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Research article

Fractional interactive fuzzy F-correlated Caputo-Katugampola differential equations

Weiqi Ding[1], **Haibo Gu**[2,*] **and Yan Chen**[3]

[1] School of Mathematics and Statistics, Qinghai Normal University, Xining 810008, China
[2] School of Mathematics Sciences, Xinjiang Normal University, Urumqi 830017, China
[3] School of Mathematics and Statistics, Shandong Normal University, Jinan 250000, China

* **Correspondence:** Email: hbgu_math@163.com.

Abstract: In this paper, we investigated the existence and uniqueness of solutions for a class of interactive fuzzy Caputo–Katugampola F-correlated fractional differential equations with time delay. First, we gave the definition of the correlated integral and derivative for interactive fuzzy Caputo–Katugampola F-correlated systems. Second, we presented an equivalent integral formulation and, under suitable assumptions, established the existence of solutions to fuzzy interactive Caputo–Katugampola F-correlated fractional-order differential equations with time delay by applying Schauder's fixed-point theorem. Finally, a numerical example was presented to validate the effectiveness of the proposed approach.

Keywords: interactive derivative; F-correlated fuzzy processes; fuzzy fractional differential equations; time delay

1. Introduction

In the theory of fuzzy fractional differential equations (FFDEs), there are at least two types of FFDEs. The first type includes functions taking values in fuzzy sets (see [1]). For such fuzzy functions, the fuzzy derivative is obtained by extending the classical derivative operator for crisp functions using Zadeh's extension principle. The second type considers fuzzy-number-valued functions, in which case we can define two types of fuzzy derivatives. One is based on the Fréchet derivative definition of the fuzzy derivative, which is given by a continuous linear operator acting on fuzzy-number-valued functions (see [2]). The alternative approach to defining the derivative of a fuzzy-number-valued function is the limit method (see [3]), where the formulation of the limit expression depends on the type of fuzzy number difference operation employed. For example, the Hukuhara difference $(-_H)$ [4] and the generalized Hukuhara difference $(-_{gH})$ [5] yield the Hukuhara derivative and the generalized Hukuhara derivative, respectively, whenever the corresponding limit exists.

In 2004, Carlsson et al. [6] stated that the interaction of fuzzy numbers and their interdependence can be described by their joint possibility distribution, and showed that the membership functions of the interactive and non-interactive sums of any pair of fully positively correlated fuzzy numbers are identical. In 2017, the interactive derivative of a fuzzy function was introduced, incorporating the concepts of joint possibility distribution and interactive arithmetic operations [7]. The interactive difference is defined using the sup-J extension principle [8]. One motivation for the development of interactive arithmetic and fuzzy derivatives is the possible interactivity or dependence between variables in a process. The idea of interactivity between two fuzzy numbers can be expressed by a joint possibility distribution [6, 9, 10]. Based on this concept, fuzzy processes have been developed (see [11–13]). In certain cases, the interactive derivative is similar to the generalized Hukuhara derivative (see [14, 15]).

In 2019, Ibanez et al. [16] solved a linear fuzzy boundary value problem (FBVP) by exploiting the concept of mutual interaction between fuzzy numbers. To provide solutions for a class of fuzzy boundary value problems involving non-interactive fuzzy boundary values, Zadeh's extension principle was implemented. In addition, based on the extension principle of the joint possibility distribution, the fuzzy solution was obtained in cases where the boundary value is given by interactive fuzzy numbers. Moreover, Pedro et al. studied a fuzzy initial value problem (FIVP) describing the evolution of an autocorrelated process. Therefore, the concept of the F-correlated derivative (F-derivative) was applied to fuzzy-number-valued functions (see [14] for more details). It has been observed that the interactivity of the process depends on the chosen derivative, which in turn provides a bidirectional measure of interactivity for the FIVP solution. The study of a population growth model governed by an interactive fuzzy differential equation shows that, for fuzzy-number-valued functions describing a locally F-correlated fuzzy process, the interactive derivative is more suitable than the gH derivative.

In 2021, Esmi et al. [17] investigated higher-order linear differential equations with additional conditions (initial or boundary conditions) expressed in terms of interactive fuzzy numbers. Their results show that the solution obtained via the Fr$\acute{e}$chet derivative for fuzzy functions corresponds to the solution obtained by the sup-J extension. Similarly, Wasques et al. stated that generalized (and generalized Hukuhara) differences are interactive, thus underlining the interactivity of generalized fuzzy derivatives.

In 2023, Pedro et al. [18] considered a fuzzy initial value problem given by the F-correlated Caputo fractional derivative of order $\alpha \in (0, 1)$:

$$\begin{cases} {}_F^C\mathcal{D}_{a^+}^{\alpha} x(t) = f(t, x(t)), \\ x(a) = x_0 \in \mathbb{R}_F, \end{cases}$$

where ${}_F^C\mathcal{D}_{a^+}^{\alpha}$ is the fuzzy F-correlated Caputo fractional derivative and $\mathbb{R}_F$ is the space of fuzzy numbers. That paper discussed fuzzy fractional differential equations that involve interactive fuzzy relations. It examined the corresponding integral equation and fractional differentiability theory from an interactive perspective, specifically introducing the concept of the fuzzy interactive derivative as a fractional derivative of Caputo type. Furthermore, the paper presented a fractional initial value problem by utilizing the interactive fuzzy Caputo fractional derivative. However, the effect of time delay was not considered in that work.

In 2024, Huang and Luo [19] investigated Ulam–Hyers stability for a class of fuzzy fractional switched implicit differential equations with non-instantaneous impulses under generalized Hukuhara differentiability. More precisely, they considered the switched impulsive system governed by a piecewise-constant switching law $r(\cdot)$, in which the non-instantaneous impulsive action persists over an interval

$(q_i, \rho_i]$ rather than acting at a single instant. From a modeling perspective, this line of work highlights a key advantage of fuzzy formulations: when uncertainty is intrinsic and cannot be faithfully encoded by crisp parameters, stability concepts in the fuzzy sense can provide robust and practically meaningful conclusions, which further motivates our investigation in an interactive fuzzy fractional setting with delay.

Recently, beyond fuzzy settings, qualitative analysis of fractional models has been actively developed in many areas. For instance, Ouyang et al. [20] investigated sign-changing and signed solutions for fractional Laplacian equations with critical or supercritical nonlinearities. Lu et al. [21] studied the solvability of a coupled system of Hadamard fractional p-Laplacian differential equations with infinite-point boundary conditions at resonance on an unbounded interval. In the context of fractional-order neural networks, synchronization criteria have been developed via intermittent control [22], and finite-time synchronization of fractional-order reaction–diffusion neural networks has been analyzed via generalized Halanay inequalities [23]. For uncertain environments, wave solutions for time-fractional geophysical KdV equations have also been studied [24]. Meanwhile, within the fuzzy fractional framework, Yan and Luo [25] considered finite-time stability for Caputo fractional fuzzy differential equations with delay in a granular sense.

Based on the above discussion, we are concerned with a class of interactive fuzzy Caputo–Katugampola F-correlated fractional differential equations with time delay of the form

$$\begin{cases} {}^{C}_{F}\mathcal{D}^{\alpha,p}_{a^+}x(t) = f(t, x(t), x(t-\tau)), & t \in J = (a, b], \\ x(s) = \Phi(s) \in \mathbb{R}_F, \quad s \in (a-\tau, a], \end{cases} \tag{1.1}$$

where ${}^{C}_{F}\mathcal{D}^{\alpha,p}_{a^+}$ is the fuzzy Caputo–Katugampola F-correlated fractional derivative of order $\alpha \in (0, 1)$, $\tau > 0$ is the time delay, $f : [a, b] \times \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$ is a continuous fuzzy mapping, $x(a) = \Phi(a) = \Phi_a$, and $\Phi : [a-\tau, a] \to \mathbb{R}_F$ is given.

Equation (1.1) is a fractional delay dynamical model with a Caputo–Katugampola–type fractional derivative. It is suitable for systems exhibiting hereditary effects together with time-delay phenomena, where uncertainty is represented by fuzzy quantities. Such structures arise naturally in engineering models involving delayed feedback and memory (e.g., hereditary material behavior or delayed control), and the fuzzy setting reflects imprecision in parameters or measurements. The F-correlated formulation further captures correlation/interaction in the uncertainty structure.

In this paper, the main objective is to investigate the Caputo–Katugampola F-correlated fractional differential equation with time delay given by system (1.1) and to obtain existence and uniqueness results for its solution. Below we highlight the main contributions of this work:

First, we provide a definition of the fuzzy Caputo–Katugampola integral and derivative in the interactive setting. We introduce two types of fuzzy processes: the original process denoted by $x(\cdot)$ and the process $x_{1-\alpha,p}(\cdot)$. We then explore the relationship between these two types of fuzzy processes, with a specific focus on the interaction type of $x_{1-\alpha,p}(\cdot)$.

Next, we investigate the properties of solutions to fuzzy Caputo–Katugampola interactive fuzzy differential equations. To account for time delay effects, we incorporate a constant delay term into the interactive fuzzy fractional model. We present two equivalent integral forms of the solutions and establish theoretical results on the existence and uniqueness of these solutions. This distinguishes our work from other articles in the field. Finally, a numerical example is given to verify the validity of the theoretical approach.

The subsequent sections of this paper are organized as follows. Section 2 introduces basic concepts of interactive fuzzy theory, such as F-correlated fuzzy numbers and calculus for correlated fuzzy processes. A new class of interactive fuzzy fractional integrals and derivatives is introduced in Section 3. In Section 4, we present two corresponding integral expressions of the solution of (1.1) for the fuzzy interactive system and prove the existence and uniqueness of a solution. In Section 5, an example is given to illustrate the practicality of the results.

