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Research article

$\mathfrak{a}$-Depth and $\mathfrak{a}$-Cohen-Macaulay modules

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Abstract: Let S be a commutative Noetherian ring and $\mathfrak{a}$ a proper ideal of S. We give several bounds of $\mathfrak{a}$-depth of S-complexes and S-modules, investigate the behavior of $\mathfrak{a}$-depth and $\mathfrak{a}$-Cohen-Macaulay modules under tensor product with a faithfully flat S-module. Furthermore, we establish the Foxby equivalence of $\mathfrak{a}$-Cohen-Macaulay S-modules.

Keywords: $\mathfrak{a}$-depth; $\mathfrak{a}$-Cohen-Macaulay S-module; Foxby equivalence

1. Introduction

The theory of Cohen-Macaulay (CM) rings and modules forms a cornerstone of modern commutative algebra, with profound implications in fields such as algebraic geometry, algebraic combinatorics, and algebraic representation theory. A fundamental theorem established by Bruns and Herzog [1] asserts the stability of Cohen-Macaulayness under flat local extensions. More precisely, for a homomorphism of local rings $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$, a finitely generated S-module M, and a finitely generated T-module N that is flat as an S-module, $M \otimes_S N$ is a CM-module over S precisely when M is a CM-module over S, and the quotient $N/\mathfrak{m}N$ is a CM-module over T.

Given a semidualizing S-module D, Enochs and Yassemi [2] proved the following category equivalence, known as Foxby equivalence:

$$\mathcal{A}_D(S) \underset{\mathrm{Hom}_S(D,-)}{\overset{D\otimes_S -}{\rightleftarrows}} \mathcal{B}_D(S),$$

where $\mathcal{A}_D(S)$ (resp. $\mathcal{B}_D(S)$) means the Auslander (resp. Bass) class with respect to D. This equivalence has subsequently been refined and extended to various subcategories of $\mathcal{A}_D(S)$ and $\mathcal{B}_D(S)$; see, for instance [3–6]. Notably, Beigi et al. [3] examined the behaviour of CM-modules under this equivalence and established the following category equivalence:

$$\mathcal{CM}(S) \cap \mathcal{A}_D(S) \underset{\mathrm{Hom}_S(D,-)}{\overset{D\otimes_S -}{\rightleftarrows}} \mathcal{CM}(S) \cap \mathcal{B}_D(S),$$

where $\mathcal{CM}(S)$ stands for the classes of CM-modules over S.

Generalizations of CM-modules go in different directions. On one hand, over commutative Noetherian local rings, there are (surjective) Buchsbaum modules, (sequentially) generalized CM-modules, sequentially CM-modules, filtered CM-modules, and so forth (see, [7–9] for details). On the other hand, Mahmood and Azam in [10] gave a generalization over commutative Noetherian (not necessarily local) rings: Let S be a commutative Noetherian ring and $\mathfrak{a}$ a proper ideal of S. As usual, A finitely generated S-module M is said to be $\mathfrak{a}$-*Cohen-Macaulay* ($\mathfrak{a}$-CM-module) if $\operatorname{depth}_S(\mathfrak{a}, M) + \dim_S(M/\mathfrak{a}M) = \dim_S M$. Note that this module exhibits many properties analogous to classical CM-modules. The aim of the paper is to further justify this viewpoint about these two kinds of modules.

Our main objectives are as follows:

In Section 1, we present a derived depth formula for the ideal $\mathfrak{a}$ under certain conditions and investigate bounds for the $\mathfrak{a}$-depth of complexes and modules.

In Section 2, we consider the behaviour of $\mathfrak{a}$-depth and $\mathfrak{a}$-CM-modules under local ring homomorphisms.

In Section 3, we establish the Foxby equivalence of $\mathfrak{a}$-CM-modules.

Throughout this work, unless otherwise stated, S and T denote two commutative Noetherian rings which are not necessarily local, and $\mathfrak{a}$ denotes a proper ideal of S. We use S-**Mod** (resp. S-**mod**) to denote the category of all S-modules (resp. the category of all finitely generated S-modules) and $\mathbf{D}(S)$ to denote the derived category of S-modules. The full subcategories $\mathbf{D}_{\sqsubset}(S)$ and $\mathbf{D}_{\sqsupset}(S)$ consist of S-complexes $\mathbf{M}$ such that $\mathrm{H}_l(\mathbf{M}) = 0$ for $l \gg 0$ and $l \ll 0$, respectively. By $\mathbf{D}^{\mathrm{f}}(S)$, we denote the full subcategory of $\mathbf{D}_{\sqsubset}(S)$ consisting of S-complexes $\mathbf{M}$ with $\mathrm{H}_l(\mathbf{X}) \in S$-**mod** for all $l \in \mathbb{Z}$. We begin by recalling some preliminary definitions that will be used in the following sections.

Associated primes, supports, dimensions, depths. As usual, we denote by $\operatorname{Spec}(S)$ the set of all prime ideals of S, and by $\mathfrak{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(S) | \mathfrak{a} \subseteq \mathfrak{p}\}$. For $\mathfrak{q} \in \operatorname{Spec}(S)$, set $\mathfrak{U}(\mathfrak{q}) = \{\mathfrak{p} \in \operatorname{Spec}(S) | \mathfrak{p} \subseteq \mathfrak{q}\}$.

Let $X \in S$-**mod**. The *associated prime* $\operatorname{Ass}_S X$ of X is defined by

$$\operatorname{Ass}_S X := \{\mathfrak{p} \in \operatorname{Spec}(S) | \ \exists \ \text{ a cyclic submodule } Z \text{ of } X, \text{ s.t. } \operatorname{Ann}_S Z = \mathfrak{p}\}.$$

The *support* $\operatorname{Supp}_S X$ of X is defined by

$$\operatorname{Supp}_S X := \{\mathfrak{p} \in \operatorname{Spec}(S) | X_\mathfrak{p} \neq 0\},$$

where $X_\mathfrak{p}$ means the localization of X at $\mathfrak{p}$.

The *Krull dimension* $\dim_S X$ of X is defined by

$$\dim_S X := \sup\{\dim S/\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp}_S X\}.$$

The $\mathfrak{a}$-*depth* $\operatorname{depth}_S(\mathfrak{a}, X)$ of X is defined by

$$\operatorname{depth}_S(\mathfrak{a}, X) := \inf\{l \in \mathbb{Z} | \operatorname{Ext}^l_S(S/\mathfrak{a}, X) \neq 0\}.$$

When $(S, \mathfrak{m})$ is a local ring, $\operatorname{depth}_S(\mathfrak{m}, X)$ is denoted simply by $\operatorname{depth}_S X$, called the *depth* of X.

2. $\mathfrak{a}$-depth of complexes

In this section, we show the derived depth formula with respect to $\mathfrak{a}$ whenever $S/\mathfrak{a}$ is semisimple and give several bounds of the $\mathfrak{a}$-depth of complexes and modules. We also investigate the behavior of $\mathfrak{a}$-depth of modules upon tensoring with a faithfully flat S-module. in what follows, we denote by $\mathrm{fd}_S M$ (resp. $\mathrm{fd}_S \mathbf{M}$) the flat dimension of an S-module M (resp. S-complex $\mathbf{M}$).

