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Research article

Discrete cosine and sine functions of matrices

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Abstract: In this paper, we introduce two methods for constructing matrix-valued cosine and sine functions defined on a discrete domain. The first method develops an algorithm to compute the $n \times n$ matrix-valued cosine and sine of a given square matrix A with some restrictions on its eigenvalues. The second method derives a formula based on the Jordan canonical form to compute each term of a Jordan block in the discrete matrix-valued cosine and sine of a given square matrix A. To illustrate the utility of our approach, we present some examples.

Keywords: matrix valued functions; Putzer algorithm; discrete cosine function; Jordan canonical form; difference equations

1. Introduction

For a certain class of functions, the formula for a function of an $n \times n$ matrix A, namely $f(A)$, has been developed by calculating $f(J_i)$ using a Taylor series of the function f centered at λ_i,

$$f(J_i) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\lambda_i)}{k!}(J_i - \lambda_i I)^k,$$

where λ_i is an eigenvalue of A, and J_i is a Jordan block. Because $J_i - \lambda_i I$ is a nilpotent matrix, it follows that

$$f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2} & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & 0 & f(\lambda) \end{bmatrix},$$

where J is the Jordan canonical form of A.

Then the natural question arises: Does this nilpotent matrix help to calculate $f(A)$ if one considers the nabla Taylor's formula (Theorem 3.48 in [1])

$$\sum_{k=0}^{\infty} \frac{\nabla^k f(\lambda_i)}{k!}(J_i - \lambda_i I)^{\overline{k}}$$

for a discrete function f?

In this article, we shall explore this question for a class of discrete cosine and sine functions. Using the nabla Taylor's formula, one can obtain the following series representations of the discrete cosine and sine functions:

$$Cos_p(t,a) = \sum_{n=0}^{\infty}(-1)^n p^{2n}\frac{(t-a)^{\overline{2n}}}{(2n)!}, \qquad Sin_p(t,a) = \sum_{n=0}^{\infty}(-1)^n p^{2n+1}\frac{(t-a)^{\overline{2n+1}}}{(2n+1)!},$$

where t and a belong to the set of whole numbers, and $|p| < 1$ is a constant. The extensive study on these functions can be found in a recent book by C. Goodrich and A. C. Peterson [1]. In the infinite series, $t^{\overline{n}}$ is known as a rising factorial for a natural number n, and it is defined as $t^{\overline{n}} = t(t+1)(t+2)\dots(t+n-1)$.

In these trigonometric functions of discrete calculus, we replace the constant p or the variable t by an $n \times n$ matrix. Then, we have two forms of matrix discrete functions, $Cos_A(t,a)$ and $Cos_p(A,a)$, where A is an $n \times n$ matrix. Motivated by the continuous analogs studied in the literature, this paper explores discrete matrix-valued cosine and sine functions and provides a direct method for computing them as $n \times n$ matrices without relying on approximation techniques.

Research regarding matrix valued cosine and sine functions in continuous time emerged after foundational developments in the calculations of matrix valued polynomials via power series expansions, such as the work by Everling and Liou in the late 1960s [2]. Over subsequent decades, algorithmic strategies for computing the matrix cosine and sine have split into two categories: recurrence-based methods using double-angle identities [3, 4] and approximation techniques employing Taylor series and/or Padé approximants [5–7]. A third approach, the spectral method utilizing Bernstein polynomials, has also been explored [8]. Recent advancements in 2024 by Qin and Lu [9] have further expanded the field with a general algorithm that works for sine, cosine, and exponential functions along with a tunable precision. Since then, most contributions to the field have focused on optimizing the algorithm to improve efficiency and reduce computational complexity. On the other hand, there has not been any work in the literature for calculating matrix-valued discrete cosine and sine functions.

We structure the paper as follows: In Section 2, we give basic definitions, some theorems, and notations in discrete calculus. In addition, we discuss the convergency of the matrix power series. Section 3 continues with an algorithm such as Putzer's algorithm in the second-order. This algorithm enables one to calculate $Cos_A(t,a)$, where A is an $n \times n$ matrix with real and positive eigenvalues less than one, by solving second-order nabla difference equations. In Section 4, we utilize a formula for discrete matrix-valued functions using Jordan canonical form to calculate each term of a Jordan block of an $n \times n$ matrix-valued discrete cosine or sine of a matrix A with integer eigenvalues. Both methods depend on the calculation of the eigenvalues of the matrix A. To illustrate the effectiveness of our approaches, we provide some illustrative examples.

For a comprehensive source in the area of functions of matrices in continuous time, we direct the reader to a book by N. J. Higham [10].

