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Research article

Existence for critical Schrödinger–Bopp–Podolsky system with p-Laplacian in $\mathbb{R}^3$

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Abstract: We intend to study a class of the critical Schrödinger–Bopp–Podolsky system with p-Laplacian in $\mathbb{R}^3$. Under different perturbation terms, the existence and multiplicity of nontrivial solutions are obtained by using critical point theorem. Considering the influence of the p-Laplacian operator and critical and nonlocal terms, which cause the loss of the compactness condition, we attempt to address this difficulty by using the concentration-compactness principle and some clever analysis.

Keywords: Schrödinger–Bopp–Podolsky system; critical growth; p-Laplacian; choquard nonlinearity; variational methods; concentration-compactness principles

1. Introduction

This article focuses on the following critical Schrödinger–Bopp–Podolsky system with p-Laplacian in $\mathbb{R}^3$:

$$\begin{cases} -\Delta_p u + b\phi|u|^{p-2}u = \alpha g(x)|u|^{q-2}u + \left(\int_{\mathbb{R}^3} \frac{|u|^{p_\mu^*}}{|x-y|^\mu} dy\right)|u|^{p_\mu^*-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi|u|^p, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Delta_p u :=\text{div}(|\nabla u|^{p-2}\nabla u)$ is a p-Laplace operator, $b, \alpha \in \mathbb{R}$ are some positive parameters, $0 < \mu < 3$, $2 \le p < 3$, $2p < q < p^* := \frac{3p}{3-p}$, $p_\mu^* = (3p - p\frac{\mu}{2})/(3 - p)$ represents the critical exponent for the Hardy–Littlewood–Sobolev inequality, $a > 0$ is the Bopp–Podolsky constant, and g is a weight function satisfies the subsequent assumption:

($\mathcal{G}$) $g(x) \ge 0$ for all $x \in \mathbb{R}^3$, and $0 < \mathcal{L}(\Lambda) < \infty$ for $\Lambda = \{x \in \mathbb{R}^3 : g(x) > 0\}$, where $g \in L^\infty(\mathbb{R}^3)$ and $\mathcal{L}$ is the Lebesgue measure in $\mathbb{R}^3$.

(1.1) is a recent version of the Schrödinger–Bopp–Podolsky system, first proposed by D'Avenia and Siciliano [1]. We can implement such a system by coupling the Schrödinger field $\psi = \psi(t, x)$ with its electromagnetic field within the Bopp–Podolsky electromagnetic theory, particularly in the electrostatic case of standing wave $\psi(t,x) = e^{iwt}u(x)$. Within the framework of electromagnetic theory, the Bopp–Podolsky theory proposed by Bopp [2] and Podolsky [3] is a second-order gauge field theory, and it can be regarded as a theory that is valid within a short distance range (see Frenkel [4]). In the case of larger distances, this theory is experimentally indistinguishable from Maxwell's theory. Therefore, the Bopp–Podolsky parameter $a > 0$, whose dimension is inversely proportional to mass, can either be regarded as the truncated distance, or it can be associated with the effective radius of the electron. Furthermore, this theory resolves the so-called infinity problem that arises in the classical Maxwell theory. For more physics content, we refer to [5–7] for details.

In recent years, a series of interesting achievements have been made in the research on the Schrödinger–Bopp–Podolsky system, and we only present some representative results in this field. Siciliano and Silva in [8] explored the following system:

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $p \in (2,6)$, $\omega > 0$ and $a > 0$. They acquired the desired results for system (1.2) depending on the value of q. More precisely, with the aid of the fibering approach, they demonstrated that when the value of q is sufficiently large, there exists no solution for system (1.2), while for small values of q, two radial solutions exist for the system. This accomplishment has exerted a profound influence on the subsequent investigations of the Schrödinger–Bopp–Podolsky system. In addition, we have discovered that a number of results for problem (1.1) have been achieved via diverse methods. For example, Damian and Siciliano [9] studied the system below using the variational method:

$$\begin{cases} -\epsilon^2\Delta u + V(x)u + Q(x)\phi u = h(x,u) + K(x)|u|^4u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

For functions V and K satisfying appropriate conditions, the authors proved that small solutions exist in the semiclassical limit for all fixed $a > 0$. Furthermore, as $a \to 0$, they demonstrated that for a fixed and sufficiently small value of ϵ, the solutions converge strongly to those corresponding to the Schrödinger–Poisson system. Let us refer to [10–12] and to the references therein on some critical problems. Recently, many mathematical researchers began to study problems with convolutional nonlinearity terms. For example, Yang and Shen [13] analyzed the following problem:

$$-\Delta u - \Delta(|u|^2)u + V(x)u = (I_\mu|u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where I_μ is the Riesz potential and $V(x) = a - \frac{b}{1+|x|^2}$. They overcame the difficulties potentially caused by the quasilinear term through the variable substitution method, and they studied the corresponding functional by means of variational methods to find the ground state solutions and positive solutions of the equation and established the nonexistence result by virtue of the Pohožaev identity.