Given an S-complex $\mathbf{M}$ over a local ring $(S, \mathfrak{m})$, its (classical) *depth* (see [11]) is defined as

$$\mathrm{depth}_S \mathbf{M} = -\sup \mathbf{R}\mathrm{Hom}_S(S/\mathfrak{m}, \mathbf{M}).$$

Definition 2.1. ([12]) The $\mathfrak{a}$-*depth* $\mathrm{depth}_S(\mathfrak{a}, \mathbf{M})$ of an S-complex $\mathbf{M}$ is defined by

$$\mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) = -\sup \mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M}).$$

When $(S, \mathfrak{m})$ is a local ring, $\mathrm{depth}_S(\mathfrak{m}, \mathbf{M})$ is exactly $\mathrm{depth}_S \mathbf{M}$.

Christensen and Foxby [13, Theorem 5.2.2] provide the following classical derived depth formula over local ring $(S, \mathfrak{m})$: If $\mathbf{M}, \mathbf{N} \in \mathbf{D}_{\sqsubset}(S)$, and $\mathrm{fd}_S \mathbf{N} < \infty$, then

$$\mathrm{depth}_S(\mathbf{M} \otimes^{\mathsf{L}}_S \mathbf{N}) = \mathrm{depth}_S \mathbf{M} + \mathrm{depth}_S \mathbf{N} - \mathrm{depth}_S S.$$

Take $\mathfrak{a} = \mathfrak{m}$ for a local ring $(S, \mathfrak{m})$. The following result was proved in [14, Theorem 16.3.1(a)].

Proposition 2.2. *Let $S/\mathfrak{a}$ be a semisimple ring. If $\mathbf{M}, \mathbf{N} \in \mathbf{D}_{\sqsubset}(S)$, and $\mathrm{fd}_S \mathbf{N} < \infty$, then*

$$\begin{aligned}\mathrm{depth}_S(\mathfrak{a}, \mathbf{M} \otimes^{\mathsf{L}}_S \mathbf{N}) &= \mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) - \sup(S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N}) \\ &= \mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) + \mathrm{depth}_S(\mathfrak{a}, \mathbf{N}) - \mathrm{depth}_S(\mathfrak{a}, S).\end{aligned}$$

Proof. For $\mathbf{M}$ and $\mathbf{N}$, we have

$$\begin{aligned}\mathrm{depth}_S(\mathfrak{a}, \mathbf{M} \otimes^{\mathsf{L}}_S \mathbf{N}) &= -\sup \mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M} \otimes^{\mathsf{L}}_S \mathbf{N}) \\ &\overset{(1)}{=} -\sup(\mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M}) \otimes^{\mathsf{L}}_S \mathbf{N}) \\ &= -\sup(\mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M}) \otimes^{\mathsf{L}}_{S/\mathfrak{a}} S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N}) \\ &\overset{(2)}{=} -\sup(\mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M}) \otimes^{\mathsf{L}}_{S/\mathfrak{a}} (S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N})) \\ &\overset{(3)}{=} -\sup(\mathrm{H}(\mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M})) \otimes_{S/\mathfrak{a}} \mathrm{H}(S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N})) \\ &= -(\sup \mathrm{H}(\mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M})) + \sup \mathrm{H}(S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N})) \\ &\overset{(4)}{=} -(\sup \mathrm{H}(\mathbf{R}\mathrm{Hom}_S(S/\mathfrak{a}, \mathbf{M})) + \sup(S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N})) \\ &= \mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) - \sup(S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N}),\end{aligned}$$

where (1) is by [13, Theorem 4.3.5], (2) is by associativity, and (3) and (4) are by [13, Proposition 2.1.19] because $S/\mathfrak{a}$ is semisimple.

Futhermore, taking $M = S$ to the above equality, one gets

$$\mathrm{depth}_S(\mathfrak{a}, \mathbf{N}) = \mathrm{depth}_S(\mathfrak{a}, S) - \sup(S/\mathfrak{a} \otimes^{\mathsf{L}}_S \mathbf{N}).$$

Now we are done. □

Remark 2.3. Let $\mathbf{M}, \mathbf{N} \in \mathbf{D}_{\sqsubset}(S)$, and $\mathrm{fd}_S \mathbf{N} < \infty$.

(1) If S is a semilocal ring or S is a semiprimary ring, or S is a semiperfect ring with Jacobson radical J, then Proposition 2.2 yields the following equality:

$$\mathrm{depth}_S(J, \mathbf{M} \otimes_S^{\mathrm{L}} \mathbf{N}) = \mathrm{depth}_S(J, \mathbf{M}) + \mathrm{depth}_S(J, \mathbf{N}) - \mathrm{depth}_S(J, S).$$

(2) If S is an Artinian ring, and $\mathfrak{a}$ is a prime ideal of S, then Proposition 2.2 yields the following equality:

$$\mathrm{depth}_S(\mathfrak{a}, \mathbf{M} \otimes_S^{\mathrm{L}} \mathbf{N}) = \mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) + \mathrm{depth}_S(\mathfrak{a}, \mathbf{N}) - \mathrm{depth}_S(\mathfrak{a}, S).$$

Let $X, Y \in S$-**Mod**. We say that X and Y are *Tor-independent* if $\mathrm{Tor}_S^i(X, Y) = 0$ for any $i > 0$. This happens if and only if $X \otimes_S^{\mathrm{L}} Y \to X \otimes_S Y$ is a quasi-isomorphism.

Corollary 2.4. *Let $S/\mathfrak{a}$ be a semisimple ring and $X, Y \in S$-**Mod**. If X and Y are Tor-independent with $\mathrm{fd}_S Y < \infty$, then*

$$\mathrm{depth}_S(\mathfrak{a}, X \otimes_S Y) = \mathrm{depth}_S(\mathfrak{a}, X) + \mathrm{depth}_S(\mathfrak{a}, Y) - \mathrm{depth}_S(\mathfrak{a}, S).$$

Let $\mathbf{M}$ be an S-complex. Following [13], the *Support* of $\mathbf{M}$ is

$$\mathrm{Supp}_S \mathbf{M} = \{\mathfrak{p} \in \mathrm{Spec}(S) \mid \mathbf{M}_{\mathfrak{p}} \not\simeq 0\},$$

where $\mathbf{M}_{\mathfrak{p}}$ means the localization of $\mathbf{M}$ at $\mathfrak{p}$.

The *(Krull) dimension* of $\mathbf{M}$ is

$$\dim_S \mathbf{M} = \sup\{\dim S/\mathfrak{p} - \inf \mathbf{M}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_S \mathbf{M}\}.$$

The next two results give bounds of $\mathfrak{a}$-depth of complexes.