2. Preliminaries

This section provides fundamental definitions in discrete calculus to ensure the reader can easily follow the notation in the subsequent sections.
Let a be a real number. We define

$$\mathbb{N}_a = \{a, a+1, a+2, \dots\}.$$

Let $f : \mathbb{N}_a \to \mathbb{R}$. The nabla derivative of the function f is defined by

$$\nabla f(t) = f(t) - f(t-1),$$

for $t \in \mathbb{N}_{a+1}$.

Definition 2.1. [1] Let p be a real number. We define the nabla sine and cosine functions as follows:

$$Cos_p(t,a) = \frac{((1-ip)^{t-a})^{-1} + ((1+ip)^{t-a})^{-1}}{2}, \quad Sin_p(t,a) = \frac{((1-ip)^{t-a})^{-1} - ((1+ip)^{t-a})^{-1}}{2i},$$

for $t \in \mathbb{N}_a$.

Hence, one can easily verify that

$$\nabla Cos_p(t,a) = -pSin_p(t,a) \text{ and } \nabla Sin_p(t,a) = pCos_p(t,a).$$

The following nabla Taylor's formula in discrete calculus and its proof can be found in [1].

Theorem 2.1. Assume $f : \mathbb{N}_{a-n} \to \mathbb{R}$ where $n \in \mathbb{N}_0$. Then,

$$f(t) = p_n(t) + R_n(t), \qquad t \in \mathbb{N}_{a-n},$$

where the nth degree nabla Taylor polynomial, $p_n(t)$ is given by

$$p_n(t) = \sum_{k=0}^{n} \nabla^k f(a) \frac{(t-a)^{\overline{k}}}{k!},$$

and the Taylor remainder, $R_n(t)$, is given by

$$R_n(t) = \sum_{s=a+1}^{n} \frac{(t-s+1)^{\overline{n}}}{n!} \nabla^{n+1} f(s)$$

for $t \in \mathbb{N}_{a-n}$.

Definition 2.2. For $k \in \mathbb{N}$, the rising factorial of an $n \times n$ matrix A is denoted by

$$A^{\overline{k}} = A(A+I_n)(A+2I_n)\cdots(A+(k-1)I_n).$$

Next, we discuss the convergency of the series

$$\sum_{n=0}^{\infty} (-1)^n p^{2n} \frac{(t-a)^{\overline{2n}}}{(2n)!} \tag{2.1}$$

and its corresponding matrix forms

$$\sum_{n=0}^{\infty}(-1)^n A^{2n}\frac{(t-a)^{\overline{2n}}}{(2n)!} \qquad \text{and} \qquad \sum_{n=0}^{\infty}(-1)^n p^{2n}\frac{(A-aI)^{\overline{2n}}}{(2n)!}.$$

The series in (2.1) is convergent for $t \in \mathbb{N}_a$ if $|p| < 1$. Let us define

$$f_p(t) = \sum_{n=0}^{\infty}(-1)^n p^{2n}\frac{(t-a)^{\overline{2n}}}{(2n)!},$$

for $t \in \mathbb{N}_a$, where $f_p : \mathbb{N}_a \to \mathbb{R}$, and $|p| < 1$.

Next, we define the function F_p on the vector space $M_{k\times k}$, which is the set of $k \times k$ matrices

$$F_p(A) = \sum_{n=0}^{\infty}(-1)^n p^{2n}\frac{(A-aI)^{\overline{2n}}}{(2n)!}, \tag{2.2}$$

for $A \in M_{k\times k}$, where $F_p : M_{k\times k} \to M_{k\times k}$, and $|p| < 1$. Let $\|\cdot\|$ be the Euclidian norm over $\mathbb{R}^k$ and $|||\cdot|||$ be a norm over the set $M_{k\times k}$ such that

$$|||A||| = max_{\|x\|=1}\|Ax\|.$$

Because $(M_{k\times k}, |||\cdot|||)$ is a complete metric space over the field $\mathbb{R}$, it is sufficient to show that the sequence

$$\left\{\sum_{i=0}^{n}(-1)^i p^{2i}\frac{(A-aI)^{\overline{2i}}}{(2i)!}\right\}_{n=0}^{\infty} \tag{2.3}$$

is a Cauchy sequence in $M_{k\times k}$. Indeed, for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\left|\left|\sum_{i=n}^{m}(-1)^i p^{2i}\frac{(A-aI)^{\overline{2i}}}{(2i)!}\right|\right|\right| \leq \sum_{i=n}^{m}(-1)^i|p|^{2i}\frac{|||(A-aI)|||^{\overline{2i}}}{(2i)!} < \epsilon$$

for $m > n \geq N$, where we used the compatible property of the norm $|||\cdot|||$. Hence, the sequence (2.3) is convergent.