2. Preliminaries

In this segment, we delineate foundational concepts and lemmas.

Definition 2.1. *(see [26]). A fuzzy subset A of $\mathbb{R}^n$ is described by its membership function $\mu_A(u) : \mathbb{R}^n \to [0,1]$, which gives the degree to which u belongs to A. Any subset Y of $\mathbb{R}^n$ can be viewed as a fuzzy subset whose membership function is given by its characteristic function $\chi_Y : \mathbb{R}^n \to \{0,1\}$, where $\chi_Y(u) = 1 \Leftrightarrow u \in Y$. The r-levels of a fuzzy subset A are classical subsets defined as*

$$\begin{aligned} [A]_r &= \{u \in \mathbb{R}^n : \mu_A(u) \geq r\}, \qquad \textit{for } 0 < r \leq 1, \\ [A]_0 &= \overline{\{u \in \mathbb{R}^n : \mu_A(u) > 0\}}, \qquad \textit{for } r = 0. \end{aligned}$$

Definition 2.2. *(see [27]). A fuzzy subset A of $\mathbb{R}^n$ is a fuzzy number if every r-level is a closed, bounded, and nonempty interval of $\mathbb{R}^n$, that is,*

$$[A]_r = [a_r^-, a_r^+], \qquad \forall\, r \in [0,1],$$

where $[A]_r = [a_r^-, a_r^+]$ denotes the r-level set of A and a_r^-, a_r^+ are the lower and upper branches of A, respectively.

The diameter of the r-level of A is defined by

$$\operatorname{diam}([A]_r) = \sup\{|a_r^-(t) - a_r^+(t)| : t \in J\}.$$

Denote by $\mathbb{R}_F$ the family of fuzzy numbers. The r-levels of a fuzzy number A are closed intervals of $\mathbb{R}$. We denote by $+$ and $-$ the traditional (Minkowski) sum and difference between fuzzy numbers, which can also be defined via Zadeh's extension principle (see [28]).

Definition 2.3. *(see [29]). The Pompeiu–Hausdorff distance $d : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ is defined by*

$$\begin{aligned} d(A,B) &= \sup_{r\in[0,1]} d_H([A]_r, [B]_r) \\ &= \sup_{r\in[0,1]} \max_{t\in[a,b]} \{|a_r^-(t) - b_r^-(t)|,\ |a_r^+(t) - b_r^+(t)|\}. \end{aligned}$$

Furthermore, the Pompeiu–Hausdorff diameter of $[A]_r \in \mathbb{R}_F$ is given by

$$d([A]_r, \hat{0}) = \max\{|a_r^- - a_r^+|\},$$

where the symbol $\hat{0}$ stands for the characteristic function of the real number 0.

If the function $t \mapsto d([x(t)]_r)$ is nondecreasing (respectively, nonincreasing) on $[a,b]$, then the fuzzy-number-valued function $x : J \to \mathbb{R}_F$ is said to be d-increasing (respectively, d-decreasing) on $[a,b]$.

Definition 2.4. *(see [30]). Let $A, B, C \in \mathbb{R}_F$. The generalized Hukuhara difference of A and B (gH-difference for short) is given by*

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} A = B + C, & \text{if } d([A]_r) \geq d([B]_r), \\ B = A + (-1)C, & \text{if } d([A]_r) \leq d([B]_r). \end{cases}$$

Definition 2.5. *(see [6, 10]). Let $A, B \in \mathbb{R}_F$, and let J be a fuzzy subset of $\mathbb{R}^2$. Then μ_J is called a joint possibility distribution of A and B if*

$$\sup_v \mu_J(u, v) = \mu_A(u) \quad \text{and} \quad \sup_u \mu_J(u, v) = \mu_B(v),$$

for all $u, v \in \mathbb{R}$. In this case, A and B are also called the marginal possibility distributions of J.

Every fuzzy number can be regarded as a possibility distribution on $\mathbb{R}$. The interactivity between fuzzy numbers is determined from a possibility distribution. If J is a possibility distribution of fuzzy numbers A and B, then the following relationship holds:

$$\mu_J(u, v) \leq \min\{\mu_A(u), \mu_B(v)\},$$

for all $u, v \in \mathbb{R}$. In addition,

$$[J]_r \subseteq [A]_r \times [B]_r,$$

for all $r \in [0, 1]$.

Definition 2.6. *(see [6, 9]). Let J be a joint possibility distribution of the fuzzy numbers A and B. The fuzzy numbers A and B are said to be non-interactive if and only if their joint possibility distribution J is given by*

$$\mu_J(u, v) = \min\{\mu_A(u), \mu_B(v)\}$$

or, equivalently,

$$[J]_r = [A]_r \times [B]_r,$$

for all $u, v \in \mathbb{R}$ and all $r \in [0, 1]$. Otherwise, they are said to be J-interactive or, simply, interactive.

Using the concept of a possibility distribution, Cabral and Barros (see [31]) introduced a special type of interactivity between fuzzy numbers, namely, F-correlation. Moreover, we consider a fuzzy-number-valued function f such that the fuzzy numbers $f(t)$ and $f(t+h)$ are interactive for all t and all h sufficiently small. This means that the value $f(t+h)$ is correlated with the value $f(t)$.

Definition 2.7. *(see [32]). The fuzzy numbers A and B are said to be F-correlated if there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that the corresponding joint possibility distribution is given by*

$$\mu_J(u, v) = \chi_{(u,v):v=F(u)}(u, v)\mu_A(u) = \chi_{(u,v):v=F(u)}(u, v)\mu_B(v),$$

where $\chi_{(u,v):v=F(u)}$ denotes the characteristic function of the curve $\{(u, F(u)) \in \mathbb{R}^2\}$.

Lemma 2.1. *(see [33, 34]). Let $A, B \in \mathbb{R}_F$, let J be a joint possibility distribution whose marginal possibility distributions are A and B, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then $f_J : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}$ is well defined and*

$$[f_J(A, B)]_r = f([J]_r) \qquad \text{for all } r \in [0, 1].$$

When F is a continuous function, then according to Nguyen's theorem (see [27]), the r-levels are given by

$$[B]_r = F([A]_r).$$

The next proposition establishes formulas for the r-levels of arithmetic operations of F-correlated fuzzy numbers when F is monotone differentiable.

Proposition 2.1. *(see [14]). Let A and B be F-correlated fuzzy numbers. If F is monotone differentiable, then for all $r \in [0, 1]$ we have*

$$[B -_F A]_r = \{F(u) - u \mid u \in [A]_r\} = \begin{cases} [b_r^- - a_r^-,\, b_r^+ - a_r^+], & \text{if } F'(u) > 1,\ \forall\, u \in [A]_r, \\ [b_r^+ - a_r^+,\, b_r^- - a_r^-], & \text{if } 0 < F'(u) \leq 1,\ \forall\, u \in [A]_r, \\ [b_r^- - a_r^+,\, b_r^+ - a_r^-], & \text{if } F'(u) \leq 0,\ \forall\, u \in [A]_r. \end{cases} \tag{2.1}$$

This can be easily verified by noticing that $F(u) - u$ is an increasing function in the first case, a decreasing function in the second case, and, in the third case, one just needs to use the fact that F is a decreasing function.

Moreover, the operation $-_F$ coincides with the usual fuzzy difference $\ominus$ when $\mathrm{diam}([A]_r) < \mathrm{diam}([B]_r)$ and $F'(u) > 1$ for all $u \in [A]_r$; it coincides with the generalized Hukuhara difference $\ominus_{gH}$ when $\mathrm{diam}([A]_r) > \mathrm{diam}([B]_r)$ and $0 < F'(u) \leq 1$ for all $u \in [A]_r$; and it coincides with the usual difference $-$ when $F'(u) < -1$ for all $u \in [A]_r$. Similarly, $+_F$ coincides with the usual sum $+$ when $F'(u) > 0$ for all $u \in [A]_r$.

Next, we introduce the fuzzy derivative for autocorrelation processes. A fuzzy autocorrelation process is a statistical process that resembles an autocorrelated statistical process. Fuzzy autocorrelated processes have been studied in several areas. If the interactivity is determined by a functional relationship F, we refer to the processes as being F-correlated.

Definition 2.8. *(see [35]) (Fuzzy process). A fuzzy process x is a fuzzy-number-valued function that, for each $t \in [a, b]$, associates a fuzzy number, that is, $x : [a, b] \to \mathbb{R}_F$. For every $r \in [0, 1]$ and $t \in [a, b]$, since $x(t)$ is a fuzzy number, we have that $[x(t)]_r$ is a nonempty closed interval of $\mathbb{R}$, denoted by $[x(t)]_r = [x_r^-(t), x_r^+(t)]$.*

When the diameter of a fuzzy-number-valued function is increasing (respectively, decreasing) in the independent variable, we say that the process is expansive (respectively, contractive).

Definition 2.9. *(see [14]) (F-autocorrelated). The fuzzy process x is said to be δ-locally F-autocorrelated at $t \in [a, b]$ (or, for short, F-autocorrelated) if for each $\delta > 0$, there exists a family of functions $F_{t,h}$ such that*

$$[x(t + h)]_r = F_{t,h}([x(t)]_r), \quad \forall\, r \in [0, 1],$$

whenever $0 < |h| < \delta$.