Later, Xiao et al. [14] dealt with the following Schrödinger–Bopp–Podolsky system with convolution nonlinearity:

$$\begin{cases} -\Delta u + V(x)u + \phi u = (I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $I_\alpha : \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential and $\alpha \in (0,2)$. A thorough examination of the nonlinear components enabled them to demonstrate the presence of ground state solutions and positive minimal energy solutions for system (1.5). Afterwards, Alves et al. [15] employed the critical point theory of non-differentiable functionals and utilized the sub-supersolution method to conduct an in-depth investigation into the existence and multiplicity of solutions for a class of generalized Schrödinger–Bopp–Podolsky systems featuring nonlinearities. Other interesting results can be found in [16–19].

It should be specially emphasized that in all the above-mentioned literature, the authors were merely interested in the $p = 2$ case for the Schrödinger–Bopp–Podolsky system. Particularly inspired by the ideas of [20–22], we propose to investigate some existing results of solutions corresponding to p-Laplacian. Moreover, once the second equation in system (1.1) contains the nonlinearity $|u|^p$, whether the nice properties of the nonlinearity ϕ are still true. In this paper, we intend to solve these problems and provide affirmative answers to them.

The main results of this work are presented as follows:

Theorem 1.1. *Let* $(\mathcal{G})$ *be satisfied. For any* $b > 0$ *and* $n \in \mathbb{N}$*, there is* $\alpha_n > 0$ *such that* $\alpha \in (\alpha_n, +\infty)$*, and system* (1.1) *has n pairs of solutions with positive energy.*

We establish these findings primarily through variational methods and topological approaches. Compared with prior research, exploring system (1.1) is more intriguing and challenging, owing to the concurrent existence of the combined effects of p-Lapalcian, convolutional nonlinearities, and non-local terms. Our contributions are mainly concentrated on the three areas below:

- (a) In the present work, we will use various critical point theorems to prove the multiplicity and existence of nontrivial solutions for system (1.1), which incorporates p-Laplacian. Our inspiration mainly stems from [1, 9, 23, 24]. However, our study does not simply replicate the approaches presented in the above-mentioned papers; instead, it involves targeted refinement and enhancement. We aim to verify the nice properties of the nonlinearity ϕ, which play a significant role in obtaining the main conclusion of this article.
- (b) The core of this paper lies in establishing the validity of the $(PS)_c$ condition. The interplay between nonlocal terms and critical nonlinearity leads to the loss of the overall spatial compactness, prompting us to employ the principle of concentrated compactness to reconstruct the compact structure of the space.
- (c) It appears to be the first application of Pereira's critical point theorem [25] to the critical Schrödinger–Bopp–Podolsky system. Although this strategy has been employed in other research problems, modifying it to the procedure for our problem is no easy task. Because of the nonlocal term, this problem requires re-evaluation and more accurate parameter estimation.

The framework layout of the article is described as follows: In Section 2, we make full preparations and give the corresponding workspace, which are useful to obtain the main conclusions. In Section 3, we prove that the energy functional I satisfies the $(PS)_c$ condition, which is established via applying relevant versions of Lion's second concentration-compactness principle. In Section 4, we prove the Theorem 1.1 by verifying the geometric hypotheses required for the above-mentioned critical point theorem due to Perera [25].

2. Preliminaries

Here and in the subsequent parts of the paper, $\|\cdot\|_p$ denotes the Sobolev space $L^p(\mathbb{R}^3)$ norm for $p > 1$. We will use the following Sobolev space $D^{1,p}(\mathbb{R}^3)$:

$$D^{1,p}(\mathbb{R}^3) := \left\{ u \in L^{p^*}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^p dx < \infty \right\},$$

which is a Banach space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^p dx \right)^{\frac{1}{p}}. \tag{2.1}$$

Moreover, the space of radially symmetric functions $D^{1,p}_{rad}(\mathbb{R}^3)$ is defined by

$$D^{1,p}_{rad}(\mathbb{R}^3) = \{u \in D^{1,p}(\mathbb{R}^3) : u(x) = u(|x|)\}$$

with the norm $\|\cdot\|$. Then there exists the continuous embedding $D^{1,p}_{rad}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in [p, p^*]$ and the compact embedding $D^{1,p}_{rad}(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in (p, p^*)$.