Next, we replace p with a $k \times k$ matrix A in (2.1) so that we have

$$\sum_{n=0}^{\infty}(-1)^n A^{2n}\frac{(t-a)^{\overline{2n}}}{(2n)!}.$$

By use of the following theorem, the series is convergent if the spectral radius of a matrix A, $\rho(A)$, is less than 1.

Theorem 2.2. [11] Let R be the radius of convergence of a scalar power series

$$\sum_{k=0}^{\infty} a_k x^k,$$

and let $A \in M_n$ be given with $\|A\| < R$. Then, the matrix power series

$$\sum_{k=0}^{\infty} a_k A^k$$

converges if $\rho(A) < R$.

For simplicity in calculations and notation, we set $a = 0$, allowing us to use $Cos_p(t)$ as a streamlined representation of $Cos_p(t, 0)$ in the following sections.

3. An algorithm to calculate $Cos_A(t)$

In this section, we develop an algorithm to derive a matrix-valued cosine function in a discrete domain. This algorithm can be considered as an extended version of the Putzer algorithm in the second order.

In [12], Putzer studied a linear first-order differential equation with a constant matrix A and discussed the calculation of e^{At}. In the following years, the algorithm named after Putzer has been used and developed further in many papers and the references therein [13, 14].

Note that for a given $n\times n$ matrix M with real and positive eigenvalues, there is a unique $n\times n$ matrix B with a real and positive spectrum such that $B^2 = M$, (see [15]). In this section, we assume that the matrix A has real and positive eigenvalues.

Definition 3.1. (Matrix-valued discrete cosine function) Let A^2 be an $n \times n$ constant matrix with a real eigenvalues less than one. The unique matrix valued solution of the initial value problem (IVP)

$$\nabla^2 Y(t) = -A^2 Y(t) \quad \text{for} \quad t \in \mathbb{N}_0 \tag{3.1}$$

$$Y(0) = I_n, \quad \nabla Y(0) = 0_n, \tag{3.2}$$

where I_n and 0_n denote the $n \times n$ identity matrix and the $n \times n$ zero matrix, respectively, is called the matrix valued discrete cosine function, $Cos_A(t)$.

Theorem 3.1. If $\lambda_1, \lambda_2, ..., \lambda_n$ are (not necessarily distinct) eigenvalues of the $n\times n$ matrix A^2, with each eigenvalue repeated as many times as its multiplicity, and $|\lambda_i| < 1$ for all $1 \leq i \leq n$, then

$$Cos_A(t) = \sum_{i=0}^{n-1} P_{i+1}(t) M_i,$$

where

$$\begin{aligned} M_0 &= I_n, \\ M_i &= (A^2 - \lambda_i I_n) M_{i-1}, \qquad (1 \leq i \leq n-1), \\ M_n &= 0, \end{aligned} \tag{3.3}$$

and the vector-valued function P defined by

$$P(t) = \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \\ \vdots \\ P_n(t) \end{bmatrix}$$

is the solution of the initial value problem

$$\nabla^2 P(t) = -\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} P(t) \qquad \text{for all} \qquad t \in \mathbb{N}_0. \tag{3.4}$$

$$P(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \nabla P(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.5}$$

Proof. Let $\Phi(t, A) = \sum_{i=0}^{n-1} P_{i+1}(t)M_i$. We first show that Φ solves the IVP (3.1) and (3.2). We note that

$$\Phi(0, A) = P_1(0)M_0 + P_2(0)M_1 + \cdots + P_n(0)M_{n-1} = I_n,$$

and

$$\nabla\Phi(0, A) = \nabla P_1(0)M_0 + \nabla P_2(0)M_1 + \cdots + \nabla P_n(0)M_{n-1} = 0_n.$$

Next, we observe that

$$\begin{aligned} \nabla^2\Phi(t, A) + A^2\Phi(t, A) &= \sum_{i=0}^{n-1} \nabla^2 P_{i+1}(t)M_i + A^2 \sum_{i=0}^{n-1} P_{i+1}(t)M_i \\ &= \nabla^2 P_1(t)M_0 + \nabla^2 P_2(t)M_1 + \cdots + \nabla^2 P_n(t)M_{n-1} + A^2 \sum_{i=0}^{n-1} P_{i+1}(t)M_i. \end{aligned}$$

Next, we use (3.4), so the last quantity equals

$$\begin{aligned} &= -\lambda_1 P_1(t)M_0 - [P_1(t) + \lambda_2 P_2(t)]M_1 - [P_2(t) + \lambda_3 P_3(t)]M_2 \\ &\quad - \cdots - [P_{n-1}(t) + \lambda_n P_n(t)]M_{n-1} + A^2 \sum_{i=0}^{n-1} P_{i+1}(t)M_i \\ &= -[\lambda_1 M_0 + M_1 - A^2 M_0]P_1(t) - [\lambda_2 M_1 + M_2 - A^2 M_1]P_2(t) \\ &\quad - \cdots - [\lambda_n M_{n-1} - A^2 M_{n-1}]P_n(t) \\ &= -[\lambda_n I_n - A^2]M_{n-1}P_n(t), \end{aligned}$$

because $M_i = (A^2 - \lambda_i I_n)M_{i-1}$ for $(1 \leq i < n)$. M_n is the zero matrix by the Cayley-Hamilton theorem.

