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Survey

Advances in cooperative game theory under uncertainty: A comprehensive survey

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Abstract: The allocation of collective benefits and costs represents a central challenge for individuals and organizations operating in uncertain environments, including interval and fuzzy data scenarios. Cooperative games under uncertainty provide a versatile game-theoretical framework and robust analytical tools to address these challenges, offering insights into decision-making and resource distribution. This survey reviews the current state of research in this emerging field, highlighting recent theoretical developments, methodological advances, and computational approaches. Furthermore, it examines how cooperative games under uncertainty extend classical cooperative game theory and discusses their practical applications and potential across economic decision-making, operations research, and strategic planning contexts. By synthesizing existing literature, this work aims to provide a comprehensive understanding of both the theoretical foundations and applied implications of these models, offering guidance for future research directions.

Keywords: operations research; cooperative games; uncertainty; fuzzy sets

1. Introduction

Cooperation among individuals and organizations is often complicated by uncertainty in the potential gains and costs associated with collective action. In practice, coalition outcomes are rarely known precisely; they are frequently subject to incomplete, imprecise, or ambiguous information, including interval-valued, gray, and fuzzy representations. Traditional cooperative game theory, although powerful, assumes precise knowledge of payoffs, limiting its applicability in such uncertain environments. Cooperative games under uncertainty address this limitation by extending classical transferable utility (TU) frameworks to accommodate payoff ranges or fuzzy estimates for each coalition. This enables players to reason strategically about possible outcomes without relying on probabilistic assumptions, thereby reflecting the realities of complex decision-making contexts.

By modeling the inherent variability in coalition values, these games provide structured methods for allocating rewards and costs, guiding cooperative strategies, and supporting the distribution of realized outcomes. Their theoretical development not only expands the scope of cooperative game theory but also delivers practical tools for contexts such as economic decision-making, organizational planning, and operations research, where ambiguity is unavoidable. As uncertainty becomes an increasingly prominent feature of real-world cooperation, the study of these models remains essential for advancing both theory and practice.

Building on this foundation, researchers have proposed a variety of solution concepts tailored to imprecise or fuzzy payoffs. These approaches, often formulated as sets of interval or fuzzy payoff vectors, help players assess the range of potential allocations within the grand coalition and inform post-cooperation sharing of realized gains or costs. Recent advances highlight not only refined theoretical frameworks but also emerging applications across diverse domains. Together, these contributions clarify the conceptual underpinnings of cooperation under uncertainty while equipping practitioners with analytical tools to navigate complex collaborative scenarios.

The remainder of this paper is organized as follows. Section 2 reviews the related literature on cooperative games under uncertainty. Section 3 introduces the mathematical preliminaries, including TU games, fuzzy intervals, and the foundations of fuzzy set theory. Section 4 develops the framework of fuzzy interval cooperative games together with core-related concepts and solution values. Section 5 presents applications and computational considerations. Finally, Section 6 concludes with a discussion of future research directions.

2. Literature review

The study of cooperative games under uncertainty has gained increasing attention over the last two decades, largely due to the growing need to capture decision-making contexts where exact payoff information is unavailable. Classical cooperative game theory, particularly transferable utility (TU) games, relies on the assumption that coalition values are known precisely, a condition rarely satisfied in practice. To overcome this limitation, researchers have proposed several extensions that incorporate uncertainty in different forms, most notably through interval, gray, and fuzzy frameworks.

Cooperative interval games constitute one of the earliest and most influential approaches to modelling uncertainty in cooperative game theory. In these frameworks, the worth of each coalition is represented by a real-valued interval, reflecting lower and upper bounds of potential outcomes when exact probabilistic information is absent. Early contributions introduced fundamental solution concepts such as the interval core, the interval Shapley value, and interval bargaining sets, thereby providing structured mechanisms to distribute uncertain gains and costs among players. Building on these foundations, subsequent research shifted attention to computational aspects and algorithmic techniques, particularly for analyzing core stability under interval uncertainty, and expanded the scope of applications to economic and operational domains.

Beyond these general models, several studies have examined domain-specific extensions and refined allocation principles. For example, interval formulations of classical cost-sharing problems such as airport models motivated the development of allocation rules, including the interval Baker–Thompson rule [1]. The Shapley value has been generalized to interval environments through axiomatic characterizations, and big boss interval games were analyzed with respect to their core properties and allocation schemes [2, 3]. Further strands of research have explored set-valued approaches, notably the interval core and interval stable sets, emphasizing their significance in convex interval games [4–8]. Applications have extended to cost-sharing contexts (e.g., mountain or connection problems) with monotonic allocation schemes, and to sequencing problems, where uncertainty in processing times or costs affects coalition formation and optimal queue orders [9–11]. These developments highlight a continuous effort to balance theoretical refinement with practical applicability, positioning cooperative interval games as a central tool for decision-making under bounded uncertainty [12].

Overall, fuzzy cooperative games extend classical cooperative game theory by representing coalition values as fuzzy sets, thereby capturing both vagueness and imprecision in payoff estimation. This has led to the development of generalized fuzzy Shapley values, fuzzy cores, and other solution concepts that incorporate degrees of membership in coalition outcomes. Applications of these models span diverse fields such as supply chain management, joint ventures, and multiagent systems, demonstrating their flexibility and effectiveness in addressing real-world decision problems where payoff information is inherently uncertain or imprecise.

Transferable utility (TU) cooperative games have been extended in several directions to address uncertainty. Within this framework, fuzzy cooperative games incorporate both coalition structures and the values of the characteristic function under fuzzy conditions. Early studies by [13, 14] pioneered the definition of characteristic functions over fuzzy subsets. Building on this idea, [15] introduced a novel class of fuzzy games based on the Choquet integral. Subsequently, [16–18] developed cooperative games with fuzzy characteristic functions, proposing fuzzy counterparts of classical concepts such as superadditivity, convexity, the core, and the Shapley value. In parallel, [19–21] investigated models where players' participation in coalitions is subject to uncertainty, expressed through fuzzy quantities. More recently, a new class of fuzzy games has been formulated that combines fuzzy coalitions with fuzzy characteristic functions. In this setting, the explicit form of the Shapley value is derived using the Hukuhara difference and the Choquet integral, as demonstrated by [22], while ensuring consistency with the four Shapley axioms originally formulated in the fuzzy context by [23]. For a comprehensive survey of the Shapley value in cooperative games under fuzzy settings, see [24].