The best embedding constant from $D^{1,p}(\mathbb{R}^3)$ into $L^{p^*}(\mathbb{R}^3)$ is defined as

$$S = \inf_{u \in D^{1,p}(\mathbb{R}^3)\setminus\{0\}} \left\{ \int_{\mathbb{R}^3} |\nabla u|^p dx : \int_{\mathbb{R}^3} |u|^{p^*} dx = 1 \right\}. \tag{2.2}$$

The Sobolev embedding theorem and Hölder inequality indicate that

$$\|u\|_{g,q}^q := \int_{\mathbb{R}^3} g(x)|u|^q dx \tag{2.3}$$

for each $u \in D^{1,p}_{rad}(\mathbb{R}^3)$. From Talenti [26], we know that $S > 0$.

In what follows, we first recall the famous Hardy–Littlewood–Sobolev inequality [27].

Lemma 2.1. *Let $\iota, \iota_* > 1$ and $0 < \mu < N$ such that $\frac{1}{\iota} + \frac{\mu}{N} + \frac{1}{\iota_*} = 2$, $h_1 \in L^\iota(\mathbb{R}^N)$ and $h_2 \in L^{\iota_*}(\mathbb{R}^N)$. Then there exists a sharp constant $C(\iota, \iota_*, \mu, N) > 0$, which is independent of h_1 and h_2 such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h_1(x)h_2(y)}{|x-y|^\mu} dxdy \le C(\iota, \iota^*, \mu, N)\|h_1\|_\iota \|h_2\|_{L^{\iota^*}}.$$

By the Hardy–Littlewood–Sobolev inequality, we get the well-defined integral

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\mu} dxdy$$

in $D^{1,p}_{rad}(\mathbb{R}^3)$, if $|u|^p \in L^t(\mathbb{R}^3)$ for $t > 1$ such that $\frac{2}{t} + \frac{\mu}{3} = 2$, that is, $t = \frac{6}{6-\mu}$. From the perspective of the Hardy–Littlewood–Sobolev inequality, the exponent p_μ^* is referred to as the upper critical exponent. In particular,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy \le \widetilde{C}(\mu)\|u\|_{p^*}^{2p_\mu^*} \tag{2.4}$$

for all $u \in D^{1,p}_{rad}(\mathbb{R}^3)$. Hence,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy \le C(\mu)\|u\|^{2p_\mu^*},$$

where $C(\mu) = \widetilde{C}(\mu)S^{2p_\mu^*} > 0$. Furthermore, we set

$$S_{H,L} = \inf_{u\in D^{1,p}(\mathbb{R}^3)\setminus\{0\}}\left\{\int_{\mathbb{R}^3}|\nabla u|^p dx : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy = 1\right\}, \tag{2.5}$$

and clearly $S_{H,L} > 0$.

On the other hand, in order to give the specific expression of Poisson's equation, let $\mathcal{X}$ represent the $C_c^\infty(\mathbb{R}^3)$ completion under the norm $\|\cdot\|_{\mathcal{X}}$ induced by the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{X}} := \int_{\mathbb{R}^3}\nabla\varphi\nabla\psi dx + a^2\int_{\mathbb{R}^3}\Delta\varphi\Delta\psi dx.$$

Then, $\mathcal{X}$ is a Hilbert space, and there exists the continuous embeddings $\mathcal{X} \hookrightarrow D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, where $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$.

Lemma 2.2. *[1] The space $\mathcal{X}$ is continuously embedded into $L^\infty(\mathbb{R}^3)$.*

For any given fixed element $u \in D^{1,p}_{rad}(\mathbb{R}^3)$ and $\phi \in \mathcal{X}$, the map $\phi \mapsto \int_{\mathbb{R}^3}\phi|u|^p$ is continuous and linear. The Riesz theorem ensures that there is an unique solution $\phi_u \in \mathcal{X}$, which solves the following equation:

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi|u|^p. \tag{2.6}$$

For the explicit expression of such a solution, we consider

$$\mathcal{K}(x) = \frac{1-e^{\frac{-|x|}{a}}}{|x|}$$

and recall the useful properties of the kernel $\mathcal{K}$.

Lemma 2.3. *[1] For $y \in \mathbb{R}^3, \mathcal{K}(\cdot - y)$ solves*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_y$$

in the sense of distributions. Moreover,

(i) if $f \in L^1_{loc}(\mathbb{R}^3)$ and, for a.e. $x \in \mathbb{R}^3$, the map $y \mapsto \frac{f(y)}{|x-y|}$ is summable, then $\mathcal{K} * f \in L^1_{loc}(\mathbb{R}^3)$;

(ii) if $f \in L^s(\mathbb{R}^3)$ with $s \in \left[1, \frac{3}{2}\right)$, then $\mathcal{K} * f \in L^q$ for all $q \in \left(\frac{3s}{3-2s}, \infty\right]$.