Definition 2.10. *(see [14]) (F-differentiable). An F-autocorrelated fuzzy process $x : [a, b] \to \mathbb{R}_F$ is F-correlated differentiable (or, for short, F-differentiable) at $t_0 \in [a, b]$ if there exists a fuzzy number $x_F'(t_0)$ such that*

$$x_F'(t_0) = \lim_{h \to 0} \frac{x(t_0 + h) -_F x(t_0)}{h},$$

where the above limit is evaluated using the metric d. When it exists, $x_F'(t_0)$ is called the F-correlated fuzzy derivative of x at t_0. At the endpoints of $[a, b]$, we consider only the one-sided derivatives.

Remark 2.1. *(see [6]). The existence of*

$$x_F'(t_0) = \lim_{h\to 0} \frac{x(t_0+h) -_F x(t_0)}{h}$$

is equivalent to the limits $\lim_{h\to 0^+} \frac{x(t_0+h)-_F x(t_0)}{h}$, $\lim_{h\to 0^-} \frac{x(t_0)-_F x(t_0-h)}{h}$, $\lim_{h\to 0^-} \frac{x(t_0+h)-_F x(t_0)}{h}$, *and* $\lim_{h\to 0^+} \frac{x(t_0)-_F x(t_0-h)}{h}$ *existing and being equal. Thus,*

$$\lim_{h\to 0} \frac{x(t_0+h) -_F x(t_0)}{h} \quad \text{is equivalent to} \quad \lim_{h\to 0} \frac{x(t_0) -_F x(t_0-h)}{h}.$$

Also, if $x_F'(t_0)$ *exists, then*

$$\begin{aligned}[x_F'(t_0)]_r &= \lim_{h\to 0^+} \frac{[x(t_0+h) -_F x(t_0)]_r}{h} = \lim_{h\to 0^-} \frac{[x(t_0) -_F x(t_0-h)]_r}{h} \\ &= \lim_{h\to 0^-} \frac{[x(t_0+h) -_F x(t_0)]_r}{h} = \lim_{h\to 0^+} \frac{[x(t_0) -_F x(t_0-h)]_r}{h},\end{aligned}$$

for any $r \in [0,1]$*, where the limits above are calculated using the Pompeiu–Hausdorff metric* d_H.

By Definition 2.9,

$$[x(t+h)]_r = F_{t,h}([x(t)]_r), \quad \forall\, r \in [0,1],$$

and according to (2.1)*, if* $[x(t)]_r = [x_r^-(t), x_r^+(t)]$*, then*

$$[x(t+h) -_F x(t)]_r = \begin{cases} [F_{t,h}(x_r^-(t)) - x_r^-(t),\ F_{t,h}(x_r^+(t)) - x_r^+(t)], \\ \qquad \text{if } F_{t,h}'(u) > 1,\ \forall\, u \in [x(t)]_r, \\ [F_{t,h}(x_r^+(t)) - x_r^+(t),\ F_{t,h}(x_r^-(t)) - x_r^-(t)], \\ \qquad \text{if } 0 < F_{t,h}'(u) \le 1,\ \forall\, u \in [x(t)]_r, \\ [F_{t,h}(x_r^-(t)) - x_r^+(t),\ F_{t,h}(x_r^+(t)) - x_r^-(t)], \\ \qquad \text{if } F_{t,h}'(u) \le 0,\ \forall\, u \in [x(t)]_r. \end{cases} \tag{2.2}$$

Lemma 2.2. *(see [14]). Let* $x : [a,b] \to \mathbb{R}_F$ *be F-differentiable at* $t_0 \in [a,b]$*, and assume that the corresponding family of functions* $F_{t,h}$ *is monotone and continuously differentiable for each h, and that* $[x(t)]_r = [x_r^-(t), x_r^+(t)]$ *for* $r \in [0,1]$*. If* x_r^- *and* x_r^+ *are differentiable at* t_0*, then*

$$[x_F'(t_0)]_r = \begin{cases} [(x_r^-)'(t_0), (x_r^+)'(t_0)], & \text{if } F_{t,h}'(u) > 1,\ \forall\, u \in [x(t)]_r, \\ \{(x_r^-)'(t_0)\} = \{(x_r^+)'(t_0)\}, & \text{if } F_{t,h}'(u) \le 0,\ \forall\, u \in [x(t)]_r, \end{cases}$$

for each $0 < |h| < \delta$ *with some* $\delta > 0$.

Definition 2.11. *(see [36]). A function* $x : [a,b] \to \mathbb{R}_F$ *is said to be strongly measurable if, for every* $r \in [0,1]$*, the set-valued mapping* $[x(\cdot)]_r$ *is (Lebesgue) measurable on* $[a,b]$ *with respect to the Hausdorff metric. Moreover,* $x : [a,b] \to \mathbb{R}_F$ *is said to be integrably bounded if there exists an integrable function* $h : [a,b] \to \mathbb{R}^+$ *such that* $\|x(t)\| \le h(t)$ *for all* $t \in [a,b]$.

A strongly measurable and integrably bounded function $x : [a, b] \to \mathbb{R}_F$ is called Aumann-integrable over $[a, b]$. Let $x : [a, b] \to \mathbb{R}_F$ be integrable over $[a, b]$. The Aumann integral of x over $[a, b]$, denoted by $\int_a^b x(t)\,dt$, is a fuzzy number whose r-levels are given by (see [34]):

$$\left[\int_a^b x(t)\,dt\right]_r = \int_a^b [x(t)]_r\,dt = \int_a^b [x_r^-(t), x_r^+(t)]\,dt$$
$$= \left\{\int_a^b y(t)\,dt \;\Big|\; y : [a, b] \to \mathbb{R} \text{ is a measurable selection of } [x(\cdot)]_r\right\},$$

for all $r \in [0, 1]$. By Remark 4.2 of [34], if the fuzzy-number-valued function x is integrable over $[a, b]$, then the integrals of the endpoint functions x_r^- and x_r^+ over $[a, b]$ exist and the following equality holds:

$$\left[\int_a^b x(t)\,dt\right]_r = \left[\int_a^b x_r^-(t)\,dt, \int_a^b x_r^+(t)\,dt\right],$$

for all $r \in [0, 1]$. We denote the set of all locally integrable functions from the interval $[a, b]$ to $\mathbb{R}_F$ by $L_{loc}([a, b], \mathbb{R}_F)$ and the set of all locally absolutely continuous functions from $[a, b]$ to $\mathbb{R}_F$ by $AC_{loc}([a, b], \mathbb{R}_F)$.

Lemma 2.3. *(see [18]). Let $x \in AC_{loc}([a, b], \mathbb{R}_F)$ be F-differentiable.*

(i) If x is expansive on $[a, b]$, that is, $\mathrm{diam}(x(t))$ is an increasing function of t and the function x_F' is Aumann-integrable over $[a, b]$, then, for all $r \in [0, 1]$, $(x_r^-)'$ and $(x_r^+)'$ are integrable over $[a, b]$ and

$$\left[\int_a^t x_F'(s)\,ds\right]_r = \left[\int_a^t (x_r^-)'(s)\,ds, \int_a^t (x_r^+)'(s)\,ds\right], \quad \forall\, t \in [a, b].$$

(ii) If x is contractive on $[a, b]$, that is, $\mathrm{diam}(x(t))$ is a decreasing function of t and the function x_F' is Aumann-integrable over $[a, b]$, then, for all $r \in [0, 1]$, $(x_r^-)'$ and $(x_r^+)'$ are integrable over $[a, b]$ and

$$\left[\int_a^t x_F'(s)\,ds\right]_r = \left[\int_a^t (x_r^+)'(s)\,ds, \int_a^t (x_r^-)'(s)\,ds\right], \quad \forall\, t \in [a, b].$$

Proposition 2.2. *(see [6]). If $x : [a, b] \to \mathbb{R}_F$ is F-differentiable on the interval $[a, b]$, then we have*

$$\int_a^b x_F'(t)\,dt = x(b) -_F x(a).$$

Theorem 2.1. *(see [37]). (Schauder's fixed-point theorem) Let X be a Banach space and let $C \subset X$ be nonempty, bounded, closed, and convex. If $T : C \to C$ is continuous and compact, that is, $T(C)$ is relatively compact in X, then T has at least one fixed point in C; namely, there exists $x^* \in C$ such that $T(x^*) = x^*$.*

3. Interactive fuzzy fractional integral and derivative

In this section, we use interactive fuzzy calculus to define fractional differential equations. Interactive calculus is capable of simulating the time-evolving uncertainty of fuzzy dynamical systems. We also define fuzzy Caputo–Katugampola fractional derivatives and integrals, which give rise to two different processes: the original process $x(\cdot)$ and the process $x_{1-\alpha,p}(\cdot)$. We then examine fuzzy processes from the viewpoint of fractional interactive derivatives. It has been shown that the Caputo–Katugampola fractional operator has the ability to change the interaction type of the autocorrelated fuzzy process $x(\cdot)$. Therefore, we investigate the relationship between $x(\cdot)$ and $x_{1-\alpha,p}(\cdot)$, and analyse the interaction type of $x_{1-\alpha,p}(\cdot)$. For instance, if $x(\cdot)$ is expansive, then $x_{1-\alpha,p}(\cdot)$ cannot be contractive. In addition, we discuss this connection, which is important for understanding the uncertainty evolution of the resulting fuzzy solutions.

The classification of processes as expansive or contractive is essential for ensuring the well-definedness of the fractional derivative in the fuzzy context, as it affects the order of the endpoint functions in the r-level sets.

Let $x \in L_{loc}([a,b],\mathbb{R}_F)$ be a fuzzy function and $[x(t)]_r = [x_r^-(t), x_r^+(t)]$ for $r \in [0,1]$.

