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https://www.aimspress.com/journal/era

ERA, 34(3): 1448–1476.
DOI: 10.3934/era.2026066
Received: 05 January 2026
Revised: 30 January 2026
Accepted: 05 February 2026
Published: 12 February 2026

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*Research article*

# Global multiplicity of solutions for a singular $p$-Laplacian quasilinear Schrödinger equation

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**Abstract:** We consider a class of $p$-Laplace quasilinear Schrödinger Equations

$$\begin{cases} -\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = \lambda u^{-\gamma} + u^q \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary, $1 < p < \infty$, $0 < \gamma < 1$, $2p-1 < q \leq 2\cdot p^*-1$ for $p \leq N$, $2p-1 < q < \infty$ for $p > N$, where $p^* = \frac{Np}{N-p}$ if $1 < p < N$, $p^* \in (p, \infty)$ is arbitrarily large if $p = N$, and $p^* = \infty$ if $p > N$. We establish global existence and multiplicity of positive solutions via a new strong comparison principle and a regularity result for weak solutions.

**Keywords:** global multiplicity; quasilinear Schrödinger equations; singular term

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## 1. Introduction and main results

In this paper, we consider the following Problem

$$\begin{cases} -\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = \lambda u^{-\gamma} + u^q \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < \gamma < 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian with $1 < p < \infty$, $2p-1 < q \leq 2\cdot p^* - 1$ for $1 < p \leq N$, $2p-1 < q < \infty$ for $p > N$, where $p^* = \frac{Np}{N-p}$ if $1 < p < N$, $p^* \in (p, \infty)$ is arbitrarily large if $p = N$, and $p^* = \infty$ if $p > N$.

For $p = 2$, Eq (1.1) is closely related to the existence of standing wave solutions for quasilinear Schrödinger equation of the form

$$i\partial_t\psi = -\Delta\psi + W(x)\psi - \widetilde{h}(x, |\psi|^2)\psi - \kappa\Delta[|\psi|^2]\psi, \tag{1.2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W(x)$ is a given potential, $\kappa$ is a real constant, and $\widetilde{h}(x, s)$ is a real-valued function. The quasilinear version of the nonlinear Schrödinger equation arises in several models of different physical phenomena and was used for the superfluid film equation in plasma physics and fluid mechanics; see [1, 2]. For the semilinear case $\kappa = 0$ with singular term, Eq (1.2) has received a lot of attention; see [3–11]. For the modified quasilinear case $\kappa > 0$, Eq (1.2) has been studied mainly in a smooth bounded domain with Dirichlet boundary conditions; see [12, 13]. The presence of the quasilinear term $u\Delta u^2$ means that the corresponding energy function is not well defined in $H_0^1(\Omega)$. Therefore, many scholars have developed new techniques to address this difficulty. By changing the variables, [2] transformed the quasilinear problem into a semilinear one and proved the existence of a positive solution. Other references on changing variables can be found in [14–20]. Using a new perturbation method, the authors of [21] studied a class of subcritical quasilinear problems and also proved the existence of solutions for the critical case in [22,23]. By this method, the authors of [24,25] obtained the existence of infinitely many sign-changing solutions. Using the Nehari manifold method, the authors of [26] proved the existence of sign-changing solutions.

For $p \neq 2$, some scholars have considered the following Equation

$$-\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = f(x, u) \text{ in } \Omega. \tag{1.3}$$

Using critical point theory, the authors of [27,28] established the existence of weak nontrivial solutions for a class of generalized quasilinear Schrödinger equations. Similarly, by applying Morse theory, [29] also demonstrated the existence of such solutions. Furthermore, by combining critical point theory with truncation arguments and the sub-super solution method, [30] obtained multiplicity results for a type of quasilinear Schrödinger Equations with combined nonlinearities, a convex term with arbitrary growth and a singular term, in a bounded smooth domain. While the studies in [27–29] dealt with general nonlinear terms, only [30] addressed a problem with a negative exponent. Furthermore, it is worth noting that in [30], $\lambda$ is in the superlinear term, in contrast to our setting where $\lambda$ belongs to the singular term. Moreover, in these works, there is no information about the non-existence of solution.

Motivated by the absence of results in the literature on Eq (1.1) and the above research results related to Eq (1.1) with $p \neq 2$, we will prove the global existence and multiplicity of positive solutions depending on the positive parameter $\lambda$. The main difficulties are the presence of the term $\frac{p}{2^{p-1}} u\Delta_p(u^2)$ and the singular/superlinear term. We overcome these difficulties mainly by changing the variables for Eq (1.1) and establishing a new sub-super solution theorem.

Our main results are as follows.

**Theorem 1.1** (Global existence)**.** *Assume that* $1 < p < \infty$, $0 < \gamma < 1$, $2p - 1 < q \leq 2 \cdot p^* - 1$ *for* $p \leq N$, *and* $2p - 1 < q < \infty$ *for* $p > N$. *Then there exists* $0 < \Lambda < \infty$ *such that Eq* (1.1) *admits at least one positive solution* $u_\lambda \in C^1(\overline{\Omega})$ *for each* $\lambda \in (0, \Lambda]$, *and has no solution for* $\lambda > \Lambda$.

**Theorem 1.2** (Global multiplicity)**.** *Assume that* $0 < \gamma < 1$, $p, q$ *satisfy* $1 < p < \infty$, *and* $2p - 1 < q < 2 \cdot p^* - 1$, *or* $2 \leq p < N$ *and* $q = 2 \cdot p^* - 1$. *Then Eq* (1.1) *admits at least two positive solutions* $u_\lambda \in C^1(\overline{\Omega})$ *for each* $\lambda \in (0, \Lambda)$.

**Remark 1.1.** *We clarify that in this work, "global" refers to the parameter* $\lambda$*: Theorem 1.1 establishes a global existence result for* $\lambda \in (0, \Lambda]$ *and a non-existence result for* $\lambda \in (\Lambda, +\infty)$*, while Theorem 1.2 provides a global multiplicity result for* $\lambda \in (0, \Lambda)$*. This contrasts with local results that hold only when* $\lambda$ *is close to a specific value.*

The paper is organized as follows. Section 2 is devoted to preliminary results, including a regularity theorem, a sub-super solution method for a modified problem of (1.1), and other related lemmas. Theorem 1.1 is proved in Section 3, and Theorem 1.2 is proved in Section 4. Finally, the regularity of the auxiliary function is verified in the Appendix.

## 2. Preliminaries

First, we introduce a variational framework for Eq (1.1). We observe that (1.1) is the Euler-Lagrange equation associated with the energy functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega (1+p|u|^p)|\nabla u|^p dx - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{1}{1+q}\int_\Omega u^{1+q}dx.$$

It is easy to see that $I_\lambda$ is not well defined for all $u \in W_0^{1,p}(\Omega)$. To overcome this difficulty, we generalize the change of variables developed in [31, 32], that is,

$$w := g^{-1}(u),$$

where $g$ is defined by

$$g'(t) = \frac{1}{(1+p|g(t)|^p)^{\frac{1}{p}}},\ t \geq 0,\ \text{and}\ g(t) = -g(-t),\ t < 0.$$

The function $g$ satisfies the following properties, as proved in [27]:

**Lemma 2.1.** *The function g defined above satisfies:*

(*i*) $g(0) = 0$;
(*ii*) $g''(t) = -p(g'(t))^{p+2}|g(t)|^{p-2}g(t), t > 0$, $g''(t) < 0$ *if* $t > 0$, *and* $g''(t) > 0$ *if* $t < 0$;
(*iii*) *g is uniquely defined,* $C^\infty$, *and invertible;*
(*iv*) $0 < g'(t) \leq 1$ *for all* $t \in \mathbb{R}$;
(*v*) $\frac{1}{2}g(t) \leq tg'(t) \leq g(t)$ *for all* $t \in \mathbb{R}$;
(*vi*) $|g(t)| \leq |t|$ *for all* $t \in \mathbb{R}$;
(*vii*) $\lim_{t\to 0}\frac{g(t)}{t} = 1$, $\lim_{t\to\infty}\frac{g(t)}{\sqrt{t}} = \sqrt{2}p^{-\frac{1}{2p}} := K_0$, $|g(t)| \leq K_0|t|^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}$;
(*viii*) *there exists* $C > 0$ *such that* $|g(t)| \geq C|t|$ *for* $|t| \leq 1$, $|g(t)| \geq C|t|^{\frac{1}{2}}$ *for* $|t| \geq 1$;
(*ix*) $|g(t)g^{'}(t)| < K_0^2$ *for all* $t \in \mathbb{R}$.

By performing the change of variable $w = g^{-1}(u)$, it follows from Lemma 2.1 that $u \in W_0^{1,p}(\Omega)$ is a solution of (1.1) if and only if $w \in W_0^{1,p}(\Omega)$ is a solution of the equation

$$-\Delta_p w = [\lambda g(w)^{-\gamma} + g(w)^q]g^{'}(w) \text{ in } \Omega, \tag{2.1}$$

with $\operatorname{ess\,inf}_{\mathcal{K}} w > 0$ for every compact subset $\mathcal{K} \subset \Omega$, that is,

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla\phi dx = \int_\Omega [\lambda g(w)^{-\gamma} + g(w)^q]g^{'}(w)\phi dx,\ \forall \phi \in C_c^\infty(\Omega). \tag{2.2}$$

The energy functional corresponding to problem (2.1) is

$$J_\lambda(w) = \frac{1}{p}\int_\Omega |\nabla w|^p dx - \frac{\lambda}{1-\gamma}\int_\Omega |g(w)|^{1-\gamma}dx - \frac{1}{1+q}\int_\Omega |g(w)|^{1+q}dx. \tag{2.3}$$

The Lemma 2.1 implies that $J_\lambda$ is well defined and continuous in $W_0^{1,p}(\Omega)$, which is endowed with the standard norm $\|w\| := (\int_\Omega |\nabla w|^p dx)^{\frac{1}{p}}$.

By definition, $\underline{w} \in W_0^{1,p}(\Omega)$ is called a sub-solution of (2.1) if $\underline{w} > 0$ in $\Omega$ and satisfies

$$\int_\Omega |\nabla \underline{w}|^{p-2}\nabla \underline{w}\nabla\phi dx \le \int_\Omega \Big[\lambda g(\underline{w})^{-\gamma} + g(\underline{w})^q\Big] g'(\underline{w})\phi dx,\ \ \forall\, 0 \le \phi \in C_c^\infty(\Omega).$$

Similarly, $\overline{w} \in W_0^{1,p}(\Omega)$ is a super-solution of (2.1) if the above inequality is reversed.

To prove the main results, it is necessary to establish the following regularity result for a weak solution of Eq (2.1). The proof is based on Theorem B.1 in [9] and the following lemmas.

**Lemma 2.2.** *Let* $1 < p < \infty$, $0 < \gamma < 1$, $2p - 1 < q \le 2 \cdot \underline{p^*} - 1$ *for* $p \le N$, *and* $2p - 1 < q < \infty$ *for* $p > N$. *Then any weak solution to Eq* (2.1) *belongs to* $C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0, 1)$.

**Lemma 2.3.** *Each positive weak solution* $w$ *of Eq* (2.1) *satisfies* $w \ge \epsilon_\lambda\phi_1$ *a.e. in* $\Omega$, *where* $\epsilon_\lambda$ *is a constant independent of* $w$. *Moreover, for any* $v \in W_0^{1,p}(\Omega)$, *we have* $g(w)^{-\gamma}g'(w)v \in L^1(\Omega)$ *and*

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v dx = \lambda \int_\Omega g(w)^{-\gamma}g'(w)v dx + \int_\Omega g(w)^q g'(w)v dx. \tag{2.4}$$

*Proof.* Let $w$ be a solution of (2.1). By definition, $w$ satisfies $\operatorname{ess\,inf}_{\mathcal{K}} w > 0$ for every compact subset $\mathcal{K} \subset \Omega$. By the density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ and Fatou's lemma, we obtain the following inequality

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v dx \ge \lambda \int_\Omega g(w)^{-\gamma}g'(w)v dx + \int_\Omega g(w)^q g'(w)v dx \tag{2.5}$$

for any $v \in W_0^{1,p}(\Omega)$ that satisfies $v \ge 0$ a.e. in $\Omega$. This implies that

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v dx \ge \lambda \int_\Omega g(w)^{-\gamma}g'(w)v dx \tag{2.6}$$

for $0 \le v \in W_0^{1,p}(\Omega)$. Note that $\underline{w}_\lambda \in W_0^{1,p}(\Omega)$ is the unique solution of (3.1) and satisfies $\underline{w}_\lambda \ge \epsilon_\lambda\phi_1$, a fact that will be proved in the following Lemma 3.1. Applying the weak comparison principle to (3.1) and (2.6), we obtain $w \ge \underline{w}_\lambda$ a.e. in $\Omega$, which implies $w \ge \epsilon_\lambda\phi_1$ a.e. in $\Omega$. Now, applying Hardy's inequality to the term $\int_\Omega g(w)^{-\gamma}g'(w)v_k dx$ as $k \to \infty$, where $v_k \subset C_c^\infty(\Omega)$ satisfies $0 \le v_k \to v$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$, we can justify passing to the limit to obtain equality (2.4) for any $v \in W_0^{1,p}(\Omega)$ that satisfies $v \ge 0$ a.e. in $\Omega$. Finally, for an arbitrary function $v \in W_0^{1,p}(\Omega)$, we write $v = v^+ - v^-$, where $v^+ = \max\{v, 0\}$ and $v^- = -\min\{v, 0\}$ are the positive and negative parts of $v$. Since $v^+, v^- \in W_0^{1,p}(\Omega)$ and $\nabla v = \nabla v^+ - \nabla v^-$, we conclude that (2.4) holds for any $v \in W_0^{1,p}(\Omega)$.