Under both circumstances, $\mathcal{K} * f$ solves

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi f. \tag{2.7}$$

Based on the above facts, for fixed $u \in D^{1,p}_{rad}(\mathbb{R}^3)$, $|u|^p \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and the solution in $\mathcal{X}$ of the second equation in the Schrödinger–Bopp–Podolsky system is the unique equation, which has the explicit form

$$\phi_u(x) := \mathcal{K} * |u|^p = \int_{\mathbb{R}^3}\frac{1-e^{\frac{-|x-y|}{a}}}{|x-y|}|u|^p(y)dy. \tag{2.8}$$

Subsequently, we give some nice properties of ϕ_u.

Lemma 2.4. *[28] For any $u \in D^{1,p}(\mathbb{R}^3)$, we have the following properties:*

(i) *For every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$;*

(ii) *$\phi_u \geq 0$ and $\phi_{tu} = t^p \phi_u$;*

(iii) *$\phi_u \in \mathcal{X}$;*

(iv) *$\|\phi_u\|_{\mathcal{X}} \leq C\|u\|^p$ with the positive C independent of u, and it holds*

$$\int_{\mathbb{R}^3} \phi_u |u|^p dx \leq C\|u\|^{2p}_{\frac{6p}{5}};$$

(v) *ϕ_u is the unique minimizer of the functional*

$$E(\phi) = \frac{1}{2}|\nabla\phi|_2^2 + \frac{a^2}{2}|\Delta\phi|_2^2 - \int_{\mathbb{R}^3} \phi|u|^p dx \ \text{ for any } \phi \in \mathcal{X};$$

(vi) *If $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{X}$ and*

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{p-2}u_n\varphi dx \to \int_{\mathbb{R}^3} \phi_u|u|^{p-2}u\varphi dx, \ \text{ for any } \varphi \in D^{1,p}(\mathbb{R}^3).$$

Using the reduction arguments introduced in d'Avenia and Siciliano [1], system (1.1) can be reduced to the single equation:

$$-\Delta_p u + b\phi_u|u|^{p-2}u = \alpha g(x)|u|^{q-2}u + \left(\int_{\mathbb{R}^3} \frac{|u|^{p^*_\mu}}{|x-y|^\mu}dy\right)|u|^{p^*_\mu-2}u, \tag{2.9}$$

where ϕ_u is defined by (2.8).

It is straightforward to get the corresponding energy functional of system (1.1) as follows:

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{b}{2p}\int_{\mathbb{R}^3} \phi_u|u|^p dx - \frac{\alpha}{q}\int_{\mathbb{R}^3} g|u|^q dx - \frac{1}{2p^*_\mu}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{p^*_\mu}|u(y)|^{p^*_\mu}}{|x-y|^\mu}dxdy. \tag{2.10}$$

By using standard proof in Rabinowitz [29], we know the energy functional $I \in C^1(D^{1,p}_{rad}(\mathbb{R}^3))$, and the Fréchet derivative of I is given by

$$\begin{aligned}\langle I'(u), v\rangle = &\int_{\mathbb{R}^3} |\nabla u|^{p-2}\nabla u\nabla v dx + b\int_{\mathbb{R}^3} \phi_u|u|^{p-2}uv dx \\ &- \alpha\int_{\mathbb{R}^3} g|u|^{q-2}uv dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u|^{p^*_\mu}}{|x-y|^\mu}|u|^{p^*_\mu-2}uv dxdy\end{aligned} \tag{2.11}$$

for any $u, v \in D^{1,p}_{rad}(\mathbb{R}^3)$.

3. $(PS)_c$ condition

In this section, we aim to establish that the energy functional I corresponding to system (1.1) satisfies the $(PS)_c$ condition. A sequence $\{u_n\} \subset D^{1,p}_{rad}(\mathbb{R}^3)$ is called a Palais–Smale sequence at the level $c \in \mathbb{R}$ for I, abbreviated as a $(PS)_c$ sequence, if

$$I(u_n) \to c \ \text{ and } \ I'(u_n) \to 0 \ \text{ as } n \to \infty. \tag{3.1}$$

If every $(PS)_c$ sequence contains a convergent subsequence, then the energy function I is said to satisfy the $(PS)_c$ condition.

Lemma 3.1. *The functional I satisfies the $(PS)_c$ condition, where*

$$0 < c < c^* := \left(\frac{1}{p} - \frac{1}{q}\right) S_{H,L}^{\frac{2p_\mu^*}{2p_\mu^*-p}}. \tag{3.2}$$

Proof. Let $\{u_n\} \subset D^{1,p}_{rad}(\mathbb{R}^3)$ be a $(PS)_c$ sequence of the functional I. Now, we shall split the proof into three claims.

