Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(3): 1506–1523.
DOI: 10.3934/era.2026068
Received: 03 December 2025
Revised: 27 January 2026
Accepted: 04 February 2026
Published: 13 February 2026

Research article

Well-posedness of the 3D MHD boundary layer equations in an analytic space

Xiaolei Dong*

School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China

* **Correspondence:** E-mail: xld0908@163.com.

Abstract: The objective of this paper is to investigate the local well-posedness of analytic solutions to the three-dimensional (3D) magnetohydrodynamic (MHD) boundary layer equations without structural assumptions. Specifically, the general initial data are required to be real-analytic in the tangential variables (x, y) and satisfy Sobolev regularity in the normal variable z. We first adopt a variable transformation involving $\phi(z)$ to homogenize the boundary conditions and eliminate resulting high-order terms. Additionally, we employ delicate energy estimates combined with Gauss weight functions to control linearly growth terms.

Keywords: MHD boundary layer equations; local well-posedness; analytic space

1. Introduction

Proposed by Prandtl in 1904 [1], the Prandtl equation is the core of boundary layer equations, describing viscous fluid flow near solid surfaces. For high Reynolds number flows (e.g., air/water around objects), a thin "boundary layer" forms where viscous forces dominate-fluid velocity rises from zero (viscous adhesion) at the surface to the free-stream velocity outside. Beyond this layer, viscosity is negligible, and flow approximates inviscid ideal flow.

Mathematically, progress has been made on the Prandtl equations' well-posedness. For 2D cases with initial tangential velocity satisfying Oleinik's monotonicity assumption, well-posedness in weighted Sobolev spaces is established. Oleinik [2, 3] first proved local-in-time well-posedness via Crocco transformation, with later reproofs using energy methods [4, 5] that leverage convective term cancellation. Inspired by [4, 5], the well-posedness for classical 2D Prandtl equations in Sobolev space was confirmed in [6, 7]. For favorable pressure ($\partial_x p \leq 0$), Xin et al. [8, 9] obtained global well-posedness and regularity of weak solutions. In addition, there have been some results regarding the well-posedness in analytic frameworks [10–13]. Subsequently, research on the well-posedness of Prandtl equations has been extended to the 3D Prandtl equations, respectively in the Gevrey class [14–16], in an analytic function space [17, 18], and in the Sobolev space [19, 20]. Furthermore, for fluid thermal coupling systems with nonzero boundary conditions, Baranovskii [21] proposed a novel approach based on Banach space isomorphism and compact perturbation theory, proving the existence of small-data strong solutions to the 3D unsteady Navier-Stokes-Boussinesq system.

After discussing the research progress on the well-posedness of the Prandtl equations, the MHD equations, as the core model describing the interaction between conducting fluids and electromagnetic fields, pose practical significance in the mathematical analysis of their boundary layer problems. So, this paper considers 3D MHD equations in $\mathbb{R}^3_+ = \{(x, y, z)|(x, y) \in \mathbb{R}^2,\ z > 0\}$ (see [34]):

$$\begin{cases} \partial_t\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} - (\tilde{\mathbf{b}} \cdot \nabla)\tilde{\mathbf{b}} + \nabla p - \frac{1}{Re}\Delta\tilde{\mathbf{u}} = 0, \\ \partial_t\tilde{\mathbf{b}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{b}} - (\tilde{\mathbf{b}} \cdot \nabla)\tilde{\mathbf{u}} - \frac{1}{Rm}\Delta\tilde{\mathbf{b}} = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = \nabla \cdot \tilde{\mathbf{b}} = 0. \end{cases} \tag{1.1}$$

Here, Re and Rm denote the hydrodynamic Reynolds number and magnetic Reynolds number, respectively. $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_h, \tilde{u}_z)$ is the velocity field and $\tilde{\mathbf{b}} = (\tilde{\mathbf{b}}_h, \tilde{b}_z)$ denotes the magnetic field. The total pressure is $p = \tilde{p} + |\tilde{\mathbf{b}}|^2/2$, where p is the fluid pressure. The boundary conditions are as follows: velocity no-slip $\tilde{\mathbf{u}}|_{z=0} = \mathbf{0}$, magnetic perfectly conducting $(\partial_z\tilde{\mathbf{b}}_h, \tilde{b}_z)\big|_{z=0} = \mathbf{0}$.

A key issue in MHD system research is understanding high Reynolds number ($Re \gg 1$) behavior in bounded domains. In this paper, we focus on the case $1/Re = 1/Rm = \epsilon$ with $\epsilon \ll 1$, which implies that the proposed model (1.2) is exclusively applicable to laminar flow scenarios, with no consideration given to the interaction mechanisms between magnetic boundary layers and turbulent flows. Using the coordinate transformation $t = t$, $x = x$, $y = y$, $\tilde{z} = \epsilon^{-1/2}z$ and

$$\begin{cases} \mathbf{u}^p_h(t, x, y, \tilde{z}) = \tilde{\mathbf{u}}_h(t, x, y, z), \quad w^p(t, x, y, \tilde{z}) = \varepsilon^{-1/2}\tilde{u}_z(t, x, y, z), \\ \mathbf{b}^p_h(t, x, y, \tilde{z}) = \tilde{\mathbf{b}}_h(t, x, y, z), \quad g^p(t, x, y, \tilde{z}) = \varepsilon^{-1/2}\tilde{b}_z(t, x, y, z), \\ P(t, x, y, \tilde{z}) = p(t, x, y, z), \end{cases}$$

the corresponding MHD boundary layer system (derived in [22]) is given as follows:

$$\begin{cases} \partial_t\mathbf{u}^p_h + (\mathbf{u}^p \cdot \nabla)\mathbf{u}^p_h - (\mathbf{b}^p \cdot \nabla)\mathbf{b}^p_h + \nabla_h P - \partial^2_z\mathbf{u}^p_h = 0, \\ \partial_t\mathbf{b}^p_h + (\mathbf{u}^p \cdot \nabla)\mathbf{b}^p_h - (\mathbf{b}^p \cdot \nabla)\mathbf{u}^p_h - \partial^2_z\mathbf{b}^p_h = 0, \\ \partial_t g^p + (\mathbf{u}^p \cdot \nabla)g^p - (\mathbf{b}^p \cdot \nabla)w^p - \partial^2_z g^p = 0, \\ \nabla \cdot \mathbf{u}^p = \nabla \cdot \mathbf{b}^p = 0, \\ (\mathbf{u}^p_h, u^p_z)\big|_{z=0} = (\partial_z\mathbf{b}^p_h, b^p_z)\big|_{z=0} = \mathbf{0},\ \ (\mathbf{u}^p_h, \mathbf{b}^p_h)\big|_{z\to+\infty} = (\mathbf{U}, \mathbf{B}). \end{cases} \tag{1.2}$$

We use the notation $\nabla = (\nabla_h, \partial_z)$, where $\nabla_h = (\partial_x, \partial_y)$ is the horizontal gradient operator. The velocity field is $\mathbf{u}^p = (\mathbf{u}^p_h, w^p)$, and the magnetic field is $\mathbf{b}^p = (\mathbf{b}^p_h, g^p)$: $\mathbf{u}^p_h = (u^p, v^p)$ (horizontal velocity, with u^p (x-direction) and v^p (y-direction)), $\mathbf{b}^p_h = (b^p, h^p)$ (horizontal magnetic field, with b^p (x-direction) and h^p (y-direction)), and w^p, g^p are the z-direction components of velocity and magnetic field, respectively. The given functions P, $\mathbf{U} = (U, V)$ and $\mathbf{B} = (B, H)$ satisfy Bernoulli's law:

$$\begin{cases} \partial_t\mathbf{U} + (\mathbf{U} \cdot \nabla_h)\mathbf{U} - (\mathbf{B} \cdot \nabla_h)\mathbf{B} + \nabla_h P = 0, \\ \partial_t\mathbf{B} + (\mathbf{U} \cdot \nabla_h)\mathbf{B} - (\mathbf{B} \cdot \nabla_h)\mathbf{U} = 0. \end{cases} \tag{1.3}$$

It is also worth noting that the third equation in (1.2) can be derived from the second equation in (1.2), and the boundary condition is $\partial_z \mathbf{b}^p|_{z=0} = \mathbf{0}$, using the relation $g^p = -\int_0^z \nabla_h \cdot \mathbf{b}^p \, d\tilde{z}$. We therefore only need to analyze the following initial-boundary value problem for the MHD boundary layer system in the domain $\mathbb{R}^3_+ = \{(x, y, z)|(x, y) \in \mathbb{R}^2, \ z > 0\}$:

$$\begin{cases} \partial_t \mathbf{u}_h^p + (\mathbf{u}_h^p \cdot \nabla_h)\mathbf{u}_h^p - (\mathbf{b}^p \cdot \nabla_h)\mathbf{b}_h^p + w^p \partial_z \mathbf{u}_h^p - g^p \partial_z \mathbf{b}_h^p + \nabla_h P - \partial_z^2 \mathbf{u}_h^p = 0, \\ \partial_t \mathbf{b}_h^p + (\mathbf{u}_h^p \cdot \nabla_h)\mathbf{b}_h^p - (\mathbf{b}_h^p \cdot \nabla_h)\mathbf{u}_h^p + w^p \partial_z \mathbf{b}_h^p - g^p \partial_z \mathbf{u}_h^p - \partial_z^2 \mathbf{b}_h^p = 0, \\ \nabla \cdot \mathbf{u}^p = \nabla \cdot \mathbf{b}^p = 0, \\ (\mathbf{u}_h^p, w^p)\big|_{z=0} = (\partial_z \mathbf{b}_h^p, g^p)\big|_{z=0} = \mathbf{0}, \ (\mathbf{u}_h^p, \mathbf{b}_h^p)\big|_{z\to+\infty} = (\mathbf{U}, \mathbf{B}), \\ \mathbf{u}_h^p\big|_{t=0} = \mathbf{u}_{h,0}^p, \ \mathbf{b}_h^p\big|_{t=0} = \mathbf{b}_{h,0}^p. \end{cases} \tag{1.4}$$

When the velocity field equation is coupled with the magnetic field equation, the boundary layer phenomenon exhibits distinct characteristics, primarily due to the emergence of magnetic boundary layers [23] that are more complex than those described by the classical Prandtl equations. Notably, issues related to the classical MHD boundary layer equations correspond to the 15th open problem in Oleinik and Samokhin's seminal work [3] (p500–503): "15. For the equations of the magnetohydrodynamic boundary layer, all problems of the above type are still open". Significantly, by adopting a non-degenerate tangential magnetic field assumption instead of Oleinik's velocity monotonicity condition, the well-posedness of MHD boundary layer equations in Sobolev spaces has been established. Liu et al. [22] proved the local existence and uniqueness of the 2D nonlinear MHD boundary layer equations in weighted Sobolev spaces via energy methods. Extending this work, they validated the Prandtl expansion with L^∞ error estimates using multiscale analysis [24]. Chen et al. [25] obtained long-time well-posedness for small initial data in lower-order weighted Sobolev spaces, showing that the solution lifespan depends on initial data to complement the result of [22].