**Lemma 2.4.** $w \in L^\infty(\Omega)$ *for any positive weak solution* $w$ *of* (2.1).

*Proof.* To prove that $w \in L^{\infty}(\Omega)$, we first show that any positive weak solution $w$ of (2.1) satisfies

$$\int_{\Omega} |\nabla(w-1)^{+}|^{p-2}\nabla(w-1)^{+} \cdot \nabla v dx \leq \int_{\Omega} (\lambda + K_0^{q+1} w^{\frac{q-1}{2}}) v dx \leq C \int_{\Omega} (\lambda + w^{\frac{q-1}{2}}) v dx \tag{2.7}$$

for $v \in C_c^{\infty}(\Omega)$ with $v \geq 0$. In fact, let $\psi : \mathbb{R} \mapsto [0,1]$ be a $C^1$ cut-off function such that $\psi(t) = 0$ for $t \leq 0$, $\psi'(t) \geq 0$ for $0 \leq t \leq 1$, and $\psi(t) = 1$ for $t \geq 1$. For any $\epsilon > 0$, we define $\psi_{\epsilon}(s) = \psi(\frac{(s-1)}{\epsilon})$ for $s \in \mathbb{R}$. Thus, we have $\psi_{\epsilon} \circ w \in W_0^{1,p}(\Omega)$ with $\nabla(\psi_{\epsilon} \circ w) = (\psi'_{\epsilon} \circ w)\nabla w$. Taking the function $\phi = (\psi_{\epsilon} \circ w)v$ as a test function in (2.2), where $v \in C_c^{\infty}(\Omega)$ with $v \geq 0$, we infer that

$$\int_{\Omega} |\nabla w|^{p-2}\nabla w \nabla[(\psi_{\epsilon} \circ w)v] dx = \int_{\Omega} [\lambda g(w)^{-\gamma} + g(w)^{q}] g'(w)(\psi_{\epsilon} \circ w) v dx.$$

Then

$$\begin{aligned}
&\int_{\Omega} |\nabla w|^{p}(\psi'_{\epsilon} \circ w) v dx + \int_{\Omega} |\nabla w|^{p-2}(\nabla w \cdot \nabla v)(\psi_{\epsilon} \circ w) \\
&= \int_{\Omega} [\lambda g(w)^{-\gamma} + g(w)^{q}] g'(w)(\psi_{\epsilon} \circ w) v dx \\
&\leq \int_{\Omega} [\lambda w^{-\gamma}(g'(w))^{1-\gamma} + K_0^{q+1} w^{\frac{q-1}{2}}](\psi_{\epsilon} \circ w) v dx \\
&\leq \int_{\Omega} [\lambda w^{-\gamma} + K_0^{q+1} w^{\frac{q-1}{2}}](\psi_{\epsilon} \circ w) v dx.
\end{aligned}$$

Letting $\epsilon \to 0^{+}$, we conclude that (2.7) is true. Then, as in the arguments of [9], we complete the proof.

**Lemma 2.5.** *Let $w$ be a positive weak solution of* (2.1)*. Then $w$ satisfies $c_{\lambda} d(x) \leq w \leq k_{\lambda} d(x)$ a.e. in $\Omega$, where* $0 < c_{\lambda} \leq k_{\lambda} < \infty$ *are constants independent of $w$.*

*Proof.* Let $w \in W_0^{1,p}(\Omega)$ be a positive weak solution of (2.1). By Lemma 2.3, we have $w(x) \geq \underline{w}_{\lambda} \geq \epsilon_{\lambda}\phi_1(x) \geq \epsilon_{\lambda} l d(x)$ for a.e. $x \in \Omega$. Taking $c_{\lambda} = \epsilon_{\lambda} l > 0$, we obtain $w \geq c_{\lambda} d(x)$ a.e. in $\Omega$.
Now, it remains to prove that there exists $k_{\lambda} \geq c_{\lambda}$ such that $w \leq k_{\lambda} d(x)$. Note that $w$ also belongs to $L^{\infty}(\Omega)$ by Lemma 2.4. It follows from Lemma 2.1 that

$$\begin{aligned}
g(w)^{q} g'(w) &= g(w)^{q-1} g(w) g'(w) \leq K_0^{q-1} w^{\frac{q-1}{2}} K_0^{2} = K_0^{q+1} w^{\frac{q-1}{2}} \\
&= K_0^{q+1} \frac{w^{\frac{q-1}{2}+\gamma}}{w^{\gamma}} \leq K_0^{q+1} \frac{\|w\|_{L^{\infty}(\Omega)}^{\frac{q-1}{2}+\gamma}}{w^{\gamma}}
\end{aligned}$$

and

$$g(w)^{-\gamma} g'(w) \leq w^{-\gamma}(g'(w))^{1-\gamma} \leq w^{-\gamma} \quad \text{for } w > 0.$$

Applying these estimates to the right-hand side of (2.1), we obtain

$$\begin{cases} -\Delta_p w \leq (\lambda + K_0^{q+1} \|w\|_{L^{\infty}(\Omega)}^{\frac{q-1}{2}+\gamma}) w^{-\gamma} & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \quad w > 0 \text{ in } \Omega. \end{cases} \tag{2.8}$$

Making the substitution

$$v = (1 + \lambda^{-1} K_0^{q+1} \|w\|_{L^{\infty}(\Omega)}^{\frac{q-1}{2}+\gamma})^{-\frac{1}{p-1+\gamma}} w,$$

we find that (2.8) is equivalent to

$$\begin{cases} -\Delta_p v \leq \lambda v^{-\gamma} \ \text{ in } \Omega, \\ v|_{\partial\Omega} = 0, \quad v > 0 \text{ in } \Omega. \end{cases}$$

Now, by arguments similar to those in [9], we obtain the desired result.

We now establish the following new strong comparison principle, which will be used frequently in our subsequent analysis.

**Lemma 2.6.** *Let* $w,\ v \in C^{1,\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$ *satisfy* $w,\ v \gneqq 0$ *and*

$$\begin{cases} -\Delta_p w - \lambda g(w)^{-\gamma} g'(w) = f \\ -\Delta_p v - \lambda g(v)^{-\gamma} g'(v) = h, \end{cases}$$

*with* $w = v = 0$ *on* $\partial\Omega$*, where* $f, h \in C(\Omega)$ *with* $0 \leq f < h$ *pointwise in* $\Omega$*. Then*

$$0 < w < v \text{ in } \Omega \ \text{ and } \ \frac{\partial v}{\partial \nu} < \frac{\partial w}{\partial \nu} < 0 \text{ on } \partial\Omega. \tag{2.9}$$

*Proof.* Assume that $\Omega_0 = \{x \in \Omega : w(x) > v(x)\} \neq \emptyset$. For $\epsilon > 0$ small enough, we define

$$w_\epsilon(x) = w(x) + \epsilon, \ \ v_\epsilon(x) = v(x) + \epsilon$$

and

$$\varphi_\epsilon = \frac{(w_\epsilon^2 - v_\epsilon^2)^+}{w_\epsilon}, \ \ \psi_\epsilon = \frac{(w_\epsilon^2 - v_\epsilon^2)^+}{v_\epsilon}.$$

Then we obtain

$$\nabla\varphi_\epsilon = \nabla w - 2\frac{v+\epsilon}{w+\epsilon}\nabla v + \frac{(v+\epsilon)^2}{(w+\epsilon)^2}\nabla w$$

and

$$\nabla\psi_\epsilon = 2\frac{w+\epsilon}{v+\epsilon}\nabla w - \frac{(w+\epsilon)^2}{(v+\epsilon)^2}\nabla v - \nabla v.$$

Taking $\varphi_\epsilon$ and $\psi_\epsilon$ as test functions and using Picone's inequality, we get

$$\begin{aligned} &\int_\Omega \Big[|\nabla w|^{p-2}\nabla w \nabla\varphi_\epsilon - |\nabla v|^{p-2}\nabla v \nabla\psi_\epsilon\Big]dx \\ &= \int_{\Omega_0} [|\nabla w|^{p-2}\nabla w \nabla\varphi_\epsilon - |\nabla v|^{p-2}\nabla v \nabla\psi_\epsilon]dx \\ &= \int_{\Omega_0} \Big[|\nabla w|^p - 2\frac{w+\epsilon}{v+\epsilon}\nabla w |\nabla v|^{p-2}\nabla v + \frac{(w+\epsilon)^2}{(v+\epsilon)^2}|\nabla v|^p\Big]dx \\ &\quad + \int_{\Omega_0} \Big[|\nabla v|^p - 2\frac{v+\epsilon}{w+\epsilon}|\nabla w|^{p-2}\nabla w \nabla v + \frac{(v+\epsilon)^2}{(w+\epsilon)^2}|\nabla w|^p\Big]dx \\ &\geq 0. \end{aligned} \tag{2.10}$$

On the other hand, from Lemma 2.1(ii),(iv), we know that $t \mapsto g(t)^{-\gamma} g'(t)$ is decreasing for $t > 0$. Then, for all $\epsilon > 0$, we have

$$\begin{aligned}
&\int_{\Omega} \Big[\lambda g(v)^{-\gamma} g'(v) \frac{(w_\epsilon^2 - v_\epsilon^2)^+}{v_\epsilon} - \lambda g(w)^{-\gamma} g'(w) \frac{(w_\epsilon^2 - v_\epsilon^2)^+}{w_\epsilon}\Big] dx \\
&= \int_{\Omega_0} \Big[\lambda g(v)^{-\gamma} g'(v) \frac{(w_\epsilon^2 - v_\epsilon^2)^+}{v_\epsilon} - \lambda g(w)^{-\gamma} g'(w) \frac{(w_\epsilon^2 - v_\epsilon^2)^+}{w_\epsilon}\Big] dx \\
&= \int_{\Omega_0} \lambda (w_\epsilon^2 - v_\epsilon^2)^+ \Big[\frac{g(v)^{-\gamma} g'(v)}{v_\epsilon} - \frac{g(w)^{-\gamma} g'(w)}{w_\epsilon}\Big] dx \\
&> \int_{\Omega_0} \lambda (w_\epsilon^2 - v_\epsilon^2)^+ g(v)^{-\gamma} g'(v) \Big(\frac{1}{v_\epsilon} - \frac{1}{w_\epsilon}\Big) dx > 0.
\end{aligned} \tag{2.11}$$

However, by assumption we have $f < h$. This fact together with (2.10) and (2.11) leads to a contradiction. Thus $\Omega_0 = \emptyset$, that is, $w(x) \le v(x)$ in $\Omega$.

Now, we prove that $w(x) < v(x)$ in $\Omega$. In fact, from the strong maximum principle of Theorem 5 in [33], it follows that $w, v > 0$ in $\Omega$ and $\frac{\partial w}{\partial \nu}, \frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$. Moreover, since $w, v \in C^1(\overline{\Omega})$, there exist $0 < l < L$ such that $ld(x) \le w(x), v(x) \le Ld(x)$ near $\partial\Omega$, where $d(x) := \text{dist}(x, \partial\Omega) \equiv \inf_{y \in \partial\Omega} |x - y|$. Together with the fact that $w(x) \le v(x)$, this yields, near $\partial\Omega$,

$$ld(x) \le w(x) \le v(x) \le Ld(x). \tag{2.12}$$

Following the approach in Proposition 2.4 of [34], we consider the open $\delta$-neighborhood $\Omega_\delta \subset \Omega$ of the boundary $\partial\Omega$,

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\} \text{ for } \delta > 0 \text{ sufficiently small.}$$

We set $z = v - w$, where $0 \le z \in C^{1,\alpha}(\overline{\Omega})$ with $z = 0$ on $\partial\Omega$. Then there exists $\delta > 0$ small enough such that

$$-\operatorname{div}(A(x)\nabla z) - \lambda B(x) z = -\sum_{i,j=1}^{N} (a_{i,j}(x) \frac{\partial z}{\partial x_j}) - \lambda B(x) z = h - f > 0, \tag{2.13}$$

where for $x \in \Omega_\delta$ and $i, j = 1, 2, \cdots, N$,

$$\begin{aligned}
a_{i,j}(x) = \int_0^1 & |(1-t)\nabla w(x) + t\nabla v(x)|^{p-2} \\
& \times \Big[\delta_{i,j} + (p-2) \frac{\frac{\partial}{\partial x_i}((1-t)w(x) + tv(x)) \frac{\partial}{\partial x_j}((1-t)w(x) + tv(x))}{|(1-t)\nabla w(x) + t\nabla v(x)|^2}\Big] dt,
\end{aligned}$$

with $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \ne j$. These coefficients belong to $C^{0,\alpha}(\overline{\Omega}_\delta)$ and form the coefficients of a uniformly elliptic operator in $\Omega_\delta$. The coefficient $B(x)$ is given by

$$\begin{aligned}
B(x) = & -\int_0^1 g^{-\gamma-1}((1-t)w(x) + tv(x))(g'((1-t)w(x) + tv(x))^2 \\
& \times \Big[\gamma + pg^p((1-t)w(x) + tv(x))(g'((1-t)w(x) + tv(x))^p\Big] < 0.
\end{aligned}$$

It is straightforward to show that $B(x)$ satisfies the conditions of Lemma 2.7 in [35] by virtue of inequalities (2.12). Applying the strong maximum principle to (2.13), we obtain $w(x) < v(x)$ in $\Omega_\delta$.