Definition 3.1. *The fuzzy fractional Riemann–Liouville–Katugampola integral of order $\alpha > 0$ of x is defined levelwise by*

$$[(I_{a+}^{\alpha,p}x)(t)]_r = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\left[\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}x_r^-(s)\,ds,\ \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}x_r^+(s)\,ds\right],\quad t \in (a,b],$$

provided that the above expression defines a fuzzy number, that is, $(I_{a+}^{\alpha,p}x)(t) \in \mathbb{R}_F$.

For the fuzzy fractional derivative, consider $x \in L_{loc}([a,b],\mathbb{R}_F)$ and the fuzzy process

$$x_{1-\alpha,p}(t) = \frac{p^{\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{-\alpha}x(s)\,ds,\ \ \text{for all } t \in (a,b], \tag{3.1}$$

where $x_{1-\alpha,p}(a) = \lim_{t\to a^+} x_{1-\alpha,p}(t)$ in the sense of the Pompeiu–Hausdorff metric. For all $0 < \alpha \le 1$, the fuzzy function $x_{1-\alpha,p} : [a,b] \to \mathbb{R}_F$ is defined by (3.1), provided that this equation satisfies Proposition 4 (see [38, 39]), that is, it defines a fuzzy number for each $t \in (a,b]$.

Definition 3.2. *The fuzzy Riemann–Liouville–Katugampola fractional derivative of order $0 < \alpha \le 1$ of x with respect to the F-derivative is defined by*

$$({}_F^{RL}D_{a^+}^{\alpha,p}x)(t) = \frac{t^{1-p}p^{\alpha}}{\Gamma(1-\alpha)}\left(\int_a^t s^{p-1}(t^p - s^p)^{-\alpha}x(s)\,ds\right)_F' = t^{1-p}\big(x_{1-\alpha,p}(t)\big)_F',$$

provided that $\int_a^t s^{p-1}(t^p - s^p)^{-\alpha}x(s)\,ds$ is an F-correlated fuzzy process and F-differentiable for all $t \in (a,b]$.

It is important to highlight that $\int_a^t s^{p-1}(t^p - s^p)^{-\alpha}x(s)\,ds$ can be an expansive or contractive fuzzy process. However, it is expansive if $x(\cdot)$ is expansive. Thus, if $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is expansive, then

$$[({}_F^{RL}D_{a+}^{\alpha,p}x)(t)]_r = \frac{t^{1-p}p^{\alpha}}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_a^t s^{p-1}(t^p - s^p)^{-\alpha}x_r^-(s)\,ds,\ \frac{d}{dt}\int_a^t s^{p-1}(t^p - s^p)^{-\alpha}x_r^+(s)\,ds\right].$$

Thus,

$$[({}_F^{RL}D_{a+}^{\alpha,p}x)(t)]_r = \begin{cases} [{}^{RL}D_{a+}^{\alpha,p}(x_r^-)(t), {}^{RL}D_{a+}^{\alpha,p}(x_r^+)(t)] & \text{if } x_{1-\alpha,p}(\cdot) \text{ or } x(\cdot) \text{ is expansive;} \\ [{}^{RL}D_{a+}^{\alpha,p}(x_r^+)(t), {}^{RL}D_{a+}^{\alpha,p}(x_r^-)(t)] & \text{if } x_{1-\alpha,p}(\cdot) \text{ is contractive.} \end{cases}$$

Definition 3.3. *Let x be an F-correlated fuzzy process. The fuzzy Caputo–Katugampola fractional derivative ${}^{C}_{F}D^{\alpha,p}_{a+}x$, with respect to the F-derivative, is defined by*

$$({}^{C}_{F}D^{\alpha,p}_{a^+}x)(t) = \left({}^{RL}_{F}D^{\alpha,p}_{a^+}[x(\cdot) -_F x(a)]\right)(t), \quad t \in (a,b].$$

Thus,

$$[({}^{C}_{F}D^{\alpha,p}_{a+}x)(t)]_r = \begin{cases} [{}^{C}D^{\alpha,p}_{a+}(x^-_r)(t), {}^{C}D^{\alpha,p}_{a+}(x^+_r)(t)], & \text{if } x_{1-\alpha,p}(\cdot) \text{ or } x(\cdot) \text{ is expansive;} \\ [{}^{C}D^{\alpha,p}_{a+}(x^+_r)(t), {}^{C}D^{\alpha,p}_{a+}(x^-_r)(t)], & \text{if } x_{1-\alpha,p}(\cdot) \text{ or } x(\cdot) \text{ is contractive.} \end{cases}$$

Definition 3.4. *Let $x \in AC_{loc}(J, \mathbb{R}_F)$ be an F-correlated fuzzy process, F-differentiable with $[x(t)]_r = [x^-_r(t), x^+_r(t)]$ for $r \in [0,1]$, and let $\alpha \in (0,1)$. Then*

$$[({}^{C}_{F}D^{\alpha,p}_{a+}x)(t)]_r = \begin{cases} \left[\dfrac{p^\alpha}{\Gamma(1-\alpha)}\displaystyle\int_a^t (t^p - s^p)^{-\alpha}(x^-_r)'(s)\,ds, \ \dfrac{p^\alpha}{\Gamma(1-\alpha)}\displaystyle\int_a^t (t^p - s^p)^{-\alpha}(x^+_r)'(s)\,ds\right], \\ \quad \text{if } x(\cdot) \text{ is expansive;} \\ \left[\dfrac{p^\alpha}{\Gamma(1-\alpha)}\displaystyle\int_a^t (t^p - s^p)^{-\alpha}(x^+_r)'(s)\,ds, \ \dfrac{p^\alpha}{\Gamma(1-\alpha)}\displaystyle\int_a^t (t^p - s^p)^{-\alpha}(x^-_r)'(s)\,ds\right], \\ \quad \text{if } x_{1-\alpha,p}(\cdot) \text{ or } x(\cdot) \text{ is contractive.} \end{cases}$$

Moreover, from the above equation, if $x \in AC_{loc}(J, \mathbb{R}_F)$, then

$$({}^{C}_{F}D^{\alpha,p}_{a^+}x)(t) = \frac{p^\alpha}{\Gamma(1-\alpha)}\int_a^t (t^p - s^p)^{-\alpha}(x_F)'(s)\,ds.$$

Theorem 3.1. *If $x \in AC(J, \mathbb{R}_F)$ is a d-monotone F-correlated fuzzy process, then*

$$I^{\alpha,p}_{a+}{}^{C}_{F}D^{\alpha,p}_{a+}x(t) = x(t) -_F x(a), \quad t \in (a,b].$$

Proof. By using Definition 3.4 and Proposition 2.2, we have

$$\begin{aligned} I^{\alpha,p}_{a+}{}^{C}_{F}D^{\alpha,p}_{a+}x(t) &= (I^{\alpha,p}_{a+}(I^{1-\alpha,p}_{a+}(s^{1-p}x')))(t) \\ &= (I^{1,p}_{a+}(s^{1-p}x'))(t) \\ &= \int_a^t x'_F(s)\,ds = x(t) -_F x(a). \end{aligned}$$

To solve problem (1.1), we suppose that every r-level of $f(t, x(t), x(t-\tau))$ can be written in terms of the corresponding r-level of $x(t)$ as follows:

$$\begin{aligned} &[f(t, x(t), x(t-\tau))]_r \\ &= [f^-_r(t, x^-_r(t), x^+_r(t), x^-_r(t-\tau), x^+_r(t-\tau)), f^+_r(t, x^-_r(t), x^+_r(t), x^-_r(t-\tau), x^+_r(t-\tau))] \end{aligned}$$

for all $r \in [0,1]$, $t \in J = (a,b]$. For convenience, let

$$[f(t, x(t), x(t-\tau))]_r = [f^-_r(t), f^+_r(t)].$$

Then the above initial value problem (1.1) becomes

$$\begin{cases} [({}^{C}_{F}D^{\alpha,p}_{a+}x)(t)]_r = [f(t, x(t), x(t-\tau))]_r, & t \in J = (a,b], \\ [x(s)]_r = [\Phi(s)]_r \in \mathbb{R}_F, & s \in (a-\tau, a]. \end{cases} \tag{3.2}$$

□

Lemma 3.1. *If an F-correlated fuzzy process $x \in C(J, \mathbb{R}_F)$ is an F-differentiable mild solution of the FFDEs (1.1) and the corresponding family of functions $F_{t,h}$ is monotone continuously differentiable, and if $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is expansive, then the r-levels of the initial value problem (3.2) are equivalent to*

$$\begin{aligned}[x(t)]_r &= [\Phi(a)]_r \\ &+ \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \left[f(s, x(s), x(s-\tau))\right]_r \, ds, \ t \in J = (a, b].\end{aligned} \tag{3.3}$$

If $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is contractive, then the r-levels of the initial value problem (3.2) are equivalent to

$$\begin{aligned}[x(t)]_r &= [\Phi(a)]_r \ominus (-1) \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \\ &\times \left[f(s, x(s), x(s-\tau))\right]_r \, ds, \quad t \in J = (a, b].\end{aligned} \tag{3.4}$$

Proof. According to Definition 3.4, if $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is expansive, then the initial value problem (3.2) can be written as

$$\begin{cases} [{}^C D_{a+}^{\alpha,p} x_r^-(t), {}^C D_{a+}^{\alpha,p} x_r^+(t)] = [f_r^-(t), f_r^+(t)], \\ [x_r^-(s), x_r^+(s)] = [\Phi_r^-(s), \Phi_r^+(s)] \in \mathbb{R}_F, \quad s \in (a-\tau, a]. \end{cases}$$