For large magnetic Reynolds numbers (neglecting resistivity), Liu et al. [26] studied the local well-posedness of the 2D MHD boundary layer equations without resistivity in Sobolev spaces and identified linear instability when the tangential magnetic field degenerates at a point. Chen and Li [27] extended this to long-time well-posedness in lower-order weighted Sobolev spaces. Addressing the absence of velocity diffusion, Li and Xu [28] proved solution existence and uniqueness via pseudo differential calculations, complementing [22, 26]. Gao et al. [29] established local well-posedness for the 2D incompressible MHD boundary layer equations in weighted conormal Sobolev spaces.

Besides, there have also been some results in analytic frameworks for the 2D MHD boundary layer equations. Xie and Yang [30] obtained global existence (with exponential decay in analytic norms) for small perturbations of Hartmann profiles in the mixed Prandtl-Hartmann regime. Using cancellation mechanisms, they also derived a lower bound for the solution lifespan in [31]. Liu and Zhang [32] established global existence and asymptotic decay for small initial data. Recently, Li and Xie [33] obtained global well-posedness in analytic spaces. Additionally, the analyticity condition has been weakened to Gevrey spaces [34–37].

In addition to the well-posedness of the 2D case, the well-posedness of the 3D MHD boundary layer equations still poses numerous challenges. For the 3D axisymmetric case, Lin and Zou [34] proved well-posedness with a Gevrey index up to 3/2 by refining the cancellation mechanism and constructing an energy functional involving polynomial weights on tangential variables. Chen et al. [38] focused on the linearized system and established local well-posedness under the condition that one tangential direction satisfies Sobolev regularity and the other satisfies analyticity. Wu [39] further considered the 3D nonlinear system and obtained local well-posedness in Sobolev spaces and linear stability results under the structural assumption that the outer flow and magnetic field share the same direction. Despite these breakthroughs under specific structural assumptions and in Gevrey spaces, open problems remain, such as the well-posedness of 3D MHD boundary layer equations in an analytic space. Compared with the well-posedness result established for the 3D axisymmetric MHD boundary layer equations in Gevrey space [34], this paper focuses on the local well-posedness of the 3D MHD boundary layer equations without structural assumptions in an analytic space. Specifically, the initial data are required to belong to the weighted real-analytic function space $X_{\tau_0,\mu}$, which is real-analytic with respect to the tangential variables (x, y) and possesses Sobolev regularity in the normal variable z. The main difficulties and the corresponding key methods of solving the well-posedness of Eq (1.4) are as follows:

1) The 3D MHD boundary layer equations exhibit strong coupling between the tangential and normal components of the velocity field. Additionally, they contain the linearly growing terms $z\nabla \cdot \mathbf{U}\partial_z\mathbf{u}$, $z\nabla \cdot \mathbf{B}\partial_z\mathbf{b}$, $z\nabla \cdot \mathbf{U}\partial_z\mathbf{b}$, and $z\nabla \cdot \mathbf{B}\partial_z\mathbf{u}$, which easily leads to the divergence of estimations, a common difficulty in the 3D MHD boundary layer problems. To address the divergence risk posed by the these linearly growing terms, a Gauss weight function θ_μ is designed. Its unique temporal and spatial derivative properties (e.g., $\partial_t\theta_\mu = -\mu Z^2\theta_\mu$, $\partial_z\theta_\mu = 2\mu Z\theta_\mu$) play a pivotal role in effectively controlling the linearly growing terms throughout Section 3.

2) In the absence of structural assumptions, it is difficult to balance the requirements of tangential analyticity and normal Sobolev regularity, and the problem of tangential derivative loss is prone to occur. To overcome this difficulty, Eq (3.1) is decomposed into linear terms, nonlinear terms, and source terms, which are individually estimated through Lemmas 3.4, 3.5, and 3.6. Specifically, we apply the tangential derivative operator ∂_h^α to Eq $(2.3)_{1,2}$ respectively, and then take the L^2 inner product with the weight function (see (3.1)). Combining tools such as Agmon's inequality and Cauchy's inequality, systematic estimations of linear terms, nonlinear terms, and source terms are created to establish stable a priori estimates. Meanwhile, a weighted real-analytic function space is constructed, requiring the initial data to be analytic in the tangential directions (x, y) and satisfy Sobolev regularity in the normal direction z. This framework not only adapts to the regularity requirements of the problem but also addresses the issue of tangential derivative loss by handling high-order derivative terms through methods such as the discrete Young's inequality.

The rest of the paper is organized as follows. In Section 2, we elaborate on the transformation process of (2.3)–(2.5) in detail and present the main results of this paper. In Section 3, we prove the existence of the solution to (2.3)–(2.5) through rigorous derivation. In Section 4, we verify the uniqueness of the solution to (2.3)–(2.5).

2. Transformation and main result

As in [11], to homogenize the boundary conditions of $(1.4)_4$, we introduce the following change of variables:

$$\begin{cases} \mathbf{u}(t, x, y, z) = \mathbf{u}_h^p(t, x, y, z) - (1 - \phi(z))\mathbf{U}(t, x, y), \\ \mathbf{b}(t, x, y, z) = \mathbf{b}_h^p(t, x, y, z) - \mathbf{B}(t, x, y), \end{cases} \tag{2.1}$$

where $\phi(z) = \frac{2\sqrt{\beta}}{\sqrt{\pi}} \int_z^{+\infty} e^{-\beta\eta^2} d\eta$, for some $\beta > 1$.

Combining the incompressibility condition $(1.4)_3$, the boundary condition $(1.4)_4$ and the substitutions (2.1), one arrives at

$$\begin{cases} w = -\partial_x \int_0^z u(t,x,y,\xi) + (1-\phi(z))U d\xi - \partial_y \int_0^z v(t,x,y,\xi) + (1-\phi(z))V d\xi \\ \quad = -\partial_x \int_0^z u(t,x,y,\xi)d\xi - \partial_y \int_0^z v(t,x,y,\xi)d\xi - \partial_x \int_0^z U(t,x,y)d\xi - \partial_y \int_0^z V(t,x,y)d\xi \\ \quad + \int_0^z \phi(z)\partial_x U(t,x,y)d\xi + \int_0^z \phi(z)\partial_y V(t,x,y)d\xi \\ \quad = -\nabla_h \cdot \mathbf{g}(\mathbf{u}) - z\nabla_h \cdot \mathbf{U} + \Phi\nabla_h \cdot \mathbf{U}, \\ g = -\partial_x \int_0^z b(t,x,y,\xi) + B d\xi - \partial_y \int_0^z h(t,x,y,\xi) + H d\xi \\ \quad = -\partial_x \int_0^z b(t,x,y,\xi)d\xi - \partial_y \int_0^z h(t,x,y,\xi)d\xi - \partial_x \int_0^z B(t,x,y)d\xi - \partial_y \int_0^z H(t,x,y)d\xi \\ \quad = -\nabla_h \cdot \mathbf{g}(\mathbf{b}) - z\nabla_h \cdot \mathbf{B}, \end{cases} \tag{2.2}$$

where $\Phi(z) = \int_0^z \phi(\xi)d\xi$, $\mathbf{u} = (u,v)$, $\mathbf{b} = (b,h)$, $\mathbf{g}(\mathbf{u}) = (g(u), g(v))$, $\mathbf{g}(\mathbf{b}) = (g(b), g(h))$, $\mathbf{U} = (U,V)$, $\mathbf{B} = (B,H)$.

Under the transformation (2.1), the original MHD boundary layer equations (1.4) can be rewritten as the following system:

$$\begin{cases} \partial_t \mathbf{u} - \partial_z^2 \mathbf{u} + N(\mathbf{u}) + L(\mathbf{u}) = F(\mathbf{U}), \\ \partial_t \mathbf{b} - \partial_z^2 \mathbf{b} + N(\mathbf{b}) + L(\mathbf{b}) = F(\mathbf{B}), \end{cases} \tag{2.3}$$

where $N(\cdot)$ represents the nonlinear terms, $L(\cdot)$ denotes the linear terms, and $F(\cdot)$ stands for the outer-flow source terms. Their specific forms are given by