Now, it remains to verify that $w(x) < v(x)$ also holds in $\Omega_\delta^c$. We first show that $w(x) < v(x)$ in $\widetilde{\Omega} = \Omega \backslash \overline{\Omega}_{\delta'}$ for some $\delta' \in (0, \delta)$. Since $z > 0$ in $\Omega_\delta$, there exists $\varepsilon > 0$ such that $z \geq \varepsilon$ on $\partial\widetilde{\Omega} \subset \Omega_\delta$. On the other hand, since $f, h \in C(\Omega)$ and $0 \leq f < h$ pointwise everywhere in $\Omega$, we can choose $\varepsilon > 0$ sufficiently small such that

$$\lambda g(w)^{-\gamma} g'(w) - \lambda g(w+\varepsilon)^{-\gamma} g'(w+\varepsilon) \leq h - f \quad \text{in } \widetilde{\Omega}.$$

Consequently, we have $w + \varepsilon \leq v$ on $\partial\widetilde{\Omega}$ and

$$-\Delta_p(w+\varepsilon) - \lambda g(w+\varepsilon)^{-\gamma} g'(w+\varepsilon) \leq f + (h - f) = h = -\Delta_p v - \lambda g(v)^{-\gamma} g'(v).$$

Applying the weak comparison principle established above, we obtain that $w + \varepsilon \leq v$ in $\widetilde{\Omega}$, which implies $w(x) < v(x)$ in $\widetilde{\Omega}$. Finally, since $\Omega = \Omega_\delta \cup \widetilde{\Omega}$, we conclude that $w(x) < v(x)$ throughout $\Omega$. This completes the proof.

## 3. The existence of solution

In order to show the existence of solutions to (2.1), we consider the auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p w = \lambda g(w)^{-\gamma} g'(w) \ \text{ in } \Omega \\ w|_{\partial\Omega} = 0, \quad w > 0 \text{ in } \Omega. \end{cases} \tag{3.1}$$

Our analysis begins by recalling key properties of the first eigenfunction of the $p$-Laplacian. Let $\phi_1 \in W_0^{1,p}(\Omega)$ be the positive first eigenfunction corresponding to the principal eigenvalue $\lambda_1$, i.e., satisfying

$$-\Delta_p \phi_1 = \lambda_1 |\phi_1|^{p-2} \phi_1$$

and normalized so that $\int_\Omega \phi_1^p dx = 1$. By the strong maximum principle and the boundary point principle (Theorem 5 in [33]), we have $\phi_1 > 0$ in $\Omega$ and $\frac{\partial\phi_1}{\partial\nu} < 0$ on $\partial\Omega$. Moreover, the regularity $\phi_1 \in C^1(\overline{\Omega})$ implies the existence of constants $0 < l < L$ such that

$$ld(x) \leq \phi_1(x) \leq Ld(x),$$

where $d(x) := \text{dist}(x, \partial\Omega)$.

With these facts at hand, we can now prove the following fundamental lemma concerning the auxiliary problem.

**Lemma 3.1.** *Eq* (3.1) *has a unique solution $\underline{w}_\lambda$ in $W_0^{1,p}(\Omega)$ in the sense of distributions, satisfying $\underline{w}_\lambda \geq \epsilon_\lambda \phi_1$ a.e. in $\Omega$, where $\epsilon_\lambda > 0$ is a constant. Moreover, $\underline{w}_\lambda$ is a weak solution according to Lemma 2.3.*

*Proof.* We define $\widetilde{J}_\lambda : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ by

$$\widetilde{J}_\lambda(w) = \frac{1}{p}\int_\Omega |\nabla w|^p dx - \frac{\lambda}{1-\gamma}\int_\Omega g(w^+)^{1-\gamma} dx.$$

It is easy to check that $\widetilde{J}_\lambda(w)$ is coercive and weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. Hence, there exists a global minimizer $w_0 \in W_0^{1,p}(\Omega)$ with $w_0 \not\equiv 0$ in $\Omega$, since $\widetilde{J}_\lambda(0) = 0 > \widetilde{J}_\lambda(\epsilon\phi_1)$ for $\epsilon$ sufficiently small. Moreover, $\widetilde{J}_\lambda(|w_0|) \leq \widetilde{J}_\lambda(w_0)$. Clearly, $\widetilde{J}_\lambda(|w_0|) = \widetilde{J}_\lambda(w_0)$ holds if and only if $w_0 \geq 0$ a.e. in $\Omega$. Thus, any global minimizer $w_0$ of $\widetilde{J}_\lambda$ must satisfy $w_0 \geq 0$ a.e. in $\Omega$. By setting

$$W_0^{1,p}(\Omega)_+ = \{w \in W_0^{1,p}(\Omega) : w \geq 0 \text{ a.e. in } \Omega\},$$

we have $w_0 \in W_0^{1,p}(\Omega)_+$.

Now, we prove that $w_0 \geq \epsilon_\lambda\phi_1$. First, we compute the Gâteaux derivative $\widetilde{J}'_\lambda(\epsilon\phi_1)$ of $\widetilde{J}_\lambda$ at $\epsilon\phi_1$. Using Lemma 2.1(iv),(viii), we obtain

$$\begin{aligned}
\widetilde{J}'_\lambda(\epsilon\phi_1) &= -\Delta_p(\epsilon\phi_1) - \lambda g(\epsilon\phi_1)^{-\gamma} g'(\epsilon\phi_1) \\
&= \lambda_1(\epsilon\phi_1)^{p-1} - \lambda g(\epsilon\phi_1)^{-\gamma} g'(\epsilon\phi_1) \\
&\leq \frac{\lambda_1}{C_1} g(\epsilon\phi_1)^{p-1} - \lambda g(\epsilon\phi_1)^{-\gamma} g'(\epsilon\phi_1) \\
&\leq g(\epsilon\phi_1)^{-\gamma}\Big[\frac{\lambda_1}{C_1} g(\epsilon\phi_1)^{p-1+\gamma} - C_2\lambda\Big] \\
&\leq g(\epsilon\phi_1)^{-\gamma}\Big[\frac{\lambda_1}{C_1} (\epsilon\phi_1)^{p-1+\gamma} - C_2\lambda\Big] \\
&\leq -\frac{\lambda}{2} C_3 g(\epsilon\phi_1)^{-\gamma} < 0,
\end{aligned}$$

where $C_1$ is given in Lemma 2.1(viii), $C_2 \in (0, 1]$, $C_3$ is a positive constant, $\epsilon > 0$ sufficiently small, and $\epsilon \leq \epsilon_\lambda$. Now, assume that $w_0 \geq \epsilon_\lambda\phi_1$ does not hold, that is, $v = (w_0 - \epsilon_\lambda\phi_1)^- = (\epsilon_\lambda\phi_1 - w_0)^+ \neq 0$. We define

$$\Omega^+ = \{x \in \Omega : v(x) > 0\}.$$

Let $\xi(t) = \widetilde{J}_\lambda(w_0 + tv)$ for $t \geq 0$. It follows from the convexity of the restriction of $\widetilde{J}_\lambda$ to $W_0^{1,p}(\Omega)_+$ that $\xi(t) \geq \xi(0)$ for $t \geq 0$. Note that $w_0 + tv \geq \max\{w_0, t\epsilon_\lambda\phi_1\} \geq t\epsilon_\lambda\phi_1$ for $t > 0$. Then $\widetilde{J}'_\lambda(w_0 + tv)$ exists and satisfies $\xi'(t) = \langle \widetilde{J}'_\lambda(w_0 + tv), v\rangle$. It is easy to see that $\xi'(t)$ is non-negative and non-decreasing for $t > 0$. We conclude that for $0 < t < 1$,

$$\begin{aligned}
0 &\leq \xi'(1) - \xi'(t) = \langle \widetilde{J}'_\lambda(w_0 + v) - \widetilde{J}'_\lambda(w_0 + tv), v\rangle \\
&= \int_{\Omega^+} \widetilde{J}'_\lambda(\epsilon_\lambda\phi_1) v dx - \xi'(t) \leq -\frac{\lambda}{2} C_3 \int_{\Omega^+} g(\epsilon_\lambda\phi_1)^{-\gamma} v dx < 0,
\end{aligned}$$

which is a contradiction. Thus we have $v \equiv 0$ in $\Omega$, i.e., $w_0 \geq \epsilon_\lambda\phi_1$ a.e. in $\Omega$. Together with the fact that $\widetilde{J}_\lambda$ is strictly convex in $W_0^{1,p}(\Omega)_+$, we conclude that $w_0$ is the only critical point of $\widetilde{J}_\lambda$ in $W_0^{1,p}(\Omega)_+$. Therefore, $\underline{w}_\lambda = w_0$ is the unique weak solution of (3.1). Moreover, $\widetilde{J}_\lambda(\underline{w}_\lambda) \leq \widetilde{J}_\lambda(\epsilon_\lambda\phi_1) < 0$. This completes the proof.

**Definition 3.1.** $\Lambda := inf\{\lambda > 0 :$ (2.1) *has no weak solution*$\}$.

**Lemma 3.2.** *Assume* $1 < p < \infty$, $0 < \gamma < 1$, $2p - 1 < q \leq 2p^* - 1$ *for* $p \leq N$, *and* $2p - 1 < q < \infty$ *for* $p > N$. *Then* $0 < \Lambda < \infty$.

*Proof.* We only consider the critical case $q = 2p^* - 1$. The proof of subcritical case $q < 2p^* - 1$ for $p \le N$ and $q < \infty$ for $p > N$ is simpler since the energy functional $\overline{J}_\lambda$ defined below is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. Set

$$f_\lambda(x,t) = \begin{cases} \lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) + g(\underline{w}_\lambda)^q g'(\underline{w}_\lambda), & \text{if } t \le \underline{w}_\lambda, \\ \lambda g(t)^{-\gamma} g'(t) + g(t)^q g'(t), & \text{if } t > \underline{w}_\lambda, \end{cases} \tag{3.2}$$

where $\underline{w}_\lambda$ is the unique solution of (3.1). Denote $F_\lambda(x,t) = \int_0^t f_\lambda(x,s)ds$. We define $\overline{J}_\lambda : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ by

$$\overline{J}_\lambda(w) = \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega F_\lambda(x,w)dx. \tag{3.3}$$

From Lemma A1 in appendix, $\overline{J}_\lambda$ is $C^1(W_0^{1,p}(\Omega),\mathbb{R})$. Now, we prove that $\Lambda > 0$. Consider the following minimization problem

$$E_\lambda = \min_{w \in \overline{B}_r} \overline{J}_\lambda(w), \tag{3.4}$$

where $B_r \subset W_0^{1,p}(\Omega)$ is the ball of radius $r$ centered at the origin. It is clear that $E_\lambda > -\infty$. Note that there exists $r > 0$ small enough such that for every $w \in \partial B_r$,

$$\int_\Omega \left(\frac{1}{p}|\nabla w|^p - \frac{1}{q+1}|g(w)|^{q+1}\right)dx \ge \int_\Omega \left(\frac{1}{p}|\nabla w|^p - \frac{K_0^{q+1}}{q+1}|w|^{\frac{q+1}{2}}\right)dx > 0.$$

Then we can fix such $r > 0$ so that the term $\frac{\lambda}{1-\gamma}\int_\Omega g(w)^{1-\gamma}dx$ can be made arbitrarily small by choosing $\lambda > 0$ small enough. Hence there exist $r$ and $\lambda$ such that

$$\min_{w \in \partial B_r} \overline{J}_\lambda(w) > 0.$$

Moreover, since $\overline{J}_\lambda(tw) < 0$ for small $t$, we obtain $E_\lambda < 0$. Let $\{w_j\}_{j=1}^\infty \subset B_r$ be a minimizing sequence, so that $\overline{J}_\lambda(w_j) \to E_\lambda$ as $j \to \infty$. It follows that $\{w_j\}_{j=1}^\infty$ satisfies $\mathrm{dist}(w_j, \partial B_r) \ge \delta_0$ for some $\delta_0 > 0$. Thus, there exists $0 < r_0 < r$ such that

$$w_j \in B_{r_0}. \tag{3.5}$$

By Ekeland's variational principle, there exist $r_0 \le r_1 < r$ and another sequence $\{v_j\}_{j=1}^\infty \subset B_{r_1}$ such that

$$\mathrm{dist}(w_j, v_j) \le \frac{1}{j}, \;\; \overline{J}_\lambda(w_j) \le \overline{J}_\lambda(v_j), \;\; \text{and} \;\; \overline{J}'_\lambda(v_j) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } j \to \infty. \tag{3.6}$$

This implies that $\{v_j\}_{j=1}^\infty$ is a minimizing sequence for $E_\lambda$ and that, up to a subsequence, $v_j \rightharpoonup \widetilde{w}_\lambda$ as $j \to \infty$ with $\widetilde{w}_\lambda \in \overline{B}_{r_1}$.