Proposition 2.5. *If $0 \not\simeq \mathbf{M} \in \mathbf{D}^{\mathrm{f}}(S)$, then*

$$\mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) \leq \mathrm{depth}_{S_{\mathfrak{q}}} \mathbf{M}_{\mathfrak{q}} + \dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p}$$

for any $\mathfrak{p} \in \mathrm{Supp}_S \mathbf{M} \cap \mathfrak{V}(\mathfrak{a})$ and $\mathfrak{q} \in \mathfrak{U}(\mathfrak{p})$.

Proof. By [14, Lemma 17.6.1], $\mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) \leq \mathrm{depth}_{S_{\mathfrak{p}}}(\mathfrak{a}_{\mathfrak{p}}, \mathbf{M}_{\mathfrak{p}})$, it follows that

$$\mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) \leq \mathrm{depth}_{S_{\mathfrak{p}}}(\mathfrak{a}_{\mathfrak{p}}, \mathbf{M}_{\mathfrak{p}}) \leq \mathrm{depth}_{S_{\mathfrak{p}}} \mathbf{M}_{\mathfrak{p}}.$$

For $\mathfrak{q} \in \mathfrak{U}(\mathfrak{p})$, one has

$$\begin{aligned} \mathrm{depth}_S(\mathfrak{a}, \mathbf{M}) &\leq \mathrm{depth}_{S_{\mathfrak{p}}} \mathbf{M}_{\mathfrak{p}} \\ &\overset{(1)}{\leq} \mathrm{depth}_{S_{\mathfrak{q}}} (\mathbf{M}_{\mathfrak{p}})_{\mathfrak{q}} + \dim(S_{\mathfrak{p}}/\mathfrak{q}S_{\mathfrak{p}}) \\ &\overset{(2)}{=} \mathrm{depth}_{S_{\mathfrak{q}}} \mathbf{M}_{\mathfrak{q}} + \dim(S_{\mathfrak{p}}/\mathfrak{q}S_{\mathfrak{p}}) \\ &\overset{(3)}{\leq} \mathrm{depth}_{S_{\mathfrak{q}}} \mathbf{M}_{\mathfrak{q}} + \dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p}, \end{aligned}$$

where (1) follows [14, E 16.4.1], (2) follows the isomorphism $(\mathbf{M}_{\mathfrak{p}})_{\mathfrak{q}} \cong \mathbf{M}_{\mathfrak{q}}$, and (3) follows [14, Proposition 14.2.8]. This shows the desired inequality. □

Proposition 2.6. *If $0 \not\simeq \mathbf{M} \in \mathbf{D}^{\mathrm{f}}_{\sqsubset}(S)$, then*

$$\operatorname{depth}_S(\mathfrak{a}, \mathbf{M}) \leq \dim_S \mathbf{M} - \dim_S S/\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Supp}_S \mathbf{M} \cap \mathfrak{V}(\mathfrak{a})$. *Moreover, if* $(S, \mathfrak{m})$ *is a local ring, then* $\operatorname{depth}_S \mathbf{M} \leq \dim_S \mathbf{M}$.

Proof. By [15, Proposition 5.2], one has $\operatorname{depth}_S(\mathfrak{a}, \mathbf{M}) \leq \operatorname{depth}_{S_\mathfrak{p}} \mathbf{M}_\mathfrak{p}$; by [13, Lemma 6.3.6], one gets $\operatorname{depth}_{S_\mathfrak{p}} \mathbf{M}_\mathfrak{p} \leq \dim_{S_\mathfrak{p}} \mathbf{M}_\mathfrak{p}$; by [13, Lemma 6.3.4], one obtains

$$\dim_{S_\mathfrak{p}} \mathbf{M}_\mathfrak{p} \leq \dim_S \mathbf{M} - \dim_S S/\mathfrak{p}.$$

Consequently,

$$\operatorname{depth}_S(\mathfrak{a}, \mathbf{M}) \leq \operatorname{depth}_{S_\mathfrak{p}} \mathbf{M}_\mathfrak{p} \leq \dim_{S_\mathfrak{p}} \mathbf{M}_\mathfrak{p} \leq \dim_S \mathbf{M} - \dim_S S/\mathfrak{p}.$$

□

Giving $0 \neq M \in S$-**mod**, it is well known that $\dim_S S/\mathfrak{q} \leq \dim_S M$.

Proposition 2.7. *For* $0 \neq M \in S$-**mod**, *one has*

$$\operatorname{depth}_S(\mathfrak{a}, M) \leq \dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p}$$

for every $\mathfrak{p} \in \operatorname{Supp}_S M \cap \mathfrak{V}(\mathfrak{a})$ *and* $\mathfrak{q} \in \operatorname{Ass}_S M \cap \mathfrak{U}(\mathfrak{p})$. *Moreover, if* $(S, \mathfrak{m})$ *is a local ring, then* $\operatorname{depth}_S M \leq \dim_S S/\mathfrak{q}$ *for all* $\mathfrak{q} \in \operatorname{Ass}_S M$.

Proof. One has

$$\begin{aligned} \operatorname{depth}_S(\mathfrak{a}, M) &\overset{(1)}{\leq} \operatorname{depth}_{S_\mathfrak{p}} M_\mathfrak{p} \\ &\overset{(2)}{\leq} \dim_{S_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{q}S_\mathfrak{p}) \text{ for all } \mathfrak{q} \in \operatorname{Ass}_{S_\mathfrak{p}} M_\mathfrak{p} \\ &\overset{(3)}{\leq} \dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p} \text{ for all } \mathfrak{q} \in \operatorname{Ass}_S M \cap \mathfrak{U}(\mathfrak{p}), \end{aligned}$$

where (1) follows by [15, Proposition 5.2], (2) holds by [1, Proposition 1.2.13], and (3) follows by [13, Lemma 6.3.4]. □

Let $M \in S$-**mod**. It follows from [12, Theorem 8.4] that $\operatorname{depth}(\mathfrak{a}, M)$ is the maximal length of an M-regular sequence in $\mathfrak{a}$.

Lemma 2.8. *Let* $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$ *be a homomorphism of local rings* $M \in S$-**mod**, $N \in T$-**mod**, *with* N *being faithfully flat as an* S*-module. Then, the following hold:*

(1) $\operatorname{Hom}_T(T/\mathfrak{a}T, M \otimes_S N) \neq 0$ *if and only if* $\operatorname{Hom}_S(S/\mathfrak{a}, M) \neq 0$.

(2) *If* $\mathbf{y}$ *is a* $N/\mathfrak{m}N$*-regular sequence in* T*, then* $\mathbf{y}$ *is an* $(M \otimes_S N)$*-regular sequence, and* $N/\mathbf{y}N$ *is a faithfully flat* S*-module.*

Proof. (1) It follows from the isomorphism $\operatorname{Hom}_T(T/\mathfrak{a}T, M \otimes_S N) \cong \operatorname{Hom}_S(S/\mathfrak{a}, M) \otimes_S N$ and faithful flatness of N.