$$\begin{cases} N(\mathbf{u}) = (\mathbf{u}\cdot\nabla_h)\mathbf{u} - (\mathbf{b}\cdot\nabla_h)\mathbf{b} - \nabla_h \cdot g(\mathbf{u})\partial_z \mathbf{u} + \nabla_h \cdot g(\mathbf{b})\partial_z \mathbf{b}, \\ N(\mathbf{b}) = (\mathbf{u}\cdot\nabla_h)\mathbf{b} - (\mathbf{b}\cdot\nabla_h)\mathbf{u} - \nabla_h \cdot g(\mathbf{u})\partial_z \mathbf{b} + \nabla_h \cdot g(\mathbf{b})\partial_z \mathbf{u}, \\ L(\mathbf{u}) = (1-\phi)(\mathbf{U}\cdot\nabla_h)\mathbf{u} + (1-\phi)(\mathbf{u}\cdot\nabla_h)\mathbf{U} - z\nabla_h \cdot \mathbf{U}\partial_z \mathbf{u} + \phi'\nabla_h \cdot g(\mathbf{u})\mathbf{U} \\ +\Phi(z)\nabla_h \cdot \mathbf{U}\partial_z \mathbf{u} - (\mathbf{b}\cdot\nabla_h)\mathbf{B} - (\mathbf{B}\cdot\nabla_h)\mathbf{b} + z\nabla_h \cdot \mathbf{B}\partial_z \mathbf{b}, \\ L(\mathbf{b}) = (1-\phi)(\mathbf{U}\cdot\nabla_h)\mathbf{b} - (1-\phi)(\mathbf{b}\cdot\nabla_h)\mathbf{U} - z\nabla_h \cdot \mathbf{U}\partial_z \mathbf{b} \\ -\phi'\nabla_h \cdot g(\mathbf{b})\mathbf{U} + \Phi(z)\nabla_h \cdot \mathbf{U}\partial_z \mathbf{b} + (\mathbf{u}\cdot\nabla_h)\mathbf{B} - (\mathbf{B}\cdot\nabla_h)\mathbf{u} + z\nabla_h \cdot \mathbf{B}\partial_z \mathbf{u}, \\ F(\mathbf{U}) = -z\phi'\mathbf{U}\nabla\cdot\mathbf{U} - \phi\nabla_h P + \phi(1-\phi)(\mathbf{U}\cdot\nabla_h)\mathbf{U} + \phi'\Phi(z)\nabla_h \cdot \mathbf{U}\mathbf{U} \\ -\phi''\mathbf{U} + \phi(\mathbf{B}\cdot\nabla_h)\mathbf{B}, \\ F(\mathbf{B}) = \phi(\mathbf{U}\cdot\nabla_h)\mathbf{B} - \phi(\mathbf{B}\cdot\nabla_h)\mathbf{U} + z\phi'(\nabla_h \cdot \mathbf{B})\mathbf{U}, \end{cases} \tag{2.4}$$

where $\mathbf{u}$ and $\mathbf{b}$ satisfy the initial conditions and homogeneous boundary conditions,

$$\begin{cases} \mathbf{u}_{h,0}(x,y,z) = \mathbf{u}(0,x,y,z) = \mathbf{u}^p(0,x,y,z) - (1-\phi(z))\mathbf{U}(0,x,y), \\ \mathbf{b}_{h,0}(x,y,z) = \mathbf{b}(0,x,y,z) = \mathbf{b}^p(0,x,y,z) - \mathbf{B}(0,x,y), \\ \mathbf{u}(t,x,y,0) = 0, \ \lim\limits_{z\to+\infty} \mathbf{u}(t,x,y,z) = 0, \\ \partial_z \mathbf{b}(t,x,y,0) = 0, \ \lim\limits_{z\to+\infty} \mathbf{b}(t,x,y,z) = 0. \end{cases} \tag{2.5}$$

To address the divergence of estimates caused by linearly growing terms such as $z\nabla\cdot\mathbf{U}\partial_z\mathbf{u}$, $z\nabla\cdot\mathbf{B}\partial_z\mathbf{b}$, $z\nabla\cdot\mathbf{U}\partial_z\mathbf{b}$, and $z\nabla\cdot\mathbf{B}\partial_z\mathbf{u}$, we introduce a Gauss weight function,

$$\theta_\mu(t,z) = \exp\Big(\frac{z^2}{1+t/\mu}\Big),$$

where $\mu > 0$ is a parameter to be determined later. By leveraging its temporal and spatial derivative properties, we directly cancel out the z-related factors in the linearly growing terms.

We present notations related to weighted norms as follows:

$$M_m = \frac{(m+1)^r}{m!}, \; r > 3 \tag{2.6}$$

and denote the tangential differential operator

$$\partial_h^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}, \; \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2. \tag{2.7}$$

For a positive time-dependent function $\tau(t)$, the Sobolev weighted norms are introduced as follows:

$$\begin{cases} X_m = X_m(\mathbf{u}, \tau) = \Big(\sum\limits_{|\alpha|=m} \|\theta_\mu \partial_h^m \mathbf{u}\|_{L^2(\mathbb{R}_+^3)}^2\Big)^{\frac{1}{2}} \tau^m M_m, \\ D_m = D_m(\mathbf{u}, \tau) = \Big(\sum\limits_{|\alpha|=m} \|\theta_\mu \partial_z \partial_h^m \mathbf{u}\|_{L^2(\mathbb{R}_+^3)}^2\Big)^{\frac{1}{2}} \tau^m M_m, \\ Y_m = Y_m(\mathbf{u}, \tau) = \Big(\sum\limits_{|\alpha|=m} \|\theta_\mu \partial_h^m \mathbf{u}\|_{L^2(\mathbb{R}_+^3)}^2\Big)^{\frac{1}{2}} \tau^{m-1/2} m^{1/2} M_m, \\ Z_m = Z_m(\mathbf{u}, \tau) = \Big(\sum\limits_{|\alpha|=m} \|Z\theta_\mu \partial_h^m \mathbf{u}\|_{L^2(\mathbb{R}_+^3)}^2\Big)^{\frac{1}{2}} \tau^m M_m, \; Z = \frac{z}{t+\mu}, \end{cases} \tag{2.8}$$

where $\|\theta_\mu \partial_h^m \mathbf{u}\|_{L^2(\mathbb{R}_+^3)}^2 = \big(\|\theta_\mu \partial_h^m u\|_{L^2(\mathbb{R}_+^3)}^2 + \|\theta_\mu \partial_h^m v\|_{L^2(\mathbb{R}_+^3)}^2\big)^{1/2}$.

We define the following analytic function space in the tangential variable (x, y) and weighted Sobolev space in the normal variable z by

$$X_{\tau,\mu} = \{\alpha \in \mathbb{N}^2, \theta_\mu \partial_h^\alpha \mathbf{u} \in L^2(\mathbb{R}_+^3; dxdydz) : \|\mathbf{u}\|_{X_{\tau,\mu}} < \infty\}$$

with the norms

$$\begin{cases} \|\mathbf{u}\|_{X_{\tau,\mu}} = \sum\limits_{m\geq 0} X_m^2, \\ \|\mathbf{u}\|_{D_{\tau,\mu}} = \sum\limits_{m\geq 0} D_m^2, \\ \|\mathbf{u}\|_{Y_{\tau,\mu}} = \sum\limits_{m\geq 1} Y_m^2, \\ \|\mathbf{u}\|_{Z_{\tau,\mu}} = \sum\limits_{m\geq 0} Z_m^2. \end{cases} \tag{2.9}$$

We now state the main result of this paper in the following theorem:

Theorem 2.1. *Let* $\mu \in (0, \frac{\sqrt{10}}{5}]$ *and* $r > 3$. *Suppose the initial data and the outer flows* $(\mathbf{U}, \mathbf{B})$ *satisfy the following conditions:*

(i) the initial datum $(\mathbf{u}_0, \mathbf{b}_0) \in X_{\tau_0,\mu}$, *for some the analytic radius* $\tau_0 > 0$,

(ii) the outer flow $(\mathbf{U}, \mathbf{B})$ *is uniformly real-analytic for* $[0, T]$.

Then there exists a time $T_* > 0$ *such that the system* (2.3)–(2.5) *has a unique real-analytic solution in* $[0, T_*]$ *with an analytic radius* $\tau \geq \frac{\tau_0}{2}$.

Remark 2.1. *A uniform analyticity requirement is imposed on the outer flows* $(\mathbf{U}, \mathbf{B})$*. This implies the existence of positive constants* C_1 *and* $\tau_1 > 0$ *such that for all* $t \in [0, T]$*,*

$$\sum_{|\alpha|=m} \|\partial_h^\alpha \mathbf{U}\|_{L_h^\infty} + \|\partial_h^\alpha \mathbf{B}\|_{L_h^\infty} \leq \frac{C_1}{\tau_1^m M_m}. \tag{2.10}$$

Remark 2.2. *Theoretically, this result can be extended to Gevrey regularity, as supported by relevant studies (e.g., [14, 34, 36]).*

Compared with Gevrey regularity, analyticity plays a crucial role in the argument: on one hand, the exponential decay of high-order tangential derivatives of analytic functions effectively controls linearly growing terms and high-order nonlinear coupling terms, eliminating the need for structural assumptions such as monotonicity or magnetic field nondegeneracy; on the other hand, the function space $X_{\tau,\mu}$ *constructed based on analyticity is highly compatible with the Gauss weight* θ_μ*, enabling rapid closure of a priori estimates and proof of uniqueness through basic inequalities.*

3. The uniform estimates on the terms u and b

In this section, we shall establishes a priori bounds for solutions $(\mathbf{u}, \mathbf{b})$ to (2.3)–(2.5) using Sobolev energy estimation methods, with the proof proceeding in three steps. First, decompose the equations into nonlinear, linear, and source terms. Second, estimate these three types of terms separately via Lemmas 3.2–3.6. Finally, collect all estimation results to prove Proposition 3.1.