In addition, from (3.6), we have

$$-\Delta_p(v_j) - f_\lambda(x, v_j) = o_j(1) \;\text{ in } W^{-1,p'}(\Omega). \tag{3.7}$$

By the properties of $g$, we have

$$\begin{aligned} (\max\{\underline{w}_\lambda, v_j\})^{-\gamma} &\ge (\max\{\underline{w}_\lambda, v_j\})^{-\gamma}[(\max\{g(\underline{w}_\lambda), g(v_j)\})']^{1-\gamma} \\ &\ge (\max\{g(\underline{w}_\lambda), g(v_j)\})^{-\gamma}(\max\{g(\underline{w}_\lambda), g(v_j)\})', \end{aligned}$$

and

$$K_0^{q+1}(\max\{\underline{w}_\lambda, v_j\})^{\frac{q-1}{2}} \geq (\max\{g(\underline{w}_\lambda), g(v_j)\})^q(\max\{g(\underline{w}_\lambda), g(v_j)\})'.$$

Then, by denoting

$$f_n(x) = (\max\{\underline{w}_\lambda, v_j\})^{-\gamma} + o_j(1),$$

$$h_n(x) = K_0^{q+1}(\max\{\underline{w}_\lambda, v_j\})^{\frac{q-1}{2}},$$

and applying arguments from Theorem 2.1 in [36] and Brezis and Lieb [37], we can derive that

$$\|v_j\|_{W_0^{1,p}(\Omega)} = \|v_j - \widetilde{w}_\lambda\|_{W_0^{1,p}(\Omega)} + \|\widetilde{w}_\lambda\|_{W_0^{1,p}(\Omega)} + o_j(1) \tag{3.8}$$

and

$$\int_\Omega g(\widetilde{w}_\lambda)^{q+1}dx = \int_\Omega |g(v_j) - g(\widetilde{w}_\lambda)|^{q+1}dx + \int_\Omega g(\widetilde{w}_\lambda)^{q+1}dx + o_j(1) \tag{3.9}$$

as $j \to \infty$. Furthermore, by using Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}\int_\Omega g(v_j)^{1-\gamma}dx &\leq \int_\Omega g(\widetilde{w}_\lambda)^{1-\gamma}dx + \int_\Omega |g(v_j) - g(\widetilde{w}_\lambda)|^{1-\gamma}dx \\ &\leq \int_\Omega g(\widetilde{w}_\lambda)^{1-\gamma}dx + C|v_j - \widetilde{w}_\lambda|_p^{1-\gamma} \\ &= \int_\Omega g(\widetilde{w}_\lambda)^{1-\gamma}dx + o_j(1).\end{aligned}$$

Similarly, we have

$$\int_\Omega g(\widetilde{w}_\lambda)^{1-\gamma}dx \leq \int_\Omega g(v_j)^{1-\gamma}dx + o_j(1).$$

Consequently,

$$\int_\Omega g(v_j)^{1-\gamma}dx = \int_\Omega g(\widetilde{w}_\lambda)^{1-\gamma}dx + o_j(1).$$

Now, from (3.5), (3.6), (3.8), and (3.9), it follows that $\widetilde{w}_\lambda, v_j - \widetilde{w}_\lambda \in B_r$. Then we obtain

$$\int_\Omega (\frac{1}{p}|\nabla v_j - \widetilde{w}_\lambda|^p - \frac{1}{q+1}|g(v_j - \widetilde{w}_\lambda)|^{q+1})dx > 0.$$

Thus, we infer that

$$\begin{aligned}E_\lambda &= \overline{J}_\lambda(v_j) + o_j(1) \\ &= \overline{J}_\lambda(\widetilde{w}_\lambda) + \frac{1}{p}\|v_j - \widetilde{w}_\lambda\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{q+1}\int_\Omega |g(v_j) - g(\widetilde{w}_\lambda)|^{q+1}dx + o_j(1) \\ &\geq \overline{J}_\lambda(\widetilde{w}_\lambda) + \frac{1}{q+1}\int_\Omega |g(v_j - \widetilde{w}_\lambda)|^{q+1}dx - \frac{1}{q+1}\int_\Omega |g(v_j) - g(\widetilde{w}_\lambda)|^{q+1}dx \\ &\geq \overline{J}_\lambda(\widetilde{w}_\lambda) + o_j(1) \geq E_\lambda + o_j(1) \quad \text{as } j \to \infty.\end{aligned}$$

Therefore, $\overline{J}_\lambda(\widetilde{w}_\lambda) = E_\lambda$ and

$$\begin{cases} -\Delta_p \widetilde{w}_\lambda = f_\lambda(x, \widetilde{w}_\lambda) \ \text{ in } \Omega, \\ \widetilde{w}_\lambda|_{\partial\Omega} = 0. \end{cases}$$

By applying Lemmas 2.2 and 2.6, we deduce that $\widetilde{w}_\lambda > \underline{w}_\lambda$ in $\Omega$. Consequently, $\widetilde{w}_\lambda$ is a weak solution of (2.1), which implies that $\Lambda > 0$.

Next, we show that $\Lambda < \infty$. If not, there exists a sequence $(\lambda_n)$ with $\lambda_n \to \infty$ such that (2.1) admits a solution $w_n$. To derive a contradiction, we define

$$h_\lambda(t) = \frac{[\lambda g(t)^{-\gamma} + g(t)^q]g'(t)}{t^{p-1}}.$$

We claim that there exists $n_0 \in \mathbb{N}$ sufficiently large such that for all $t > 0$,

$$h_{\lambda_{n_0}}(t) \geq \lambda_1 + \varepsilon, \tag{3.10}$$

where $\varepsilon > 0$ and $\lambda_1$ is the first eigenvalue of $-\Delta_p$.
Indeed, for any $s > 1$, let $t_s = t_{s,n} \in [\frac{1}{s}, s]$ be such that $h_{\lambda_n}(t_s) \leq h_{\lambda_n}(t)$ for all $t \in [\frac{1}{s}, s]$. Then we see that $t_s \to t_{0,n} \in (0, \infty)$ as $s \to \infty$, up to a subsequence. If $t_s \to 0$, Lemma 2.1(vi),(vii) imply that $h_{\lambda_n}(t) \geq \infty$ for all $n \in \mathbb{N}$, which is impossible. For similar reasons, it is also impossible for $t_s \to \infty$. Thus, $t_s \to t_{0,n} \in (0, \infty)$ and

$$h_{\lambda_n}(t) \geq \frac{[\lambda_n g(t_{0,n})^{-\gamma} + g(t_{0,n})^q]g'(t_{0,n})}{t_{0,n}^{p-1}}.$$

Then, by repeating the above arguments, we get that $t_n = t_{0,n} \to t_0 \in (0, \infty)$, up to a subsequence. Consequently, $\lim_{n\to\infty} h_{\lambda_n}(t) > \lambda_1$, which shows (3.10). From (3.10), we obtain that

$$\lambda_{n_0} g(t)^{-\gamma} g'(t) + g(t)^q g'(t) \geq (\lambda_1 + \varepsilon)t^{p-1}.$$

By choosing $\lambda_n > \lambda_{n_0}$, we infer that $w_n$ is a super-solution of

$$\begin{cases} -\Delta_p w = (\lambda_1 + \epsilon)w^{p-1} \ \text{ in } \Omega, \\ \quad w > 0, \quad w|_{\partial\Omega} = 0. \end{cases} \tag{3.11}$$

Now, take $\mu < \lambda_1 + \epsilon$ small enough such that $\mu\phi_1(x) < w_{n_0}(x)$ and such that $\mu\phi_1$ is a sub-solution of (3.11). By the method of monotone iteration, we obtain a solution to (3.11) for any $\epsilon > 0$. This contradicts the fact that $\lambda_1$ is an isolated point in the spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$. Therefore, $\Lambda < \infty$. This completes the proof.

**Proposition 3.1.** *There exists a positive weak solution $w_\lambda$ to* (2.1) *for any $\lambda \in (0, \Lambda)$. Moreover, $J_\lambda(w_\lambda) < 0$.*

*Proof.* Given $0 < \lambda < \Lambda$, there exist $\lambda_2 \in (\lambda, \Lambda)$ and $w_{\lambda_2} \in W_0^{1,p}(\Omega)$, is a solution of

$$\begin{cases} -\Delta_p w = \lambda_2 g(w)^{-\gamma} g'(w) + g(w)^q g'(w) \ \text{ in } \Omega, \\ \quad w > 0, \quad w|_{\partial\Omega} = 0. \end{cases} \tag{3.12}$$

Then $\overline{w} = w_{\lambda_2}$ is a super-solution of (2.1). As in Lemma 3.1, $\underline{w}_\lambda$ is the solution of (3.1), that is,

$$-\Delta_p \underline{w}_\lambda = \lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda). \tag{3.13}$$

By Lemma 2.2, we have $\underline{w}_\lambda, \overline{w} \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Moreover, we can prove that $\underline{w}_\lambda \leq \overline{w}$ in $\Omega$. In fact, let $\Omega_0 = \{x \in \Omega : \underline{w}_\lambda > \overline{w}\}$. From Lemma 2.1, (3.12), and (3.13), we obtain

$$\begin{aligned}
&\int_{\Omega_0} (+\Delta_p \overline{w} - \Delta_p \underline{w}_\lambda)(\underline{w}_\lambda - \overline{w})dx \\
&\leq \lambda \int_{\Omega_0} (g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) - g(\overline{w})^{-\gamma} g'(\overline{w}))(\underline{w}_\lambda - \overline{w})dx \\
&\leq \lambda \int_{\Omega_0} g'(\overline{w})(g(\underline{w}_\lambda)^{-\gamma} - g(\overline{w})^{-\gamma})(\underline{w}_\lambda - \overline{w})dx \\
&\leq 0
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\int_{\Omega_0} (+\Delta_p \overline{w} - \Delta_p \underline{w}_\lambda)(\underline{w}_\lambda - \overline{w})dx \\
&\geq \int_{\Omega_0} (|\nabla \underline{w}_\lambda|^{p-2} \nabla \underline{w}_\lambda - |\nabla \overline{w}|^{p-2} \nabla \overline{w})(\nabla \underline{w}_\lambda - \nabla \overline{w})dx \\
&\geq \begin{cases} C_p \int_{\Omega_0} \frac{|\nabla(\underline{w}_\lambda - \overline{w})|^2}{(|\nabla \underline{w}_\lambda| + |\nabla \overline{w}|)^{2-p}} dx & \text{if } 1 < p < 2 \\ C_p \int_{\Omega_0} |\nabla(\underline{w}_\lambda - \overline{w})|^p dx & \text{if } p \geq 2 \end{cases} \\
&\geq 0.
\end{aligned} \tag{3.15}$$

Then (3.14) and (3.15) imply that $\underline{w}_\lambda \leq \overline{w}$. Furthermore, by Lemma 2.6, we get $\underline{w}_\lambda < \overline{w}$ in $\Omega$ and $\frac{\partial \underline{w}_\lambda}{\partial \nu} > \frac{\partial \overline{w}}{\partial \nu}$ on $\partial\Omega$.
Now, let us define

$$\widetilde{f_\lambda}(x,t) = \begin{cases} \lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) + g(\underline{w}_\lambda)^q g'(\underline{w}_\lambda), & \text{if } t < \underline{w}_\lambda(x); \\ \lambda g(t)^{-\gamma} g'(t) + g(t)^q g'(t), & \text{if } \underline{w}_\lambda(x) \leq t \leq \overline{w}(x); \\ \lambda g(\overline{w})^{-\gamma} g'(\overline{w}) + g(\overline{w})^q g'(\overline{w}), & \text{if } t \geq \overline{w}(x). \end{cases}$$

Let $\widetilde{F_\lambda}(x,t) = \int_0^t \widetilde{f_\lambda}(x,s)ds$. Define the functional $\widetilde{J_\lambda} : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ by

$$\widetilde{J_\lambda}(w) = \frac{1}{p} \int_\Omega |\nabla w|^p dx - \int_\Omega \widetilde{F_\lambda}(x,w)dx.$$

Then $\widetilde{J_\lambda}$ is bounded below in $W_0^{1,p}(\Omega)$ and is weakly lower semicontinuous. Indeed, from Lemma 2.1, we have

$$\widetilde{J_\lambda}(w) \geq \frac{1}{p}\|w\|^p - \lambda\|\overline{w}\|^{1-\gamma} - C\|\overline{w}\|^{\frac{q+1}{2}}.$$

Thus, $\widetilde{J_\lambda}$ is bounded below in $W_0^{1,p}(\Omega)$. On the other hand, it suffices to show that $\widetilde{J_\lambda}$ is weakly lower semicontinuous in $M := \{w \in W_0^{1,p}(\Omega) : \underline{w}_\lambda \leq w \leq \overline{w}$ a.e. in $\Omega\}$. Let $M \ni w_n \to w$ weakly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Then we may assume that $w_n \to w$ pointwise a.e. in $\Omega$, up to a subsequence. It follows from

$$\int_\Omega |g(w_n)|^{q+1} dx < +\infty \quad \text{and} \quad \int_\Omega |g(w_n)|^{1-\gamma} dx < +\infty,$$

and by Lebesgue's dominated convergence theorem, that

$$\int_\Omega |g(w_n)|^{q+1}dx \to \int_\Omega |g(w)|^{q+1}dx \text{ and } \int_\Omega |g(w_n)|^{1-\gamma}dx \to \int_\Omega |g(w)|^{1-\gamma}dx.$$