Because $\Phi(t, A)$ satisfies the IVP (3.1) and (3.2), we have

$$\Phi(t, A) = Cos_A(t)$$

by the uniqueness of the solution.

Next, we will give an example to illustrate the use of the algorithm for a 3×3 matrix.

Example 3.1. Let us calculate $Cos_A(t)$ for a given matrix $A = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 1 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}$ with eigenvalues $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = \frac{3}{4}$. Then, we have

$$Cos_A(t) = \sum_{i=0}^{n-1} P_{i+1}(t) M_i$$

with

$$M_0 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_1 = A^2 - \frac{1}{16} I_3 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{16} & \frac{5}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$M_2 = (A^2 - \frac{1}{4} I_3) M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{5}{8} \\ 0 & 0 & \frac{5}{32} \end{bmatrix},$$

$$M_3 = 0.$$

We also set initial value problems

$$\nabla^2 P_1(t) = -\frac{1}{16} P_1(t),$$
$$P_1(0) = 1, \quad \nabla P_1(0) = 0,$$

$$\nabla^2 P_2(t) = -P_1(t) - \frac{1}{4} P_2(t),$$
$$P_2(0) = 0, \quad \nabla P_2(0) = 0,$$

and

$$\nabla^2 P_3(t) = -P_2(t) - \frac{9}{16} P_3(t),$$

$$P_3(0) = 0, \quad \nabla P_3(0) = 0.$$

After solving each initial value problem, we have

$$P_1(t) = Cos_{\frac{1}{4}}(t),$$

$$P_2(t) = \frac{16}{3}Cos_{\frac{1}{2}}(t) - \frac{16}{3}Cos_{\frac{1}{4}}(t),$$

and

$$P_3(t) = \frac{32}{5}Cos_{\frac{3}{4}}(t) - \frac{16^2}{15}Cos_{\frac{1}{2}}(t) + \frac{32}{3}Cos_{\frac{1}{4}}(t).$$

Finally, we have

$$Cos_A(t) = P_1(t)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + P_2(t)\begin{bmatrix}0 & 0 & 0\\ \frac{3}{4} & \frac{3}{16} & \frac{5}{4}\\ 0 & 0 & \frac{1}{2}\end{bmatrix} + P_3(t)\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \frac{5}{8}\\ 0 & 0 & \frac{5}{32}\end{bmatrix}$$

$$= \begin{bmatrix}P_1(t) & 0 & 0\\ \frac{3}{4}P_2(t) & P_1(t) + \frac{3}{16}P_2(t) & \frac{5}{4}P_2(t) + \frac{5}{8}P_3(t)\\ 0 & 0 & P_1(t) + \frac{1}{2}P_2(t) + \frac{5}{32}P_3(t)\end{bmatrix}.$$

.

Remark 3.1. If the initial conditions (3.2) in the matrix IVP (3.1) and (3.2) are replaced by

$$Y(0) = 0_n, \quad \nabla Y(0) = I_n,$$

then we define the solution of the IVP as $Sin_A(t)$. Hence, the associated Putzer algorithm can be stated for the calculation of the matrix-valued discrete sine function.

4. Calculating $Cos_p(A)$ with Jordan blocks

In this section, we develop a method to calculate $Cos_p(A)$ for a given square matrix A using the nabla Taylor's formula

$$\sum_{k=0}^{\infty} \frac{\nabla^k f(\lambda)}{k!}(A - \lambda I_n)^{\overline{k}},$$

where $f(t) = Cos_p(t)$, and $\lambda \in \mathbb{N}_0$ is an eigenvalue of A.