Claim 1. We claim that $\{u_n\}$ is bounded in $D^{1,p}_{rad}(\mathbb{R}^3)$.

Indeed, by (2.7) and (2.8) and together with (2.2), for every $(PS)_c$ sequence $\{u_n\}$ of I, we have

$$\begin{aligned}
1 + c + \|u_n\| &\geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^3} |\nabla u_n|^p dx + b\left(\frac{1}{2p} - \frac{1}{q}\right)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p dx \\
&\quad + \left(\frac{1}{q} - \frac{1}{2p_\mu^*}\right)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy.
\end{aligned} \tag{3.3}$$

Since $2p < q < p^* < 2p_\mu^*$, we know that the $(PS)_c$ sequence $\{u_n\}$ is bounded in $D^{1,p}_{rad}(\mathbb{R}^3)$. Then there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and $u \in D^{1,p}_{rad}(\mathbb{R}^3)$ such that

$$u_n \to u \text{ a.e. in } \mathbb{R}^3, \;\; u_n \rightharpoonup u \text{ in } D^{1,p}_{rad}(\mathbb{R}^3), \;\; \text{and} \;\; u_n \to u \text{ in } L^\theta_{loc}(\mathbb{R}^3) \text{ for all } \theta \in [p, \frac{3p}{3-p}). \tag{3.4}$$

Claim 2. We claim that $\Lambda = \emptyset$.

By Claim 1, together with the concentration-compactness principle of Lions [30, Theorem 1.3] (see also [31, Lemma 4.5]), passing to a subsequence, we may assume that there exists $u \in D^{1,p}_{rad}(\mathbb{R}^3)$ such that

$$|\nabla u_n|^p dx \rightharpoonup \omega \geq |\nabla u_n|^p + \sum_{i\in\Lambda} \omega_i \delta_{x_i}$$

and

$$\left(\int_{\mathbb{R}^3} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right)|u_n(x)|^{p_\mu^*} dx \rightharpoonup \zeta = \left(\int_{\mathbb{R}^3} \frac{|u_n|^{p_\mu^*}}{|x-y|^\mu} dy\right)|u_n|^{p_\mu^*} + \sum_{i\in\Lambda} \zeta_i \delta_{x_i}$$

in the sense of measures, and

$$S_{H,L}\zeta_i^{\frac{p}{2p_\mu^*}} \leq \omega_i, \tag{3.5}$$

where Λ is an index set that is at most countable, $S_{H,L}$ represents the best Sobolev constant, δ_{x_i} is the Dirac-mass of mass 1 concentrated at $x_i \in \mathbb{R}^3$ and $\{\omega_i\}_{i\in\Lambda}, \{\zeta_i\}_{i\in\Lambda}$ denotes two families of positive numbers where ζ_i are contants.

In the following, we prove that $\Lambda = \emptyset$. Assume by contradiction that $\Lambda \neq \emptyset$, then there exist some $i \in \Lambda$ with $\zeta_i > 0$. For any $\epsilon > 0$, we define a smooth cut-off function $\varphi_\epsilon \in C_0^\infty(\mathbb{R}^3)$ satisfying $\varphi_\epsilon(x) = 1$

in $B(x_i,\epsilon)$, $\varphi_\epsilon(x)=0$ in $\mathbb{R}^3\setminus B(x_i,2\epsilon)$, and $|\nabla\varphi_\epsilon|\le\frac{2}{\epsilon}$ in $\mathbb{R}^3$. We can easily know that $\{u_n\varphi_\epsilon\}$ is bounded in $D^{1,p}_{rad}(\mathbb{R}^3)$, then $\lim_{n\to\infty}\langle I'(u_n),(u_n\varphi_\epsilon)\rangle=0$. Therefore, we have

$$\begin{aligned} o_n(1)=&\int_{\mathbb{R}^3}|\nabla u_n|^p\varphi_\epsilon dx+\int_{\mathbb{R}^3}u_n|\nabla u_n|^{p-1}\nabla\varphi_\epsilon dx+\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\varphi_\epsilon dx\\ &-\alpha\int_{\mathbb{R}^3}g|u_n|^q\varphi_\epsilon dx-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^{p^*_\mu}|u_n(y)|^{p^*_\mu}}{|x-y|^\mu}\varphi_\epsilon dxdy. \end{aligned} \tag{3.6}$$