This implies that $\liminf_{n\to\infty} \widetilde{J_\lambda}(w_n) \geq \widetilde{J_\lambda}(w)$. Hence, $\widetilde{J_\lambda}$ is weakly lower semicontinuous. Thus, there exists a global minimizer $w_\lambda \in M$. Moreover, $w_\lambda$ solves the equation $-\Delta_p w_\lambda = \widetilde{f_\lambda}(x, w_\lambda)$ in $\Omega$ since $\widetilde{J_\lambda}$ is $C^1$ by Lemma A1. It follows from the strong maximum principle of Theorem 5 in [33] and Lemma 2.2 that $w_\lambda > 0$ and $w_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Applying Lemma 2.6, we obtain $\underline{w}_\lambda < w_\lambda$ in $\Omega$ and $\frac{\partial \underline{w}_\lambda}{\partial \nu} > \frac{\partial w_\lambda}{\partial \nu}$ on $\partial\Omega$. Similarly, we can get $w_\lambda < \overline{w}$ in $\Omega$ and $\frac{\partial w_\lambda}{\partial \nu} > \frac{\partial \overline{w}_\lambda}{\partial \nu}$ on $\partial\Omega$. Hence $\underline{w}_\lambda < w_\lambda < \overline{w}$ in $\Omega$, and then

$$\widetilde{f_\lambda}(x, w_\lambda) = \lambda g(w_\lambda)^{-\gamma} g'(w_\lambda) + g(w_\lambda)^q g'(w_\lambda),$$

which means that $w_\lambda$ is a weak solution to (2.1). Moreover,

$$\widetilde{J_\lambda}(w_\lambda) \leq \widetilde{J_\lambda}(\underline{w}_\lambda) = J_\lambda(\underline{w}_\lambda) < \frac{1}{p}\int_\Omega |\nabla \underline{w}_\lambda|^p dx - \frac{1}{1-\gamma}\int_\Omega g(\underline{w}_\lambda)^{1-\gamma}dx < 0.$$

This completes the proof.

**Proposition 3.2.** *For $\lambda = \Lambda$, there exists at least one positive weak solution to* (2.1).

*Proof.* Let $(\lambda_k) \subset (0, \Lambda)$ be an increasing sequence such that $\lambda_k \to \Lambda$ as $k \to \infty$. Then there exists $w_k = w_{\lambda_k} \geq \underline{w}_\lambda$, which is a weak positive solution to (2.1) for $\lambda = \lambda_k$. Hence, for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega |\nabla w_k|^{p-2}\nabla w_k \nabla\varphi dx = \lambda_k \int_\Omega g(w_k)^{-\gamma} g'(w_k)\varphi dx + \int_\Omega g(w_k)^q g'(w_k)\varphi dx. \tag{3.16}$$

From the fact that $w_k \in W_0^{1,p}(\Omega)$ and $w_k \geq \underline{w}_\lambda$, we conclude that (3.16) also holds for $\varphi \in W_0^{1,p}(\Omega)$. It follows from Proposition 3.1 that $J_{\lambda_k}(w_k) < 0$, and then we have

$$\sup_k \|w_k\|_{W_0^{1,p}(\Omega)} < \infty.$$

Therefore, there exists $w_\Lambda \in W_0^{1,p}(\Omega)$ such that $w_k \to w_\Lambda$ weakly in $W_0^{1,p}(\Omega)$ as $k \to \infty$ and $w_\Lambda \geq \underline{w}_\lambda$. Applying the Sobolev imbedding theorem, we have $w_k \to w_\Lambda$ weakly in $L^q(\Omega)$ and pointwise a.e. in $\Omega$ as $k \to \infty$. Together with the fact that

$$|g(w_k)^{-\gamma} g'(w_k)| \leq |g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda)|$$

and

$$|g(w_k)^q g'(w_k)| \leq K_0^{q+1} w_k^{\frac{q-1}{2}},$$

Lebesgue's dominated convergence theorem can be used to get that

$$\int_\Omega g(w_k)^{-\gamma} g'(w_k)\varphi dx \to \int_\Omega g(w_\Lambda)^{-\gamma} g'(w_\Lambda)\varphi dx \text{ as } k \to \infty$$

and

$$\int_\Omega g(w_k)^q g'(w_k)\varphi dx \to \int_\Omega g(w_\Lambda)^q g'(w_\Lambda)\varphi dx \text{ as } k \to \infty.$$

Thus, for any $\varphi \in W_0^{1,p}(\Omega)$, taking the limit as $k \to \infty$ in (3.16) yields

$$\int_\Omega |\nabla w_k|^{p-2}\nabla w_k \nabla\varphi dx = \Lambda \int_\Omega g(w_\Lambda)^{-\gamma} g'(w_\Lambda)\varphi dx + \int_\Omega g(w_\Lambda)^q g'(w_\Lambda)\varphi dx.$$

This completes the proof.

**Corollary 3.1.** *Assume* $1 < p \le \frac{\ln q}{\ln 2}$ *and* $max\{2, 2p-1\} < q \le 2p^* - 1$, $0 < \gamma < 1$, $0 < \lambda \le \Lambda$. *Then* (2.1) *has a minimal solution.*

*Proof.* Consider the following monotone iterative problem

$$\begin{cases} -\Delta_p w_n - \lambda g(w_n)^{-\gamma} g'(w_n) = g(w_{n-1})^q g'(w_{n-1}) \ \text{ in } \Omega, \\ w_n|_{\partial\Omega} = 0, \end{cases} \tag{3.17}$$

with $w_0 = \underline{w}_\lambda$ as given in Lemma 3.1, which is a sub-solution to (3.17). Note that $w_0 \le w_\Lambda$, where $w_\Lambda$ is the solution to (2.1) obtained in Proposition 3.2. From Lemma 2.1 and $1 < p \le \frac{\ln q}{\ln 2}$, we obtain

$$\begin{aligned} [g(t)^q g'(t)]' &= qg(t)^{q-1}(g'(t))^2 + g(t)^q g''(t) \\ &= g(t)^{q-1}\left(q(g'(t))^2 - pg(t)(g'(t))^{p+2}|g(t)|^{p-2}g(t)\right) \\ &= g(t)^{q-1}(g'(t))^2\left(q - p|g(t)g'(t)|^p\right) \\ &\ge 0. \end{aligned}$$

Then $g(t)^q g'(t)$ is non-decreasing for $1 < p \le \frac{\ln q}{\ln 2}$ and $w_\Lambda$ is a super-solution to (3.17). By monotone iteration, we obtain that the sequence $\{w_n\}_{n=1}^\infty$ is non-decreasing. Thus, there exists a weak solution $\widehat{w}_\lambda$ of (2.1) such that $w_n \to \widehat{w}_\lambda$ weakly in $W_0^{1,p}(\Omega)$ and pointwise a.e. in $\Omega$. If $v_\lambda$ is another weak solution to (2.1), we have $w_n \le v_\lambda$ for all $n$ by the weak comparison principle. Then $\widehat{w}_\lambda \le v_\lambda$. Moreover, $\widehat{w}_\lambda$ is the minimal solution to (2.1) for any $0 < \lambda \le \Lambda$, which completes the proof.

### *3.1. Existence of a local minimizer*

**Lemma 3.3.** *Assume* $w_\lambda$ *is the solution to* (2.1) *obtained by Proposition 3.1. Then* $w_\lambda$ *is a local minimizer of* $\overline{J}_\lambda$ *in* $W_0^{1,p}(\Omega)$ *for* $0 < \lambda < \Lambda$.

*Proof.* We first claim that $w_\lambda$ is a local minimizer in the $C^1$-topology. Let $c > 0$ be small enough. For any $w \in C^1(\overline{\Omega})$ with $\|w - w_\lambda\|_{C^1(\overline{\Omega})} \le c$, since $\underline{w}_\lambda < w_\lambda < \overline{w}$ in Proposition 3.1, we have $\underline{w}_\lambda \le w \le \overline{w}$. Then we obtain $\overline{J}_\lambda(w_\lambda) = \widetilde{J}_\lambda(w_\lambda) \le \widetilde{J}_\lambda(w) = \overline{J}_\lambda(w)$. Thus the claim is proved.

Now, we are ready to prove that $w_\lambda$ is a local minimizer of $\overline{J}_\lambda$ in $W_0^{1,p}(\Omega)$. We argue by contradiction. We first consider the case $q < 2p^* - 1$. It is easy to show that $\overline{J}_\lambda$ is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$ in this case, and then $\overline{J}_\lambda$ achieves a minimizer on bounded subsets of $W_0^{1,p}(\Omega)$. For $\varepsilon > 0$, assume that there exists $\eta_\varepsilon$ satisfying $0 < \|\eta_\varepsilon\|_{W_0^{1,p}(\Omega)} \le \varepsilon$ and

$$\overline{J}_\lambda(w_\lambda + \eta_\varepsilon) < \overline{J}_\lambda(w_\lambda), \ \ \overline{J}_\lambda(w_\lambda + \eta_\varepsilon) = \inf_{\|\eta\|_{W_0^{1,p}(\Omega)} \le \varepsilon} \overline{J}_\lambda(w_\lambda + \eta). \tag{3.18}$$

From the Lagrange multiplier rule, there exists $\theta_\varepsilon \leq 0$ such that for any $y \in W_0^{1,p}(\Omega)$,

$$\langle \overline{J}_\lambda'(w_\lambda + \eta_\varepsilon), y \rangle = \theta_\varepsilon \int_\Omega |\nabla \eta_\varepsilon|^{p-2} \nabla \eta_\varepsilon \cdot \nabla y \, dx.$$

This implies that in the weak sense,

$$-\Delta_p(w_\lambda + \eta_\varepsilon) - f_\lambda(x, w_\lambda + \eta_\varepsilon) = -\theta_\varepsilon \Delta_p \eta_\varepsilon, \tag{3.19}$$

where $f_\lambda$ is given by (3.2). To analyze this equation, we define the following applications

$$B_\varepsilon(x, z) = |\nabla w_\lambda(x) + z|^{p-2}(\nabla w_\lambda(x) + z) - |\nabla w_\lambda(x)|^{p-2} \nabla w_\lambda(x) - \theta_\varepsilon |z|^{p-2} z,$$

and

$$\begin{aligned} h_\lambda(x,t) &= f_\lambda(x, w_\lambda(x) + t) - f_\lambda(x, w_\lambda(x)) \\ &= \lambda(\max\{g(w_\lambda(x) + t), g(\underline{w}_\lambda(x))\})^{-\gamma}(\max\{g(w_\lambda(x) + t), g(\underline{w}_\lambda(x))\})' \\ &\quad - \lambda g(w_\lambda(x))^{-\gamma} g'(w_\lambda(x)) \\ &\quad + (\max\{g(w_\lambda(x) + t), g(\underline{w}_\lambda(x))\})'(\max\{g(w_\lambda(x) + t), g(\underline{w}_\lambda(x))\})^q \\ &\quad - g(w_\lambda(x))^q g'(w_\lambda(x)). \end{aligned}$$

Then we can rewrite (3.19) as

$$\begin{cases} -\nabla \cdot (B_\varepsilon(x, \nabla \eta_\varepsilon)) = h_\lambda(x, \eta_\varepsilon) \ \text{ in } \Omega, \\ \eta_\varepsilon|_{\partial\Omega} = 0. \end{cases} \tag{3.20}$$

Combining arguments from [38], the fact that the singular terms in (3.20) are non-increasing due to the monotonicity of $s \mapsto g(s)^{-\gamma} g'(s)$, and techniques from [39] (Theorem A1), we obtain $\sup_\varepsilon \|\eta_\varepsilon\|_{L^\infty(\Omega)} < \infty$. Now, we will prove that $\sup_\varepsilon \|\frac{\eta_\varepsilon}{d(x)}\|_{L^\infty(\Omega)} < \infty$, where $d(x) := \text{dist}(x, \partial\Omega)$. To this end, it suffices to estimate $\eta_\varepsilon$ near $\partial\Omega$. We take $\overline{\eta}_\varepsilon$ as the unique solution of the following singular problem

$$\begin{cases} -\Delta_p(w_\lambda + \overline{\eta}_\varepsilon) + \theta_\varepsilon \Delta_p \overline{\eta}_\varepsilon = \lambda(\max\{g(\underline{w}_\lambda), g(w_\lambda + \overline{\eta}_\varepsilon)\})^{-\gamma}(\max\{g(\underline{w}_\lambda), g(w_\lambda + \overline{\eta}_\varepsilon)\})' \ \text{ in } \Omega, \\ \overline{\eta}_\varepsilon|_{\partial\Omega} = 0. \end{cases} \tag{3.21}$$

Note that $(\xi - 1)w_\lambda$ is a sub-solution for $\xi > 0$ sufficiently small and $Kw_\lambda$ is a super-solution for $K > 0$ sufficiently large to (3.21). Thus, we have $(\xi - 1)w_\lambda \leq \overline{\eta}_\varepsilon \leq Kw_\lambda$. Moreover, we can get $\overline{\eta}_\varepsilon \leq \eta_\varepsilon$. In fact, if $\overline{\eta}_\varepsilon > \eta_\varepsilon$, from the properties of $g$, we have

$$\begin{aligned} 0 &\leq \int_{\overline{\eta}_\varepsilon > \eta_\varepsilon} (-\Delta_p(w_\lambda + \overline{\eta}_\varepsilon) + \Delta_p(w_\lambda + \eta_\varepsilon))(\overline{\eta}_\varepsilon - \eta_\varepsilon) dx \\ &\quad + \theta_\varepsilon \int_{\overline{\eta}_\varepsilon > \eta_\varepsilon} (-\Delta_p \overline{\eta}_\varepsilon + \Delta_p \eta_\varepsilon)(\overline{\eta}_\varepsilon - \eta_\varepsilon) dx \\ &\leq \int_{\overline{\eta}_\varepsilon > \eta_\varepsilon} \Big[\lambda(\max\{g(\underline{w}_\lambda), g(w_\lambda + \overline{\eta}_\varepsilon)\})^{-\gamma}(\max\{g(\underline{w}_\lambda), g(w_\lambda + \overline{\eta}_\varepsilon)\})' \\ &\quad - \lambda(\max\{g(\underline{w}_\lambda), g(w_\lambda + \overline{\eta}_\varepsilon)\})^{-\gamma}(\max\{g(\underline{w}_\lambda), g(w_\lambda + \overline{\eta}_\varepsilon)\})'\Big](\overline{\eta}_\varepsilon - \eta_\varepsilon) dx \leq 0, \end{aligned}$$

which implies a contradiction. Thus $\overline{\eta}_\varepsilon \le \eta_\varepsilon$ and $\xi w_\lambda \le w_\lambda + \eta_\varepsilon$.