Proposition 3.1. *Under the assumptions of Theorem 2.1, there exists a solution* $(\mathbf{u}, \mathbf{b})$ *to problem* (2.3)–(2.5) *such that*

$$(\|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{b}\|_{X_{\tau,\mu}}^2) + \int_0^t (\|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \|\mathbf{b}\|_{D_{\tau,\mu}}^2) \lesssim (\|\mathbf{u}\|_{X_{\tau_0,\mu}}^2 + \|\mathbf{b}\|_{X_{\tau_0,\mu}}^2)$$

for $\tau(t) \geq \tau_0/2$, $t \in [0, T_*]$*. Hereafter, we write* $A \lesssim B$ *to denote that there exists a constant* C *independent of* m *and* α *such that* $A \leq CB$*.*

Fix $m \geq 0$ and $|\alpha| = m$, applying the tangential differential operator ∂_h^m to equations $(2.3)_{1,2}$, multiplying the resulting equations by $\theta_\mu^2 \partial_h^m \mathbf{u}$ and $\theta_\mu^2 \partial_h^m \mathbf{b}$, respectively, integrating it over $\mathbb{R}^2$, and then sum the results. This yields

$$\begin{aligned} &\langle \partial_h^\alpha(\partial_t \mathbf{u} - \partial_z^2 \mathbf{u}), \theta_\mu^2 \partial_h^\alpha \mathbf{u}\rangle + \langle \partial_h^\alpha(\partial_t \mathbf{b} - \partial_z^2 \mathbf{b}), \theta_\mu^2 \partial_h^\alpha \mathbf{b}\rangle \\ &= \langle \partial_h^\alpha(\mathbf{F}(\mathbf{U}) - L(\mathbf{u}) - N(\mathbf{u})), \theta_\mu^2 \partial_h^\alpha \mathbf{u}\rangle + \langle \partial_h^\alpha(\mathbf{F}(\mathbf{B}) - L(\mathbf{b}) - N(\mathbf{b})), \theta_\mu^2 \partial_h^\alpha \mathbf{b}\rangle. \end{aligned} \tag{3.1}$$

We now establish the estimates of each term in (3.1) as follows. For the first and third terms on the left-hand side of (3.1), we obtain

$$\begin{aligned} \langle \partial_h^\alpha \partial_t \mathbf{u}, \theta_\mu^2 \partial_h^\alpha \mathbf{u}\rangle + \langle \partial_h^\alpha \partial_t \mathbf{b}, \theta_\mu^2 \partial_h^\alpha \mathbf{b}\rangle = &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_\mu \partial_h^\alpha \mathbf{b}\|_{L^2}^2 \\ &+ \mu\|Z\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 + \mu\|Z\theta_\mu \partial_h^\alpha \mathbf{b}\|_{L^2}^2, \end{aligned} \tag{3.2}$$

where we have used the fact that $\partial_t \theta_\mu = -\mu Z^2 \theta_\mu$.

For the second and fourth terms on the left-hand side of (3.1), integrating it by parts over $\mathbb{R}^3_+$ with respect to variable z and using the boundary condition (2.5), we arrive at

$$\begin{aligned}
&-\langle \partial_h^\alpha \partial_z^2 \mathbf{u}, \theta_\mu^2 \partial_h^\alpha \mathbf{u}\rangle - \langle \partial_h^\alpha \partial_z^2 \mathbf{b}, \theta_\mu^2 \partial_h^\alpha \mathbf{b}\rangle \\
&= \|\theta_\mu \partial_z \partial_h^\alpha \mathbf{u}\|^2 + \langle \partial_h^\alpha \partial_z \mathbf{u}, 2Z\theta_\mu^2 \partial_h^\alpha \mathbf{u}\rangle + \|\theta_\mu \partial_z \partial_h^\alpha \mathbf{b}\|^2 + \langle \partial_h^\alpha \partial_z \mathbf{b}, 2Z\theta_\mu^2 \partial_h^\alpha \mathbf{b}\rangle \\
&\geq \frac{1}{2}\|\theta_\mu \partial_z \partial_h^\alpha \mathbf{u}\|^2 - 2\mu^2 \|Z\theta_\mu \partial_h^\alpha \mathbf{u}\|^2 + \frac{1}{2}\|\theta_\mu \partial_z \partial_h^\alpha \mathbf{b}\|^2 - 2\mu^2 \|Z\theta_\mu \partial_h^\alpha \mathbf{b}\|^2,
\end{aligned} \tag{3.3}$$

where we have used the fact that $\partial_z \theta_\mu = 2\mu Z\theta_\mu$.

Substituting (3.2)–(3.3) into (3.1), we can derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|_{L^2}^2 + \|\theta_\mu \partial_h^\alpha \partial_z (\mathbf{u}, \mathbf{b})\|_{L^2}^2 + 2\|\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|_{L^2}^2 + 2(1-2\mu^2)\|Z\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|^2 \\
&\leq 2|\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{U}) - L(\mathbf{u}) - N(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u}\rangle| + 2|\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{B}) - L(\mathbf{b}) - N(\mathbf{b})), \theta_\mu \partial_h^\alpha \mathbf{b}\rangle|,
\end{aligned} \tag{3.4}$$

where the notation $\|\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|_{L^2}^2 = \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 + \|\theta_\mu \partial_h^\alpha \mathbf{b}\|_{L^2}^2$.

Moreover, multiplying both sides of the inequality (3.4) by $\tau^{2m} M_m^2$ and summing over $|\alpha| = m$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|_{L^2}^2\right)\tau^{2m} M_m^2 + \sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \partial_z (\mathbf{u}, \mathbf{b})\|_{L^2}^2 \tau^{2m} M_m^2 \\
&\quad + 2\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|_{L^2}^2 \tau^{2m} M_m^2 + 2(1-2\mu^2)\sum_{|\alpha|=m} \|Z\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|^2 \\
&\leq 2\sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{U}) - L(\mathbf{u}) - N(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u}\rangle| \tau^{2m} M_m^2 \\
&\quad + 2\sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{B}) - L(\mathbf{b}) - N(\mathbf{b})), \theta_\mu \partial_h^\alpha \mathbf{b}\rangle| \tau^{2m} M_m^2,
\end{aligned} \tag{3.5}$$

which, together with the Definition (2.8), causes Ineq (3.5) to imply

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha (\mathbf{u}, \mathbf{b})\|_{L^2}^2\right)\tau^{2m} M_m^2 + D_m^2(\mathbf{u}, \mathbf{b}) + 2X_m^2(\mathbf{u}, \mathbf{b}) + 2(1-2\mu^2) Z_m^2(\mathbf{u}, \mathbf{b}) \\
&\leq 2\sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{U}) - L(\mathbf{u}) - N(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u}\rangle| \tau^{2m} M_m^2 \\
&\quad + 2\sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{B}) - L(\mathbf{b}) - N(\mathbf{b})), \theta_\mu \partial_h^\alpha \mathbf{b}\rangle| \tau^{2m} M_m^2.
\end{aligned} \tag{3.6}$$

Summing (3.6) over $m \geq 0$ and using the notation from (2.9), we conclude

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{u}, \mathbf{b})\|_{X_{\tau,\mu}}^2 - \dot{\tau}\|(\mathbf{u}, \mathbf{b})\|_{Y_{\tau,\mu}}^2 + \|(\mathbf{u}, \mathbf{b})\|_{D_{\tau,\mu}}^2 + 2\|(\mathbf{u}, \mathbf{b})\|_{X_{\tau,\mu}}^2 + 2(1-2\mu^2)\|(\mathbf{u}, \mathbf{b})\|_{Z_{\tau,\mu}}^2 \\
&\leq 2\sum_{m\geq 0}\sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{U}) - L(\mathbf{u}) - N(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u}\rangle| \tau^{2m} M_m^2 \\
&\quad + 2\sum_{m\geq 0}\sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{F}(\mathbf{B}) - L(\mathbf{b}) - N(\mathbf{b})), \theta_\mu \partial_h^\alpha \mathbf{b}\rangle| \tau^{2m} M_m^2.
\end{aligned} \tag{3.7}$$

To estimate the terms on the right-hand side of Ineq (3.7), we shall analyze its components individually in the following three subsections.

3.1. Estimates on the nonlinear terms $N(\mathbf{u})$ and $N(\mathbf{b})$

In this subsection, we intend to establish the estimates for the each nonlinear term in the terms $N(\mathbf{u})$ and $N(\mathbf{b})$. To control the estimates of the nonlinear term $N(\mathbf{u})$, we first present Lemma 3.2, which has been proved in [40], and which shall be used throughout the rest of the paper.

Lemma 3.2. *Let $\mu > 0$. For any integers $|\alpha| = m \geq 0$, the following inequalities hold*

$$\begin{cases} \|\partial_h^\alpha \mathbf{u}\|_{L_z^\infty L_h^2} \lesssim \|\theta_\mu \partial_z \partial_h^\alpha \mathbf{u}\|_{L_{h,z}^2}, \\ \left\|\partial_h^\alpha \mathbf{u}\right\|_{L_h^\infty L_z^\infty} \lesssim \left\|\theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e_2} \partial_z \mathbf{u}\right\|_{L^2}^{1/2} + \left\|\theta_\mu \partial_h^{\alpha+e_1} \partial_z \mathbf{u}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e} \partial_z \mathbf{u}\right\|_{L^2}^{1/2}, \\ \|\partial_h^\alpha \mathbf{g}(\mathbf{u})\|_{L_z^\infty L_h^2} \lesssim \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L_{x,y,z}^2}, \\ \left\|\partial_h^\alpha \mathbf{g}(\mathbf{u})\right\|_{L_h^\infty L_z^\infty} \lesssim \left\|\theta_\mu \partial_h^\alpha \mathbf{u}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e_2} \mathbf{u}\right\|_{L^2}^{1/2} + \left\|\theta_\mu \partial_h^{\alpha+e_1} \mathbf{u}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e} \mathbf{u}\right\|_{L^2}^{1/2}, \end{cases} \tag{3.8}$$

for some $e_1 = (1,0),\ e_2 = (0,1),\ e = (1,1)$.

Building on Lemma 3.2, we further establish the relationship between the L^∞ and L^2 bounds of magnetic field components, providing support for the energy estimation of nonlinear term $N(\mathbf{b})$.

