, we get $s_{i < j}(A_3) = n$, and then $\alpha_{23} = \alpha_{11} - n$.

Substituting $\alpha_{22} = \alpha_{13} - n$, $\alpha_{33} = \alpha_{12} - n$, and $\alpha_{23} = \alpha_{11} - n$ into the integral, we obtain

$$\begin{aligned}
& |\mathcal{F}(f_1, f_2, f_3)| \\
&= \left| \int_{\mathbb{R}^{3n}} G(x_1, x_2, x_3) \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \right| \\
&\leq \|f_1\|_\infty \int_{\mathbb{R}^{3n}} G(x_1, x_2, x_3) |f_2(x_2)| |f_3(x_3)| \prod_{i=1}^3 |x_i|^{-\alpha_{ii}} dx_1 dx_2 dx_3 \\
&= \|f_1\|_\infty \int_{\mathbb{R}^{2n}} \frac{|f_2(x_2)| |f_3(x_3)|}{|x_2|^{\alpha_{22}} |x_2 - x_3|^{\alpha_{23}} |x_3|^{\alpha_{33}}} \int_{\mathbb{R}^n} |x_1|^{-\alpha_{11}} |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} dx_1 dx_2 dx_3 \\
&= \phi(\cdot) \|f_1\|_\infty \int_{\mathbb{R}^{2n}} \frac{|f_2(x_2)| |f_3(x_3)| |x_2|^{\alpha_{13}-n} |x_2 - x_3|^{\alpha_{11}-n} |x_3|^{\alpha_{12}-n}}{|x_2|^{\alpha_{22}} |x_2 - x_3|^{\alpha_{23}} |x_3|^{\alpha_{33}}} dx_2 dx_3 \\
&= \phi(\cdot) \|f_1\|_\infty \int_{\mathbb{R}^{2n}} |f_2(x_2)| |f_3(x_3)| dx_2 dx_3 \\
&= \phi(\cdot) \|f_1\|_\infty \|f_2\|_1 \|f_3\|_1.
\end{aligned}$$

For sharpness, take $f_1 \equiv 1$ and $f_2, f_3 \geq 0$. Then the equality in the above-derived inequality holds, which proves that $\phi(\alpha_{11}, \alpha_{12}, \alpha_{13}, n)$ is optimal.

Theorem 5. Let $f_i \in L^1(\mathbb{R}^n)$ for $i = 1, 2, 3$. The operator T defined by

$$T(f_1, f_2, f_3)(x_1, x_2, x_3) := \prod_{1 \leq i < j \leq 3} |x_i - x_j|^{-\alpha_{ij}} \prod_{i=1}^3 f_i(x_i) |x_i|^{-\alpha_{ii}}$$

is bounded from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ if and only if $\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{12} = \alpha_{13} = \alpha_{23} = 0$. Moreover, the norm of operator T is 1, that is,

$$\|T\|_{L^1 \times L^1 \times L^1 \rightarrow L^1} = 1.$$

Proof. The sufficiency of the conditions in Theorem 5 is evident; therefore, we only need to establish the necessity of these conditions.

The proof of necessity hinges on condition (v) of Lemma 5. When $I_i = \{i\}$ for $i = 1, 2, 3$, invoking Remark 2 and condition (v), we arrive at

$$\frac{1}{p_i} + \frac{\alpha_{ii}}{n} \leq |I_i| = 1. \quad (3.26)$$

Given that $p_i = 1$, from (3.26), we infer that $\alpha_{ii} \leq 0$ for $i = 1, 2, 3$. Considering $\sum_{i=1}^3 \alpha_{ii} \geq 0$, we conclude that $\sum_{i=1}^3 \alpha_{ii} = 0$. Consequently, we have $\alpha_{ii} = 0$ for $i = 1, 2, 3$.

When $I^i = \{1, 2, 3\} \setminus \{i\}$ with $i = 1, 2, 3$, condition (v) implies

$$\sum_{k, l \in I^i, k < l} \frac{\alpha_{kl}}{n} + \sum_{j \in I^i} \left(\frac{1}{p_j} + \frac{\alpha_{jj}}{n} \right) \leq |I^i| = 2. \quad (3.27)$$

From (3.27), we deduce that $\alpha_{ij} \leq 0$, where $i, j \in \{1, 2, 3\}$ and $i < j$. Since

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{\alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_{12} + \alpha_{13} + \alpha_{23}}{n} = 3,$$

we obtain

$$\alpha_{12} + \alpha_{13} + \alpha_{23} = 0. \quad (3.28)$$

Combining (3.28) with $\alpha_{ij} \leq 0$, we find that $\alpha_{ij} = 0$, where $i, j \in \{1, 2, 3\}$ and $i < j$.

Since it has been shown that $\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{12} = \alpha_{13} = \alpha_{23} = 0$, clearly, we have

$$\|T\|_{L^1 \times L^1 \times L^1 \rightarrow L^1} = 1,$$

which concludes the proof of Theorem 5.