(2) By [1, Lemma 1.2.17], the first statement holds, and $N/\mathbf{y}N$ is flat over S. Next, we show that $N/\mathbf{y}N$ is faithful. Take an S-module L. If $L \in S$-**mod**, then $N \otimes_S L \in T$-**mod**. Thus, if $(N \otimes_S L)/\mathbf{y}(N \otimes_S L) \cong N/\mathbf{y}N \otimes_S L = 0$, then Nakayama's lemma implies that $N \otimes_S L = 0$. But N is faithful, so $L = 0$. If $L \in S$-**Mod** and $N/\mathbf{y}N \otimes_S L = 0$. Then, $L = \bigcup_{i \in I} L_i$, where each L_i is a finitely generated submodule of L. Hence, $N/\mathbf{y}N \otimes_S L = \bigcup_{i \in I}(N/\mathbf{y}N \otimes_S L_i) = 0$, which implies that $N/\mathbf{y}N \otimes_S L_i = 0$ for all $i \in I$. Consequently, $L = 0$. This shows that $N/\mathbf{y}N$ is a faithfully flat S-module. □

Theorem 2.9. *Let* $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$ *be a homomorphism of local rings,* $M \in S$-**mod**, $N \in T$-**mod**, *with* N *being faithfully flat as an* S*-module. Then,*

$$\operatorname{depth}_T(\mathfrak{a}T, M \otimes_S N) = \operatorname{depth}_S(\mathfrak{a}, M) + \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N).$$

Proof. As [15, Proposition 5.2], one has $\operatorname{depth}_T(\mathfrak{a}T, M \otimes_S N) = \operatorname{depth}_S(\mathfrak{a}, M \otimes_S N)$ and $\operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N) = \operatorname{depth}_S(\mathfrak{a}, N/\mathfrak{m}N)$. Set $\operatorname{depth}_S(\mathfrak{a}, M) = k$ and $\operatorname{depth}_S(\mathfrak{a}, N/\mathfrak{m}N) = l$. It suffices to show that $\operatorname{depth}_S(\mathfrak{a}, M \otimes_S N) = k + l$. Let $\boldsymbol{w} = w_1, \ldots, w_k \in \mathfrak{a}$ be a maximal M-regular sequence and $\boldsymbol{z} = z_1, \ldots, z_l \in \mathfrak{a}$ a maximal $N/\mathfrak{a}N$-regular sequence. Due to Lemma 2.8(2), $\boldsymbol{z}$ is a $(M/\boldsymbol{w}M \otimes_S N)$-regular sequence. Set $\overline{M} = M/\boldsymbol{w}M$. Because $\overline{M} \otimes_S N \cong (M \otimes_S N)/\boldsymbol{w}(M \otimes_S N)$, $\boldsymbol{w}, \boldsymbol{z} \in \mathfrak{a}$ is an $(M \otimes_S N)$-regular sequence. So, $\operatorname{depth}_S(\mathfrak{a}, M \otimes_S N) \geq k + l$. Put $\overline{N} = N/\boldsymbol{z}N$. One has

$$\overline{N}/\mathfrak{m}\overline{N} \cong (N/\mathfrak{m}N)/\boldsymbol{z}(N/\mathfrak{m}N),$$

$$\overline{M} \otimes_S \overline{N} \cong (M \otimes_S N)/(\boldsymbol{w}, \boldsymbol{z})(M \otimes_S N).$$

It follows that

$$\operatorname{Ext}_S^{k+l}(S/\mathfrak{a}, M \otimes_S N) \cong \operatorname{Hom}_S(S/\mathfrak{a}, \overline{M} \otimes_S \overline{N}) \cong \operatorname{Hom}_S(S/\mathfrak{a}, \overline{M}) \otimes_S \overline{N}.$$

By [1, Proposition 1.2.3] and Lemma 2.8(2), $\operatorname{Ext}_S^{k+l}(S/\mathfrak{a}, M \otimes_R N) \neq 0$. Thus, $\operatorname{depth}_S(\mathfrak{a}, M \otimes_S N) \leq k+l$, as desired. □

By Theorem 2.9, we have the following standard formula of $\mathfrak{a}$-depth of modules:

Corollary 2.10. ([1, Proposition 1.2.16 (a)]) *Let* $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$ *be a homomorphism of local rings* $M \in S$-**mod**, $N \in T$-**mod**, *with* N *being faithfully flat as an* S*-module. Then,*

$$\operatorname{depth}_T(M \otimes_S N) = \operatorname{depth}_S M + \operatorname{depth}_T(N/\mathfrak{m}N).$$

3. $\mathfrak{a}$-CM-modules

In this section, the behavior of $\mathfrak{a}$-CM-modules over homomorphism of local rings is discovered.

Definition 3.1. ([10]) Let $M \in S$-**mod**. We say that M is an $\mathfrak{a}$-*Cohen-Macaulay module* ($\mathfrak{a}$-CM-module) provided that

$$\operatorname{depth}_S(\mathfrak{a}, M) + \dim_S(M/\mathfrak{a}M) = \dim_S M.$$

The ring S is said to be $\mathfrak{a}$-CM if it is an $\mathfrak{a}$-CM-module.

Remark 3.2. (1) For any $M \in S$-**mod**, one has

$$\begin{aligned} \dim_S M &= \sup\{\dim_S S/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_S M\} \\ &= \sup\{\dim_S S/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_S(M/\mathfrak{a}M) \cup \operatorname{Supp}_S(\mathfrak{a}M)\} \\ &= \max\{\dim_S(M/\mathfrak{a}M), \dim_S(\mathfrak{a}M)\}. \end{aligned}$$

(i) $\dim_S(M/\mathfrak{a}M) \geq \dim_S(\mathfrak{a}M)$ whenever M is $\mathfrak{a}$-CM and $\operatorname{depth}_S(\mathfrak{a}, M) = 0$.

(ii) $\dim_S(M/\mathfrak{a}M) < \dim_S(\mathfrak{a}M)$ whenever M is $\mathfrak{a}$-CM and $\operatorname{depth}_S(\mathfrak{a}, M) > 0$.

(2) Suppose that $(S, \mathfrak{m})$ is a local ring, and $M \in S$-**mod**.

(iii) If $\mathfrak{a}$ is generated by an M-regular sequence, then M is $\mathfrak{a}$-CM by [1, Theorem 2.1.2(c)].

(iv) $\mathfrak{m}$-CM-module is exactly the CM-module (see [1, Chapter 2]). In particular, if M is a CM-module, then M is an $\mathfrak{a}$-CM-module by [1, Theorem 2.1.2(b)].

Definition 3.3. ([16]) Let $\mathfrak{p} \in \operatorname{Spec}(S)$. We say that $\mathfrak{a}$ is *primary ideal* if for $ab \in S$ with $a \notin S$, there exists $n \geq 1$ such that $b^n \in \mathfrak{a}$. In particular, if $\mathfrak{a}$ is a primary ideal with $\mathfrak{p} = \sqrt{\mathfrak{a}}$, then $\mathfrak{a}$ is called $\mathfrak{p}$-primary.