On the other hand, we get that $\eta_\varepsilon \le K w_\lambda$ for $K$ large enough near $\partial\Omega$ by using the weak comparison principle in a small neighborhood of $\partial\Omega$. As a consequence, we have $\sup_\varepsilon \|\frac{\eta_\varepsilon}{d(x)}\|_{L^\infty(\Omega)} < \infty$. It follows from Lemma 2.2 that $\sup_\varepsilon \|\eta_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} < \infty$ for some $0 < \alpha < 1$. Now, from the Arzelà-Ascoli theorem, we obtain $\eta_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in $C^1(\overline{\Omega})$, which is a contradiction since $w_\lambda$ is a $C^1$-minimizer of $\overline{J}_\lambda$. Thus $w_\lambda$ is a local minimizer of $\overline{J}_\lambda$ in $W_0^{1,p}(\Omega)$ for $q < 2p^* - 1$.

By a similar argument for the critical case $q = 2p^* - 1$ as in Proposition 3.7 in [9], we can obtain a contradiction. Thus we complete the proof.

## 4. Existence of a second solution

**Definition 4.1.** *Suppose $\mathcal{F} \subset W_0^{1,p}(\Omega)$ is a closed set. We say that a sequence $\{v_k\}_{k=1}^\infty \subset W_0^{1,p}(\Omega)$ is a Palais-Smale sequence for $\overline{J}_\lambda$ at the level c around $\mathcal{F}$, denoted by $(PS)_{\mathcal{F},c}$, if*

$$\lim_{k\to\infty} \operatorname{dist}(v_k, \mathcal{F}) = 0, \quad \lim_{k\to\infty} \overline{J}_\lambda(v_k) = c, \quad \lim_{k\to\infty} \|\overline{J}_\lambda'(v_k)\|_{W^{-1,p'}(\Omega)} = 0.$$

Now, we give the following compactness result.

**Lemma 4.1.** *Let $\lambda \in (0, \Lambda)$. Suppose $\{v_k\}_{k=1}^\infty \subset W_0^{1,p}(\Omega)$ is a $(PS)_{\mathcal{F},c}$ sequence for $\overline{J}_\lambda$. Then $\{v_k\}_{k=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$, and there exists a subsequence (still denoted by $\{v_k\}_{k=1}^\infty$) such that $v_k \to v_\lambda$ weakly in $W_0^{1,p}(\Omega)$, where $v_\lambda$ is a weak solution to* (2.1).

*Proof.* First, we claim that $\{v_k\}_{k=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$. Indeed, by Definition 4.1, there exists a positive constant $D$ such that

$$\begin{aligned} \frac{1}{p}\int_\Omega |\nabla v_k|^p dx - \int_{v_k > \underline{w}_\lambda} \Big[ \big(\frac{\lambda}{1-\gamma} g(v_k)^{1-\gamma} + \frac{1}{q+1} g(v_k)^{q+1}\big) - \big(\frac{\lambda}{1-\gamma} g(\underline{w}_\lambda)^{1-\gamma} \\ + \frac{1}{q+1} g(\underline{w}_\lambda)^{q+1}\big)\Big] dx - \int_{v_k \le \underline{w}_\lambda} \Big[\lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) + g(\underline{w}_\lambda)^q g'(\underline{w}_\lambda)\Big] v_k dx \le D, \end{aligned}$$

which implies that there exists a positive constant $D'$ such that

$$\frac{1}{p}\int_\Omega |\nabla v_k|^p dx - \int_{v_k > \underline{w}_\lambda} \left[\frac{\lambda}{1-\gamma} g(v_k)^{1-\gamma} + \frac{1}{q+1} g(v_k)^{q+1}\right] dx \le D'.$$

Together with the fact that

$$\begin{aligned} \int_\Omega |\nabla v_k|^p dx = & \int_{v_k > \underline{w}_\lambda} \Big[\lambda g(v_k)^{-\gamma} g'(v_k) v_k + g(v_k)^q g'(v_k) v_k\Big] dx \\ & + \int_{v_k \le \underline{w}_\lambda} \Big[\lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) v_k + g(\underline{w}_\lambda)^q g'(\underline{w}_\lambda) v_k\Big] dx + o(1)\|v_k\|_{W_0^{1,p}(\Omega)}, \end{aligned}$$

we can derive that

$$\begin{aligned} \|v_k\|^p_{W_0^{1,p}(\Omega)} + O_k(\|v_k\|_{W_0^{1,p}(\Omega)}) & \ge \int_{v_k > \underline{w}_\lambda} g(v_k)^q g'(v_k) v_k dx \\ & \ge \frac{1}{2}\int_{v_k > \underline{w}_\lambda} g(v_k)^{q+1} dx \ge \frac{q+1}{2p}\|v_k\|^p_{W_0^{1,p}(\Omega)} - D'. \end{aligned}$$

This shows that $\{v_k\}_{k=1}^{\infty}$ is bounded in $W_0^{1,p}(\Omega)$, and then there exists $v_0$ such that for some subsequence, $v_k \to v_0$ weakly in $W_0^{1,p}(\Omega)$. By Definition 4.1, given $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla\varphi dx = \int_\Omega f_\lambda(x, v_k)\varphi dx + o_k(1). \tag{4.1}$$

Moreover, the nonlinear term satisfies the estimate

$$\begin{aligned} f_\lambda(x, v_k) &\leq \lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) + \max\{g(\underline{w}_\lambda)^q g'(\underline{w}_\lambda), g(v_k)^q g'(v_k)\} \\ &\leq \lambda g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda) + \max\{K_0^{q+1}\underline{w}_\lambda^{\frac{q-1}{2}}, K_0^{q+1} v_k^{\frac{q-1}{2}}\}. \end{aligned}$$

Passing to the limit as $k \to \infty$ in (4.1) and applying Lebesgue's dominated convergence theorem, we obtain

$$\int_\Omega |\nabla v_0|^{p-2}\nabla v_0 \nabla\varphi dx = \int_\Omega f_\lambda(x, v_0)\varphi dx.$$

Applying Lemma 2.6 again, we obtain that $v_0$ is a weak solution of (2.1). This completes the proof.

We observe that $\lim\limits_{t\to\infty}\overline{J}_\lambda(t\phi) = -\infty$ for some fixed $0 \leq \phi \in W_0^{1,p}(\Omega) \setminus \{0\}$. In fact, let $\phi \in W_0^{1,p}(\Omega)$ be a function that satisfies $\phi > 0$ in $\Omega$ and $\|\phi\| = 1$. Then we have

$$\begin{aligned} \overline{J}_\lambda(t\phi) &\leq \frac{1}{p}\|t\phi\|^p - \frac{\lambda}{1-\gamma}\int_\Omega g(t\phi)^{1-\gamma}dx - \frac{1}{q+1}\int_\Omega g(t\phi)^{q+1}dx \\ &\leq \frac{1}{p}t^p - K_0^{q+1} t^{\frac{q+1}{2}}\int_\Omega \phi^{\frac{q+1}{2}}dx \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$, since $\frac{q+1}{2} > p$. Combined with Proposition 3.1, it follows that $\overline{J}_\lambda$ has a mountain pass geometry near $w_\lambda$. Thus, there exists $\tau \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\overline{J}_\lambda(\tau) < \overline{J}_\lambda(w_\lambda)$.

Now, let $R_0 = \|\tau - w_\lambda\|_{W_0^{1,p}(\Omega)}$, and choose $l_0 > 0$ sufficiently small such that $w_\lambda$ is a minimizer of $\overline{J}_\lambda$ on $\overline{B_{l_0}(w_\lambda)}$. Define

$$\Gamma = \{\eta \in C([0,1], W_0^{1,p}(\Omega)) | \eta(0) = w_\lambda, \eta(1) = \tau\}$$

and the mountain pass level

$$\gamma_0 = \inf_{\eta\in\Gamma}\max_{t\in[0,1]}\overline{J}_\lambda(\eta(t)).$$

Then we have the following two cases:

$(P_1)$: $\inf\{\overline{J}_\lambda(w) : w \in W_0^{1,p}(\Omega), \|w - w_\lambda\|_{W_0^{1,p}(\Omega)} = l\} \leq \overline{J}_\lambda(w_\lambda)$ for all $l < R_0$;

$(P_2)$: there exists $l_1 < R_0$ such that $inf\{\overline{J}_\lambda(w) : w \in W_0^{1,p}(\Omega), \|w - w_\lambda\|_{W_0^{1,p}(\Omega)} = l_1\} > \overline{J}_\lambda(w_\lambda)$.

It is easy to see that $(P_1)$ implies $\gamma_0 = \overline{J}_\lambda(w_\lambda)$ and $(P_2)$ implies $\gamma_0 > \overline{J}_\lambda(w_\lambda)$. When case $(P_1)$ occurs, a $(PS)_{\mathcal{F},\gamma_0}$ sequence with $\mathcal{F} = \partial B_l(w_\lambda)$ for $l \leq l_0$ can be constructed, and a second weak solution of (2.1) can be obtained as follows.

**Proposition 4.1.** *Assume that* $1 < p < \infty$, $2p < q + 1 \leq 2p^*$ *for* $p \leq N$, $2p < q + 1 < \infty$ *for* $p > N$, $0 < \gamma < 1$, $0 < \lambda < \Lambda$, *and that* $(P_1)$ *holds. Then there exists a weak solution* $v_\lambda$ *of* (2.1) *such that* $v_\lambda \neq w_\lambda$.

*Proof.* It follows from Theorem 1 in [40] that there exists a $(PS)_{\mathcal{F},\gamma_0}$ sequence $\{v_n\}_{n=1}^{\infty}$ for some $l \leq l_0$. Lemma 4.1 implies that $\{v_n\}_{n=1}^{\infty}$ is bounded and, up to a subsequence, weakly convergent to $v_\lambda$ in $W_0^{1,p}(\Omega)$, where $v_\lambda$ is a weak solution of (2.1).