Lemma 3.3. *Let $\mu > 0$. For any integers $|\alpha| = m \geq 0$, it holds that*

$$\begin{cases} \|\partial_h^\alpha \mathbf{b}\|_{L_z^\infty L_h^2} \lesssim \|\theta_\mu \partial_z \partial_h^\alpha \mathbf{b}\|_{L_{h,z}^2}, \\ \left\|\partial_h^\alpha \mathbf{b}\right\|_{L_h^\infty L_z^\infty} \lesssim \left\|\theta_\mu \partial_h^\alpha \partial_z \mathbf{b}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e_2} \partial_z \mathbf{b}\right\|_{L^2}^{1/2} + \left\|\theta_\mu \partial_h^{\alpha+e_1} \partial_z \mathbf{b}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e} \partial_z \mathbf{b}\right\|_{L^2}^{1/2}, \\ \|\partial_h^\alpha \mathbf{g}(\mathbf{b})\|_{L_z^\infty L_h^2} \lesssim \|\theta_\mu \partial_h^\alpha \mathbf{b}\|_{L_{x,y,z}^2}, \\ \left\|\partial_h^\alpha \mathbf{g}(\mathbf{b})\right\|_{L_h^\infty L_z^\infty} \lesssim \left\|\theta_\mu \partial_h^\alpha \mathbf{b}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e_2} \mathbf{b}\right\|_{L^2}^{1/2} + \left\|\theta_\mu \partial_h^{\alpha+e_1} \mathbf{b}\right\|_{L^2}^{1/2} \left\|\theta_\mu \partial_h^{\alpha+e} \mathbf{b}\right\|_{L^2}^{1/2}. \end{cases} \tag{3.9}$$

Proof. For any $|\alpha| = m \geq 0$, because $\partial_h^\alpha \mathbf{b} \to 0$ as $z \to +\infty$, we have $\mathbf{b}(z) = -\int_z^{+\infty} \partial_z \mathbf{b}(\zeta) d\zeta$. Taking the L^∞-norm in z and applying Hölder's inequality, we get

$$|\partial_h^\alpha \mathbf{b}(x,y,z)| \leq \int_z^{+\infty} |\partial_h^\alpha \partial_z \mathbf{b}(x,y,\zeta)| d\zeta \leq \left(\int_z^{+\infty} \theta_\mu^{-2} d\zeta\right)^{1/2} \left(\int_z^{+\infty} \theta_\mu^2 |\partial_h^\alpha \partial_z \mathbf{b}|^2 d\zeta\right)^{1/2}.$$

Because $\theta_\mu^{-1} \in L_z^2$ for $\mu > 0$, the integral $\int_z^{+\infty} \theta_\mu^{-2} d\zeta \leq C$ holds. Taking the L^2-norm in (x,y) on both sides completes the proof of the first inequality. The second inequality follows by applying the following 2D Agmon's inequality,

$$\|f\|_{L_{x,y}^\infty} \leq C\|f\|_{L_{x,y}^2}^{1/2} \|\partial_y f\|_{L_{x,y}^2}^{1/2} + C\|\partial_x f\|_{L_{x,y}^2}^{1/2} \|\partial_x \partial_y f\|_{L_{x,y}^2}^{1/2}.$$

Similarly, we can deduce the last two inequalities. □

Combining the Lemmas 3.2–3.3, we now establish the energy estimate for nonlinear terms $N(\mathbf{u})$ and $N(\mathbf{b})$, whose results will be directly substituted into the Ineq (3.7).

Lemma 3.4. *Let $r > 3$ and $\mu > 0$. It holds that*

$$\begin{aligned} \mathcal{N} = &\sum_{m \geq 0} \sum_{|\alpha| = m} |\langle \theta_\mu \partial_h^\alpha N(\mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 + |\langle \theta_\mu \partial_h^\alpha N(\mathbf{b}), \theta_\mu \partial_h^\alpha \mathbf{b} \rangle| \tau^{2m} M_m^2 \\ \lesssim &\frac{1}{\tau^4} (\|\mathbf{u}\|_{X_{\tau,\mu}}^4 + \|\mathbf{b}\|_{X_{\tau,\mu}}^4) + \frac{1}{8}\Big(\|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \|\mathbf{b}\|_{D_{\tau,\mu}}^2\Big) \\ &+ \frac{1}{\tau^2}\Big(\|\mathbf{u}\|_{D_{\tau,\mu}} + \|\mathbf{b}\|_{D_{\tau,\mu}}\Big)\Big(\|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + \|\mathbf{b}\|_{Y_{\tau,\mu}}^2\Big). \end{aligned} \tag{3.10}$$

Proof. Using Lemmas 3.2 and 3.3 along with the Cauchy-Schwarz inequality and the discrete Young's inequality,

$$\|\zeta \cdot (\xi * \eta)\|_{l^1} \leq \|\zeta\|_{l^2} \|\xi\|_{l^1} \|\eta\|_{l^2}, \tag{3.11}$$

we estimate the nonlinear terms by decomposing the derivatives of the product terms. For example, for $\mathbf{u} \cdot \nabla_h \mathbf{u}$, the derivative $\partial_h^\alpha(\mathbf{u} \cdot \nabla_h \mathbf{u})$ can be expanded into a sum of products of lower-order derivatives of $\mathbf{u}$. By summing over all α and weighting with $\tau^{2|\alpha|}$, we obtain the desired bound. Specifically, similar to the proof method of Lemma 3.3 in [40] and combined with the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}
\mathcal{N} =& \sum_{m\geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha N(\mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 + |\langle \theta_\mu \partial_h^\alpha N(\mathbf{b}), \theta_\mu \partial_h^\alpha \mathbf{b} \rangle| \tau^{2m} M_m^2 \\
\lesssim& \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 \\
&+ \frac{1}{\tau^2} \|\mathbf{b}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \frac{1}{\tau^2} \|\mathbf{b}\|_{D_{\tau,\mu}} \|\mathbf{b}\|_{Y_{\tau,\mu}}^2 \\
&+ \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{b}\|_{X_{\tau,\mu}}^2 + \frac{1}{\tau^2} \|\mathbf{b}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 \\
&+ \frac{1}{\tau^2} \|\mathbf{b}\|_{D_{\tau,\mu}} \|\mathbf{b}\|_{X_{\tau,\mu}}^2 + \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{b}\|_{Y_{\tau,\mu}}^2 \\
\lesssim& \frac{1}{\tau^4} (\|\mathbf{u}\|_{X_{\tau,\mu}}^4 + \|\mathbf{b}\|_{X_{\tau,\mu}}^4) + \frac{1}{8} \Big(\|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \|\mathbf{b}\|_{D_{\tau,\mu}}^2 \Big) \\
&+ \frac{1}{\tau^2} \Big(\|\mathbf{u}\|_{D_{\tau,\mu}} + \|\mathbf{b}\|_{D_{\tau,\mu}} \Big) \Big(\|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + \|\mathbf{b}\|_{Y_{\tau,\mu}}^2 \Big).
\end{aligned}$$

□

3.2. *Estimates on the linear terms $L(\mathbf{u})$ and $L(\mathbf{b})$*

Following the nonlinear term estimates, we now handle the linear terms $L(\mathbf{u})$ and $L(\mathbf{b})$, which include critical linearly growing components (e.g., $z\nabla_h \cdot \mathbf{U}\partial_z \mathbf{u}$) and field outer–flow couplings. Controlling these terms is pivotal to avoiding estimation divergence. The lemma below leverages the Gaussian weight θ_μ to tame growth and establish stable bounds.

Lemma 3.5. *Let $\mu > 0$. We have the following estimate:*

$$\begin{aligned}
\mathcal{L} =& \sum_{m\geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha L(\mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 + \sum_{m\geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha L(\mathbf{b}), \theta_\mu \partial_h^\alpha \mathbf{b} \rangle| \tau^{2m} M_m^2 \\
=& \mathcal{L}(\mathbf{u}) + \mathcal{L}(\mathbf{b}) \\
\lesssim& (1 + \frac{1}{\tau})(\|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{b}\|_{X_{\tau,\mu}}^2) + (\|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + \|\mathbf{b}\|_{Y_{\tau,\mu}}^2) \\
&+ \frac{1}{8} \Big(\|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \|\mathbf{b}\|_{D_{\tau,\mu}}^2 \Big) + \mu^2 \Big(\|\mathbf{u}\|_{Z_{\tau,\mu}}^2 + \|\mathbf{b}\|_{Z_{\tau,\mu}}^2 \Big).
\end{aligned} \tag{3.12}$$

Proof. In view of the definitions of $L(\mathbf{u})$ that

$$L(\mathbf{u}) = (1 - \phi)(\mathbf{U} \cdot \nabla_h)\mathbf{u} + (1 - \phi)(\mathbf{u} \cdot \nabla_h)\mathbf{U} + \phi' \nabla_h \cdot g(\mathbf{u})\mathbf{U} + \Phi(z)\nabla_h \cdot \mathbf{U}\partial_z \mathbf{u}$$

$$
\begin{aligned}
&-(\mathbf{b}\cdot\nabla_h)\mathbf{B}-(\mathbf{B}\cdot\nabla_h)\mathbf{b}-z\nabla_h\cdot\mathbf{U}\partial_z\mathbf{u}+z\nabla_h\cdot\mathbf{B}\partial_z\mathbf{b}\\
&=:\sum_{i=1}^{8}L_i,
\end{aligned}
$$

denote

$$
I_i=\sum_{m\geq 0}\sum_{|\alpha|=m}\left|\langle\theta_\mu\partial_h^\alpha L_i,\theta_\mu\partial_h^\alpha\mathbf{u}\rangle\right|\tau^{2m}M_m^2,\ i=1,2,\cdots,8.
$$

Thus, we have

$$
\mathcal{L}(\mathbf{u})=\sum_{m\geq 0}\sum_{|\alpha|=m}\left|\langle\theta_\mu\partial_h^\alpha\left(\sum_{i=1}^{8}L_i\right),\theta_\mu\partial_h^\alpha\mathbf{u}\rangle\right|\tau^{2m}M_m^2\leq\sum_{i=1}^{8}I_i. \tag{3.13}
$$

The estimate of the first four terms of (3.13) have been derived in [40]; thus, we only rewrite them as follows:

$$
\sum_{i=1}^{4}I_i\lesssim(1+\frac{1}{\tau})\|\mathbf{u}\|_{X_{\tau,\mu}}^2+\|\mathbf{u}\|_{Y_{\tau,\mu}}^2+\|\mathbf{u}\|_{D_{\tau,\mu}}\|\mathbf{u}\|_{X_{\tau,\mu}}. \tag{3.14}
$$

According to the definition of $\phi(z)=\dfrac{2\sqrt{\beta}}{\sqrt{\pi}}\displaystyle\int_z^{+\infty}e^{-\beta\eta^2}d\eta$, $\phi(z)\in L_z^\infty$ as $\beta>0$. In other words, $\|1-\phi\|_{L_z^\infty}\leq C$. Hence, the estimates of terms I_5 and I_6 are similar to the first two terms, so we have

$$
\sum_{i=5}^{6}I_i\lesssim(1+\frac{1}{\tau})\|\mathbf{u}\|_{X_{\tau,\mu}}\|\mathbf{b}\|_{X_{\tau,\mu}}+\|\mathbf{u}\|_{Y_{\tau,\mu}}\|\mathbf{b}\|_{Y_{\tau,\mu}}. \tag{3.15}
$$