In addition to coalition-based cooperative games, bargaining models constitute another important class of cooperative frameworks. In particular, Nash bargaining provides a principled way to determine mutually acceptable agreements when agents negotiate over surplus division. Bargaining problems can be classified as symmetric or asymmetric depending on whether players have equal bargaining power and as single-factor or multifactor depending on the number of negotiation dimensions involved. Such models have been widely applied in energy trading, networked systems, and multiagent environments under uncertainty, where issues such as cheating, privacy, and strategic misreporting may arise [25–27].

Recent literature on cooperative decision-making under uncertainty has increasingly focused on multiagent systems, where autonomous agents interact and coordinate their actions in complex environments such as intelligent transportation systems, smart grids, and sensor networks. Various modeling paradigms have been proposed, including Markov decision processes, game-theoretic frameworks, and swarm-based approaches, often supported by learning-based and evolutionary algorithms. In parallel, logic-based and fuzzy frameworks have been introduced to represent strategic interaction, incomplete information, and imprecision in a more flexible manner. These strands of research highlight the growing need for integrated models that jointly address cooperation, uncertainty, and computational tractability [28–31].

In addition to standard single-layer cooperative games, more complex organizational settings may give rise to multilayer or nested cooperative games, where cooperation takes place at different hierarchical levels. For example, subsidiaries may form coalitions at a local level while their headquarters simultaneously engage in cooperative agreements at a higher strategic level. Such nested structures naturally arise in large organizations, supply chains, and multilevel governance systems, and they allow one to capture how interactions, incentives, and uncertainty at one level influence cooperation and stability at another [32, 33].

Although cooperative game theory focuses on coalition stability and fairness, it is worth noting that noncooperative analysis is traditionally anchored in Nash equilibrium. Nash's notion of equilibrium [34–37] provides a self-enforcing stability concept at the level of individual strategies, whereas cooperative solution concepts such as the core and the Shapley value address stability and fairness at the coalition level. This distinction highlights the complementary roles of noncooperative and cooperative approaches in modeling strategic interactions under uncertainty.

Beyond interval and fuzzy specifications, stochastic frameworks such as regime-switching and jump-diffusion models offer powerful tools for capturing structural breaks and abrupt changes in dynamic systems. Although these approaches are often developed in noncooperative settings, they inform cooperative allocations through value-function characterizations, sensitivity analyses, and equilibrium-based reasoning. In particular, stochastic differential games in Markov regime-switching environments enable the analysis of zero-sum and nonzero-sum decision problems via Hamilton–Jacobi–Bellman–Isaacs equations, Feynman–Kac representations, and stochastic maximum principles, including extensions with delays, jumps, and partial information. These advances highlight the flexibility and depth of stochastic analysis for modeling complex, evolving systems under uncertainty [38, 39].

The practical relevance of cooperative games under uncertainty has been demonstrated across numerous economic and operations research contexts. Interval, fuzzy, and gray models have been applied to cost allocation in collaborative projects, profit sharing in uncertain markets, and joint planning problems under incomplete or ambiguous data. By providing structured mechanisms to evaluate alternative outcomes and allocate uncertain gains or costs, these models equip decision-makers with rigorous tools for designing equitable allocation schemes and supporting strategic collaboration even when precise information is lacking.

Despite these advances, important research gaps remain. Much of the literature continues to address only one form of uncertainty, such as interval, fuzzy, or gray without integrating multiple representations in a unified framework. Moreover, scalability and computational efficiency for large-scale cooperative games remain challenging. Current research trends point toward hybrid approaches that combine interval, fuzzy, and probabilistic uncertainty as well as dynamic cooperative game models that capture evolving environments. Promising application areas include digital economies, renewable energy systems, and multiagent networks. Taken together, these developments suggest that cooperative games under uncertainty will remain a vibrant field, offering opportunities for both theoretical innovation and practical impact.

3. Mathematical background

3.1. Cooperative game theory

A cooperative game in coalitional form is an ordered pair (N, v), where $N = \{1, 2, ..., n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is a characteristic function assigning to each coalition $S \in 2^N$ a real number such that $v(\emptyset) = 0$. We refer to such a game as a TU (transferable utility) game. We denote by G^N the family of all classical (crisp) cooperative games with player set N [40,41].

A cooperative game (N, v) is said to be superadditive if, for any $S, T \in 2^N$ such that $S \cap T = \emptyset$, it holds that

$$v(S \cup T) \geq v(S) + v(T).$$

A cooperative game (N, v) is said to be a big boss game with player n as the big boss if the following conditions hold [42–44]:

(i) $v \in G^N$ is monotonic, that is, $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$;
(ii) $v(S) = 0$ if $n \notin S \subseteq N$;
(iii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\}))$ for all S with $n \in S \subseteq N$.

Let $\Pi(N)$ denote the set of all permutations $\sigma : N \to N$ of N. The set $P^\sigma(i) := \left\{r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i)\right\}$ consists of all predecessors of i with respect to the permutation σ. Let $v \in G^N$ and $\sigma \in \Pi(N)$. The marginal contribution vector $m^\sigma(v) \in \mathbb{R}^N$ with respect to σ and v has the ith coordinate the value $m_i^\sigma(v) := v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ for each $i \in N$. The Shapley value $\phi(v) = (\phi_i(v))_{i \in N}$ of a game $v \in G^N$ is the average of the marginal vectors of the game as follows [45]:

$$\phi_i(v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

It is known that for each $i \in N$, $\phi_i(v)$ is given by

$$\phi_i(v) := \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} |S|!\,(n - |S| - 1)!\,(v(S \cup \{i\}) - v(S)),$$

where $|S|$ denotes the number of players in S.

The Shapley value allocates the total worth of a coalition by distributing payoffs according to the average marginal contribution of each player across all possible coalition orders. It satisfies axioms such as efficiency, symmetry, dummy player, and additivity, and is often interpreted as a fairness-based allocation rule.