In order to prove $v_\lambda \neq w_\lambda$, it suffices to show that $v_n \to v_\lambda$ strongly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Given that $v_n \to v_\lambda$ weakly in $W_0^{1,p}(\Omega)$, the Brezis-Lieb lemma implies that as $n \to \infty$,

$$\|v_n\|_{W_0^{1,p}(\Omega)} = \|v_n - v_\lambda\|_{W_0^{1,p}(\Omega)} + \|v_\lambda\|_{W_0^{1,p}(\Omega)} + o(1)$$

and

$$\int_\Omega g(v_n)^q g'(v_n)v_n dx = \int_\Omega g(v_n - v_\lambda)^q g'(v_n - v_\lambda)dx + \int_\Omega g(v_\lambda)^q g'(v_\lambda)v_\lambda dx + o(1).$$

Since $|g(v_n)^{-\gamma} g'(v_n)| \leq |g(\underline{w}_\lambda)^{-\gamma} g'(\underline{w}_\lambda)|$ for $v_n \geq \underline{w}_\lambda$ and $\{v_n\}_{n=1}^{\infty}$ is bounded in $W_0^{1,p}(\Omega)$, Lebesgue's dominated convergence theorem implies that

$$\int_{v_n \geq \underline{w}_\lambda} g(v_n)^{-\gamma} g'(v_n)v_n dx \to \int_{v_n \geq \underline{w}_\lambda} g(v_\lambda)^{-\gamma} g'(v_\lambda)v_\lambda dx.$$

Similarly,

$$\int_{v_n \geq \underline{w}_\lambda} g(v_n)^{-\gamma} g'(v_n)v_\lambda dx \to \int_{v_n \geq \underline{w}_\lambda} g(v_\lambda)^{-\gamma} g'(v_\lambda)v_\lambda dx.$$

Notice that $v_\lambda$ is a weak solution of (2.1), that is,

$$\int_\Omega |\nabla v_\lambda|^p dx - \lambda \int_\Omega g(v_\lambda)^{-\gamma} g'(v_\lambda)v_\lambda dx - \int_\Omega g(v_\lambda)^q g'(v_\lambda)v_\lambda dx = 0.$$

Hence, as $n \to \infty$,

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v_\lambda)dx = \lambda \int_{v_n \geq \underline{w}_\lambda} g(v_n)^{-\gamma} g'(v_n)(v_n - v_\lambda)dx + \int_\Omega g(v_n)^q g'(v_n)(v_n - v_\lambda)dx + o(1).$$

Consequently, we have

$$\int_\Omega |\nabla v_n - \nabla v_\lambda|^p dx = \int_\Omega g(v_n - v_\lambda)^q g'(v_n - v_\lambda)(v_n - v_\lambda)dx + o(1) \tag{4.2}$$

as $n \to \infty$. Without loss of generality, we may assume that $\overline{J}_\lambda(w_\lambda) = \overline{J}_\lambda(v_\lambda)$; otherwise, the proof is complete. By Hölder's inequality and Lemma 2.1, we have as $n \to \infty$,

$$\begin{aligned}
\int_{v_n \geq \underline{w}_\lambda} g(v_n)^{1-\gamma} dx &\leq \int_{v_n \geq \underline{w}_\lambda} g(v_\lambda)^{1-\gamma} dx + \int_{v_n \geq \underline{w}_\lambda} |g(v_n) - g(v_\lambda)|^{1-\gamma} dx \\
&\leq \int_{v_n \geq \underline{w}_\lambda} g(v_\lambda)^{1-\gamma} dx + C|v_n - v_\lambda|_s^{1-\gamma} \\
&= \int_{v_n \geq \underline{w}_\lambda} g(v_\lambda)^{1-\gamma} dx + o(1).
\end{aligned}$$

Similarly,

$$\int_{v_n \geq \underline{w}_\lambda} g(v_\lambda)^{1-\gamma} dx \leq \int_{v_n \geq \underline{w}_\lambda} g(v_n)^{1-\gamma} dx + o(1).$$

It follows that

$$\int_{v_n \geq \underline{w}_\lambda} |g(v_n)^{1-\gamma} - g(v_\lambda)^{1-\gamma}| dx = o(1), \quad \text{as } n \to \infty.$$

By the Brezis-Lieb lemma again, as $n \to \infty$,

$$\int_\Omega |g(v_n)|^{q+1} dx = \int_\Omega |g(v_n) - g(v_\lambda)|^{q+1} dx + \int_\Omega |g(v_\lambda)|^{q+1} dx + o(1).$$

Therefore, we have

$$\overline{J}_\lambda(v_n - v_\lambda) \leq \overline{J}_\lambda(v_n) - \overline{J}_\lambda(v_\lambda) + o(1) \leq o(1).$$

Then

$$\frac{1}{p} \|v_n - v_\lambda\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{q+1} \int_\Omega |g(v_n - v_\lambda)|^{q+1} dx \leq o(1), \quad \text{as } n \to \infty. \tag{4.3}$$

By Lemma 2.1(v), we obtain

$$\int_\Omega g(v_n - v_\lambda)^q g'(v_n - v_\lambda)(v_n - v_\lambda) dx \geq \frac{1}{2} \int_\Omega g(v_n - v_\lambda)^{q+1} dx + o(1). \tag{4.4}$$

From (4.2) to (4.4), we conclude that $\|v_n - v_\lambda\|_{W_0^{1,p}(\Omega)} \to 0$ as $n \to \infty$. Thus, $\|w_\lambda - v_\lambda\|_{W_0^{1,p}(\Omega)} = l$ and $w_\lambda \neq v_\lambda$. This completes the proof.

In what follows, we consider the case $(P_2)$:

**Proposition 4.2.** *Assume that p and q satisfy either* $1 < p < \infty$, $2p - 1 < q < 2p^* - 1$ *or* $2 \leq p < N$, $q = 2p^* - 1$. *Given* $\lambda \in (0, \Lambda)$ *and that* $(P_2)$ *holds, there exists a weak solution* $v_\lambda$ *of* (2.1) *such that* $w_\lambda \neq v_\lambda$.

*Proof.* We only consider the case $q = 2p^* - 1$, as the subcritical case follows from Lemma 4.1 and a standard argument. We may assume that $\overline{J}_\lambda$ attains its minimal energy at $w_\lambda$ among all weak solutions; otherwise, a second solution would be found. Let $\mathcal{F} = W_0^{1,p}(\Omega)$. We consider

$$U_\epsilon(x) = \frac{\epsilon^{\frac{N-p}{2p(p-1)}}}{(\epsilon^{\frac{p}{p-1}} + |x - y|^{\frac{p}{p-1}})^{\frac{N-p}{2p}}} \varphi(x),$$

where $\epsilon > 0$, $y \in \Omega$, and $\varphi \in C_c^\infty(\Omega)$ is a cut-off function satisfying $\varphi = 1$ in a neighborhood of $y$.

We claim that there exist $\epsilon_0 > 0$ and $R_0 \geq 1$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$\overline{J}_\lambda(w_\lambda + RU_\epsilon) = J_\lambda(w_\lambda + RU_\epsilon) < J_\lambda(w_\lambda), \quad \forall R \geq R_0,$$

$$\overline{J}_\lambda(w_\lambda + tR_0U_\epsilon) = J_\lambda(w_\lambda + tR_0U_\epsilon) < J_\lambda(w_\lambda) + \frac{1}{2N} S^{\frac{N}{p}}, \quad \forall t \in [0, 1].$$

In fact, according to the arguments in [40], we get

$$\int_\Omega |U_\epsilon|^{q+1} dx = \int_\Omega \frac{\epsilon^{\frac{N}{p-1}} \varphi^{2 \cdot p^*}}{(\epsilon^{\frac{p}{p-1}} + |x - y|^{\frac{p}{p-1}})^N} dx = \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^{\frac{p}{p-1}})^N} dx + O(\epsilon^{\frac{N}{p-1}}),$$

$$\int_\Omega |U_\epsilon|^p |\nabla U_\epsilon|^p dx = \int_\Omega \frac{\epsilon^{\frac{N-p}{2(p-1)}}\varphi^p}{(\epsilon^{\frac{p}{p-1}} + |x-y|^{\frac{p}{p-1}})^{\frac{N-p}{2}}} \times \Big| \frac{\epsilon^{\frac{N-p}{2p(p-1)}}\nabla\varphi}{(\epsilon^{\frac{p}{p-1}} + |x-y|^{\frac{p}{p-1}})^{\frac{N-p}{2p}}} - \frac{(N-p)\epsilon^{\frac{N-p}{2p(p-1)}}|x-y|^{\frac{2-p}{p-1}}(x-y)\varphi}{2(p-1)(\epsilon^{\frac{p}{p-1}} + |x-y|^{\frac{p}{p-1}})^{\frac{N+p}{2p}}}\Big|^p dx$$
$$= \Big[\frac{N-p}{2(p-1)}\Big]^p \int_\Omega \frac{\epsilon^{\frac{N-p}{p-1}}|x-y|^{\frac{p}{p-1}}\varphi^{2p}}{(\epsilon^{\frac{p}{p-1}} + |x-y|^{\frac{p}{p-1}})^N} dx + O(\epsilon^{\frac{N}{p-1}})$$
$$= \Big[\frac{N-p}{2(p-1)}\Big]^p \int_{\mathbb{R}^N} \frac{|x|^{\frac{p}{p-1}}}{1+|x|^{\frac{p}{p-1}}}^N dx + O(\epsilon^{\frac{N}{p-1}}).$$

Set

$$B := (\frac{N-p}{p-1})^p \int_{\mathbb{R}^N} \frac{|x|^{\frac{p}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^N} dx, \quad A := \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^{\frac{p}{p-1}})^N},$$

and

$$S := \frac{B}{A^{\frac{N-p}{N}}}.$$

Since $w_\lambda \in L^\infty(\Omega)$, we have

$$I_\lambda(u_\lambda) = \frac{1}{p}\int_\Omega (1+p|u_\lambda|^p)|\nabla u_\lambda|^p dx - \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma} dx - \frac{1}{q+1}\int_\Omega |u_\lambda|^{q+1} dx,$$

where $u_\lambda = g(w_\lambda) \in L^\infty(\Omega)$. For $s, R > 0$, define

$$I_\lambda(u_\lambda + sRU_\epsilon) = \frac{1}{p}\int_\Omega (1+p|u_\lambda + sRU_\epsilon|^p)|\nabla(u_\lambda + sRU_\epsilon)|^p dx - \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda + sRU_\epsilon|^{1-\gamma} dx - \frac{1}{q+1}\int_\Omega |u_\lambda + sRU_\epsilon|^{q+1} dx.$$

Note that $J_\lambda(w_\lambda) = I_\lambda(u_\lambda)$. Moreover, for $t > 0$,

$$J_\lambda(w_\lambda + tRU_\epsilon) = I_\lambda(g(w_\lambda + tRU_\epsilon)) = I_\lambda(g(w_\lambda) + g'(w_\lambda + \zeta RU_\epsilon)tRU_\epsilon) = I_\lambda(u_\lambda + sRU_\epsilon),$$

where $\zeta \in (0, t)$ and $s = g'(w_\lambda + \zeta RU_\epsilon)t$. Therefore, following the argument in [40], for $p \in (2, 3)$, we obtain

$$
\begin{aligned}
&J_\lambda(w_\lambda + tRU_\epsilon) - J_\lambda(w_\lambda) = I_\lambda(u_\lambda + sRU_\epsilon) - I_\lambda(u_\lambda)\\
&= \frac{1}{p}\int_\Omega (1 + p|u_\lambda + sR_0U_\epsilon|^p)|\nabla(u_\lambda + sRU_\epsilon)|^p dx - \frac{1}{p}\int_\Omega (1 + p|u_\lambda|^p)|\nabla u_\lambda|^p dx\\
&\quad + \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma}dx - \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda + sRU_\epsilon|^{1-\gamma}dx + \frac{1}{q+1}\int_\Omega |u_\lambda|^{q+1}dx\\
&\quad - \frac{1}{q+1}\int_\Omega |u_\lambda + sRU_\epsilon|^{q+1}dx\\
&\le \int_\Omega (1 + p|u_\lambda|^p + ps^pR^pU_\epsilon^p + p^2sR|u_\lambda|^{p-2}u_\lambda U_\epsilon)sR|\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla U_\epsilon dx\\
&\quad + \int_\Omega (s^{p-1}R^{p-1}U_\epsilon^{p-1} + p|u_\lambda|^{p-2}u_\lambda)sRU_\epsilon|\nabla u_\lambda|^p dx\\
&\quad + \int_\Omega (\frac{1}{p} + |u_\lambda|^p + psR|u_\lambda|^{p-2}u_\lambda U_\epsilon)(sR)^p|\nabla U_\epsilon|^p dx + \int_\Omega (sR)^{2p}U_\epsilon^p|\nabla U_\epsilon|^p dx\\
&\quad + \frac{1}{q+1}\int_\Omega \Big[|u_\lambda|^{q+1} - |u_\lambda + sRU_\epsilon|^{q+1}\Big]dx + o(\epsilon^{\frac{N-p}{p}})\\
&= \frac{1}{2^p}(sR)^{2p}B - \frac{(sR)^{q+1}}{q+1}A + o(\epsilon^{\frac{N-p}{p}})\\
&\le \frac{1}{2p}(sR)^{2p}B - \frac{(sR)^{q+1}}{q+1}A + o(\epsilon^{\frac{N-p}{p}}).
\end{aligned}
$$

Arguing as in [40] for $p \in [2, 3)$ and noting that the case $p = 2$ was considered in [23], we find $R_0 > 0$ such that

$$
\sup_{t\in[0,1]} J_\lambda(w_\lambda + tR_0U_\epsilon) < J_\lambda(w_\lambda) + \frac{1}{2N}S^{\frac{N}{p}}.
$$