Using the definition of φ_ϵ and the Hölder inequality, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3}u_n|\nabla u_n|^{p-1}\nabla\varphi_\epsilon dx\right|&=\left|\int_{B(x_i,2\epsilon)}u_n|\nabla u_n|^{p-1}\nabla\varphi_\epsilon dx\right|\\ &\le\left(\int_{B(x_i,2\epsilon)}|\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_{B(x_i,2\epsilon)}|u_n\nabla\varphi_{2\epsilon}|^p dx\right)^{\frac{1}{p}}\\ &\le C\left(\int_{B(x_i,2\epsilon)}|u_n\nabla\varphi_\epsilon|^p dx\right)^{\frac{1}{p}}, \end{aligned}$$

where $C>0$ is independent of ϵ. Since for any fixed ϵ, together with the boundness $\{u_n\}$ in $D^{1,p}_{rad}(\mathbb{R}^3)$, one has

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}u_n|\nabla u_n|^{p-1}\nabla\varphi_\epsilon dx\right|=0. \tag{3.7}$$

By Lemma 2.4(iv) and definition of φ_ϵ, we get

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p\varphi_\epsilon dx\right|=0. \tag{3.8}$$

Moreover, we also have

$$\begin{aligned} &\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^p\varphi_\epsilon dx=\int_{\mathbb{R}^3}\varphi_\epsilon d\omega\ge\int_{\mathbb{R}^3}|\nabla u|^p\varphi_\epsilon dx+\omega_i,\\ &\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}\frac{|u_n|^{p^*_\mu}}{|x-y|^\mu}dy\right)|u_n|^{p^*_\mu}\varphi_\epsilon dx=\int_{\mathbb{R}^3}\varphi_\epsilon d\zeta=\left(\int_{\mathbb{R}^3}\frac{|u|^{p^*_\mu}}{|x-y|^\mu}dy\right)|u|^{p^*_\mu}\varphi_\epsilon dx+\zeta_i \end{aligned} \tag{3.9}$$

and

$$\left|\int_{\mathbb{R}^3}g(x)|u_n|^q\varphi_\epsilon dx\right|\le\|g(x)\|_\infty\int_{B(x_i,2\varepsilon)}|u_n|^q dx.$$

Therefore, we obtain that

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}g(x)|u_n|^q\varphi_\epsilon dx\right|=0. \tag{3.10}$$

Putting (3.7)–(3.10) into (3.6), and taking $\varepsilon\to0$ and $n\to\infty$ in (3.6), we have $\zeta_i\ge\omega_i$.

Together with the fact $S_{H,L}\zeta_i^{\frac{p}{2p^*_\mu}}\le\omega_i$ for $i\in\Lambda$, we have

$$(I)\ \omega_i\ge S_{H,L}^{\frac{2p^*_\mu}{2p^*_\mu-p}}\ \text{ or }\ (II)\ \omega_i=0. \tag{3.11}$$

If (I) is true, then by $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$, together with (3.11), this implies

$$\begin{aligned} c &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle\right) \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^3} |\nabla u_n|^p dx + b\left(\frac{1}{2p} - \frac{1}{q}\right)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p dx \\ &\quad + \left(\frac{1}{q} - \frac{1}{2p_\mu^*}\right)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^3} |\nabla u_n|^p \varphi_\varepsilon dx \geq \left(\frac{1}{p} - \frac{1}{q}\right)\omega_i \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right) S_{H,L}^{\frac{2p_\mu^*}{2p_\mu^*-p}}, \end{aligned} \tag{3.12}$$

which is a contradiction. Therefore, $\Lambda = \emptyset$.

Claim 3. We claim that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy. \tag{3.13}$$

Taking the similar arguments in Claim 2, we also employ the concentration-compactness principle at infinity proposed by Chabrowski [32] (see also [24, 33]). We define the following limits:

$$\omega_\infty := \lim_{R\to\infty}\lim_{n\to\infty}\int_{B_R^c} |\nabla u_n|^p dx \;\text{ and }\; \zeta_\infty := \lim_{R\to\infty}\lim_{n\to\infty}\int_{B_R^c}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy.$$

Thus, there exist well-defined quantities ω_∞ and ζ_∞ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} |\nabla u_n|^p dx = \omega_\infty + \int_{\mathbb{R}^3} d\omega \;\text{ and }\; \lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy = \zeta_\infty + \int_{\mathbb{R}^3} d\zeta.$$

Taking the same arguments as Claim 2, we also obtain that

$$(III)\; S_{H,L}\zeta_\infty^{\frac{p}{2p_\mu^*}} \leq \omega_\infty \;\text{ or }\; (IV)\; \omega_\infty = 0. \tag{3.14}$$