Applying $I_{a+}^{\alpha,p}$ to both sides of the above equation and using Theorem 3.1, we obtain

$$\begin{aligned}&[I_{a+}^{\alpha,p} \, {}^C D_{a+}^{\alpha,p} x_r^-(t), I_{a+}^{\alpha,p} \, {}^C D_{a+}^{\alpha,p} x_r^+(t)] \\ &= [I_{a+}^{\alpha,p} f_r^-(t), I_{a+}^{\alpha,p} f_r^+(t)] \\ &= [x_r^-(t) -_F x_r^-(a), x_r^+(t) -_F x_r^+(a)] \\ &= \Big[\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s) \, ds, \ \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s) \, ds\Big].\end{aligned}$$

Since $(x_r^-)'(t) \leq (x_r^+)'(t)$, we have $x_r^-(t) -_F x_r^-(a) \leq x_r^+(t) -_F x_r^+(a)$. According to Proposition 2.1, in this case $-_F = \ominus$, and therefore

$$\begin{aligned}&[x_r^-(t) \ominus x_r^-(a), x_r^+(t) \ominus x_r^+(a)] \\ &= [I_{a+}^{\alpha,p} f(t, x(t), x(t-\tau))]_r \\ &= [x_r^-(t), x_r^+(t)] \\ &= [\Phi_r^-(a), \Phi_r^+(a)] \\ &+ \Big[\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s) \, ds, \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s) \, ds\Big],\end{aligned}$$

$$[x(t)]_r = [\Phi(a)]_r + \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \left[f(s, x(s), x(s-\tau))\right]_r \, ds.$$

If $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is contractive, according to Definition 3.3, the initial value problem (3.2) can be written as

$$\begin{cases} [{}^C D_{a+}^{\alpha,p} x_r^+(t), {}^C D_{a+}^{\alpha,p} x_r^-(t)] = [f_r^-(t), f_r^+(t)], \quad t \in J = (a, b], \\ [x_r^-(s), x_r^+(s)] = [\Phi_r^-(s), \Phi_r^+(s)] \in \mathbb{R}_F, \quad s \in (a-\tau, a]. \end{cases}$$

The above equation can also be transformed into the following two sets of initial value problems:

$$\begin{cases} {}^{C}D_{a+}^{\alpha,p} x_r^+(t) = f_r^-(t), & t \in J = (a,b], \\ x_r^+(s) = \Phi_r^+(s) \in \mathbb{R}_F, & s \in (a-\tau, a], \end{cases} \tag{3.5}$$

and

$$\begin{cases} {}^{C}D_{a+}^{\alpha,p} x_r^-(t) = f_r^+(t), & t \in J = (a,b], \\ x_r^-(s) = \Phi_r^-(s) \in \mathbb{R}_F, & s \in (a-\tau, a]. \end{cases} \tag{3.6}$$

Integrating both sides of the first equation of (3.5) and (3.6) and using Theorem 3.1, we obtain

$$x_r^+(t) -_F x_r^+(a) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s)\, ds,$$

and

$$x_r^-(t) -_F x_r^-(a) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s)\, ds.$$

Since $(x_r^+)'(t) \le (x_r^-)'(t)$, it follows that $x_r^+(t) -_F x_r^+(a) \le x_r^-(t) -_F x_r^-(a)$. According to Proposition 2.1, in this case $-_F = \ominus_{gH}$, and hence

$$\begin{aligned} x_r^+(t) \ominus_{gH} x_r^+(a) &= \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s)\, ds, \\ x_r^+(t) &= x_r^+(a) \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s)\, ds, \end{aligned}$$

and

$$\begin{aligned} x_r^-(t) \ominus_{gH} x_r^-(a) &= \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s)\, ds, \\ x_r^-(t) &= x_r^-(a) \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s)\, ds. \end{aligned}$$

Therefore, if $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is contractive, then the r-levels of the initial value problem (3.2) are equivalent to

$$\begin{aligned} & [x_r^-(t), x_r^+(t)] \\ & = [x_r^-(a), x_r^+(a)] \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \left[f(s, x(s), x(s-\tau))\right]_r\, ds \\ & = [\Phi_r^-(a), \Phi_r^+(a)] \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \left[f(s, x(s), x(s-\tau))\right]_r\, ds \\ [x(t)]_r & = [\Phi(a)]_r \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \left[f(s, x(s), x(s-\tau))\right]_r\, ds. \end{aligned}$$

□

4. The existence of a solution of fuzzy interactive fractional differential equations

By taking r-level sets and using the standard level-set representation of fuzzy-valued functions, problem (1.1) is equivalent to the parametric r-level formulation given in (4.1). Thus, (4.1) is the level-wise representation used for the fixed-point analysis.

Consider the following fuzzy Caputo-Katugampola F-correlated fractional differential equations:

$$\begin{cases} [({}^{C}_{F}D^{\alpha,p}_{a+}x)(t)]_r = [f(t,x(t),x(t-\tau))]_r, & t \in J = (a,b], \\ [x(s)]_r = [\Phi(s)]_r \in \mathbb{R}_F, \quad s \in (a-\tau, a]. \end{cases} \tag{4.1}$$

$f : [a,b] \times \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$ is a continuous fuzzy mapping, $[x(s)]_r = [\Phi(s)]_r$, and $\Phi : [a-\tau, b] \to \mathbb{R}_F$. According to Lemma 3.1, if $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is expansive, the mild solution $[x(t)]_r$ of problem (4.1) is equivalent to

$$[x(t)]_r = \begin{cases} [\Phi(a)]_r + \dfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} \displaystyle\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} [f(s,x(s),x(s-\tau))]_r\, ds, \quad t \in J = (a,b], \\ [\Phi(t)]_r, \quad t \in (a-\tau, a]. \end{cases}$$

The solution to problem (1.1) in the above equation is

$$x(t) = \begin{cases} \Phi(a) + \dfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} \displaystyle\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f(s,x(s),x(s-\tau)) ds, \quad t \in J = (a,b], \\ \Phi(t), \quad t \in (a-\tau, a]. \end{cases}$$

If $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is contractive, then

$$[x(t)]_r = \begin{cases} [\Phi(a)]_r \ominus (-1) \dfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} \displaystyle\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} [f(s,x(s),x(s-\tau))]_r\, ds, \quad t \in J = (a,b], \\ [\Phi(t)]_r, \quad t \in (a-\tau, a]. \end{cases}$$

The solution to problem (1.1) in the above equation is

$$x(t) = \begin{cases} \Phi(a) \ominus (-1) \dfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} \displaystyle\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f(s,x(s),x(s-\tau)) ds, \quad t \in J = (a,b], \\ \Phi(t), \quad t \in (a-\tau, a]. \end{cases}$$

Next consider the following assumptions:

(H1) (Non-Lipschitz condition). For each fixed $t \in [-\tau, b]$, $f(t,x,y) \in \mathbb{R}_F$ is a continuous function and, for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_F$, the following inequality holds:

$$d([f(t,x_1,y_1)]_r, [f(t,x_2,y_2)]_r) \leq \lambda(t) G(d([x_1]_r, [x_2]_r), d([y_1]_r, [y_2]_r)),$$

where $I^{\alpha,p}_{a+}\lambda \in C(J, \mathbb{R}^+)$ and $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $G(0,0) = 0$, and we denote $G(t,t)$ simply by $G(t)$.

(H2) For any $x_1(t), x_2(t) \in C(J, \mathbb{R}_F)$, we suppose that there exists a function g such that for $G(d(x_1(t), x_2(t))) : \mathbb{R}^+ \to \mathbb{R}^+$, we have

$$G(d([x_1(t)]_r, [x_2(t)]_r)) \leq g(t) d([x_1(t)]_r, [x_2(t)]_r),$$

where $g \in C(J, \mathbb{R}^+)$.

Theorem 4.1. *Assume $x \in C(J, \mathbb{R}_F)$ is an F-correlated fuzzy process and that, under the conditions (H1) and (H2), there exists at least one solution $[x(t)]_r$ to Eq (4.1).*

Proof. First consider the case where $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is contractive, and define the operator T : $C([a,b], \mathbb{R}_F) \to C([a,b], \mathbb{R}_F)$ by

$$\begin{aligned}
&T[x(t)]_r \\
&= \left[\Phi_r^-(a) \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s)ds,\right. \\
&\quad \left.\Phi_r^+(a) \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s)ds\right] \\
&= [\Phi(a)]_r \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}\left[f(s, x(s), x(s-\tau))\right]_r ds.
\end{aligned}$$

Step 1. T is completely continuous. For this, we need to show that T is continuous. Let $\{x_n(t), n = 1, 2, \cdots\}$ be a sequence converging to $x(t)$ in $C(J, \mathbb{R}_F)$ and $x_n(t) = \Phi(t)$ for $t \in [a-\tau, a]$. For each $x_n(t) \in C(J, \mathbb{R}_F)$ and for all $t \in J$, by Lemma 3.1 and condition (H1), we get

$$\begin{aligned}
&d(T[x_n(t)]_r, T[x(t)]_r) \\
&= d\Bigg([\Phi(a)]_r \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}[f(s, x_n(s), x_n(s-\tau))]_r ds, \\
&\qquad [\Phi_a]_r \ominus (-1)\tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\le d\Bigg(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}\left[f(s, x_n(s), x_n(s-\tau))\right]_r ds, \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \\
&\qquad \times \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds\Bigg) \\
&\le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} d\left([f(s, x_n(s), x_n(s-\tau))]_r,\right. \\
&\qquad \left.[f(s, x(s), x(s-\tau))]_r\right) ds \\
&\le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}\lambda(s) G\Big(d([x_n(s)]_r, [x(s)]_r), \\
&\qquad d([x_n(s-\tau)]_r, [x(s-\tau)]_r)\Big) ds.
\end{aligned}$$