For $p \ge 3$, we have

$$
\begin{aligned}
&J_\lambda(w_\lambda + tRU_\epsilon) - J_\lambda(w_\lambda) = I_\lambda(u_\lambda + sRU_\epsilon) - I_\lambda(u_\lambda)\\
&= \frac{1}{p}\int_\Omega (1 + p|u_\lambda + sRU_\epsilon|^p)|\nabla(u_\lambda + sRU_\epsilon)|^p dx - \frac{1}{p}\int_\Omega (1 + p|u_\lambda|^p)|\nabla u_\lambda|^p dx\\
&\quad + \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma}dx - \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda + sRU_\epsilon|^{1-\gamma}dx + \frac{1}{q+1}\int_\Omega |u_\lambda|^{q+1}dx\\
&\quad - \frac{1}{q+1}\int_\Omega |u_\lambda + sRU_\epsilon|^{q+1}dx\\
&\le \int_\Omega (1 + p|u_\lambda|^p + ps^pR^pU_\epsilon^p + p^2sR|u_\lambda|^{p-2}u_\lambda U_\epsilon)sR|\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla U_\epsilon dx\\
&\quad + \int_\Omega (s^{p-1}R^{p-1}U_\epsilon^{p-1} + p|u_\lambda|^{p-2}u_\lambda)sRU_\epsilon|\nabla u_\lambda|^p dx\\
&\quad + \int_\Omega \Big(\frac{1}{p} + |u_\lambda|^p + psR|u_\lambda|^{p-2}u_\lambda U_\epsilon\Big)(sR)^p|\nabla U_\epsilon|^p dx + \int_\Omega (sR)^{2p}U_\epsilon^p|\nabla U_\epsilon|^p dx\\
&\quad + C_1\epsilon^{\frac{4(N-p)}{p(p-1)}} - \frac{1}{q+1}\int_\Omega (sR)^{q+1}U_\epsilon^{q+1}dx - C_2\epsilon^{\frac{2(N-p)}{p(p-1)}} + o(\epsilon^{\frac{2(N-p)}{p(p-1)}})
\end{aligned}
$$

$$\leq \frac{1}{2^p}(sR)^{2p}B - \frac{(sR)^{q+1}}{q+1}A - C_3\epsilon^{\frac{2(N-p)}{p(p-1)}} + o(\epsilon^{\frac{2(N-p)}{p(p-1)}})$$
$$\leq \frac{1}{2p}(sR)^{2p}B - \frac{(sR)^{q+1}}{q+1}A - C_3\epsilon^{\frac{2(N-p)}{p(p-1)}} + o(\epsilon^{\frac{2(N-p)}{p(p-1)}}).$$

Thus, we obtain

$$\sup_{t\in[0,1]} J_\lambda(w_\lambda + tR_0U_\epsilon) < J_\lambda(w_\lambda) + \frac{1}{2N}S^{\frac{N}{p}}.$$

Hence, the claim is proved. Now, we show that the sequence $\{v_n\}_{n=1}^{\infty}$ is compact. First, we prove that $\{v_n\}_{n=1}^{\infty}$ is bounded in $W_0^{1,p}(\Omega)$. Define $\eta(t) = w_\lambda + tR_0U_\varepsilon$ for $t \in [0,1]$. Then we have

$$\overline{J_\lambda}(w_\lambda) = J_\lambda(w_\lambda) < \gamma_0 < J_\lambda(w_\lambda) + \frac{1}{2N}S^{\frac{N}{p}}.$$

Applying Ekeland's variational principle to the functional

$$\Phi(\eta) = \max_{t\in[0,1]} J_\lambda(\eta(t)), \quad \eta \in \Gamma,$$

there exists a sequence $\{\eta_k\}_{k=1}^{\infty} \subset \Gamma$ such that

$$\max_{t\in[0,1]} J_\lambda(\eta_k(t)) \leq \gamma_0 + \frac{1}{k},$$

$$\max_{t\in[0,1]} J_\lambda(\eta_k(t)) \leq \max_{t\in[0,1]} J_\lambda(\eta(t)) + \frac{1}{k}\max_{t\in[0,1]}\|\eta(t) - \eta_k(t)\|, \ \forall \eta \in \Gamma.$$

By a similar proof of Lemma 3.5 in [41], one can show that there exists $v_k \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} J_\lambda(v_k) \to \gamma_0 \ as\ k \to \infty, \\ \int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla(w - v_k)dx - \lambda\int_\Omega g(v_k)^{-\gamma}g'(v_k)(w - v_k)dx - \int_\Omega g(v_k)^q g'(v_k)(w - v_k)dx \\ \geq -\frac{C}{k}(1 + \|w\|), \ \ \forall \omega \in W_0^{1,p}(\Omega). \end{cases} \tag{4.5}$$

Taking $w = 2v_k$ in (4.5), we obtain

$$\int_\Omega |\nabla v_k|^p dx - \lambda\int_\Omega g(v_k)^{-\gamma}g'(v_k)v_k dx - \int_\Omega g(v_k)^q g'(v_k)v_k dx \geq -\frac{C}{k}(1 + \|2v_k\|).$$

From Lemma 2.1(v), it follows that

$$\int_\Omega |\nabla v_k|^p dx - \frac{\lambda}{2}\int_\Omega g(v_k)^{1-\gamma}dx - \frac{1}{2}\int_\Omega g(v_k)^{q+1}dx \geq -\frac{C}{k}(1 + \|2v_k\|).$$

Therefore,

$$\begin{aligned} \gamma_0 + o(1) &= \frac{1}{p}\|v_k\|^p - \frac{\lambda}{1-\gamma}\int_\Omega g(v_k)^{1-\gamma}dx - \frac{1}{q+1}\int_\Omega g(v_k)^{q+1}dx \\ &\geq (\frac{1}{p} - \frac{2}{q+1})\|v_k\|^p - \lambda(\frac{1}{1-\gamma} - \frac{1}{q+1})\int_\Omega |v_k|^{1-\gamma}dx - \frac{C}{k}(1 + \|2v_k\|). \end{aligned}$$

Thus, $\{v_k\}_{k=1}^{\infty}$ is bounded in $W_0^{1,p}(\Omega)$. By Lemma 4.1, we obtain that $v_k \to v_\lambda$ weakly in $W_0^{1,p}(\Omega)$, where $v_\lambda$ is a weak solution of (2.1), and

$$\int_\Omega |\nabla(v_k - v_\lambda)|^p dx = \int_\Omega g(v_k - v_\lambda)^q g'(v_k - v_\lambda)(v_k - v_\lambda)dx + o(1).$$

Moreover, for small $\xi > 0$, we have

$$\begin{aligned} &\frac{1}{p}\|v_k - v_\lambda\|^p - \frac{1}{q+1}\int_\Omega |g(v_k - v_\lambda)|^{q+1} dx \\ &= J_\lambda(v_k) - J_\lambda(v_\lambda) + o(1) \\ &\leq \gamma_0 - J_\lambda(w_\lambda) + o(1) \\ &< \frac{1}{2N} S^{\frac{N}{p}} - \xi. \end{aligned} \tag{4.6}$$

By Lemma 2.1, we have

$$\int_\Omega g(v_k - v_\lambda)^q g'(v_k - v_\lambda)(v_k - v_\lambda)dx = \int_\Omega |g(v_k - v_\lambda)|^{q+1} dx + o(1). \tag{4.7}$$

In addition, it follows from the definition of $g(t)$ and Lemma 2.1 that $g(t)g'(t) \to 0$ as $t \to 0$. Therefore, there exists a large constant $A \geq (\frac{p}{p+N})^{p/(N-p)} K_0^{2-2p^*}$ such that

$$g(t)g'(t) \leq \frac{1}{A}.$$

This yields

$$\begin{aligned} \int_\Omega g(v_k - v_\lambda)^q g'(v_k - v_\lambda)(v_k - v_\lambda)dx &\leq \frac{1}{A}\int_\Omega |g(v_k - v_\lambda)|^{q-1}(v_k - v_\lambda)dx \\ &\leq \frac{K_0^{q-1}}{A}\int_\Omega |v_k - v_\lambda|^{\frac{q+1}{2}} dx \\ &= \frac{K_0^{2p^*-2}}{A}\int_\Omega |v_k - v_\lambda|^{p^*} dx. \end{aligned} \tag{4.8}$$

Now suppose that $\int_\Omega |\nabla(v_k - v_\lambda)|^p dx \to b$ and $\int_\Omega g(v_k - v_\lambda)^q g'(v_k - v_\lambda)(v_k - v_\lambda)dx \to b$ as $k \to \infty$. Then (4.8) implies that

$$\frac{K_0^{2p^*-2}}{A}\int_\Omega |v_k - v_\lambda|^{p^*} dx \geq b.$$

From the Sobolev imbedding theorem and the best constant $S$, we get

$$b \geq S(AK_0^{2-2p^*} b)^{\frac{p}{p^*}}.$$

Hence, either $b = 0$ or

$$b \geq A^{\frac{N-p}{p}} K_0^{\frac{2N-2pN}{p}-2} S^{\frac{N}{p}}.$$

If $b \geq A^{\frac{N-p}{p}} K_0^{\frac{2N-2pN}{p}-2} S^{\frac{N}{p}}$, then from (4.6) and (4.7), we derive

$$\frac{N+p}{2pN} b \leq \frac{1}{2N} S^{\frac{N}{p}} - \xi,$$

which implies

$$b \leq \frac{p}{N+p} S^{\frac{N}{p}} - \varrho \leq A^{\frac{N-p}{p}} K_0^{\frac{2N-2pN}{p}-2} S^{\frac{N}{p}} - \varrho,$$

provided that $A \geq (\frac{p}{p+N})^{p/(N-p)} K_0^{2-2p^*}$, where $\varrho = \frac{2pN}{N+p}\xi > 0$. This leads to a contradiction. Therefore, we must have $b = 0$, which means $v_k \to v_\lambda$ strongly in $W_0^{1,p}(\Omega)$. Consequently, $J_\lambda(v_\lambda) = \gamma_0 > J_\lambda(w_\lambda)$, implying that $v_\lambda \neq w_\lambda$. This completes the proof.

## Appendix

In this section, we will prove a result that has been cited before.

**Lemma A1.** *Suppose that* $0 < \gamma < 1$, $1 < p < \infty$, $2p - 1 < q < \infty$, *and that there exists a function* $\underline{\omega} \in W_0^{1,p}(\Omega)$ *such that* $\underline{\omega} \geq \epsilon\phi_1$ *for some* $\epsilon > 0$. *Define*

$$f_\lambda(x,t) = \begin{cases} \lambda g(\underline{\omega})^{-\gamma} g'(\underline{\omega}) + g(\underline{\omega})^q g'(\underline{\omega}), & \textit{if } t \leq \underline{\omega}, \\ \lambda g(t)^{-\gamma} g'(t) + g(t)^q g'(t), & \textit{if } t > \underline{\omega}, \end{cases}$$

$F_\lambda(x,t) = \int_0^t f_\lambda(x,s)ds$, *and for* $w \in W_0^{1,p}(\Omega)$,

$$\widehat{J_\lambda}(w) = \frac{1}{p}\int_\Omega |\nabla w|^p \, dx - \int_\Omega F_\lambda(x,w)\, dx.$$

*Then* $\widehat{J_\lambda}$ *belongs to* $C^1(W_0^{1,p}(\Omega), \mathbb{R})$.

*Proof.* We first consider the singular term; the other terms can be handled by standard methods. Define

$$h(x,t) = \begin{cases} g(\underline{\omega})^{-\gamma} g'(\underline{\omega}), & \text{if } t \leq \underline{\omega}, \\ \lambda g(t)^{-\gamma} g'(t), & \text{if } t > \underline{\omega}, \end{cases}$$

$$H(x,t) = \int_0^t h(x,s)\, ds, \quad \text{and} \quad S(w) = \int_\Omega H(x,w)\, dx.$$

For any $w \in W_0^{1,p}(\Omega)$, the Gâteaux derivative $S'(w)$ of $S(w)$ is given by

$$\langle S'(w), v\rangle = \int_\Omega (\max\{g(\underline{\omega}), g(w)\})^{-\gamma} (\max\{g(\underline{\omega}), g(w)\})' v(x) dx.$$

Let $w_k \in W_0^{1,p}(\Omega)$ with $w_k \to w_0$. Then

$$\begin{aligned}|\langle S'(w_k) - S'(w_0), v\rangle| &= \Big| \int_\Omega \Big[(\max\{g(\underline{\omega}), g(w_k)\})^{-\gamma}(\max\{g(\underline{\omega}), g(w_k)\})' v \\ &\qquad - (\max\{g(\underline{\omega}), g(w_0)\})^{-\gamma}(\max\{g(\underline{\omega}), g(w_0)\})' v\Big]\, dx\Big| \\ &\le 2 \int_\Omega g(\underline{\omega})^{-\gamma} g'(\underline{\omega})|v|\, dx \\ &\le 2 \int_\Omega \underline{\omega}^{-\gamma}(g'(\underline{\omega}))^{1-\gamma}|v|\, dx \\ &\le 2\epsilon^{-\gamma} \int_\Omega \phi_1^{-\gamma}|v|\, dx\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. By Hardy's inequality, $\phi_1^{-\gamma} v \in L^1(\Omega)$ for $0 < \gamma < 1$. Hence, applying Lebesgue's dominated convergence theorem, we conclude that $S'(w)$ is continuous, that is, $S \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Consequently, $\widehat{J_\lambda} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. This completes the proof.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

The research of Siyu Chen was supported by the National Natural Science Foundation of China (Grant No. 12501142), the Zhejiang Provincial Natural Science Foundation of China (Grant No. LQN25A010017), and the China Scholarship Council (Grant No. 202409610002). The research of Yu Zheng was supported by the National Natural Science Foundation of China (Grant No. 12101257). The research of Jiazheng Zhou was supported by CNPq/Brazil (Grant No. 304627/2023-2).

## Conflict of interest

Professor Jiazheng Zhou is the Guest Editor of special issue "Nonlinear PDEs and Dynamical Systems: New Trends and Applications in Mathematical Biology" for Electronic Research Archive. Prof. Zhou was not involved in the editorial review and the decision to publish this article. The authors declare there are no conflicts of interest.

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