In a cooperative game (N, v), where N is the set of players, and v is the characteristic function, the marginal vector (upper vector) $M(v) = (M_i(v))_{i \in N}$ is defined as

$$M_i(v) := v(N) - v(N \setminus \{i\}),$$

and the minimal right vector (lower vector) $m(v) = (m_i(v))_{i \in N}$ is given by

$$m_i(v) := \max_{S:\, i \in S} (v(S) - \sum_{j \in S \setminus \{i\}} M_j(v)),$$

where $v(S) - \sum_{j \in S \setminus \{i\}} M_j(v)$ represents the remainder function for player i in coalition S.

The class of quasibalanced games with the player set N is denoted by Q^N, where

$$Q^N := \{v \in G^N | m(v) \leq M(v), \sum_{i=1}^{n} m_i(v) \leq v(N) \leq \sum_{i=1}^{n} M_i(v)\}.$$

The τ-value, defined on Q^N, maps each game $v \in Q^N$ to a vector $\tau(v) \in \mathbb{R}^N$ as a convex combination of the lower vector $m(v) = (m_i(v))_{i \in N}$ and upper vector $M(v) = (M_i(v))_{i \in N}$:

$$\tau(v) := m(v) + \alpha(M(v) - m(v)).$$

Here, α is chosen to satisfy the efficiency condition, ensuring that $\sum_{i=1}^{n} \tau_i(v) = v(N)$ [44,46].

Next, we give some solution concepts from cooperative game theory such as the centre of gravity of the imputation set value, which is shortly denoted by "CIS-value", and the equal division solution, which is shortly denoted by "ED-solution" [47].

The CIS-value assigns to every player its individual worth and distributes the remainder of $v(N)$ equally among all players and is defined by

$$CIS_i(v) = v(\{i\}) + \tfrac{1}{|N|}(v(N) - \sum_{j \in N} v(\{j\})) \text{ for all } i \in N.$$

The ED-solution distributes $v(N)$ equally among all players and is defined by

$$ED_i(v) = \frac{v(N)}{|N|}, \text{ for all } i \in N.$$

In addition to the solution concepts explicitly discussed in this paper, several other classical allocation rules have been proposed in the cooperative game theory literature, including the nucleolus, the Owen value, and the Banzhaf value. These concepts are based on different normative principles and provide complementary perspectives on payoff allocation in coalition games. For comprehensive definitions and theoretical background, the interested reader is referred to [21,40,41]

3.2. Fuzzy intervals

In many decision-making problems, uncertainty does not solely arise from randomness but rather from imprecision, vagueness, or incomplete knowledge. Classical interval representations capture uncertainty by specifying lower and upper bounds; however, they implicitly assume that all values within these bounds are equally relevant. This assumption may be somewhat restrictive in practical settings, as some values are often considered more representative than others.

Fuzzy intervals overcome this limitation by assigning to each real value a degree of membership in the unit interval $[0, 1]$, which reflects its level of compatibility with the uncertain quantity. In this way, fuzzy intervals provide a richer and more informative representation of uncertainty than classical intervals, as they incorporate not only admissible bounds but also graded relevance within those bounds.

A fuzzy set $\mathcal{F}$ on $\mathbb{R}$ is characterized by a membership function $u_{\mathcal{F}} : \mathbb{R} \to [0, 1]$, where $u_{\mathcal{F}}(x)$ represents the degree of membership of x in $\mathcal{F}$.

For any $\alpha \in [0, 1]$, the α-level set (α − cut) of $\mathcal{F}$ is defined as [48]

$$[u_{\mathcal{F}}]_\alpha = \{x \in \mathbb{R} : u_{\mathcal{F}}(x) \geq \alpha\}.$$

If $\alpha = 0$, then $[u_{\mathcal{F}}]_0 = cl\{x \in \mathbb{R} : u_{\mathcal{F}}(x) > 0\}$. Here, $cl\{x \in \mathbb{R} : u_{\mathcal{F}}(x) > 0\}$ is the closure of $\{x \in \mathbb{R} : u_{\mathcal{F}}(x) > 0\}$.

If the following conditions are satisfied, a fuzzy set $\mathcal{F}$ in $\mathbb{R}$ is said to be a *fuzzy interval* [49]:

i. $[u_{\mathcal{F}}]_\alpha$ is compact for any $\alpha \in [0, 1]$;
ii. $[u_{\mathcal{F}}]_\alpha$ is convex for any $\alpha \in [0, 1]$;
iii. $[u_{\mathcal{F}}]_\alpha$ is normal, that is, there exist $x \in \mathbb{R}$ such that $u_{\mathcal{F}}(x) = 1$.

The set of fuzzy intervals is denoted by $\mathcal{F}(\mathbb{R})$. For any $\mathcal{F} \in \mathcal{F}(\mathbb{R})$, there exist $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $\mathcal{L} : [x_1, x_2] \to \mathbb{R}$ nondecreasing and $\mathcal{R} : [x_3, x_4] \to \mathbb{R}$ nonincreasing such that the *membership function* $u_{\mathcal{F}}$ is given as below [49]:

$$u_{\mathcal{F}}(x) = \begin{cases} \mathcal{L}(x), & x_1 \le x \le x_2, \\ 1, & x_2 \le x \le x_3, \\ \mathcal{R}(x), & x_3 \le x \le x_4, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{L}$ and $\mathcal{R}$ are linear, then $u_{\mathcal{F}}(x)$ is called a *trapezoidal fuzzy interval.* Such a trapezoidal fuzzy interval is denoted by (x_1, x_2, x_3, x_4). The set of all trapezoidal fuzzy intervals are denoted by $\mathcal{F}_{\mathcal{T}}(\mathbb{R})$. In this case, if $x_1 = x_2$, and $x_3 = x_4$, then (x_1, x_2, x_3, x_4) is a compact interval.