According to the definition of d, we obtain

$$\begin{aligned}
& d([x_n(t-\tau)]_r, [x(t-\tau)]_r) \\
&= \sup_{r\in[0,1]} \max_{t\in[a,b]} \{|x^-_{nr}(t-\tau) - x^-_r(t-\tau)|, |x^+_{nr}(t-\tau) - x^+_r(t-\tau)|\} \\
&\le \sup_{r\in[0,1]} \max_{s\in[a-\tau,b-\tau]} \{|x^-_{nr}(s) - x^-_r(s)|, |x^+_{nr}(s) - x^+_r(s)|\} \\
&\le \sup_{r\in[0,1]} \max_{s\in[a-\tau,b]} \{|x^-_{nr}(s) - x^-_r(s)|, |x^+_{nr}(s) - x^+_r(s)|\} \\
&\le \sup_{r\in[0,1]} \max_{a-\tau\le s\le a} \{|x^-_{nr}(s) - x^-_r(s)|, |x^+_{nr}(s) - x^+_r(s)|\} \\
&\quad + \sup_{r\in[0,1]} \max_{a\le s\le b} \{|x^-_{nr}(s) - x^-_r(s)|, |x^+_{nr}(s) - x^+_r(s)|\} \\
&= \sup_{r\in[0,1]} \max_{a\le s\le b} \{|x^-_{nr}(s) - x^-_r(s)|, |x^+_{nr}(s) - x^+_r(s)|\} \\
&= \sup_{r\in[0,1]} \max_{a\le t\le b} \{|x^-_{nr}(t) - x^-_r(t)|, |x^+_{nr}(t) - x^+_r(t)|\} \\
&= d([x_n(t)]_r, [x(t)]_r).
\end{aligned}$$

Then

$$\begin{aligned}
d(T[x_n(t)]_r, T[x(t)]_r) \le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}\lambda(s) G\Big(& d([x_n(s)]_r, [x(s)]_r), \\
& d([x_n(s)]_r, [x(s)]_r)\Big) ds.
\end{aligned}$$

By conditions (H1) and (H2), we have

$$\begin{aligned}
d(T[x_n(t)]_r, T[x(t)]_r) &\le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}\lambda(s) G\Big(d([x_n(s)]_r, [x(s)]_r)\Big) ds \\
&\le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}\lambda(s) g(s) d([x_n(s)]_r, [x(s)]_r) ds.
\end{aligned}$$

Since $I^{\alpha,p}_{a+}\lambda \in C(J, \mathbb{R}^+)$ and g is continuous, it follows that $I^{\alpha,p}_{a+}\lambda$ and g are bounded. Let $I^{\alpha,p}_{a+}\lambda \le M$ and $N = \sup_{t\in[a,b]} g(t)$, and then

$$d(T[x_n(t)]_r, T[x(t)]_r) \le \frac{M p^{-\alpha}(b-a)^{p\alpha} N}{\alpha\Gamma(1-\alpha)} d([x_n]_r, [x]_r).$$

We can see that $d(T[x_n(t)]_r, T[x(t)]_r) \to 0$ as $n \to \infty$. So T is continuous.

Next, T maps bounded sets into bounded sets in $C(J, \mathbb{R}_F)$. We prove that there exists a positive constant β, and for every $\delta > 0$ satisfying $\forall x(t) \in B_\delta = \{x(t) \in C(J, \mathbb{R}_F) : d(x(t), \hat{0}) \le \delta\}$, one has $d(T(x(t)), \hat{0}) \le \beta$. Now, for all $t \in J$ and all $x(t) \in B_{\delta_1}$, we obtain

$$\begin{aligned}
& d(T[x(t)]_r, \hat{0}) \\
&= d\left([\Phi(a)]_r \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds, \hat{0}\right) \\
&\le d([\Phi(a)]_r, \hat{0}) + (-1) d\left(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds, \hat{0}\right) \\
&\le d\left([\Phi(a)]_r, \hat{0}\right) + (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} d\left([f(s, x(s), x(s-\tau))]_r, \hat{0}\right) ds \\
&\le d\left([\Phi(a)]_r, \hat{0}\right) + (-1)\frac{(b-a)^{p\alpha} p^{-\alpha}}{\alpha\Gamma(1-\alpha)} d\left([f(t, x(t), x(t-\tau))]_r, \hat{0}\right).
\end{aligned}$$

By the condition that $f \in C(J, \mathbb{R}_F)$, there is a constant $C_h > 0$ such that $d([f(t, v, w)]_r, \hat{0}) \leq C_h$ for all $(t, v, w) \in [a, b] \times \mathbb{R}_F \times \mathbb{R}_F$. Therefore $d([f(t, x(t), x(t-\tau))]_r, \hat{0}) \leq C_h$. Then we get

$$d(T[x(t)]_r, \hat{0}) \leq d([\Phi(a)]_r, \hat{0}) + (-1)\frac{C_h(b-a)^{p\alpha}p^{-\alpha}}{\alpha\Gamma(1-\alpha)} := \beta_1.$$

Hence, for every $x(t) \in B_{\beta_1}$, we have $d(T[x(t)]_r, \hat{0}) \leq \beta_1$, which means that $TB_{\beta_1} \subset B_{\beta_1}$.

Next, T maps bounded sets into equi-continuous sets. For each $x(t) \in B_{\beta_1}$ and $t_1, t_2 \in J$ with $t_1 < t_2$, using Lemma 3.1, we obtain

$$\begin{aligned}
&d(T[x(t_1)]_r, T[x(t_2)]_r)\\
&\leq d\bigg(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^{t_1} s^{p-1}(t_1^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds,\\
&\qquad \tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^{t_2} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds\bigg)\\
&= d\bigg(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^{t_1} s^{p-1}(t_1^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds,\\
&\qquad \tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^{t_1} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds\\
&\qquad + \tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_{t_1}^{t_2} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds\bigg)\\
&\leq d\bigg(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^{t_1} s^{p-1}(t_1^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds,\\
&\qquad \tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_a^{t_1} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds\bigg)\\
&\qquad + d\bigg(\tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)}\int_{t_1}^{t_2} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r ds, \hat{0}\bigg)\\
&\leq \int_a^{t_1} d\bigg(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} s^{p-1}(t_1^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r,\\
&\qquad \tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s, x(s), x(s-\tau))]_r\bigg) ds
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_{t_1}^{t_2} d\left(\tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s,x(s),x(s-\tau))]_r, \hat{0}\right) ds \\
&= \int_a^{t_1} \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} s^{p-1} |(t_1^p - s^p)^{\alpha-1} - (t_2^p - s^p)^{\alpha-1}| d\left([f(s,x(s),x(s-\tau))]_r, \hat{0}\right) ds \\
&\qquad + \int_{t_1}^{t_2} d\left(\tfrac{p^{1-\alpha}}{\Gamma(1-\alpha)} s^{p-1}(t_2^p - s^p)^{\alpha-1}[f(s,x(s),x(s-\tau))]_r, \hat{0}\right) ds \\
&= \int_a^{t_1} \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} s^{p-1} |(t_1^p - s^p)^{\alpha-1} - (t_2^p - s^p)^{\alpha-1}| d\left([f(s,x(s),x(s-\tau))]_r, \hat{0}\right) ds \\
&\qquad + \tfrac{p^{-\alpha}(t_2^p - t_1^p)^\alpha}{\alpha\Gamma(1-\alpha)} d([f(t,x(t),x(t-\tau))]_r, \hat{0}) \\
&\le \frac{p^{-\alpha}}{\alpha\Gamma(1-\alpha)} |(t_2^p - t_1^p)^\alpha + (t_1^p - a^p)^\alpha - (t_2^p - a^p)^\alpha| d\left([f(t,x(t),x(t-\tau))]_r, \hat{0}\right) \\
&\qquad + \tfrac{p^{-\alpha}(t_2^p - t_1^p)^\alpha}{\alpha\Gamma(1-\alpha)} d([f(t,x(t),x(t-\tau))]_r, \hat{0}) \\
&\le \frac{p^{-\alpha}}{\alpha\Gamma(1-\alpha)} |2(t_2 - t_1)^{p\alpha} + (t_1 - a)^{p\alpha} - (t_2 - a)^{p\alpha}| d\left([f(t,x(t),x(t-\tau))]_r, \hat{0}\right) \\
&\le \frac{p^{-\alpha}}{\alpha\Gamma(1-\alpha)} |2(t_2 - t_1)^{p\alpha} + (t_1 - a)^{p\alpha} - (t_2 - a)^{p\alpha}| C_h.
\end{aligned}
$$

We can see that $\frac{p^{-\alpha}}{\alpha\Gamma(1-\alpha)}|2(t_2 - t_1)^{p\alpha} + (t_1 - a)^{p\alpha} - (t_2 - a)^{p\alpha}|$ is independent of $x(t) \in B_{\delta_2}$ and $\frac{p^{-\alpha}}{\alpha\Gamma(1-\alpha)}|2(t_2 - t_1)^{p\alpha} + (t_1 - a)^{p\alpha} - (t_2 - a)^{p\alpha}| \to 0$ when $t_2 \to t_1$. It means that $d(T[x(t_1)]_r, T[x(t_2)]_r) \to 0$. Therefore, the set TB_{δ_2} is equicontinuous. According to the Arzelà–Ascoli theorem, it is easy to see that T is completely continuous.

Step 2. We need to prove that there is a bounded, convex, and closed subset $B_{\delta'} = \{x(t) \in C(J, \mathbb{R}_F) : d([x(t)]_r, \hat{0}) \le \delta'\}$ such that $TB_{\delta'} \subseteq B_{\delta'}$.