Now, we establish the estimate of last two terms,

$$
\begin{aligned}
I_7&=\sum_{m\geq 0}\sum_{|\alpha|=m}\left|\langle\theta_\mu\partial_h^\alpha(z\nabla_h\cdot\mathbf{U}\partial_z\boldsymbol{u}),\theta_\mu\partial_h^\alpha\boldsymbol{u}\rangle\right|\tau^{2m}M_m^2\\
&\leq(t+\mu)\sum_{m\geq 0}\sum_{|\alpha|=m}\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\|\partial_h^\beta\nabla_h\cdot\mathbf{U}\|_{L_h^\infty}\|\theta_\mu\partial_h^{\alpha-\beta}\partial_z\boldsymbol{u}\|_{L^2}\|Z\theta_\mu\partial_h^\alpha\boldsymbol{u}\|_{L^2}\\
&\leq(t+\mu)\sum_{m\geq 0}\sum_{j=0}^{m}\binom{m}{j}\left(\sum_{|\beta|=j}\|\partial_h^\beta\nabla_h\cdot\mathbf{U}\|_{L_h^\infty}\right)\left(\sum_{|\gamma|=m-j}\|\theta_\mu\partial_h^\gamma\partial_z\boldsymbol{u}\|_{L^2}^2\right)^{1/2}\\
&\quad\times\left(\sum_{|\alpha|=m}\|Z\theta_\mu\partial_h^\alpha\boldsymbol{u}\|_{L^2}^2\right)^{1/2}\tau^{2m}M_m^2\\
&\leq C_1(t+\mu)\sum_{m\geq 0}\sum_{j=0}^{m}\left(\frac{\tau}{\tau_1}\right)^j a_{m,r,j}D_{m-j}Z_m,
\end{aligned} \tag{3.16}
$$

where

$$
a_{m,r,j}=\frac{m!(m+1)^r j!(m-j)!}{j!(m-j)!m!(j+1)^r(m-j+1)^r}=\frac{(m+1)^r}{(j+1)^r(m-j+1)^r}\leqslant 1. \tag{3.17}
$$

Choosing $\tau_0 \le \min\{\tau_1/2, \tau_1/(1+8C_1)\}$, then under the assumption that $\tau(t)$ is decreasing, implies

$$C_1 \sum_{j\ge1} \left(\frac{\tau}{\tau_1}\right)^j \le C_1 \sum_{j\ge1} \left(\frac{\tau_0}{\tau_1}\right)^j = C_1 \frac{\tau_0}{\tau_1-\tau_0} \le \frac{1}{8}. \tag{3.18}$$

Assume that there exists a small enough time T such that $T \le \mu$. Inserting (3.17) and (3.18) into (3.16) and using (3.11), we have

$$I_7 \le \frac{\mu}{4} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Z_{\tau,\mu}}. \tag{3.19}$$

Analogously, we can derive that

$$I_8 \le \frac{\mu}{4} \|\mathbf{b}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Z_{\tau,\mu}}. \tag{3.20}$$

Together with the estimates of I_1-I_8, we conclude

$$\begin{aligned}\mathcal{L}(\mathbf{u}) \lesssim{}& (1+\frac{1}{\tau})\|\mathbf{u}\|^2_{X_{\tau,\mu}} + \|\mathbf{u}\|^2_{Y_{\tau,\mu}} + \|\mathbf{u}\|_{D_{\tau,\mu}}\|\mathbf{u}\|_{X_{\tau,\mu}} \\ &+ (1+\frac{1}{\tau})\|\mathbf{u}\|_{X_{\tau,\mu}}\|\mathbf{b}\|_{X_{\tau,\mu}} + \|\mathbf{u}\|_{Y_{\tau,\mu}}\|\mathbf{b}\|_{Y_{\tau,\mu}} \\ &+ \frac{\mu}{4}\|\mathbf{u}\|_{D_{\tau,\mu}}\|\mathbf{u}\|_{Z_{\tau,\mu}} + \frac{\mu}{4}\|\mathbf{b}\|_{D_{\tau,\mu}}\|\mathbf{u}\|_{Z_{\tau,\mu}}.\end{aligned} \tag{3.21}$$

Similar to the estimate of $\mathcal{L}(\mathbf{u})$, we can derive that

$$\begin{aligned}\mathcal{L}(\mathbf{b}) \lesssim{}& (1+\frac{1}{\tau})\|\mathbf{b}\|^2_{X_{\tau,\mu}} + \|\mathbf{b}\|^2_{Y_{\tau,\mu}} + \|\mathbf{b}\|_{D_{\tau,\mu}}\|\mathbf{b}\|_{X_{\tau,\mu}} \\ &+ (1+\frac{1}{\tau})\|\mathbf{b}\|_{X_{\tau,\mu}}\|\mathbf{u}\|_{X_{\tau,\mu}} + \|\mathbf{b}\|_{Y_{\tau,\mu}}\|\mathbf{u}\|_{Y_{\tau,\mu}} \\ &+ \frac{\mu}{4}\|\mathbf{b}\|_{D_{\tau,\mu}}\|\mathbf{b}\|_{Z_{\tau,\mu}} + \frac{\mu}{4}\|\mathbf{u}\|_{D_{\tau,\mu}}\|\mathbf{b}\|_{Z_{\tau,\mu}}.\end{aligned} \tag{3.22}$$

Combing the estimates of (3.21) with (3.22) and using Cauchy–Schwarz inequality, we get

$$\begin{aligned}\mathcal{L} \lesssim{}&(1+\frac{1}{\tau})(\|\mathbf{u}\|^2_{X_{\tau,\mu}} + \|\mathbf{b}\|^2_{X_{\tau,\mu}}) + (\|\mathbf{u}\|^2_{Y_{\tau,\mu}} + \|\mathbf{b}\|^2_{Y_{\tau,\mu}}) \\ &+ \frac{1}{8}\Big(\|\mathbf{u}\|^2_{D_{\tau,\mu}} + \|\mathbf{b}\|^2_{D_{\tau,\mu}}\Big) + \frac{\mu^2}{2}\Big(\|\mathbf{u}\|^2_{Z_{\tau,\mu}} + \|\mathbf{b}\|^2_{Z_{\tau,\mu}}\Big).\end{aligned}$$

□

3.3. Estimates on the outer-flows U *and* B

Having completed the systematic estimation of the nonlinear terms $\mathcal{N}(\cdot)$ and linear terms $\mathcal{L}(\cdot)$ in the previous subsections, we next focus on the estimation of the outer-flow source terms $F(\mathbf{U})$ and $F(\mathbf{B})$. The estimate for the outer-flow $\mathbf{U}$ in $F(\mathbf{U})$ has been established in [40]. Notably, the outer-flow $\mathbf{B}$ in $F(\mathbf{B})$ exhibits an identical structural form to that of $\mathbf{U}$ in $F(\mathbf{U})$. As such, the analytical approach employed to derive the estimates for $\mathbf{U}$ can be directly adapted to obtain the corresponding estimates for $\mathbf{B}$. For this reason, the detailed proof of the following lemma is omitted herein.

Lemma 3.6. *Let $\mu > 0$. We have the following estimates:*

$$\begin{cases}\mathcal{F}(\mathbf{U}) = \sum\limits_{m\ge0}\sum\limits_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha F(\mathbf{U}), \theta_\mu \partial_h^\alpha \mathbf{u}\rangle| \tau^{2m} M_m^2 \lesssim \|\mathbf{u}\|_{X_{\tau,\mu}} + \|\mathbf{b}\|_{X_{\tau,\mu}}, \\ \mathcal{F}(\mathbf{B}) = \sum\limits_{m\ge0}\sum\limits_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha F(\mathbf{B}), \theta_\mu \partial_h^\alpha \mathbf{b}\rangle| \tau^{2m} M_m^2 \lesssim \|\mathbf{u}\|_{X_{\tau,\mu}} + \|\mathbf{b}\|_{X_{\tau,\mu}}.\end{cases} \tag{3.23}$$

3.4. The proof of Proposition 3.1

In this subsection, we may conclude the a priori estimates needed to prove the local existence of solutions to Eqs (2.3)–(2.4). First, inserting (3.10), (3.12), and (3.23) into (3.7), we get

$$\begin{aligned}
&\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|^2_{X_{\tau,\mu}} + \frac{3}{4}\|(\mathbf{u},\mathbf{b})\|^2_{D_{\tau,\mu}} + 2\|(\mathbf{u},\mathbf{b})\|^2_{X_{\tau,\mu}} + 2(1-\frac{5\mu^2}{2})\|(\mathbf{u},\mathbf{b})\|^2_{Z_{\tau,\mu}} \\
&\leq \Big(\dot{\tau} + \frac{C}{\tau^2}\|(\mathbf{u},\mathbf{b})\|_{D_{\tau,\mu}} + C\Big)\|(\mathbf{u},\mathbf{b})\|^2_{Y_{\tau,\mu}} + \frac{C}{\tau^4}\|(\mathbf{u},\mathbf{b})\|^4_{X_{\tau,\mu}} \\
&\quad + C(1+\frac{1}{\tau})\|(\mathbf{u},\mathbf{b})\|^2_{X_{\tau,\mu}} + C\|(\mathbf{u},\mathbf{b})\|_{X_{\tau,\mu}},
\end{aligned} \tag{3.24}$$

where we have used the parameter $\mu \in (0, \frac{\sqrt{10}}{5}]$.