For any $\alpha \in [0, 1]$ the α-level set of a fuzzy interval of $\mathcal{F}$ is given by [49]

$$\begin{aligned} [u_{\mathcal{F}}]_\alpha &= \left[\mathcal{L}^{-1}(0)(1-\alpha) + \alpha\mathcal{L}^{-1}(1), \mathcal{R}^{-1}(0)(1-\alpha) + \alpha\mathcal{R}^{-1}(1)\right], \\ &= \left[(u_{\mathcal{F}})^-_\alpha, (u_{\mathcal{F}})^+_\alpha\right]. \end{aligned}$$

If $\mathcal{F} \in \mathcal{F}(\mathbb{R})$, then the decomposition of $\mathcal{F}$ is as follows [49]:

$$u_{\mathcal{F}}(x) = \bigcup_{0<\alpha\le 1} \left[(u_{\mathcal{F}})^-_\alpha, (u_{\mathcal{F}})^+_\alpha\right].$$

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}(\mathbb{R})$, and $*$ be a binary operation on $\mathcal{F}(\mathbb{R})$. The $*$ operation can be extended to fuzzy intervals by means of *Zadeh's extension principle* [50] in the following way:

$$u_{\mathcal{F}_1 * \mathcal{F}_2}(z) := \sup_{z=x*y} \min\{u_{\mathcal{F}_1}(x), u_{\mathcal{F}_2}(y)\},$$

where $\mathcal{F}_1 * \mathcal{F}_2$ is a fuzzy interval with the membership function $u_{\mathcal{F}_1 * \mathcal{F}_2}$. Here we obtain [51]:

i) $[u_{\mathcal{F}_1}]_\alpha + [u_{\mathcal{F}_2}]_\alpha = \left[(u_{\mathcal{F}_1})^-_\alpha + (u_{\mathcal{F}_2})^-_\alpha, (u_{\mathcal{F}_1})^+_\alpha + (u_{\mathcal{F}_2})^+_\alpha\right]$,
ii) $k \cdot [u_{\mathcal{F}}]_\alpha = \left[k \cdot (u_{\mathcal{F}})^-_\alpha, k \cdot (u_{\mathcal{F}})^+_\alpha\right]$, $k \in \mathbb{R}^+$.

Let $\mathcal{F}_1 = (x_1, x_2, x_3, x_4)$ and $\mathcal{F}_2 = (y_1, y_2, y_3, y_4) \in \mathcal{F}_{\mathcal{T}}(\mathbb{R})$ be two trapezoidal fuzzy intervals and $k \in \mathbb{R}^+$ so that the following conditions hold:

(*i*) $\mathcal{F}_1 + \mathcal{F}_2 = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$,
(*ii*) $k \cdot \mathcal{F}_1 = (kx_1, kx_2, kx_3, kx_4)$,

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}(\mathbb{R})$, so that the binary relation $\trianglerighteq$ is defined on $\mathcal{F}(\mathbb{R})$ as follows [51]. The relation $\trianglelefteq$ is defined analogously. For all $\alpha \in [0, 1]$,

$$\mathcal{F}_1 \trianglerighteq \mathcal{F}_2 \Longleftrightarrow (u_{\mathcal{F}_1})^-_\alpha \ge (u_{\mathcal{F}_2})^-_\alpha \text{ and } (u_{\mathcal{F}_1})^+_\alpha \ge (u_{\mathcal{F}_2})^+_\alpha.$$

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}(\mathbb{R})$. If there exist $\mathcal{F}_3 \in \mathcal{F}(\mathbb{R})$ such that $\mathcal{F}_1 = \mathcal{F}_2 + \mathcal{F}_3$, then $\mathcal{F}_3$ is said to be a *Hukuhara difference* between $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by

$$\mathcal{F}_3 = \mathcal{F}_1 -_H \mathcal{F}_2 = [u_{\mathcal{F}_1}]_\alpha -_H [u_{\mathcal{F}_2}]_\alpha = \left[(u_{\mathcal{F}_1})_\alpha^- - (u_{\mathcal{F}_2})_\alpha^-, (u_{\mathcal{F}_1})_\alpha^+ - (u_{\mathcal{F}_2})_\alpha^+\right].$$

A Hukuhara difference between two fuzzy intervals does not always exist. Regarding the existence of the Hukuhara difference, there is an extensive study provided in [51]. Here, we review some existing facts on the Hukuhara difference [51–53], which are interesting for our paper.

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}(\mathbb{R})$. $\mathcal{F}_1 -_H \mathcal{F}_2$ exists if and only if

$$(u_{\mathcal{F}_1})_\alpha^- - (u_{\mathcal{F}_2})_\alpha^- \leq (u_{\mathcal{F}_1})_\beta^- - (u_{\mathcal{F}_2})_\beta^- \leq (u_{\mathcal{F}_1})_\beta^+ - (u_{\mathcal{F}_2})_\beta^+ \leq (u_{\mathcal{F}_1})_\alpha^+ - (u_{\mathcal{F}_2})_\alpha^+$$

for all $\alpha, \beta \in (0,1]$ and $\beta > \alpha$.

Let $\mathcal{F}_1 = (x_1, x_2, x_3, x_4)$ and $\mathcal{F}_2 = (y_1, y_2, y_3, y_4) \in \mathcal{F}_\mathcal{T}(\mathbb{R})$ be two trapezoidal fuzzy intervals. If $\mathcal{F}_1 -_H \mathcal{F}_2$ exists, then

$$\mathcal{F}_1 -_H \mathcal{F}_2 = (x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4).$$

Throughout this paper, whenever the Hukuhara difference exists, we write $\mathcal{F}_1 - \mathcal{F}_2$ instead of $\mathcal{F}_1 -_H \mathcal{F}_2$ for $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}(\mathbb{R})$.

Defuzzification is the conversion of a fuzzy quantity to a precise value. In the literature, various defuzzification techniques exist.

The *signed distance* of fuzzy set $\mathcal{F}$ is given as follows [54]:

$$d^*(\mathcal{F}) = \frac{1}{2}\int_0^1 \left((u_\mathcal{F})_\alpha^- + (u_\mathcal{F})_\alpha^+\right) d\alpha.$$

If $\mathcal{F} \in \mathcal{F}_\mathcal{T}(\mathbb{R})$, then the signed distance of $\mathcal{F} = (a,b,c,d)$ is

$$d^*(\mathcal{F}) = \frac{1}{4}(a + b + c + d).$$

For other specific cases, an expansion of the signed distance formula is proposed in [55].