Let A be an $n \times n$ matrix and $A = PJP^{-}$, in which J is the $n \times n$ Jordan canonical form of matrix A. After performing the matrix multiplication and arranging the terms, we have

$$\begin{aligned} A^{\overline{k}} &= A(A + I_n)(A + 2I_n) \cdots (A + (k-1)I_n) \\ &= a_1 A^k + a_2 A^{k-1} + \cdots + a_k A \\ &= a_1 PJ^k P^{-} + a_2 PJ^{k-1} P^{-} + \cdots + a_k PJP^{-}, \end{aligned}$$

where the constants $a_1, \cdots, a_k$ are the coefficients of the expansion of the rising factorial. Then, we can factor out the P and P^- on either side to get

$$A^{\overline{k}} = P(a_1 J^k + a_2 J^{k-1} + \cdots + a_k J)P^- = PJ^{\overline{k}}P^-. \tag{4.1}$$

We also observe that

$$\begin{aligned}
f(A) &= \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!}(A - \lambda I_n)^{\overline{k}} \\
&= \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!}(P(J - \lambda I_n)P^-)^{\overline{k}} \\
&= \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!}P(J - \lambda I_n)^{\overline{k}}P^- \\
&= Pf(J)P^-.
\end{aligned}$$

Definition 4.1. The unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined as the coefficients of x^k after performing the n rising factorial of x,

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Proposition 4.1. Let J_i be a Jordan block of some matrix A, and let λ_i be the corresponding eigenvalue. Then,

$$(J_i - \lambda_i I)^{\overline{s}}_{r,c} = \begin{bmatrix} s \\ c - r \end{bmatrix}$$

is valid for all $s \in \mathbb{N}_0$.

Proof. We will prove this formula using the mathematical induction principle. When $s = 0$, the matrix $(J_i - \lambda_i I)^{\overline{s}}$ is equal to the $n \times n$ identity matrix. Indeed, we have

$$\begin{aligned}
(J_i - \lambda_i I)^{\overline{0}} &= \begin{bmatrix}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ n-2 \end{bmatrix} & \begin{bmatrix} 0 \\ n-1 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ n-3 \end{bmatrix} & \begin{bmatrix} 0 \\ n-2 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ n-4 \end{bmatrix} & \begin{bmatrix} 0 \\ n-3 \end{bmatrix} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\begin{bmatrix} 0 \\ -(n-2) \end{bmatrix} & \begin{bmatrix} 0 \\ -(n-3) \end{bmatrix} & \begin{bmatrix} 0 \\ -(n-4) \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -(n-1) \end{bmatrix} & \begin{bmatrix} 0 \\ -(n-2) \end{bmatrix} & \begin{bmatrix} 0 \\ -(n-3) \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}.
\end{aligned}$$

Next, we consider $(J_i - \lambda_i I)^{\overline{d+1}}$ assuming that $(J_i - \lambda_i I)^{\overline{d}}_{r,c} = \begin{bmatrix} d \\ c-r \end{bmatrix}$ is true. We set

$$(J_i - \lambda_i I)^{\overline{d}} = \begin{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix} & \begin{bmatrix} d \\ 1 \end{bmatrix} & \begin{bmatrix} d \\ 2 \end{bmatrix} & \cdots & \begin{bmatrix} d \\ n-2 \end{bmatrix} & \begin{bmatrix} d \\ n-1 \end{bmatrix} \\ \begin{bmatrix} d \\ -1 \end{bmatrix} & \begin{bmatrix} d \\ 0 \end{bmatrix} & \begin{bmatrix} d \\ 1 \end{bmatrix} & \cdots & \begin{bmatrix} d \\ n-3 \end{bmatrix} & \begin{bmatrix} d \\ n-2 \end{bmatrix} \\ \begin{bmatrix} d \\ -2 \end{bmatrix} & \begin{bmatrix} d \\ -1 \end{bmatrix} & \begin{bmatrix} d \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} d \\ n-4 \end{bmatrix} & \begin{bmatrix} d \\ n-3 \end{bmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \begin{bmatrix} d \\ -(n-2) \end{bmatrix} & \begin{bmatrix} d \\ -(n-3) \end{bmatrix} & \begin{bmatrix} d \\ -(n-4) \end{bmatrix} & \cdots & \begin{bmatrix} d \\ 0 \end{bmatrix} & \begin{bmatrix} d \\ 1 \end{bmatrix} \\ \begin{bmatrix} d \\ -(n-1) \end{bmatrix} & \begin{bmatrix} d \\ -(n-2) \end{bmatrix} & \begin{bmatrix} d \\ -(n-3) \end{bmatrix} & \cdots & \begin{bmatrix} d \\ -1 \end{bmatrix} & \begin{bmatrix} d \\ 0 \end{bmatrix} \end{bmatrix}.$$

Then, we have

$$\begin{aligned} (J_i - \lambda_i I)^{\overline{d+1}} &= (J_i - \lambda_i I)^{\overline{d}}(J_i - \lambda_i I + dI) \\ &= (J_i - \lambda_i I)^{\overline{d}} \begin{bmatrix} d & 1 & 0 & \cdots & 0 & 0 \\ 0 & d & 1 & \cdots & 0 & 0 \\ 0 & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d & 1 \\ 0 & 0 & 0 & \cdots & 0 & d \end{bmatrix}. \end{aligned}$$