If (III) is satisfied, then for each $R > 0$, let $\Psi \in C_0^\infty(\mathbb{R}^3, [0, 1])$ be a smooth cut-off function satisfying

$$\Psi_R(x) = \begin{cases} 0 & \text{for } \; |x| < R, \\ 1 & \text{for } \; |x| > R + 1. \end{cases}$$

Clearly, the sequence $\{u_n\Psi_R\}$ is bounded in $D_{rad}^{1,p}(\mathbb{R}^3)$, then

$$\begin{aligned} o_n(1) = &\int_{\mathbb{R}^3} |\nabla u_n|^p \Psi_R dx + \int_{\mathbb{R}^3} u_n|\nabla u_n|^{p-1}\nabla\Psi_R dx + \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p\Psi_R dx \\ &- \alpha\int_{\mathbb{R}^3} g|u_n|^q\Psi_R dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\Psi_R dxdy. \end{aligned} \tag{3.15}$$

Taking the similar arguments in Claim 2, we have that $\omega_\infty = 0$. Therefore, based on the above facts, we obtain that Claim 3 is true.

In the following, by $(\mathcal{G})$, we have that

$$\alpha \int_{\mathbb{R}^3} g(x)|u_n|^q dx \to \alpha \int_{\mathbb{R}^3} g(x)|u|^q dx. \tag{3.16}$$

Taking the same arguments as Lemma 5.1 in [1], we obtain

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p dx \to \int_{\mathbb{R}^3} \phi_u|u|^p dx. \tag{3.17}$$

Indeed, we define the linear and continuous operators on $\mathcal{X}$

$$\Phi_n(\varphi) = \int_{\mathbb{R}^3} \varphi|u_n|^p dx \ \text{ and } \ \Phi(\varphi) = \int_{\mathbb{R}^3} \varphi|u_n|^p dx.$$

According to the Riesz theorem, they are represented by ϕ_{u_n} and ϕ_u, respectively. It follows from the Hölder inequality that

$$\|\phi_{u_n} - \phi_u\|_{\mathcal{X}} = \|\Phi_n - \Phi\|_{\mathcal{X}'} \le C\|u_n - u\|_{6p/5}^{2p} \to 0.$$

Together with the Hölder inequality and Sobolev embeddings $\mathcal{X} \hookrightarrow D^{1,2}(\mathbb{R}^3) \hookrightarrow L^{\tilde{q}}$ for all $\tilde{q} \in [2, 6]$, and $D^{1,p}_{rad}(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^{\vartheta}(\mathbb{R}^3)$ for all $\vartheta \in (p, p^*)$, we have that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p dx - \int_{\mathbb{R}^3} \phi_u|u|^p dx\Big| \\
&= \Big|\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^p dx - \int_{\mathbb{R}^3} \phi_u|u_n|^p dx + \int_{\mathbb{R}^3} \phi_u|u_n|^p dx - \int_{\mathbb{R}^3} \phi_u|u|^p dx\Big| \\
&\le \int_{\mathbb{R}^3} |\phi_{u_n} - \phi_u||u_n|^p dx + \int_{\mathbb{R}^3} |\phi_u|(|u_n|^p - |u|^p) dx \\
&\le \Big(\int_{\mathbb{R}^3} |\phi_{u_n} - \phi_u|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3} |u_n|^{2p} dx\Big)^{\frac{1}{2}} + \Big(\int_{\mathbb{R}^3} |\phi_u|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3} \big(|u_n|^p - |u|^p\big)^2 dx\Big)^{\frac{1}{2}} \\
&\le C\|\phi_{u_n} - \phi_u\|_{D^{1,2}(\mathbb{R}^3)}\Big(\int_{\mathbb{R}^3} |u_n|^{2p} dx\Big)^{\frac{1}{2}} + \|\phi_u\|_{D^{1,2}(\mathbb{R}^3)}\||u_n|^p - |u|^p\|_2 \\
&\le C\|\phi_{u_n} - \phi_u\|_{\mathcal{X}}\|u_n\| + C\|\phi_{u_n}\|_{\mathcal{X}}\||u_n|^p - |u|^p\|_2 \\
&\to 0.
\end{aligned}$$

Therefore, we obtain that (3.17) is true.

By $I'(u_n)u_n = o_n(1)$ and Lemma 2.4, we have

$$\lim_{n\to\infty} \|u_n\|^p = -b\int_{\mathbb{R}^3} \phi_u|u|^p dx + \alpha\int_{\mathbb{R}^3} g(x)|u|^q dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{p^*_\mu}|u(y)|^{p^*_\mu}}{|x-y|^\mu} dxdy. \tag{3.18}$$

Since $I'(u_n)u \to I'(u)u$ and $I'(u_n)u \to 0$, then

$$\|u\|^p = -b\int_{\mathbb{R}^3} \phi_u|u|^p dx + \alpha\int_{\mathbb{R}^3} g(x)|u|^q dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{p^*_\mu}|u(y)|^{p^*_\mu}}{|x-y|^\mu} dxdy. \tag{3.19}$$

Together with (3.13)–(3.17), we obtain that

$$\lim_{n\to\infty} \|u_n\|^p = \|u\|^p.$$

This implies that $u_n \to u$ in $D^{1,p}_{rad}(\mathbb{R}^3)$. Therefore, the proof is completed.