We choose a suitable function $\tau(t)$ such that the following ordinary differential equation holds

$$\begin{cases} \frac{d}{dt}(\tau(t))^{\frac{3}{2}} + \frac{3C}{2}(\|\mathbf{u}\|_{D_{\tau,\mu}} + \|\mathbf{b}\|_{D_{\tau,\mu}}) + \frac{3C}{2}\tau^2 = 0, \\ \tau(0) = \tau_0, \end{cases} \tag{3.25}$$

which yields

$$\begin{aligned}
\tau^3 &\geq \tau_0^3 - 3C\tau_0^2 t - 3C\int_0^t (\|\mathbf{u}(s)\|_{D_{\tau,\mu}} + \|\mathbf{b}(s)\|_{D_{\tau,\mu}})ds \\
&\geq \tau_0^3 - 3C\tau_0^2 t - 3Ct^{1/2}\left(\int_0^t (\|\mathbf{u}(s)\|^2_{D_{\tau,\mu}} + \|\mathbf{b}(s)\|^2_{D_{\tau,\mu}})ds\right)^{1/2}.
\end{aligned} \tag{3.26}$$

We assume that there exists a suitable small time $0 < T^* < T$ such that

$$\int_0^t (\|\mathbf{u}(s)\|^2_{D_{\tau,\mu}} + \|\mathbf{b}(s)\|^2_{D_{\tau,\mu}})ds \leq C\Big(\|\mathbf{u}(0)\|^2_{X_{\tau_0,\mu}} + \|\mathbf{b}(0)\|^2_{X_{\tau_0,\mu}}\Big). \tag{3.27}$$

Therefore, the estimate (3.26) shows that at least for some suitable short time $0 < T_1 < T^*$, we have

$$\tau(t) \geq \frac{\tau_0}{2}. \tag{3.28}$$

Finally, we verify (3.27). Combining (3.24)–(3.25) and (3.28), one concludes that

$$\begin{aligned}
&\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|^2_{X_{\tau,\mu}} + \|(\mathbf{u},\mathbf{b})\|^2_{D_{\tau,\mu}} \\
&\leq C\left(1+\frac{2}{\tau_0}\right)\|(\mathbf{u},\mathbf{b})\|^2_{X_{\tau,\mu}} + \frac{16C}{\tau_0^4}\|(\mathbf{u},\mathbf{b})\|^4_{X_{\tau,\mu}} + C\|(\mathbf{u},\mathbf{b})\|_{X_{\tau,\mu}}.
\end{aligned} \tag{3.29}$$

Hence, if T_* is chosen small enough, (3.27) follows from the comparison principle of ordinary differential equations. This completes the proof of Proposition 3.1. □

4. The proof of Theorem 2.1

With the estimates on solutions to (2.3)–(2.5) from Proposition 3.1 in hand, we may establish local existence in the analytic function space by adapting the argument of [10]. The proof of Theorem 2.1 is then completed by verifying the uniqueness of solutions to (2.3)–(2.5). Next, we are going to prove the uniqueness of solutions to (2.3)–(2.5). Let $(\mathbf{u}^1, \mathbf{b}^1)$ and $(\mathbf{u}^2, \mathbf{b}^2)$ be two pairs of solutions to (2.3)–(2.5) with the same initial data, $(\mathbf{u}^1, \mathbf{b}^1)(0, x, y) = (\mathbf{u}^2, \mathbf{b}^2)(0, x, y) \in X_{\tau_0,\mu}$. Represent the tangential radii of analytic regularity of $(\mathbf{u}^1, \mathbf{b}^1)$ and $(\mathbf{u}^2, \mathbf{b}^2)$ by $\tau^1(t)$ and $\tau^2(t)$, respectively, which satisfy the bounds in Theorem 2.1.

We define $\tau(t)$ as the solution to the following equation:

$$\begin{cases} \frac{d}{dt}\tau^{\frac{3}{2}} + \frac{3C}{2}\Big(\|\mathbf{u}_1\|_{D_{\tau^1,\mu}} + \|\mathbf{b}_1\|_{D_{\tau^1,\mu}}\Big) + \frac{3C}{2}\tau^2 = 0, \\ \tau(0) = \frac{\tau_0}{4}. \end{cases} \tag{4.1}$$

From the estimates given in Section 3, we derive

$$\frac{\tau_0}{8} \le \tau(t) \le \frac{\tau_0}{4} \le \frac{\min\{\tau^1(t), \tau^2(t)\}}{2}$$

for all $t \in [0, T_*]$.

We set $\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$ and $\mathbf{b} = \mathbf{b}^1 - \mathbf{b}^2$, which satisfy

$$\begin{cases} \partial_t \mathbf{u} - \partial_z^2 \mathbf{u} + \mathbb{L}_1(\mathbf{u}) + \mathbb{N}_2(\mathbf{u}) = 0, \\ \partial_t \mathbf{b} - \partial_z^2 \mathbf{b} + \mathbb{L}_1(\mathbf{b}) + \mathbb{N}_2(\mathbf{b}) = 0, \end{cases} \tag{4.2}$$

where

$$\begin{cases} \mathbb{N}_1(\mathbf{u}) = (\mathbf{u}^1 \cdot \nabla_h)\mathbf{u} + (\mathbf{u} \cdot \nabla_h)\mathbf{u}^2 - (\mathbf{b}^1 \cdot \nabla_h)\mathbf{b} - (\mathbf{b} \cdot \nabla_h)\mathbf{b}^2 \\ \qquad -\nabla_h \cdot g(\mathbf{u})\partial_z \mathbf{u}^1 - \nabla_h \cdot g(\mathbf{u}^2)\partial_z \mathbf{u} + \nabla_h \cdot g(\mathbf{b})\partial_z \mathbf{b}^1 + \nabla_h \cdot g(\mathbf{b}^2)\partial_z \mathbf{b}, \\ \mathbb{N}_2(\mathbf{b}) = (\mathbf{u}^1 \cdot \nabla_h)\mathbf{b} + (\mathbf{u} \cdot \nabla_h)\mathbf{b}^2 - (\mathbf{b}^1 \cdot \nabla_h)\mathbf{u} - (\mathbf{b} \cdot \nabla_h)\mathbf{u}^2 \\ \qquad -\nabla_h \cdot g(\mathbf{u})\partial_z \mathbf{b}_h^1 - \nabla_h \cdot g(\mathbf{u}^2)\partial_z \mathbf{b} + \nabla_h \cdot g(\mathbf{b})\partial_z \mathbf{u}_h^1 + \nabla_h \cdot g(\mathbf{b}_h^2)\partial_z \mathbf{u}, \\ \mathbb{L}_1(\mathbf{u}) = (1-\phi)(\mathbf{U} \cdot \nabla_h)\mathbf{u} + (1-\phi)(\mathbf{u} \cdot \nabla_h)\mathbf{U} - z\nabla_h \cdot \mathbf{U}\partial_z \mathbf{u} + \phi'\nabla_h \cdot g(\mathbf{u})\mathbf{U} \\ \qquad +\Phi(z)\nabla_h \cdot \mathbf{U}\partial_z \mathbf{u} - (\mathbf{b} \cdot \nabla_h)\mathbf{B} - (\mathbf{B} \cdot \nabla_h)\mathbf{b} + z\nabla_h \cdot \mathbf{B}\partial_z \mathbf{b}, \\ \mathbb{L}_2(\mathbf{b}) = (1-\phi)(\mathbf{U} \cdot \nabla_h)\mathbf{b} - (1-\phi)(\mathbf{b} \cdot \nabla_h)\mathbf{U} - \phi'\nabla_h \cdot g(\mathbf{b})\mathbf{U} - z\nabla_h \cdot \mathbf{U}\partial_z \mathbf{b} \\ \qquad +\Phi(z)\nabla_h \cdot \mathbf{U}\partial_z \mathbf{b} + (\mathbf{u} \cdot \nabla_h)\mathbf{B} - (\mathbf{B} \cdot \nabla_h)\mathbf{u} + z\nabla_h \cdot \mathbf{B}\partial_z \mathbf{u}. \end{cases}$$

The corresponding initial data and the boundary condition for $\mathbf{u}$ and $\mathbf{b}$ are as follows

$$\begin{cases} \mathbf{u}(0, x, y, z) = \mathbf{b}(0, x, y, z) = 0, \\ \mathbf{u}(t, x, y, 0) = 0, \ \lim\limits_{z \to +\infty} \mathbf{u}(t, x, y, z) = 0, \\ \partial_z \mathbf{b}(t, x, y, 0) = 0, \ \lim\limits_{z \to +\infty} \mathbf{b}(t, x, y, z) = 0. \end{cases} \tag{4.3}$$

Analogous to the a priori estimates in Section 3, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{b}\|_{X_{\tau,\mu}}^2\Big) + \Big(\|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \|\mathbf{b}\|_{D_{\tau,\mu}}^2\Big)$$

$$\leq \Big(\dot{\tau} + \frac{C}{\tau^2}(\|\mathbf{u}^1\|_{D_{\tau,\mu}} + \|\mathbf{b}^1\|_{D_{\tau,\mu}}) + C\Big)\Big(\|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + \|\mathbf{b}\|_{Y_{\tau,\mu}}^2\Big) + \Big(\|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{b}\|_{X_{\tau,\mu}}^2\Big).$$

Because of (4.1), we obtain

$$\dot{\tau} + \frac{C}{\tau^2}(\|\mathbf{u}^1\|_{D_{\tau,\mu}} + \|\mathbf{b}^1\|_{D_{\tau,\mu}}) + C \leq 0.$$

Furthermore, with the help of the initial conditions (4.3) for **u** and **b** and by applying the Gronwall inequality, we can conclude that

$$\|\mathbf{u}\|_{X_{\tau,\mu}} = \|\mathbf{b}\|_{X_{\tau,\mu}} \equiv 0, \text{ for all } t \in [0, T_*]. \tag{4.4}$$

Thus, the proof of the uniqueness of solutions is complete. Combining with Proposition 3.1, we have finished the proof of Theorem 2.1. □

5. Conclusions

This paper aims to investigate the local well-posedness of analytic solutions to the 3D MHD boundary layer equations without structural assumptions. By using a variable transformation involving $\phi(z)$, designing a Gauss weight function θ_μ to control linearly growing terms, and combining various inequality tools to establish a priori estimates, we prove the existence and uniqueness of a local analytic solution for initial data that are real-analytic in tangential variables and satisfy Sobolev regularity in the normal variable (with an analytic radius $\geq \tau_0/2$). To a certain extent, this study breaks through the reliance of traditional methods on structural assumptions, providing theoretical reference for the prediction and control of laminar boundary layers in relevant engineering scenarios. Future research can be extended to global well-posedness, the relaxation of initial data regularity requirements, and more.