4. Discussion

The complete characterization of the trilinear Stein-Weiss inequality established in this work has several important implications. Theoretically, it resolves a fundamental boundedness problem in multilinear harmonic analysis and provides a precise benchmark for future studies on higher-order weighted multilinear operators. In terms of applicability, such inequalities are pivotal tools in the analysis of partial differential equations with multilinear nonlinearities, where they can lead to sharp existence and regularity results. They also find relevance in mathematical physics, particularly in the rigorous study of models involving three-body interactions. Furthermore, the necessary and sufficient conditions we derive elucidate the delicate interplay between the singular weights and function spaces, offering deeper insight for problems in geometric analysis involving singular measures. The techniques of weighted norm inequalities and singular integrals, as exemplified in this work, also serve as foundational tools in other advanced areas of analysis, such as the study of operator semigroups on function spaces [15] and the qualitative theory of fractional differential equations [16].

5. Conclusions

This paper systematically establishes the trilinear Stein-Weiss inequality for $1 \leq p_1, p_2, p_3 \leq \infty$. In particular, when $1 < p_1, p_2, p_3 < \infty$, it provides the necessary and sufficient conditions characterizing this inequality. For future research, we plan to extend the results to the general k -linear ($k \geq 4$) case and to investigate problems such as the attainability of the optimal constant in the trilinear Stein-Weiss inequality.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. E. Stein, G. Weiss, Fractional integrals on n -dimensional Euclidean space, *J. Math. Mech.*, **7** (1958), 503–514. <https://doi.org/10.1512/iumj.1958.7.57030>
2. Z. Shi, D. Wu, D. Yan, Necessary and sufficient conditions of doubly weighted Hardy-Littlewood-Sobolev inequality, *Anal. Theory Appl.*, **30** (2014), 193–204. <https://doi.org/10.4208/ata.2014.v30.n2.5>
3. B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. Am. Math. Soc.*, **192** (1974), 261–274. <https://doi.org/10.1090/S0002-9947-1974-0340523-6>
4. E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. Math.*, **118** (1983), 349–374. <https://doi.org/10.2307/2007032>
5. W. Chen, C. Li, The best constant in a weighted Hardy-Littlewood-Sobolev inequality, *Proc. Am. Math. Soc.*, **136** (2008), 955–962. <https://doi.org/10.1090/S0002-9939-07-09232-5>
6. W. Beckner, Pitt's inequality with sharp convolution estimates, *Proc. Am. Math. Soc.*, **136** (2008), 1871–1885. <https://doi.org/10.1090/S0002-9939-07-09216-7>
7. P. L. De Nápoli, I. Drelichman, R. G. Durán, On weighted inequalities for fractional integrals of radial functions, *Ill. J. Math.*, **55** (2011), 575–587. <https://doi.org/10.1215/ijm/1359762403>
8. T. Chen, W. C. Sun, Geometric inequalities in harmonic analysis, *Sci. China Math.*, **48** (2018), 1–32. <https://doi.org/10.1360/N012018-00081>
9. L. Chen, G. Lu, C. Tao, Existence of extremal functions for the Stein–Weiss inequalities on the Heisenberg group, *J. Funct. Anal.*, **277** (2019), 1112–1138. <https://doi.org/10.1016/j.jfa.2019.01.002>
10. P. L. De Nápoli, T. Picon, Stein-Weiss inequality in L^1 norm for vector fields, *Proc. Am. Math. Soc.*, **151** (2023), 1663–1679. <https://doi.org/10.1090/proc/16241>
11. Z. Shi, D. Wu, D. Yan, On the multilinear fractional integral operators with correlation kernels, *J. Fourier Anal. Appl.*, **25** (2019), 538–587. <https://doi.org/10.1007/s00041-017-9591-1>
12. Y. Zhou, Y. Deng, D. Wu, D. Yan, Necessary and sufficient conditions on weighted multilinear fractional integral inequality, *Commun. Pure Appl. Anal.*, **21** (2022), 727–747. <https://doi.org/10.3934/cpaa.2021196>
13. L. Grafakos, C. Morpurgo, A Selberg integral formula and applications, *Pac. J. Math.*, **191** (1999), 85–94. <https://doi.org/10.2140/pjm.1999.191.85>
14. L. Grafakos, *Classical Fourier Analysis*, 3rd edition, Springer, New York, 2014. <https://doi.org/10.1007/978-1-4939-1194-3>
15. D. Chalishajar, M. Somasundaram, P. Sethuraman, Analyticity of weighted composition semigroups on the space of holomorphic functions, *Bull. Iran. Math. Soc.*, **51** (2025). <https://doi.org/10.1007/s41980-024-00923-7>

16. D. Chalişhajar, K. Karthikeyan, P. Karthikeyan, D. S. Raja, P. Sundararajan, Qualitative analysis of Ψ -Hilfer fractional impulsive differential equations involving almost sectorial operators with nonlocal multi-point condition, *Eur. J. Math. Appl.*, **2** (2022). <https://doi.org/10.28919/ejma.2022.2.13>



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