The following result provides a link between $\mathfrak{a}$-CM-modules and $\mathfrak{p}$-CM-modules by primary ideals.

Proposition 3.4. *Let $M \in S$-**mod** and $\mathfrak{a}$ a $\mathfrak{p}$-primary ideal of S with $\mathfrak{p} \in \operatorname{Supp}_S M$. Then, M is $\mathfrak{a}$-CM if and only if it is $\mathfrak{p}$-CM.*

Proof. By [17, Proposition 4.1], $\mathfrak{p}$ is the smallest prime ideal containing $\mathfrak{a}$. So,

$$\mathfrak{V}(\mathfrak{a}) = \mathfrak{V}(\mathfrak{p}), \quad \text{and} \quad \operatorname{Supp}_S(M/\mathfrak{a}M) = \operatorname{Supp}_S(M/\mathfrak{p}M).$$

Thus, $\dim_S(M/\mathfrak{a}M) = \dim_S(M/\mathfrak{p}M)$, and $\operatorname{depth}_S(\mathfrak{a}, M) = \operatorname{depth}_S(\mathfrak{p}, M)$ by [1, Proposition 1.2.10(b)]. The result then follows. □

The next result is inspired by an analogous statement for CM-modules (see [1, Theorem 2.1.2]), which is a more general version of ([10, Proposition 2.9, and Theorem 2.11]).

Proposition 3.5. *Let $\mathfrak{a}$ be a $\mathfrak{p}$-primary ideal of S and M be an $\mathfrak{a}$-CM-module with $\mathfrak{p} \in \operatorname{Supp}_S M$. Then,*

(1) $\operatorname{depth}_S(\mathfrak{a}, M) = \dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p}$ *for all* $\mathfrak{q} \in \operatorname{Ass}_S M \cap \mathfrak{U}(\mathfrak{p})$.

(2) $\dim_S M = \dim_S S/\mathfrak{q}$ *for all* $\mathfrak{q} \in \operatorname{Ass}_S M \cap \mathfrak{U}(\mathfrak{p})$.

(3) $\operatorname{depth}_S(\mathfrak{b}, M) = \dim_S M - \dim_S(M/\mathfrak{b}M)$ *for all ideals* $\mathfrak{b} \subseteq \mathfrak{a}$.

Proof. (1) By Proposition 2.7, $\operatorname{depth}_S(\mathfrak{a}, M) \leq \dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p}$ for all $\mathfrak{q} \in \operatorname{Ass}_S M \cap \mathfrak{U}(\mathfrak{p})$. Note that $\mathfrak{p} \in \operatorname{Supp}_S M \cap \mathfrak{V}(\mathfrak{a}) = \operatorname{Supp}_S(M/\mathfrak{a}M) = \operatorname{Supp}_S(M/\mathfrak{p}M)$. Therefore, $\dim_S(M/\mathfrak{a}M) = \dim_S S/\mathfrak{p}$. By $\operatorname{Ass}_S M \subseteq \operatorname{Supp}_S M$, $\dim_S S/\mathfrak{q} - \dim_S S/\mathfrak{p} \leq \dim_S S/\mathfrak{q} - \dim_S(M/\mathfrak{a}M) \leq \dim_S M - \dim_S(M/\mathfrak{a}M) = \operatorname{depth}_S(\mathfrak{a}, M)$. This shows the desired equality.

(2) By (1), $\dim_S(M/\mathfrak{a}M) = \dim_S S/\mathfrak{p}$. Because M is an $\mathfrak{a}$-CM-module, $\dim_S M = \dim_S S/\mathfrak{q}$.

(3) There are the following two cases: the first one, if $\operatorname{depth}_S(\mathfrak{b}, M) = 0$, then $\operatorname{Hom}_S(S/\mathfrak{b}, M) \neq 0$. Thus, $\emptyset \neq \operatorname{Ass}_S \operatorname{Hom}_S(R/\mathfrak{b}, M) \subseteq \operatorname{Supp}_S M \cap \mathfrak{V}(\mathfrak{b})$. Because $\mathfrak{p} \in \operatorname{Supp}_S M \cap \mathfrak{V}(\mathfrak{b})$, there is $\mathfrak{q} \in \operatorname{Ass}_S \operatorname{Hom}_S(S/\mathfrak{b}, M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Hence, $\mathfrak{q} \in \operatorname{Ass}_S M \cap \mathfrak{U}(\mathfrak{p})$ with $\mathfrak{b} \subseteq \mathfrak{q}$. It follows from (2) that $\dim_S(M/\mathfrak{b}M) \leq \dim_S M = \dim_S S/\mathfrak{q} \leq \dim_S(M/\mathfrak{b}M)$. The second one: if $\operatorname{depth}_S(\mathfrak{b}, M) > 0$, we choose $x \in \mathfrak{b}$ regular on M. Then, $\dim_S(M/xM) = \dim_S M - 1$, and $\operatorname{depth}_S(\mathfrak{b}, M/xM) = \operatorname{depth}_S(\mathfrak{b}, M) - 1$, so the result follows from induction. □

Let $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$ be a homomorphism of local rings, $M \in S$-**mod**, $N \in T$-**mod**, with N being flat as an S-module. A classical result by Bruns and Herzog [1, Theorem 2.1.7] tells us that $M \otimes_S N$ is a CM-module over S if and only if M is a CM-module over S, and the quotient $N/\mathfrak{m}N$ is a CM-module over T. Our main theorem below will investigate the behavior of $\mathfrak{a}$-Cohen-Macaulayness, which improves Mahmood and Azam [10, Theorem 2.7].

Theorem 3.6. *Let $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$ be a homomorphism of local rings, $M \in S$-**mod**, $N \in T$-**mod**, with N being faithfully flat as an S-module. Then, $M \otimes_S N$ is $\mathfrak{a}S$-CM over T if and only if M is $\mathfrak{a}$-CM over S, and $N/\mathfrak{m}N$ is $\mathfrak{a}S$-CM over T.*

Proof. Let $\boldsymbol{y} = y_1, \ldots, y_k$ be a maximal $N/\mathfrak{m}N$-regular sequence in $\mathfrak{a}T$. Because $(\boldsymbol{y}) \subseteq \mathfrak{a}T$,

$$\begin{aligned} T/\mathfrak{a}T &\cong (T/\mathfrak{a}T)/\boldsymbol{y}(T/\mathfrak{a}T) \\ &\cong (T/\mathfrak{a}T) \otimes_T (T/(\boldsymbol{y})). \end{aligned}$$