Using the recursive formula $\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$, after multiplying the matrices, we obtain each element of $(J_i - \lambda_i I)^{\overline{d+1}}$ as

$$\begin{aligned} (J_i - \lambda_i I)^{\overline{d+1}}_{r,c} &= d\begin{bmatrix} d \\ c-r \end{bmatrix} + \begin{bmatrix} d \\ c-r-1 \end{bmatrix} \\ &= \begin{bmatrix} d+1 \\ c-r \end{bmatrix}, \end{aligned}$$

which coincides with the formula for $s = d + 1$. Therefore, $(J_i - \lambda_i I)^{\overline{d+1}}$ is

$$\begin{bmatrix} \begin{bmatrix} d+1 \\ 0 \end{bmatrix} & \begin{bmatrix} d+1 \\ 1 \end{bmatrix} & \begin{bmatrix} d+1 \\ 2 \end{bmatrix} & \cdots & \begin{bmatrix} d+1 \\ n-2 \end{bmatrix} & \begin{bmatrix} d+1 \\ n-1 \end{bmatrix} \\ \begin{bmatrix} d+1 \\ -1 \end{bmatrix} & \begin{bmatrix} d+1 \\ 0 \end{bmatrix} & \begin{bmatrix} d+2 \\ 1 \end{bmatrix} & \cdots & \begin{bmatrix} d+1 \\ n-3 \end{bmatrix} & \begin{bmatrix} d+1 \\ n-2 \end{bmatrix} \\ \begin{bmatrix} d+1 \\ -2 \end{bmatrix} & \begin{bmatrix} d+1 \\ -1 \end{bmatrix} & \begin{bmatrix} d+1 \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} d+1 \\ n-4 \end{bmatrix} & \begin{bmatrix} d+1 \\ n-3 \end{bmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \begin{bmatrix} d+1 \\ -(n-2) \end{bmatrix} & \begin{bmatrix} d+1 \\ -(n-3) \end{bmatrix} & \begin{bmatrix} d+1 \\ -(n-4) \end{bmatrix} & \cdots & \begin{bmatrix} d+1 \\ 0 \end{bmatrix} & \begin{bmatrix} d+1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} d+1 \\ -(n-1) \end{bmatrix} & \begin{bmatrix} d+1 \\ -(n-2) \end{bmatrix} & \begin{bmatrix} d+1 \\ -(n-3) \end{bmatrix} & \cdots & \begin{bmatrix} d+1 \\ -1 \end{bmatrix} & \begin{bmatrix} d \\ 0 \end{bmatrix} \end{bmatrix}.$$

Hence, we have that

$$(J_i - \lambda_i I)^{\overline{s}}_{r,c} = \begin{bmatrix} s \\ c-r \end{bmatrix}$$

is true for all $s \geq 0$ by the mathematical induction principle.

Remark 4.1. Alternatively, we can use the falling factorial and signed Stirling numbers of the first kind to derive a formula with the delta Δ operator, where $\Delta f(t) = f(t+1) - f(t)$. Noting the delta Taylor series (see Definition 1.63 in [1])

$$\sum_{k=0}^{\infty} \frac{(\Delta^k f)(\lambda)}{k!} (A - \lambda I_n)^{\underline{k}},$$

we can use a similar process of mathematical induction, the signed Stirling number of the first kind $s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$, and the recurrence relation

$$s(n+1,k) = -n \cdot s(n,k) + s(n,k-1)$$

to prove that

$$(J_i - \lambda_i I)^{\underline{s}} = \begin{bmatrix} s(s,0) & s(s,1) & s(s,2) & \cdots & s(s,n-1) \\ s(s,-1) & s(s,0) & s(s,1) & \cdots & s(s,n-2) \\ s(s,-2) & s(s,-1) & s(s,0) & \cdots & s(s,n-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s(s,-n+1) & s(s,-n+2) & s(s,-n+3) & \cdots & s(s,0) \end{bmatrix}$$

with element $(J_i - \lambda_i I)^{\overline{s}}_{r,c} = s(s, c-r)$. We claim that

$$f(J_i)_{r,c} = \sum_{k=0}^{\infty} \frac{(\Delta^k f)(\lambda_i)}{k!} s(k, c-r).$$

Example 4.1. We now calculate the discrete cosine $Cos_p(A)$ for the 2×2 nondiagonalizable matrix $A = \begin{bmatrix} 6 & 2 \\ -2 & 2 \end{bmatrix}$ and some $|p| < 1$. We can factor A in the following way:

$$A = PJP^{-} = \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

Then, we have $Cos_p(A) = Cos_p(PJP^{-}) = PCos_p(J)P^{-}$, $P = \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, and the only eigenvalue is $\lambda = 4$. It follows that

$$Cos_p(J)_{r,c} = \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!} \begin{bmatrix} k \\ c-r \end{bmatrix}$$

with $f(x) = Cos_p(x)$ and $\lambda = 4$.