4. Fuzzy interval cooperative games

A fuzzy-interval cooperative game [56] is a pair $(N, \mathcal{U})$, where $N = \{1, 2, ..., n\}$ is the set of players, and $\mathcal{U} : 2^N \to \mathcal{F}(\mathbb{R})$ maps the coalitions $S \in 2^N$ into fuzzy intervals $\mathcal{U}(S) \in \mathcal{F}(\mathbb{R})$ with $\mathcal{U}(\emptyset) = 0$.

We denote by $\mathcal{F}(\mathbb{R})^N$ the set of all such fuzzy payoff vectors and $\mathcal{F}G^N$ the family of all fuzzy-interval cooperative games.

A game $(N, \mathcal{U})$ is said to be *convex* if for any $S, T \in 2^N$,

$$\mathcal{U}_\alpha(S \cup T) + \mathcal{U}_\alpha(S \cap T) \trianglerighteq \mathcal{U}_\alpha(S) + \mathcal{U}_\alpha(T).$$

A fuzzy-interval game $(N, \mathcal{U})$ is called *superadditive* if for any $S, T \in 2^N$ such that $S \cap T = \emptyset$,

$$\mathcal{U}_\alpha(S \cup T) \trianglerighteq \mathcal{U}_\alpha(S) + \mathcal{U}_\alpha(T).$$

These conditions generalize crisp *convexity* and crisp *superadditivity* to the fuzzy interval setting. In this paper, we mainly focus on the class of superadditive fuzzy-interval cooperative games.

Moreover, if for any coalition $T \subseteq S$ and any $\alpha, \beta \in (0, 1]$ such that $\beta > \alpha$, the superadditive game $(N, \mathcal{U})$ also satisfies

$$\mathcal{U}_\alpha^- (S) - \mathcal{U}_\alpha^- (T) \leq \mathcal{U}_\beta^- (S) - \mathcal{U}_\beta^- (T) \leq \mathcal{U}_\beta^+ (S) - \mathcal{U}_\beta^+ (T) \leq \mathcal{U}_\alpha^+ (S) - \mathcal{U}_\alpha^+ (T).$$

A fuzzy interval game $(N, \mathcal{U})$ is called *size-monotonic* if its associated length game $(N, |\mathcal{U}|)$ is size-monotonic, that is, $|\mathcal{U}| (T) \geq |\mathcal{U}| (S)$ $S, T \in 2^N$ with $S \subseteq T$, where $|\mathcal{U} (S)| = \mathcal{U}_\alpha^+ (S) - \mathcal{U}_\alpha^- (S)$. We will denote the class of fuzzy-interval games that are size-monotonic and superadditive by $SM\mathcal{F}G^N$. In this class, the Hukuhara difference is always well-defined.

For each $\mathcal{U} \in SM\mathcal{F}G^N$ and each $i \in N$, the *fuzzy marginal contribution* of player i in the game $\mathcal{U}$ is defined by $M_i(N, \mathcal{U}) = \mathcal{U}(N) -_H \mathcal{U}(N \setminus \{i\})$.

Within this framework, the Shapley value can be naturally extended to fuzzy-interval games by employing the Hukuhara difference. We now briefly recall the Shapley value in the fuzzy interval setting.

Let $(N, \mathcal{U})$ be a fuzzy-interval cooperative game. Then, the *Hukuhara-Shapley value* is defined as follows [22]:

$$\phi(N, \mathcal{U}) = (\phi_1(\mathcal{U}), \phi_2(\mathcal{U}), ..., \phi_n(\mathcal{U}))$$

with

$$\phi_i(N, \mathcal{U}) := \sum_{S \subset N \setminus \{i\}} \frac{|S|! (N - 1 - |S|)!}{N!} (\mathcal{U} (S \cup \{i\}) -_H \mathcal{U} (S)).$$

Alternative Shapley-value extensions for fuzzy cooperative games can be found in [57, 58].

Next, we introduce the fuzzy core.

Let $(N, \mathcal{U})$ be a fuzzy-interval cooperative game. The fuzzy-interval core ($\mathcal{F}$-core) is defined as the following set [56]:

$$C (N, \mathcal{U}) = \left\{ (\mathcal{F}_1, ..., \mathcal{F}_n) \in \mathcal{F} (\mathbb{R})^N : \sum_{i=1}^n \mathcal{F}_i = \mathcal{U} (N) \text{ and } \sum_{i \in S} \mathcal{F}_i \trianglerighteq \mathcal{U} (S) \ \forall S \subset N \right\}.$$

Unlike the approach in [16], the $\mathcal{F}$-core generalizes the interval core introduced in [5]. Under $\alpha-$cuts, border games induce crisp cores whose intersection yields feasibility conditions for the $\mathcal{F}$-core. Building on these definitions, several allocation rules such as the $\mathcal{F}CIS$-value and the $\mathcal{F}ED$-solution have been introduced in the fuzzy interval setting, extending their crisp counterparts in a natural way [59].

The $\mathcal{F}CIS$-value is defined by

$$\mathcal{F}CIS : SM\mathcal{F}G^N \to \mathcal{F} (\mathbb{R})^N$$

$$\mathcal{F}CIS_i (N, \mathcal{U}) = \mathcal{U} (\{i\}) + \frac{1}{|N|} \left(\mathcal{U} (N) -_H \sum_{j \in N} \mathcal{U} (\{j\}) \right)$$

for all $i \in N$ and $\mathcal{U} \in SM\mathcal{F}G^N$.

Finally, the $\mathcal{F}ED$-solution is given by

$$\begin{aligned} \mathcal{F}ED &: \mathcal{F}G^N \to \mathcal{F} (\mathbb{R})^N \\ \mathcal{F}ED_i (\mathcal{U}) &= \frac{\mathcal{U} (N)}{|N|}. \end{aligned}$$

Remark 4.1. *The $\mathcal{FCIS}$-value is well-defined only in $SM\mathcal{F}G^N$.*

Before introducing big boss fuzzy interval cooperative games, it is important to note that structured subclasses of cooperative games are often employed to address scalability and computational tractability. In large-scale settings, evaluating all possible coalitions is generally infeasible. By imposing hierarchical or veto-based structures, such as those characterizing big boss games, the effective coalition space is significantly reduced. This structural restriction allows cooperative solution concepts to be computed more efficiently while preserving essential stability and fairness properties under uncertainty.