By Lemma 2.8(2), $N/\boldsymbol{y}N$ is flat over S. Hence, [1, Theorem A.11] implies that

$$\begin{aligned} \dim_T((M \otimes_S N)/\mathfrak{a}T(M \otimes_S N)) &= \dim_T((T/\mathfrak{a}T) \otimes_T M \otimes_S N) \\ &= \dim_T(((T/\mathfrak{a}T) \otimes_T (T/(\boldsymbol{y})) \otimes_T M \otimes_S N) \\ &= \dim_T(((S/\mathfrak{a}) \otimes_S M) \otimes_S (S/(\boldsymbol{y})) \otimes_T N) \\ &= \dim_T((M/\mathfrak{a}M) \otimes_S (N/\boldsymbol{y}N)) \\ &= \dim_S(M/\mathfrak{a}M) + \dim_T(N'/\mathfrak{m}N'), \end{aligned}$$

where $N' = N/\boldsymbol{y}N$. Also,

$$\dim_T(M \otimes_S N) = \dim_S M + \dim_T(N/\mathfrak{m}N),$$

and

$$\dim_T(M/\mathfrak{a}M \otimes_{S/\mathfrak{a}} N/\mathfrak{a}N) = \dim_S M/\mathfrak{a}M + \dim_T(N/\mathfrak{a}N)/\mathfrak{m}(N/\mathfrak{a}N).$$

"If" part. Because $\boldsymbol{y}$ is a maximal $(N/\mathfrak{m}N)$-regular sequence in $\mathfrak{a}T$, and $N/\mathfrak{m}N$ is an $\mathfrak{a}T$-CM-module, one has

$$\begin{aligned} \dim_T(N'/\mathfrak{m}N') &= \dim_T(N/\mathfrak{m}N)/\boldsymbol{y}(N/\mathfrak{m}N)) \\ &= \dim_T N/\mathfrak{m}N - k \\ &= \dim_T N/\mathfrak{m}N - \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N) \\ &= \dim_T(N/\mathfrak{m}N)/\mathfrak{a}T(N/\mathfrak{m}N)). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\dim_T(M \otimes_S N) - \operatorname{depth}_T(\mathfrak{a}T, M \otimes_S N) \\ &\overset{(1)}{=} \dim_S M + \dim_T N/\mathfrak{m}N - (\operatorname{depth}_S(\mathfrak{a}, M) + \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N)) \\ &\overset{(2)}{=} (\dim_S M - \operatorname{depth}_S(\mathfrak{a}, M)) + (\dim_T N/\mathfrak{m}N - \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N)) \\ &\overset{(3)}{=} \dim_S M/\mathfrak{a}M + \dim_T(N/\mathfrak{m}N)/\mathfrak{a}T(N/\mathfrak{m}N) \\ &\overset{(4)}{=} \dim_S M/\mathfrak{a}M + \dim_T N'/\mathfrak{m}N' \\ &\overset{(5)}{=} \dim_T(M \otimes_R N)/\mathfrak{a}T(M \otimes_R N). \end{aligned}$$

In the sequence, (1) is by the above equalities and Theorem 2.9, (2) is obvious, (3) is by assumption, and (4) and (5) are by the above equalities. Thus, $M \otimes_S N$ is $\mathfrak{a}T$-CM.

"Only If" part. By assumption, we have

$$
\begin{aligned}
&\dim_T(M \otimes_S N) - \dim_T((M \otimes_S N)/\mathfrak{a}T(M \otimes_S N))\\
&\overset{(6)}{=} \operatorname{depth}_T(\mathfrak{a}T, M \otimes_S N)\\
&\overset{(7)}{=} \operatorname{depth}_S(\mathfrak{a}, M) + \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N)\\
&\overset{(8)}{\leq} \dim_S M - \dim_S M/\mathfrak{a}M + \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N)\\
&\overset{(9)}{\leq} \dim_S M - \dim_S M/\mathfrak{a}M + (\dim_T N/\mathfrak{m}N - \dim_T(N/\mathfrak{m}N)/\mathfrak{a}T(N/\mathfrak{m}N))\\
&\overset{(10)}{=} \dim_S M + \dim_T N/\mathfrak{m}N - (\dim_S M/\mathfrak{a}M + \dim_T(N/\mathfrak{m}N)/\mathfrak{a}T(N/\mathfrak{m}N))\\
&\overset{(11)}{=} \dim_S M + \dim_T N/\mathfrak{m}N - (\dim_S M/\mathfrak{a}M + \dim_T(N/\mathfrak{a}N)/\mathfrak{m}(N/\mathfrak{a}N))\\
&\overset{(12)}{=} \dim_T(M \otimes_S N) - \dim_T(M/\mathfrak{a}M \otimes_{S/\mathfrak{a}} N/\mathfrak{a}N)\\
&\overset{(13)}{=} \dim_T(M \otimes_S N) - \dim_T(M \otimes_S N)/\mathfrak{a}T(M \otimes_S N)\\
&\overset{(14)}{\leq} \dim_S M - \dim_T(M/\mathfrak{a}M) + \operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N).
\end{aligned}
$$

Here, (6) follows by assumption, (7) is by Theorem 2.9, (8) and (9) are by Proposition 2.6, (10) is ovbious, (11) follows by $(N/\mathfrak{m}N)/\mathfrak{a}T(N/\mathfrak{m}N) \cong (N/\mathfrak{a}N)/\mathfrak{m}(N/\mathfrak{a}N)$, (12) is by the above equalities, (13) is by $M/\mathfrak{a}M \otimes_{S/\mathfrak{a}} N/\mathfrak{a}N \cong (M \otimes_S N)/\mathfrak{a}T(M \otimes_S N)$, and (14) is by (6)–(8). It follows that

$$\operatorname{depth}_T(\mathfrak{a}T, N/\mathfrak{m}N) = \dim_T(N/\mathfrak{m}N) - \dim_T((N/\mathfrak{m}N)/\mathfrak{a}T(N/\mathfrak{m}N)),$$

and $\operatorname{depth}_S(\mathfrak{a}, M) = \dim_S M - \dim_T(M/\mathfrak{a}M)$. Thus, M is $\mathfrak{a}$-CM over S, and $N/\mathfrak{m}N$ is $\mathfrak{a}S$-CM over T. □

As a consequence of Theorem 3.6, we have the following:

Corollary 3.7. *Let $f : (S, \mathfrak{m}) \to (T, \mathfrak{n})$ be a faithfully flat homomorphism of local rings. Then, T is an $\mathfrak{a}T$-CM-ring if and only if S is an $\mathfrak{a}$-CM-ring, and $T/\mathfrak{m}T$ is an $\mathfrak{a}S$-CM-ring.*

The $\mathfrak{a}$-Cohen-Macaulayness is stable under specialization.

Proposition 3.8. *Let $(S, \mathfrak{m})$ be a local ring and $M \in S$-**mod**. Assume that $\boldsymbol{w} = w_1, \ldots, w_k$ is an M-regular sequence in $\mathfrak{a}$. Then, M is an $\mathfrak{a}$-CM-module over S if and only if $M/\boldsymbol{w}M$ is an $\mathfrak{a}/(\boldsymbol{w})$-CM-module over $S/(\boldsymbol{w})$.*

Proof. Let $\overline{M} = M/\boldsymbol{w}M$, $\overline{S} = S/(\boldsymbol{w})$, and $\overline{\mathfrak{a}} = \mathfrak{a}/(\boldsymbol{w})$. "Only if" part by [10, Theorem 2.10]. "If" part is by the equalities $\operatorname{depth}_{\overline{S}}(\overline{\mathfrak{a}}, \overline{M}) = \operatorname{depth}_S(\mathfrak{a}, M) - k$, $\dim_{\overline{S}} \overline{M} = \dim_S M - k$, and $\dim_{\overline{S}}(\overline{M}/\overline{\mathfrak{a}}\overline{M}) = \dim_S(M/\mathfrak{a}M)$. □

The following proposition provides a height formula of ideal by the $\mathfrak{p}$-Cohen-Macaulayness.