We now calculate $Cos_p(J)_{r,c}$ for $c - r = 0$ and $c - r = 1$. We do not have to calculate $Cos_p(J)_{r,c}$ for $c - r < 0$ because $\begin{bmatrix} k \\ c-r \end{bmatrix} = 0$ for all negative $c - r$. For $c - r = 0$,

$$Cos_p(J)_{1,1} = Cos_p(J)_{2,2} = \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!} \begin{bmatrix} k \\ 0 \end{bmatrix} = Cos_p(4),$$

because $\begin{bmatrix} k \\ 0 \end{bmatrix} = 0$ for all $k > 0$. For $c - r = 1$, we get

$$Cos_p(J)_{1,2} = \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!} \begin{bmatrix} k \\ 1 \end{bmatrix}.$$

To simplify this result, we must first pay attention to the $\frac{1}{k!}\begin{bmatrix} k \\ 1 \end{bmatrix}$ portion. Let us use the recursion formula for unsigned Stirling numbers of the first kind. We know that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

Setting $k = 1$ gives us

$$\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = n \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 0 \end{bmatrix}.$$

$\begin{bmatrix} n \\ 0 \end{bmatrix} = 0$ for all $n > 0$, so we get

$$\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = n \begin{bmatrix} n \\ 1 \end{bmatrix}.$$

Because $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$$

for all $n > 0$, and

$$\frac{1}{k!}\begin{bmatrix} k \\ 1 \end{bmatrix} = \frac{1}{k}$$

for all $k > 0$. Then, taking into account that $\frac{1}{0!}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$, we have

$$Cos_p(J)_{1,2} = \sum_{k=0}^{\infty} \frac{(\nabla^k f)(\lambda)}{k!}\begin{bmatrix} k \\ 1 \end{bmatrix} = \sum_{k=1}^{\infty} \frac{(\nabla^k f)(\lambda)}{k}.$$

This implies that

$$Cos_p(J)_{1,2} = \sum_{k=1}^{\infty} \frac{(\nabla^k f)(\lambda)}{k} = -pSin_p(4) - \frac{p^2 Cos_p(4)}{2} + \frac{p^3 Sin_p(4)}{3} + \frac{p^4 Cos_p(4)}{4} + \cdots .$$

We can make a general formula for this summation by grouping every two terms to get

$$Cos_p(J)_{1,2} = \sum_{k=1}^{\infty} (-1)^k (\frac{p^{2k-1} Sin_p(\lambda)}{2k-1} + \frac{p^{2k} Cos_p(\lambda)}{2k}).$$

Using the root test, the series is absolutely convergent, and hence it can be written as a sum of two series:

$$\sum_{k=1}^{\infty} (-1)^k (\frac{p^{2k-1} Sin_p(\lambda)}{2k-1} + \frac{p^{2k} Cos_p(\lambda)}{2k}) = \sum_{k=1}^{\infty} (-1)^k \frac{p^{2k-1} Sin_p(\lambda)}{2k-1} + \sum_{k=1}^{\infty} (-1)^k \frac{p^{2k} Cos_p(\lambda)}{2k}.$$

Factoring out $Sin_p(\lambda)$ and $Cos_p(\lambda)$ and calculating the alternating power series gives us

$$Cos_p(J)_{1,2} = -\tan^{-1}(p) Sin_p(\lambda) - \frac{1}{2}\ln(1+p^2) Cos_p(\lambda).$$

Then, noting that $\lambda = 4$, we have

$$Cos_p(J) = \begin{bmatrix} Cos_p(4) & -\tan^{-1}(p) Sin_p(4) - \frac{1}{2}\ln(1+p^2) Cos_p(4) \\ 0 & Cos_p(4) \end{bmatrix},$$

and

$$\begin{aligned} Cos_p(A) &= PCos_p(J)P^{-} \\ &= \begin{bmatrix} -1 & \frac{-1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Cos_p(4) & -\tan^{-1}(p) Sin_p(4) - \frac{1}{2}\ln(1+p^2) Cos_p(4) \\ 0 & Cos_p(4) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}. \end{aligned}$$

We perform the matrix multiplications on the right-hand side of the above equality to obtain a 2×2 matrix

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix},$$

where $\alpha = -2(-(Cos_p(4)/2) + Sin_p(4)\tan^{-1}(p) + 1/2Cos_p(4)\ln(1 + p^2))$,
$\beta = -Cos_p(4) - 2(-(Cos_p(4)/2) + Sin_p(4)\tan^{-1}(p) + 1/2Cos_p(4)\ln(1 + p^2))$,
$\gamma = -2(-Sin_p(4)\tan^{-1}(p) - 1/2Cos_p(4)\ln(1 + p^2))$,
$\delta = Cos_p(4) - 2(-Sin_p(4)\tan^{-1}(p) - 1/2Cos_p(4)\ln(1 + p^2))$.