Some classical TU games associated with a fuzzy-interval game $\mathcal{U} \in \mathcal{F}G^N$ play a key role, namely the *border* games $(N, \mathcal{U}_\alpha^-)$, $(N, \mathcal{U}_\alpha^+)$ and the *length* game $(N, |\mathcal{U}|)$, where $|\mathcal{U}|(S) = \mathcal{U}_\alpha^+(S) - \mathcal{U}_\alpha^-(S)$ for each $S \in 2^N$. A fuzzy interval game $(N, \mathcal{U})$ is called a *big boss fuzzy-interval game* if its border games and its length game are crisp big boss games. We denote by $BB\mathcal{F}G^N$ the set of all such games.

The following theorem characterizes such games in terms of crisp big boss conditions.

Theorem 4.1. *Let $\mathcal{U} \in SM\mathcal{F}G^N$. Then, the following two conditions are equivalent [60]:*

(i) $\mathcal{U} \in BB\mathcal{F}G^N$;

(ii) $(N, \mathcal{U})$ *satisfies the following:*

- *(a) Veto power property:*
  $\mathcal{U}(S) = 0$ *for each* $S \in 2^N$ *with* $n \notin S$;
- *(b) Monotonicity property:*
  $\mathcal{U}(S) \trianglelefteq \mathcal{U}(T)$ *for each* $S, T \in 2^N$ *with* $n \in S \subseteq T$;
- *(c) Union property:*

$$\mathcal{U}(T) -_H \mathcal{U}(S) \trianglerighteq \sum_{i \in T \setminus S} (\mathcal{U}(T) -_H \mathcal{U}(T \setminus \{i\})) \text{ for all } S, T \text{ with } n \in S \subset T,$$

- *(d) n-concavity property:*

$$\mathcal{U}(S \cup \{i\}) -_H \mathcal{U}(S) \trianglerighteq \mathcal{U}(T \cup \{i\}) -_H \mathcal{U}(T).$$

Based on this characterization, a dedicated allocation rule, *the fuzzy $\mathcal{T}$-value*, has been proposed as a natural extension of the classical τ-value to fuzzy-interval big boss games [60].

Corollary 4.1. *Let $\mathcal{U} \in BB\mathcal{F}G^N$ and Player k be a big boss. The fuzzy $\mathcal{T}$-value of $(N, \mathcal{U})$ is defined as follows:*

$$\mathcal{T}_j(N, \mathcal{U}) = \begin{cases} \mathcal{U}(N) -_H \frac{1}{2} \sum_{i \in N \setminus \{k\}} M_i(\mathcal{U}), & j = k \\ \frac{1}{2} M_j(\mathcal{U}), & j \in N \setminus \{k\} \end{cases}.$$

Remark 4.2. *Let $\mathcal{U} \in BB\mathcal{F}G^N$. Then, $\mathcal{T}(N, \mathcal{U})$ belongs to $C(N, \mathcal{U})$.*

Next, we reinterpret the $\mathcal{FCIS}$ value for the class $BB\mathcal{F}G^N$.

Corollary 4.2. *Let $\mathcal{U} \in BB\mathcal{F}G^N$ and Player k be a big boss. The fuzzy $\mathcal{FCIS}$-value of $(N, \mathcal{U})$ is defined as follows:*

$$\mathcal{FCIS}_i(N, \mathcal{U}) = \begin{pmatrix} \mathcal{U}(\{k\}) + \frac{1}{|N|} (\mathcal{U}(N) -_H \mathcal{U}(\{k\})), & i = k \\ \frac{1}{|N|} (\mathcal{U}(N) -_H \mathcal{U}(\{k\})), & i \in N \setminus \{k\} \end{pmatrix}.$$

Next, before closing this section, we present a numerical example to illustrate these results.

Example 4.1. *Let* $(N, \mathcal{U}) \in BB\mathcal{F}G^N$ *and Player* 1 *be a big boss with* $N = \{1, 2, 3\}$ *[71].* $\mathcal{U}(S) = (0, 0, 0, 0)$ *if* $\{1\} \notin S$*,* $\mathcal{U}(\{1\}) = (30, 35, 50, 55)$*,* $\mathcal{U}(\{1, 2\}) = (130, 140, 160, 190)$*,* $\mathcal{U}(\{1, 3\}) = (75, 90, 125, 160)$*, and* $\mathcal{U}(N) = (175, 195, 235, 270)$*. Here, border games* $(N, \mathcal{U}^-_\alpha)$*,* $(N, \mathcal{U}^+_\alpha)$ *and the length game* $(N, |\mathcal{U}|)$ *are calculated as follows:*

N	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mathcal{U}^-_\alpha$	0	$5\alpha + 30$	0	0	$10\alpha + 130$	$15\alpha + 75$	0	$20\alpha + 175$
$\mathcal{U}^+_\alpha$	0	$55 - 5\alpha$	0	0	$190 - 30\alpha$	$160 - 35\alpha$	0	$270 - 35\alpha$
$\lvert\mathcal{U}\rvert$	0	$25 - 10\alpha$	0	0	$60 - 40\alpha$	$85 - 50\alpha$	0	$95 - 55\alpha$