Proposition 3.9. *Let $(S, \mathfrak{m})$ be a local ring and $\mathfrak{p}$ a prime ideal of S. If S is a $\mathfrak{p}$-CM-ring, then*

$$\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} + \operatorname{ht}\mathfrak{p}/\mathfrak{q}$$

for any prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$.

Proof. By [10, Theorem 2.14], $S_\mathfrak{p}$ is a CM-ring. Therefore, $\operatorname{ht}\mathfrak{p} = \dim S_\mathfrak{p} = \operatorname{ht}\mathfrak{q}S_\mathfrak{p} + \dim(S_\mathfrak{p}/\mathfrak{q}S_\mathfrak{p}) = \operatorname{ht}\mathfrak{q} + \operatorname{ht}(\mathfrak{p}/\mathfrak{q})$, as desired. □

4. Foxby equivalence

In this section, we establish the Foxby equivalance of $\mathfrak{a}$-CM-modules over S.

Definition 4.1. ([18]) Let $D \in S$-**mod**. D is said to be *semidualizing* if the canonical map

$$S \longrightarrow \mathrm{Hom}_S(D, D)$$

is an isomorphism, and

$$\mathrm{Ext}^i_S(D, D) = 0$$

for all $i \geq 1$.

In the remainder of this section, let D be a fixed semidualizing S-module.

Definition 4.2. ([2]) (1) The *Auslander class* $\mathcal{A}_D(S)$ with respect to D consists of all S-modules N such that the canonical map

$$N \longrightarrow \mathrm{Hom}_S(D, D \otimes_S N)$$

is an isomorphism, and

$$\mathrm{Tor}^S_i(D, N) = 0 = \mathrm{Ext}^i_S(D, D \otimes_S N)$$

for any $i \geq 1$.

(2) The *Bass class* $\mathcal{B}_D(S)$ with respect to D consists of all S-modules N such that the canonical map

$$D \otimes_S \mathrm{Hom}_D(D, N) \longrightarrow N$$

is an isomorphism, and

$$\mathrm{Ext}^i_S(D, N) = 0 = \mathrm{Tor}^S_i(D, \mathrm{Hom}_S(D, N))$$

for any $i \geq 1$.

The following lemma gives some characterizations of associated primes and support of modules with respect to semidualizing module, which was already proved in [3, Lemma 3.1] under the condition that $(S, \mathfrak{m})$ is a local ring.

Lemma 4.3. *Let* $M \in S$-**mod**. *Then,*

(1) $\mathrm{Supp}_S D = \mathrm{Spec}(S)$.

(2) $\mathrm{Supp}_S(D \otimes_S M) = \mathrm{Supp}_S M = \mathrm{Supp}_S \mathrm{Hom}_S(D, M)$. *Particularly,* $\dim_S(D \otimes_S M) = \dim_S M = \dim_S \mathrm{Hom}_S(D, M)$.

(3) $C \otimes_S M \neq 0$ *if and only if* $M \neq 0$ *if and only if* $\mathrm{Hom}_S(D, M) \neq 0$.

Proof. (1) follows from [19, Lemma 2.5], and (3) holds by (1).

(2) By (1), we have

$$\mathrm{Supp}_S(D \otimes_S M) = \mathrm{Supp}_S D \cap \mathrm{Supp}_S M = \mathrm{Supp}_S M.$$

By (1) and [20, Proposition 10], one obtains that

$$\mathrm{Ass}_S \mathrm{Hom}_S(D, M) = \mathrm{Supp}_S D \cap \mathrm{Ass}_S M = \mathrm{Ass}_S M.$$

Thus, $\mathrm{Ann}_S M = \mathrm{Ann}_S \mathrm{Hom}_S(D, M)$ and $\mathrm{Supp}_S M = \mathrm{Supp}_S \mathrm{Hom}_S(D, M)$. □

The next result provides a dimension formula for S-modules with respect to D.

Lemma 4.4. *Let $M \in S$-**mod**. Then,*

$$\dim_S (D \otimes_S M)/\mathfrak{a}(D \otimes_S M) = \dim_S M/\mathfrak{a}M = \dim_S \mathrm{Hom}_S(D, M)/\mathfrak{a}(\mathrm{Hom}_S(D, M)).$$

Proof. (1) It follows from Lemma 4.3 that $\mathrm{Supp}_S(D \otimes_S M) = \mathrm{Supp}_S M = \mathrm{Supp}_S \mathrm{Hom}_S(D, M)$. Thus, $\mathrm{Supp}_S(D \otimes_S M) \cap \mathfrak{V}(\mathfrak{a}) = \mathrm{Supp}_S M \cap \mathfrak{V}(\mathfrak{a}) = \mathrm{Supp}_S \mathrm{Hom}_S(D, M) \cap \mathfrak{V}(\mathfrak{a})$; that is to say, $\mathrm{Supp}_S(D \otimes_S M)/\mathfrak{a}(D \otimes_S M) = \mathrm{Supp}_S M/\mathfrak{a}M = \mathrm{Supp}_S \mathrm{Hom}_S(D, M)/\mathfrak{a}(\mathrm{Hom}_S(D, M))$. Hence, $\dim_S(D \otimes_S M)/\mathfrak{a}(D \otimes_S M) = \dim_S M/\mathfrak{a}M = \dim_S \mathrm{Hom}_S(D, M)/\mathfrak{a}(\mathrm{Hom}_S(D, M))$. □

The following lemma was proved in [3, Lemma 3.2] under the condition that $(S, \mathfrak{m})$ is a local ring.

Lemma 4.5. *Let $M, N \in S$-**mod**. If $\boldsymbol{w} = w_1, \ldots, w_k \in \mathfrak{a}$, $M \in \mathcal{A}_D(S), N \in \mathcal{B}_D(S)$, then*

(1) $\mathrm{Ass}_S M = \mathrm{Ass}_S(D \otimes_S M)$.

(2) *$\boldsymbol{w}$ is an M-regular sequence if and only if it is a $D \otimes_S M$-regular sequence. Particularly,* $\mathrm{depth}_S(\mathfrak{a}, M) = \mathrm{depth}_S(\mathfrak{a}, D \otimes_S M)$.

(3) $\mathrm{Ass}_S N = \mathrm{Ass}_S \mathrm{Hom}_S(D, N)$.