We know that $B_{\delta'}$ is a bounded, convex, closed subset of $C(J, \mathbb{R}_F)$ for every $\delta' \in \mathbb{R}^+$. If for some $\delta' \in \mathbb{R}^+$, there exists $x_{\delta'}(t) \in B_{\delta'}$ such that $T([x_{\delta'}(t)]_r) \notin B_{\delta'}$, that is, $d(T([x_{\delta'}(t)]_r), \hat{0}) > \delta'$, then

$$
\begin{aligned}
\delta' &< d(T[x_{\delta'}(t)]_r, \hat{0}) \\
&= d\left([\Phi(a)]_r \ominus (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1}[f(s, x_{\delta'}(s), x_{\delta'}(s-\tau))]_r ds, \hat{0}\right) \\
&\le d([\Phi(a)]_r, \hat{0}) + (-1)\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} d([f(s, x_{\delta'}(s), x_{\delta'}(s-\tau))]_r, \hat{0}) ds \\
&\le d\left([\Phi(a)]_r, \hat{0}\right) + (-1)\frac{p^{\alpha-1}(b-a)^{p\alpha}}{\alpha\Gamma(1-\alpha)} C_h.
\end{aligned}
$$

Taking the limit as $\delta' \to +\infty$, it is immediate to obtain

$$
d\left([\Phi(a)]_r, \hat{0}\right) + (-1)\frac{p^{\alpha-1}(b-a)^{p\alpha}}{\alpha\Gamma(1-\alpha)} C_h \to +\infty,
$$

which is in contradiction with the fact that $d([\Phi(a)]_r, \hat{0}) + (-1)\frac{p^{\alpha-1}(b-a)^{p\alpha}}{\alpha\Gamma(1-\alpha)} C_h$ is bounded. Consequently, for every positive constant δ', we get $TB_{\delta'} \subseteq B_{\delta'}$.

By Theorem 2.1, there exists at least one solution of Eq (4.1). If $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is expansive, the method of proving the existence of the horizontal level solution is similar, and is therefore omitted here. To sum up, the proof of Theorem 4.1 has been completed. □

Theorem 4.2. *Suppose that conditions (H1) and (H2) hold. If*

$$\sup_{t\in[a,b]} g(t) < \frac{\alpha\Gamma(1-\alpha)}{Mp^{\alpha-1}(b-a)^{p\alpha}},$$

then the solution to Eq (4.1) is unique.

Proof. If $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is contractive, then $[x_r^+(t), x_r^-(t)]$ is a solution to Eq (4.1) if

$$\begin{aligned}
&[x_r^-(t), x_r^+(t)] \\
&= \Big[\Phi_r^-(a) \ominus (-1)(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^-(s)ds), \\
&\quad \Phi_r^+(a) \ominus (-1)(\frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} f_r^+(s)ds)\Big]
\end{aligned}$$

holds. If $x(t) \in C(J, \mathbb{R}_F)$ is a fixed point of T defined as in Theorem 4.1, then $x(t)$ is the solution of Eq (4.1). Let $x(t), y(t) \in C(J, \mathbb{R}_F)$ and $t \in [-\tau, a]$, and suppose $x(t) = y(t) = \Phi(t)$. For all $t \in [a, b]$, we can get

$$\begin{aligned}
& d(T[x(t)]_r, T[y(t)]_r) \\
& \le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} d([f(s, x(s), x(s-\tau))]_r, [f(s, y(s), y(s-\tau))]_r)ds \\
& \le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \lambda(s) \\
& \qquad \times G\Big(d([x(s)]_r, [y(s)]_r), d([x(s-\tau)]_r, [y(s-\tau)]_r)\Big)ds \\
& \le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \lambda(s) G\Big(d([x(s)]_r, [y(s)]_r), d([x(s)]_r, [y(s)]_r)\Big)ds \\
& \le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \lambda(s) G\Big(d([x(s)]_r, [y(s)]_r)\Big)ds \\
& \le \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \int_a^t s^{p-1}(t^p - s^p)^{\alpha-1} \lambda(s) g(s) d([x(s)]_r, [y(s)]_r)ds \\
& \le \sup_{t\in[a,b]} g(t) \frac{Mp^{\alpha-1}(b-a)^{p\alpha}}{\alpha\Gamma(1-\alpha)} d([x]_r, [y]_r).
\end{aligned}$$

Due to $\sup_{t\in[a,b]} g(t) < \frac{\alpha\Gamma(1-\alpha)}{Mp^{\alpha-1}(b-a)^{p\alpha}}$, we know that

$$d(T[x(t)]_r, T[y(t)]_r) < d([x]_r, [y]_r).$$

Hence, T has a unique fixed point $[x(t)]_r \in C(J, \mathbb{R}_F)$ based on the Banach contraction principle. Therefore, the theorem is proved. □

If $x_{1-\alpha,p}(\cdot)$ or $x(\cdot)$ is expansive, the method of proving the uniqueness of the horizontal level solution is similar, and is therefore omitted here.

5. Example

Consider the following problem for the fuzzy fractional differential equation

$$\begin{cases} {}^{C}_{F}\mathcal{D}_{0^+}^{\frac{1}{2},\frac{1}{2}} x(t) = \lambda x(t-\tau), \quad t \in (a,b], \\ x(s) = \Phi(s), \quad s \in (a-\tau, a], \end{cases} \tag{5.1}$$

where $\alpha = p = \frac{1}{2}, a = 1, b = 2, \tau = 1$, $x_0 = (1;2;3)$, and $\Phi(s) = (1-s^2; 2-s^2; 3-s^2)$.

First, let us consider $\lambda > 0$ in (5.1). If $x(\cdot)$ is an expansive process (i.e., diam$[x(t)]_r$ is increasing) on $(a,b]$, then for $x_1 = (1;2;3)$ and for all $r \in [0,1]$, we have

$$\begin{aligned} {}^{C}D_{a+}^{\alpha,p} x_r^-(t) &= \lambda x_r^-(t-\tau); \qquad x_r^-(s) = \Phi_r^-(s) = 2 - s^2 + r, \\ {}^{C}D_{a+}^{\alpha,p} x_r^+(t) &= \lambda x_r^+(t-\tau); \qquad x_r^+(s) = \Phi_r^+(s) = 4 - s^2 - r. \end{aligned}$$

It is easy to check that (H1) and (H2) are satisfied and that $\dfrac{\alpha\Gamma(1-\alpha)}{kp^{\alpha-1}(b-a)^{p\alpha}} \approx 0.3989422 < 1$. Therefore, Theorem 4.2 applies and the solution exists and is unique. The numerical simulation results are shown in Figures 1 and 2.

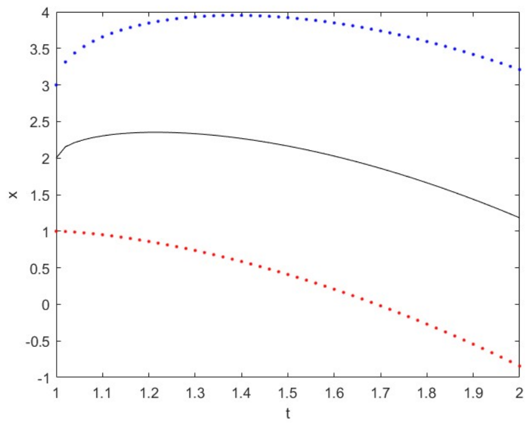

Figure 1. Fuzzy solutions at the 0-level set (dashed blue and red lines) and 1-level set (solid line).

This corresponds to the case of fuzzy solutions at the 0-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 3$. This also corresponds to the case of fuzzy solutions at the 1-level set (solid line), where the initial value of the solution is $x_r^-(1) = x_r^+(1) = 2$. When the maximum solution coincides with the minimum solution, the fuzzy solution becomes an exact (crisp) solution.

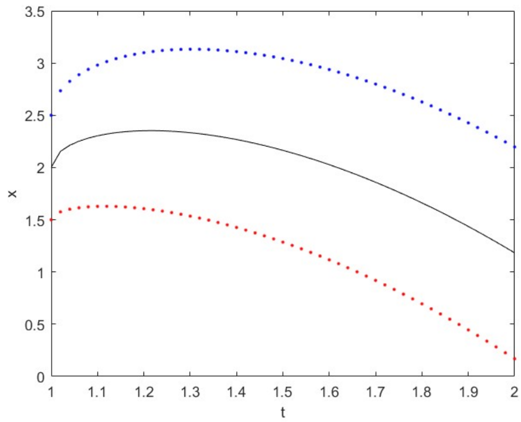

Figure 2. Fuzzy solutions at the 0.5-level set (dashed blue and red lines) and 1-level set (solid line).

This corresponds to the case of fuzzy solutions at the 0.5-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1.5$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 2.5$. The solid line is the same as in Figure 1.

If $x(\cdot)$ is a contractive process (i.e., diam$[x(t)]_r$ is decreasing) on $(a, b]$, then for $x_1 = (1; 2; 3)$ and for all $r \in [0, 1]$, we have

$$
\begin{aligned}
{}^{C}D_{a+}^{\alpha,p} x_r^-(t) &= \lambda x_r^+(t-\tau); \qquad & x_r^-(s) &= \Phi_r^-(s) = 2 - s^2 + r, \\
{}^{C}D_{a+}^{\alpha,p} x_r^+(t) &= \lambda x_r^-(t-\tau); \qquad & x_r^+(s) &= \Phi_r^+(s) = 4 - s^2 - r.
\end{aligned}
$$

It is easy to check that (H1) and (H2) are satisfied. Therefore, Theorem 4.2 applies and the solution exists and is unique. The numerical simulation results are shown in Figures 3 and 4.