This game is a big boss fuzzy-interval game because border games $(N, \mathcal{U}^-_\alpha)$*,* $(N, \mathcal{U}^+_\alpha)$ *and the length game* $(N, |\mathcal{U}|)$ *are crisp big boss games for all* $\alpha \in [0, 1]$*. Finally, we can obtain related solutions as follows:*

$$\begin{aligned}
\phi(N, \mathcal{U}) &= \begin{pmatrix} (102.5, 115, 142.5, 166.67), (50, 52.5, 55, 59.167), \\ (22.5, 27.5, 37.5, 44.167) \end{pmatrix}, \\
\mathcal{T}(N, \mathcal{U}) &= \begin{pmatrix} (102.5, 115, 142.5, 175), (50, 52.5, 55, 55), \\ (22.5, 27.5, 37.5, 40) \end{pmatrix}, \\
\mathcal{F}CIS(N, \mathcal{U}) &= \begin{pmatrix} (78.33, 88.33, 111.67, 126.67), (48.33, 53.33, 61.67, 71.67), \\ (48.33, 53.33, 61.67, 71.67) \end{pmatrix}, \\
\mathcal{F}ED(N, \mathcal{U}) &= \begin{pmatrix} (58.333, 65, 78.333, 90), (58.333, 65, 78.333, 90), \\ (58.333, 65, 78.333, 90) \end{pmatrix}.
\end{aligned}$$

These results illustrate that different solution concepts lead to markedly different payoff allocations, reflecting distinct fairness and stability principles in the presence of fuzzy interval uncertainty.

5. Applications

This section summarizes representative application domains of cooperative games under fuzzy and interval uncertainty. We group the literature into organizational and industrial contexts, digital and information technology related domains, and financial settings and highlight how different sources of uncertainty such as structural, informational, and behavioral factors motivate fuzzy interval formulations and related allocation rules.

Organizational and industrial applications

In real-world logistics and organizational systems, decision-making is fundamentally shaped by pervasive and multisource uncertainty, arising from volatile demand, incomplete information, and dynamically evolving constraints. More generally, cooperative game models have been applied to a broad range of logistics-related problems, including inventory management, production planning, transportation planning, and waste management, where key parameters are inherently uncertain. These frameworks have also been extended to human resource management (HRM) and marketing, capturing behavioral and organizational sources of uncertainty. By explicitly accounting for such uncertainty in coalition values, these models enable fair and stable decision-making across interconnected organizational functions [61–63].

In organizational and industrial environments, uncertainty often stems from hierarchical dependencies, vague responsibilities, and imprecise performance measurement, making fuzzy coalition values a natural modeling choice. In such settings, uncertainty-based cooperative game models have proven particularly effective where coalition structures are inherently vague. Fuzzy cooperative frameworks have been used to model hierarchical organizations with leaders, intermediaries, and agents, where outcomes are not precisely measurable. In such contexts, fuzzy payoffs capture ambiguity in responsibilities and benefits, ensuring that rewards are distributed equitably across coalition members. Applications in auction environments further highlight how these models account for hierarchical dependencies and uncertainty, producing stable and fair allocations [64]. Another strand of research has generalized bubbly cooperative games to fuzzy bubbly games, where the fuzzy bubbly core provides a stability concept under incomplete or noisy information [65]. Similarly, cooperative fuzzy models have been applied to strategic alliances between firms in competitive industries. By representing coalition values as fuzzy sets, these models reveal how cooperative arrangements influence individual payoffs and demonstrate the practical relevance of fuzzy interval solutions in industries where cooperation and competition coexist [66].

Recent survey studies highlight the growing role of game-theoretic models in real-world energy systems, particularly for strategic resource allocation under uncertainty, providing valuable context for the application of cooperative game frameworks in complex decision-making environments. Moreover, evolutionary and game-theoretic approaches have been shown to be effective in decentralized renewable energy systems operating under uncertainty, complementing interval and fuzzy cooperative models discussed in this survey [67, 68].

Digital and IT applications

Digital and information technology (IT) systems introduce uncertainty through dynamic interactions, cybersecurity risks, and infrastructure complexity; consequently, fuzzy cooperative models provide a flexible way to assess joint benefits and cost-sharing under incomplete information.

In technological domains, cooperative fuzzy models have been integrated with modern digital infrastructures to address uncertainty in complex systems. A notable example arises in cloud computing and cybersecurity, where crypto-cloud frameworks based on Amazon Web Services have been analyzed using a combination of cryptographic schemes and fuzzy cooperative games. Results demonstrate that uncertainty-based approaches can reduce operational costs while improving system efficiency, outperforming deterministic benchmarks [69, 70]. Fuzzy interval cooperative games have also been applied to IT project management, particularly in joint ventures between capital-linked firms. By employing the $BB\mathcal{F}G^N$ model, researchers were able to evaluate project profitability and design profit-sharing mechanisms that balance fairness and strategic incentives [71, 72]. Beyond IT collaborations, sequencing problems in operations research have been modeled through fuzzy and interval cooperative games, allowing efficiency gains from scheduling to be distributed equitably even when task durations and costs are uncertain [73, 74]. Furthermore, fuzzy matrix games with generalized trapezoidal payoffs have been proposed to reduce computational complexity. By applying linear ranking functions, fuzzy matrix games can be transformed into crisp equivalents, enabling simpler yet rigorous solutions [75].

Financial applications

Financial decision-making represents one of the most prominent domains where cooperative games under uncertainty have been applied. Recent work has extended cooperative formulations to Gaussian type-2 fuzzy environments, where mean reduction and critical value methods provide tools for handling higher-order vagueness. These models have been tested in complex contexts such as multilevel drug resistance problems, illustrating their robustness and flexibility in uncertain environments [76]. On the stochastic side, regime-switching and jump-diffusion frameworks have been employed to model systemic changes in financial markets. Applications include portfolio optimization, optimal consumption under time delays, and the analysis of systemic crises such as the 2008–2009 global financial crash, showing the relevance of stochastic hybrid systems with jumps [38, 39, 77, 78]. Robust approaches have also been incorporated into stochastic factor models and insider Black–Scholes markets, where optimal strategies differ significantly from nonrobust counterparts, emphasizing the importance of robustness in practical investment planning [79–81]. More recently, two-player zero-sum stochastic differential games in Markov-switching jump-diffusion environments have been studied, yielding Isaacs-type partial differential equations (PDE) characterizations that not only strengthen the stochastic game literature but also open pathways toward cooperative extensions [82].