(4) *$\boldsymbol{w}$ is an N-regular sequence if and only if it is a* $\mathrm{Hom}_S(D, N)$*-regular sequence. Particularly,* $\mathrm{depth}_S(\mathfrak{a}, N) = \mathrm{depth}_S(\mathfrak{a}, \mathrm{Hom}_S(D, N))$.

Proof. (1) One has

$$\begin{aligned} \mathrm{Ass}_S M &= \mathrm{Ass}_S \mathrm{Hom}_S(D, D \otimes_S M) \\ &= \mathrm{Supp}_S D \cap \mathrm{Ass}_S(D \otimes_S M) \\ &= \mathrm{Ass}_S(D \otimes_S M), \end{aligned}$$

where the first is by $M \in \mathcal{A}_D(S)$, the second is by [20, Proposition 10], the third is by Lemma 4.3. Thus, $\mathrm{Ann}_S M = \mathrm{Ann}_S(D \otimes_S M)$.

(2) If $M/\mathfrak{a}M = 0$, then $\mathrm{depth}_S(\mathfrak{a}, M) = \infty$. By Lemma 4.3, $D \otimes_S M/\mathfrak{a}(D \otimes_S M) = 0$, and $\mathrm{depth}_S(\mathfrak{a}, D \otimes_S M) = \infty$. If $M/\mathfrak{a}M \neq 0$, then $M/wM \neq 0$. By Lemma 4.3, $D \otimes_S (M/wM) \cong (D \otimes_S M)/w(D \otimes_S M) \neq 0$. So, we need enough to prove that $\boldsymbol{w}$ is weak M-regular sequence if and only if $\boldsymbol{w}$ is a weak $D \otimes_S M$-regular sequence. We proceed by induction on k. If $k = 1$, then

$$\begin{aligned} w_1 \text{ is a weak M-regular element} &\Longleftrightarrow w_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_S M} \mathfrak{p} \\ &\Longleftrightarrow w_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_S(D \otimes_S M)} \mathfrak{p} \\ &\Longleftrightarrow w_1 \text{ is a weak } D \otimes_S M\text{-regular element}, \end{aligned}$$

where the second is by (1).

Next, we assume that $k > 1$, and the result holds for $k - 1$. Suppose that w_1 is a weak M-regular element and it is also a weak $D \otimes_S M$-regular element. Set $\overline{M} := M/w_1 M$. Considering the following exact sequence,

$$0 \longrightarrow M \xrightarrow{w_1} M \longrightarrow \overline{M} \longrightarrow 0,$$

it follows from [18, Proposition 3.1.7 (a)] that $\overline{M} \in \mathcal{A}_D(S)$. Using the induction hypothesis, $w_2, \dots, w_k$ is a weak M-regular sequence if and only if it is a weak $D \otimes_S M$-regular sequence. Note that

$$D \otimes_S \overline{M} \cong (D \otimes_S M)/\boldsymbol{w}(D \otimes_S M) \neq 0.$$

Thus, $w_1, w_2, \dots, w_k \in \mathfrak{a}$ is weak M-regular sequence iff it is a weak $D \otimes_S M$-regular sequence.

From $N \in \mathcal{B}_D(S)$, we know that $N \cong D \otimes_S \mathrm{Hom}_S(D, N)$, and $\mathrm{Hom}_S(D, N) \in \mathcal{A}_D(S)$. So, one can prove (3) and (4) by (1) and (2). □

Now, we establish Foxby equivalance of $\mathfrak{a}$-CM-modules, which is our main theorem in this section.

Theorem 4.6. *Let $M, N \in S$-**mod**. If $M \in \mathcal{A}_D(S), N \in \mathcal{B}_D(S)$, then*

(1) *M is $\mathfrak{a}$-CM if and only if $D \otimes_S M$ is $\mathfrak{a}$-CM.*
(2) *N is $\mathfrak{a}$-CM if and only if $\mathrm{Hom}_S(D, N)$ is $\mathfrak{a}$-CM.*

Proof. (1) In view of Lemmas 4.3–4.5, one obtains that

$$\begin{aligned} & M \text{ is } \mathfrak{a}\text{-CM} \\ \Longleftrightarrow\ & \mathrm{depth}_S(\mathfrak{a}, M) + \dim_R M/\mathfrak{a}M = \dim_R M \\ \Longleftrightarrow\ & \mathrm{depth}_S(\mathfrak{a}, D \otimes_S M) + \dim_S (D \otimes_S M)/\mathfrak{a}(D \otimes_S M) = \dim_S (D \otimes_S M) \\ \Longleftrightarrow\ & D \otimes_S M \text{ is } \mathfrak{a}\text{-CM}. \end{aligned}$$

(2) From Lemmas 4.3–4.5, we see that

$$\begin{aligned} & N \text{ is } \mathfrak{a}\text{-CM} \\ \Longleftrightarrow\ & \mathrm{depth}_S(\mathfrak{a}, N) + \dim_S N/\mathfrak{a}N = \dim_S N \\ \Longleftrightarrow\ & \mathrm{depth}_S(\mathfrak{a}, \mathrm{Hom}_S(D, N)) + \dim_S \mathrm{Hom}_S(D, N)/\mathfrak{a}(\mathrm{Hom}_S(D, N)) = \dim_S \mathrm{Hom}_S(D, N) \\ \Longleftrightarrow\ & \mathrm{Hom}_S(D, N) \text{ is } \mathfrak{a}\text{-CM}. \end{aligned}$$

□

Corollary 4.7. *There exists an equivalence of categories:*

$$\mathfrak{a}\text{–}\mathcal{CM}(S) \cap \mathcal{A}_D(S) \underset{\mathrm{Hom}_S(D,-)}{\overset{D \otimes_S -}{\rightleftarrows}} \mathfrak{a}\text{–}\mathcal{CM}(S) \cap \mathcal{B}_D(S),$$

where $\mathfrak{a}\text{–}\mathcal{CM}(S)$ stands for the class of all $\mathfrak{a}$-CM-modules over S.

Remark 4.8. Let $(S, \mathfrak{m})$ be a local ring, and take $\mathfrak{a} = \mathfrak{m}$. Then, the equivalence in Corollary 4.7 is just the equivalence established in [3, Corollary 4.10].

We end the paper by an example to give an $\mathfrak{a}$-CM ring in which the ideal $\mathfrak{a}$ is not maximal.

Example 4.9. Consider the polynomial ring $T = \mathcal{K}[x_1, x_2, x_3, y_1, y_2, y_3]$ in which $\mathcal{K}$ is a field. Let $\mathfrak{b} = \langle x_1, x_2, x_3 \rangle \cap \langle y_1, y_2, y_3 \rangle$ (resp. $\mathfrak{a} = \langle x_1, x_2, x_3 \rangle S$) be the ideal of T (resp. S), where $S = T/\mathfrak{b}$. It follows from [10, Example 2.3(2)] that $\dim_S S/\mathfrak{a} = 3$. Then, $\mathfrak{a}$ is not a maximal ideal of S by the definition of Krull dimension. However, S is an $\mathfrak{a}$-CM ring (see [10, Example 2.3(2)]).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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