Example 4.2. Let us consider the nondiagonalizable 3×3 matrix

$$A = \begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 1\\ 0 & 0 & 3\end{bmatrix}$$

to calculate $f(A) = Cos_p(A)$ for $|p| < 1$.

Factoring gives us

$$A = PJP^{-} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 1\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Therefore, we have two Jordan blocks, $J_1 = \begin{bmatrix}1\end{bmatrix}$ and $J_2 = \begin{bmatrix}3 & 1\\ 0 & 3\end{bmatrix}$, with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, respectively.

We start by calculating $f(J_1)$. We only have one element and one diagonal with $c-r = 0$. Therefore,

$$Cos_p(J_1) = \frac{f(\lambda)}{0!}\begin{bmatrix}0\\ 0\end{bmatrix} = Cos_p(1).$$

The second Jordan block calculates similarly as in the previous example, but with a different eigenvalue:

$$f(J_2) = \begin{bmatrix}Cos_p(3) & -\tan^{-1}(p)Sin_p(3) - \frac{1}{2}\ln(1 + (p)^2)Cos_p(3)\\ 0 & Cos_p(3)\end{bmatrix}.$$

Hence, we have

$$f(J) = \begin{bmatrix}Cos_p(1) & 0 & 0\\ 0 & Cos_p(3) & -\tan^{-1}(p)Sin_p(3) - \frac{1}{2}\ln(1 + (p)^2)Cos_p(3)\\ 0 & 0 & Cos_p(3)\end{bmatrix},$$

and

$$\begin{aligned}Cos_p(A) &= PCos_p(J)P^{-}\\ &= P\begin{bmatrix}Cos_p(1) & 0 & 0\\ 0 & Cos_p(3) & -\tan^{-1}(p)Sin_p(3) - \frac{1}{2}\ln(1 + (p)^2)Cos_p(3)\\ 0 & 0 & Cos_p(3)\end{bmatrix}P^{-}\\ &= \begin{bmatrix}Cos_p(1) & 0 & -Cos_p(1) + Cos_p(3)\\ 0 & Cos_p(3) & -\tan^{-1}(p)Sin_p(3) - \frac{1}{2}\ln(1 + p^2)\\ 0 & 0 & Cos_p(3)\end{bmatrix}.\end{aligned}$$

Remark 4.2. The process for calculation is very similar for $Sin_p(A)$; we replace the initial $f(x)$ with $Sin_p(A)$ for $Cos_p(A)$. Then, the series for $Sin_p(J)_{1,2}$ is

$$\sum_{k=1}^{\infty} \frac{(\nabla^k f)(\lambda)}{k} = pCos_p(4) - \frac{p^2 Sin_p(4)}{2} - \frac{p^3 Cos_p(4)}{3} + \frac{p^4 Sin_p(4)}{4} + \cdots .$$

In the same process as before, we can write the result as a sum of two series to get

$$\sum_{k=1}^{\infty}(-1)^k \frac{-p^{2k-1}Cos_p(\lambda)}{2k-1} - \sum_{k=1}^{\infty}(-1)^k \frac{p^{2k}Sin_p(\lambda)}{2k} = \tan^{-1}(p)Cos_p(\lambda) - \frac{1}{2}\ln(1+p^2)Sin_p(\lambda).$$

Then, $Sin_p(J)_{1,1}$ and $Sin_p(J)_{2,2}$ are $Sin_p(\lambda)$ in the same fashion as with $Cos_p(A)$, and

$$Sin_p(J) = \begin{bmatrix} Sin_p(4) & \tan^{-1}(p)Cos_p(4) - \frac{1}{2}\ln(1+p^2)Sin_p(4) \\ 0 & Sin_p(4) \end{bmatrix}.$$

$Sin_p(A)$ can be further calculated using $Sin_p(A) = PSin_p(J)P^{-}$.

5. Conclusions

In this paper, we introduce matrix-valued nabla cosine and sine functions within the framework of discrete calculus. We develop two calculation methods that serve as techniques for evaluating matrix-valued discrete trigonometric functions. Whereas the matrix-valued sine and cosine functions in continuous time have applications in vibration of mechanical structures, control systems with oscillatory dynamics, and scientific computing, the discrete counterparts developed here promise comparable impact in numerical algorithms, fast transforms, and especially image and signal processing. The results presented here therefore open new research directions in several areas.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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