In addition to purely numerical or probabilistic uncertainty, recent studies emphasize behavioral and cognitive drivers, indicating that uncertainty in socioeconomic and financial systems is inherently multidimensional. In this direction, [83] analyzed the dynamic interplay between investor sentiment and market behavior through coupled stochastic systems, showing that sentiment-driven effects significantly influence stability and systemic risk. From a cognitive perspective, neuroscience-inspired fuzzy models have been proposed to formalize human reasoning under vagueness, for instance by representing similarity judgments through max–min fuzzy decompositions that reflect neural activation mechanisms [84]. Building on this view, [85] argued that operations research frameworks can be enriched by such cognitive and neuro-inspired models to capture bounded rationality, and [86] further demonstrated how fuzzy neuromatrix representations support formal modeling of human similarity perception. More recently, [87] proposed a hybrid structure combining financial indicators with human factor indices, improving both robustness and interpretability. Collectively, these works motivate cooperative game-theoretic frameworks that can incorporate behavioral and cognitive uncertainty into allocation, stability, and fairness analyses.

Alternative value concepts

Beyond applications of core-based or Shapley-based allocations, recent studies have sought to broaden the range of solution concepts in uncertain cooperative environments. In particular, fuzzy Banzhaf values have been introduced for cooperative games with fuzzy characteristic functions, providing an alternative to the Shapley value. This approach emphasizes different fairness principles and offers more flexibility in contexts where Shapley-based allocations may not capture the full spectrum of coalition dynamics [88]. Such extensions ensure that cooperative game theory under uncertainty continues to provide versatile tools for real-world applications.

6. Conclusions and future directions

The principal contribution of this study is to provide a comprehensive and systematically organized synthesis of cooperative game theory under uncertainty with a particular focus on fuzzy-interval formulations. In contrast to classical cooperative game models, which rely on precisely defined coalition values, fuzzy-interval cooperative games offer a flexible and mathematically rigorous framework for representing both imprecision and vagueness inherent in real-world decision-making environments. The survey demonstrates how well-established solution concepts, including the core, the Shapley value, and related allocation rules, can be consistently extended to fuzzy-interval settings while preserving fundamental properties such as stability, fairness, and efficiency. By integrating theoretical foundations with computational considerations and application-oriented studies, this work clarifies the added value of fuzzy-interval cooperative games over traditional approaches and situates them within the broader literature on decision-making under uncertainty.

While cooperative game theory primarily focuses on stability and payoff allocation once cooperation is achieved, real-world collaboration processes may also involve negotiation failures, incomplete coalition formation, or strategic misreporting. In particular, issues related to bargaining, coalition formation, and the dynamics of cooperation under uncertainty represent complementary research streams that address how cooperation emerges, evolves, or breaks down. Incorporating such perspectives into uncertainty-based cooperative game models constitutes a promising direction for future research.

It is important to note that the present survey implicitly assumes truthful and unrestricted information sharing among cooperating agents. In many real-world collaborative settings, however, agents may have partially misaligned interests and may be unwilling to disclose sensitive or strategic information. Under such circumstances, issues related to information privacy, selective disclosure, and privacy-preserving cooperation become highly relevant. Investigating how cooperative solution concepts can be adapted to settings with privacy constraints constitutes an important and complementary direction for future research.

Despite substantial progress in modeling interval, fuzzy, and gray uncertainty, several promising avenues for future research remain open. A first priority lies in the development of hybrid frameworks that combine fuzzy, interval, and probabilistic representations, enabling multiple layers of uncertainty to be captured simultaneously. Such hybrid models may yield richer and more realistic representations of coalition values in complex systems, where uncertainty is not only imprecise but also stochastic or time dependent.

In addition to the uncertainty representations discussed in this survey, future research may further benefit from incorporating alternative methodological paradigms such as stochastic optimization, chance-constrained programming, robust optimization, and risk-based measures such as conditional value-at-risk (CVaR). These approaches have been widely used to analyze how different quantifications of uncertainty affect optimization outcomes, and they offer complementary perspectives to fuzzy and interval-based models. Integrating such methods into cooperative game-theoretic frameworks could provide deeper insights into the robustness, stability, and sensitivity of allocation rules under different forms of uncertainty.

Closely related to this direction is the study of dynamic cooperative games in which coalition values and uncertainty evolve over time. Many real-world cooperative systems are governed by stochastic dynamics, motivating extensions toward dynamic and control-based formulations. Addressing temporal variations may provide deeper insights into strategic adaptation and real-time decision-making in domains such as renewable energy trading, supply chain coordination, and financial networks. In addition, uncertainty sets may be generalized from one-dimensional intervals to higher-dimensional objects such as cubes or parallelepipeds, together with inner and outer approximation schemes (e.g., via ellipsoidal representations), to enhance both modeling expressiveness and computational tractability. To ensure practical viability, future research must also confront challenges of scalability and computational efficiency by advancing approximation algorithms, heuristic methods, parallel computing strategies, and optimization-based techniques.

Another important frontier concerns the integration of data-driven and machine learning approaches with cooperative game theory. Embedding learning mechanisms into fuzzy and interval frameworks would allow coalition valuations to be updated adaptively based on observed data, thereby strengthening the empirical grounding of cooperative solutions. Application-oriented studies in areas such as health-care resource allocation, blockchain-based collaborations, smart cities, and IT project evaluation may provide valuable validation while simultaneously refining theoretical models through exposure to real-world datasets.

Beyond methodological and application-oriented advances, it is essential to acknowledge the ethical and societal dimensions of cooperation under uncertainty. Cooperative structures may be undermined by hidden or overt egoism, irresponsibility, or asocial behavior, threatening fairness and stability. In extreme cases, coalition models may even be misused to reconstruct socially undesirable or harmful networks, representing the negative or "shadow" side of cooperation. Future research should therefore also consider cooperation among fundamentally different agents, including humans and intelligent machines, and reflect on the ethical boundaries and responsibilities associated with such interactions.

Taken together, these perspectives underscore a vibrant and expanding research landscape for cooperative games under uncertainty. By linking hybrid theoretical models with dynamic, computational, data-driven, and ethical considerations, future studies can not only advance the mathematical foundations of cooperative game theory but also deliver robust decision-support tools for real-world collaborative environments in economics, operations research, and beyond.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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