Use of AI tools declaration

The author declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author is grateful to the anonymous referees and editors for their valuable suggestions and comments, which have greatly improved the presentation of an earlier version of the manuscript. This paper was in part supported by the Natural Science Foundation of Henan with contract numbers 242300420673 and 252300420940 and the Key Scientific Research Projects of Henan Higher Education Institutions with contract number 26A110022.

Conflict of interest

The author declare there is no conflicts of interest.

References

1. L. Prandtl, *ü*ber Fl*ü*ssigkeitsbewegung bei sehr kleiner Reibung, *Intern. Math. Kongresses*, (1904), 484–491.

2. O. A. Oleinik, On the system of Prandtl equations in boundary layer theory, *Dokl. Akad. Nauk SSSR*, **150** (1963), 28–31.

3. O. A. Oleinik, V. N. Samokhin, *Mathematical Models in Boundary Layer Theory*, Routledge, 1999. https://doi.org/10.1201/9780203749364

4. R. Alexandre, Y. G. Wang, C. J. Xu, T. Yang, Well-posedness of the Prandtl equation in sobolev spaces, *J. Am. Math. Soc.* **28** (2015), 745–784. http://doi.org/10.1090/S0894-0347-2014-00813-4

5. N. Masmoudi, T. K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, *Comm. Pure Appl. Math.*, **68** (2015), 1683–1741. https://doi.org/10.1002/cpa.21595

6. D. X. Chen, Y. X. Wang, Z. F. Zhang, Well-posedness of the Prandtl equation with monotonicity in Sobolev spaces, *J. Differ. Equations*, **264** (2018), 5870–5893. https://doi.org/10.1016/j.jde.2018.01.024

7. C. J. Xu, X. Zhang, Long time well-posedness of Prandtl equations in Sobolev space, *J. Differ. Equations*, **263** (2017), 8749–8803. https://doi.org/10.1016/j.jde.2017.08.046

8. Z. P. Xin, L. Q. Zhang, On the global existence of solutions to the Prandtl's system, *Adv. Math.*, **181** (2004), 88–133. https://doi.org/10.1016/S0001-8708(03)00046-X

9. Z. P. Xin, L. Q. Zhang, J. N. Zhao, Global well-posedness and regularity of weak solutions to the Prandtl's system, *SIAM J. Math. Anal.*, **56** (2024), 3042–3081. https://doi.org/10.1137/23M1547238

10. M. Ignatova, V. Vicol, Almost global existence for the Prandtl boundary layer equations, *Arch. Ration. Mech. Anal.*, **220** (2016), 809–848. https://doi.org/10.1007/s00205-015-0942-2

11. I. Kukavica, V. Vicol, On the local existence of analytical solution to the Prandtl boundary layer equations, *Commun. Math. Sci.*, **11** (2013), 269–292. https://dx.doi.org/10.4310/CMS.2013.v11.n1.a8

12. M. Paicu, P. Zhang, Global existence and the decay of solutions to the Prandtl system with small analytic data, *Arch. Ration. Mech. Anal.*, **241**(2021), 403–446. https://doi.org/10.1007/s00205-021-01654-3

13. P. Zhang, Z. F. Zhang, Long time well-posedness of Prandtl system with small and analytic initial data, *J. Funct. Anal.*, **270** (2016), 2591–2615. https://doi.org/10.1016/j.jfa.2016.01.004

14. W. X. Li, N. Masmoudi, T. Yang, Well-posedness in Gevrey function space for 3D Prandtl equations without structural assumption, *Comm. Pure Appl. Math.*, **75**(2022), 1755–1797. https://doi.org/10.1002/cpa.21989

15. X. H. Pan, C. J. Xu, Global Gevrey-2 solutions of the 3D axially symmetric Prandtl equations, *Anal. Appl.*, **22** (2024), 1195–1253. https://doi.org/10.1142/S0219530524500167

16. X. H. Pan, C. J. Xu, Long-time existence of Gevrey-2 solutions to the 3D Prandtl boundary layer equations, *Commun. Math. Sci.*, **22**(2024), 1203–1250. https://dx.doi.org/10.4310/CMS.2024.v22.n5.a3

17. X. H. Pan, C. J. Xu, Global tangentially analytical solutions of the 3D axially symmetric Prandtl equations, *Chinese Ann. Math. Ser. B*, **45**(2024), 573–596. https://doi.org/10.1007/s11401-024-0029-1

18. X. Y. Lin, T. Zhang, Almost global existence for the 3D Prandtl boundary layer equations, *Acta. Appl. Math.*, **169** (2020), 383–410. https://doi.org/10.1007/s10440-019-00303-y

19. C. J. Liu, Y. G. Wang, T. Yang, Global existence of weak solutions to the three-dimensional Prandtl equations with a special structure, *Discrete Cont. Dyn. Syst.*, **9**(2016), 2011–2029. https://doi.org/10.3934/dcdss.2016082

20. C. J. Liu, Y. G. Wang, T. Yang, A well-posedness theory for the Prandtl equations in three space variables, *Adv. Math.*, **308** (2017), 1074–1126. https://doi.org/10.1016/j.aim.2016.12.025

21. E. S. Baranovskii, The unsteady Navier-Stokes-Boussinesq system with non-zero boundary conditions, *Discrete Contin. Dyn. Syst.*, **45** (2025), 3955–3968. http://doi.org/10.3934/dcds.2025044

22. C. J. Liu, F. Xie, T. Yang, MHD boundary layers theory in Sobolev spaces without monotonicity I: Well-posedness theory, *Commun. Pure Appl. Math.*, **72** (2019), 63–121. https://doi.org/10.1002/cpa.21763

23. D. Gérard-Varet, M. Prestipino, Formal derivation and stability analysis of boundary layer models in MHD, *Z. Angw. Math. Phys.*, **68** (2017), 16. https://doi.org/10.1007/s00033-017-0820-x

24. C. J. Liu, F. Xie, T. Yang, Justification of Prandtl ansatz for MHD boundary layer, *SIAM J. Math. Anal.*, **51** (2019), 2748–2791. https://doi.org/10.1137/18M1219618

25. D. X. Chen, S. Q. Ren, Y. X. Wang, Z. F. Zhang, Long time well-posedness of the MHD boundary layer equation in Sobolev space, *Anal. Theory Appl.*, **36** (2020), 1–18. https://doi.org/10.4208/ata.OA0015

26. C. J. Liu, D. H. Wang, F. Xie, T. Yang, Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces, *J. Funct. Anal.*, **279** (2020), 108637. https://doi.org/10.1016/j.jfa.2020.108637

27. D. X. Chen, X. L. Li, Long time well-posedness of two dimensional magnetohydrodynamic boundary layer equation without resistivity, *Math. Meth. Appl. Sci.*, **46** (2023), 10186–10202. https://doi.org/10.1002/mma.9110

28. W. X. Li, R. Xu, Well-posedness in Sobolev spaces of the two-dimensional MHD boundary layer equations without viscosity, *Electron. Res. Arch.*, **29** (2021), 4243–4255. https://doi.org/10.3934/era.2021082

29. J. C. Gao, D. W. Huang, Z. A. Yao, Boundary layer problems for the two-dimensional inhomogeneous incompressible magnetohydrodynamics equations, *Calc. Var. Partial Differ. Equations*, **60** (2021), 67. https://doi.org/10.1007/s00526-021-01958-y

30. F. Xie, T. Yang, Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime, *SIAM. J. Math. Anal.*, **50** (2018), 5749–5760. https://doi.org/10.1137/18M1174969

31. F. Xie, T. Yang, Lifespan of solutions to MHD boundary layer equations with analytic perturbation of general shear flow, *Acta Math. Appl. Sin E*, **35** (2019), 209–229. https://doi.org/10.1007/s10255-019-0805-y

32. N. Liu, P. Zhang, Global small analytic solutions of MHD boundary layer equations, *J. Differ. Equations*, **281** (2021), 199–257. https://doi.org/10.1016/j.jde.2021.02.003

33. S. X. Li, F. Xie, Global solvability of 2D MHD boundary layer equations in analytic function spaces, *J. Differ. Equations*, **299** (2021), 362–401. https://doi.org/10.1016/j.jde.2021.07.025

34. X. Y. Lin, L. Zou, Well-posedness in Gevrey function space for the 3D axially symmetric MHD boundary layer equations without structural assumption, *Results Math.*, **79** (2024), 91. https://doi.org/10.1007/s00025-023-02112-0

35. W. X. Li, R. Xu, T. Yang, Global well-posedness of a Prandtl model from MHD in Gevrey function spaces, *Acta Math. Sci. Ser. B*, **42** (2022), 2343–2366. https://doi.org/10.1007/s10473-022-0609-7

36. W. X. Li, T. Yang, Well-posedness of the MHD boundary layer system in Gevrey function space without structural assumption, *SIAM J. Math. Anal.*, **53** (2021), 3236–3264. https://doi.org/10.1137/20M1367027

37. Z. Tan, Z. E. Wu, Global small solutions of MHD boundary layer equations in Gevrey function space, *J. Differ. Equations*, **366** (2023), 444–517. https://doi.org/10.1016/j.jde.2023.04.019

38. K. Chen, W. X. Li, T. Yang, Local well-posedness of the three dimensional linearized MHD boundary layer system, *Commun. Math. Anal. Appl.*, **3** (2024), 483–500. https://doi.org/10.4208/cmaa.2024-0020

39. Z. E. Wu, A well-posedness theory for the MHD boundary layer equations in three space variables, *J. Differ. Equations*, **447** (2025), 113650. https://doi.org/10.1016/j.jde.2025.113650

40. Y. F. Chen, J. N. Li, L. Z. Ruan, The local well-posedness of the three-dimensional compressible isentropic boundary layer equations, *Sci. Sin. Math.*, **56** (2026), 1–22. https://doi.org/10.1360/SSM-2025-0201

AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)