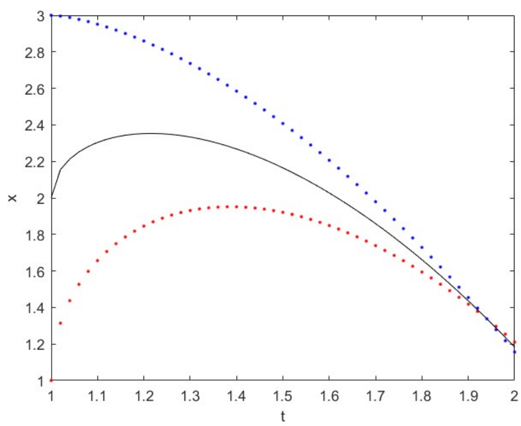

Figure 3. Fuzzy solutions at the 0-level set (dashed blue and red lines) and 1-level set (solid line) for the contractive case.

This corresponds to the case of fuzzy solutions at the 0-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1$ and the initial value of the upper solution (dashed red line) is

$x_r^+(1) = 3$. This also corresponds to the case of fuzzy solutions at the 1-level set (solid line), where the initial value of the solution is $x_r^-(1) = x_r^+(1) = 2$. When the maximum solution coincides with the minimum solution, the fuzzy solution becomes an exact solution.

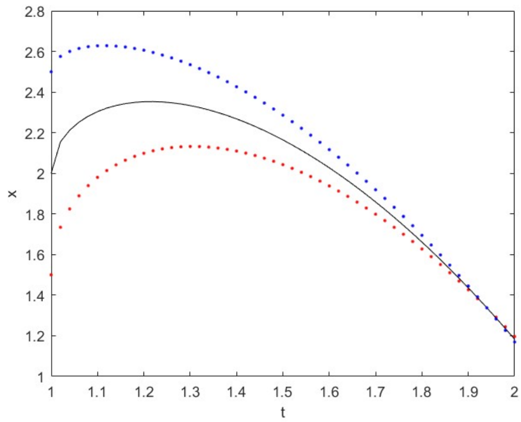

Figure 4. Fuzzy solutions at the 0.5-level set (dashed blue and red lines) and 1-level set (solid line) for the contractive case.

This corresponds to the case of fuzzy solutions at the 0.5-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1.5$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 2.5$. The solid line is the same as in Figure 3.

Now let us consider $\lambda < 0$ in (5.1). If $x(\cdot)$ is an expansive process (i.e., $\mathrm{diam}[x(t)]_r$ is increasing) on $(a, b]$, then for $x_1 = (1; 2; 3)$ and for all $r \in [0, 1]$, we have

$$
\begin{aligned}
{}^C D^{\alpha,p}_{a+} x_r^-(t) &= \lambda x_r^+(t-\tau); \qquad & x_r^-(s) &= \Phi_r^-(s) = 2 - s^2 + r, \\
{}^C D^{\alpha,p}_{a+} x_r^+(t) &= \lambda x_r^-(t-\tau); \qquad & x_r^+(s) &= \Phi_r^+(s) = 4 - s^2 - r.
\end{aligned}
$$

It is easy to check that (H1) and (H2) are satisfied. Therefore, Theorem 4.2 applies and the solution exists and is unique. The numerical simulation results are shown in Figures 5 and 6.

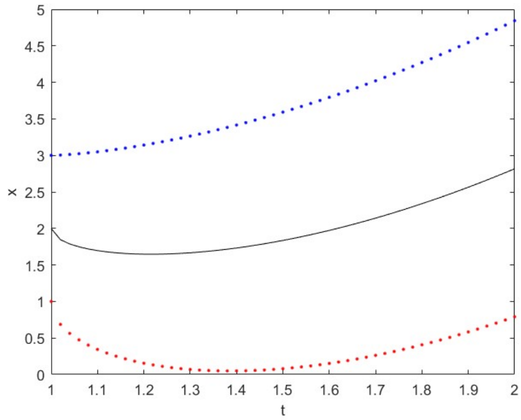

Figure 5. Fuzzy solutions at the 0-level set (dashed blue and red lines) and 1-level set (solid line) for $\lambda < 0$, the expansive case.

This corresponds to the case of fuzzy solutions at the 0-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 3$. This also corresponds to the case of fuzzy solutions at the 1-level set (solid line), where the initial value of the solution is $x_r^-(1) = x_r^+(1) = 2$. When the maximum solution coincides with the minimum solution, the fuzzy solution becomes an exact solution.

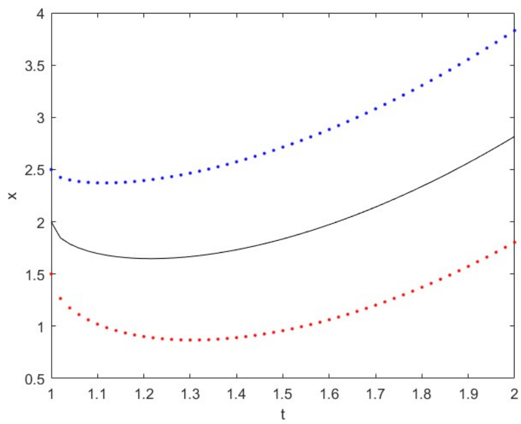

Figure 6. Fuzzy solutions at the 0.5-level set (dashed blue and red lines) and 1-level set (solid line) for $\lambda < 0$, the expansive case.

This corresponds to the case of fuzzy solutions at the 0.5-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1.5$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 2.5$, which corresponds to the case of the lower branch set of solutions. The solid line is the same as in Figure 5.

If $x(\cdot)$ is a contractive process (i.e., $\text{diam}[x(t)]_r$ is decreasing) on $(a, b]$, then for $x_1 = (1; 2; 3)$ and for all $r \in [0, 1]$, we have

$$
\begin{aligned}
{}^{C}D_{a+}^{\alpha,p} x_r^-(t) &= \lambda x_r^-(t-\tau); \qquad x_r^-(s) = \Phi_r^-(s) = 2 - s^2 + r,\\
{}^{C}D_{a+}^{\alpha,p} x_r^+(t) &= \lambda x_r^+(t-\tau); \qquad x_r^+(s) = \Phi_r^+(s) = 4 - s^2 - r.
\end{aligned}
$$

It is easy to check that (H1) and (H2) are satisfied. Therefore, Theorem 4.2 applies and the solution exists and is unique. The numerical simulation results are shown in Figures 7 and 8.

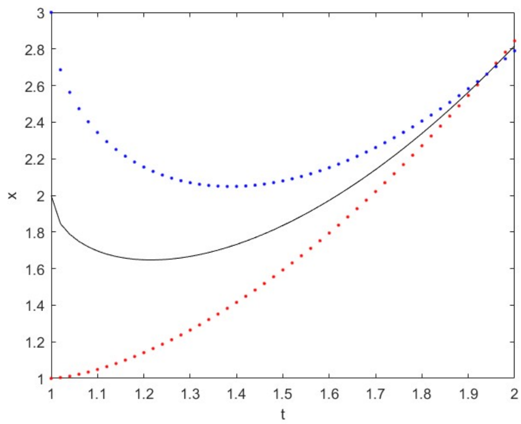

Figure 7. Fuzzy solutions at the 0-level set (dashed blue and red lines) and 1-level set (solid line) for $\lambda < 0$, the contractive case.

This corresponds to the case of fuzzy solutions at the 0-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 3$. This also corresponds to the case of fuzzy solutions at the 1-level set (solid line), where the initial value of the solution is $x_r^-(1) = x_r^+(1) = 2$. When the maximum solution coincides with the minimum solution, the fuzzy solution becomes an exact solution.

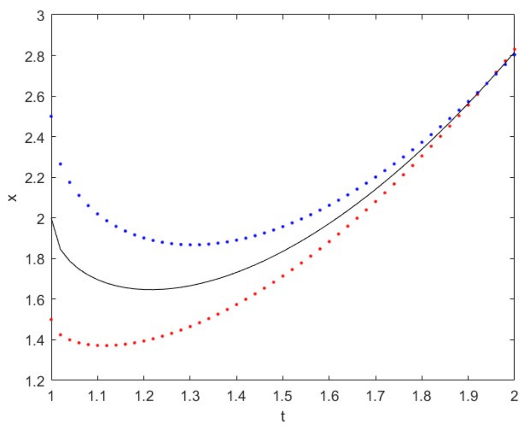

Figure 8. Fuzzy solutions at the 0.5-level set (dashed blue and red lines) and 1-level set (solid line) for $\lambda < 0$, the contractive case.

This corresponds to the case of fuzzy solutions at the 0.5-level set. The initial value of the lower solution (dashed blue line) is $x_r^-(1) = 1.5$ and the initial value of the upper solution (dashed red line) is $x_r^+(1) = 2.5$. The solid line is the same as in Figure 7.

6. Conclusions

Our research focuses on a class of fractional interactive fuzzy Caputo-Katugampola F-correlated time-delayed differential equations, with an emphasis on the existence and uniqueness of solutions. First, we give the definitions of the correlated integral and derivative for interactive fuzzy Caputo-Katugampola F-correlated systems. Second, we derive an equivalent integral formulation and establish the existence of solutions for fuzzy interactive Caputo-Katugampola F-correlated fractional differential equations with time delay by applying Schauder's fixed-point theorem. Finally, the behavior of fuzzy solutions at different r-levels is illustrated, and their convergence to exact solutions under specific conditions is highlighted. For future research, we may consider investigating the properties of solutions to fractional interactive fuzzy differential equations in the sense of Hilfer, as well as exploring the stability and boundary value problems of the interactive solutions proposed herein.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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