4. Proof of Theorem 1.1

Let Y be a Banach space. For a symmetric subset E of Y with nonzero points, we denote the cohomological index of E by $i(E)$ in Fadell and Rabinowitz [34]. When E is homeomorphic to the unit sphere S^{n-1} in $\mathbb{R}^n$, then $i(E) = n$. We require the following result to prove that Theorem 1.1 is satisfied.

Proposition 4.1. *([25]) Let Y be a Banach space, and $I : Y \to \mathbb{R}$ be an even C^1-functional satisfying the Palais–Smale condition at level c, where $c \in (0, c^*)$ and c^* is a positive constant. Assuming that 0 is a strict local minimizer of I and there exists a compact symmetric set $E \subset \partial B_R$ and $R > 0$, where $B_R = \{u \in Y | \|u\| < R\}$, such that $i(E) = n$,*

$$\max_E I \leq 0 \ \ and \ \ \max_H I \leq c^*,$$

where $H = \{\xi u | \xi \in [0, 1], u \in E\}$, then the functional I possesses m pairs of nonzero critical points, and all critical points with positive critical values.

Proof of Theorem 1.1. Letting

$$c^* = \left(\frac{1}{p} - \frac{1}{q}\right) S_{H,L}^{\frac{2p_\mu^*}{2p_\mu^* - p}}$$

be satisfied in Lemma 3.1, I satisfies $(PS)_c$ for $c \in (0, c^*)$. Since $2p < q < p^*$, it is clear that $u = 0$ is a strict local minimizer of I. Let $Z = \{u \in D_{rad}^{1,p}(\mathbb{R}^3) | \operatorname{supp} u \subset \Lambda\}$, where Λ is as in $(\mathcal{G})$, then Z is an infinite-dimensional subspace of $D_{rad}^{1,p}(\mathbb{R}^3)$. Let Z_n denote the n-dimensional subspace of Z. On Z_n, $\|u\|_{g,q}$ given in (2.3) is the norm of $u \in Z_n$. All norms on Z_n are equivalent because $\dim Z_n < \infty$. Therefore, for $u \in Z_n$, we get

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{b}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p dx - \frac{\alpha}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx - \frac{1}{2p_\mu^*}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy \\ &\leq c_1\|u\|^p + bc_2\|u\|^{2p} - \alpha c_3\|u\|^q - c_4\|u\|^{2p_\mu^*}. \end{aligned} \tag{4.1}$$

Take $M > 0$ such that

$$f(M) := c_1 M^p + c_2 M^{2p} - c_4 M^{2p_\mu^*} < 0. \tag{4.2}$$

Let $\mathcal{E} := Z_n \cap \partial B_R$, then $i(\mathcal{E}) = n$. For all positive α, if $u \in \mathcal{E}$, then $I(u) \leq f(M) < 0$ by (4.1) and (4.2). Hence, $\max_{\mathcal{E}} I \leq 0$. From (4.2), if f is continuous and $f(0) = 0$, then there exists $\varrho \in (0, M)$ satisfying $f(t) < c^*$ for all $t \in [0, \varrho]$. Let

$$\alpha_n = 1 + \max_{t \in [\varrho, M]} \left| \frac{f(t) - c^*}{c_3 t^q} \right|.$$

If $\alpha > \alpha_n$, it follows that

$$f(t) - \alpha c_3 t^q < c^* \quad \text{for } t \in [\varrho, M].$$

Thus, for $u \in \mathcal{E}$,

1). if $\xi \in \left[0, \frac{\varrho}{M}\right]$, we have $\|\xi u\| \leq \varrho$, thus

$$I(\xi u) \leq f(\|\xi u\|) < c^*;$$

2). if $\xi \in \left[\frac{\varrho}{M}, 1\right]$, then $\|\xi u\| \in [\varrho, M]$, thus

$$I(\xi u) \leq f(\|\xi u\|) - \alpha c_3 \cdot \|\xi u\|^q < c^*.$$

Consequently, for $H = \{\xi u \mid \xi \in [0,1], u \in E\}$, we can get

$$\max_H I < c^*.$$

By Proposition 4.1, we obtain n pairs of nonzero critical points, which correspond to nontrivial solutions of system (1.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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