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Research article

Stability analysis of neutral-type stochastic delayed neural networks with Markov switching

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Abstract: This paper investigated the stability of a class of neutral-type stochastic delayed neural networks with Markov switching. Under a general decay rate and weaker conditions on the neutral term, sufficient conditions for stability in the p-th moment, almost sure stability, and actual stability were established by constructing appropriate Lyapunov functions and applying the nonnegative semimartingale convergence theorem. The theoretical analysis was validated via MATLAB simulations using the Euler-Maruyama method.

Keywords: neutral stochastic delayed neural networks; p-th moment stability; Markov switching; general decay; Lyapunov function

1. Introduction

In recent years, stochastic delayed neural networks (SDNNs) have emerged as one of the core models in key fields, owing to their remarkable capabilities in simulating biological learning mechanisms, fault diagnosis, and intelligent information processing [1, 2]. These applications rely critically on the stability analysis of the equilibrium points in neural networks [3, 4]. However, time delays and random disturbances, commonly present in practical engineering systems, often serve as primary sources of instability. For example, time delays in high-speed integrated circuits can readily induce oscillations [5–7]. Moreover, system parameters and structures are susceptible to external environmental interference, often accompanied by random fluctuations and noise caused by probabilistic factors. Gaussian noise processes provide effective mathematical representations for such stochastic perturbations in physical systems [8–10]. Consequently, a systematic investigation of dynamical behaviors in SDNNs, with particular emphasis on stability properties, bears substantial theoretical significance.

To accurately characterize structural and parametric transitions induced by stochastic abrupt variations, including subsystem reconfiguration and component failures, Kac and Krasovskii pioneered the incorporation of Markov switching mechanisms within system modeling paradigms [11]. By coupling continuous dynamics with discrete modes, systems with Markov switching significantly enhance the modeling capability for complex practical systems. Their successful applications in military, electronic, and automation fields demonstrate significant theoretical and practical value [12–15]. Furthermore, neutral-type stochastic delayed systems, as a more complex class of dynamic models, can precisely describe physical processes such as signal propagation in lossless transmission lines. Recent years have witnessed considerable advances in stability analysis for SDNNs incorporating Markov switching, with numerous significant contributions documented in the literature [16–23]. In [17], using Lyapunov–Krasovskii functionals and stochastic analysis, Zhu and Cao derived sufficient conditions for mean-square global asymptotic stability in neutral-type stochastic neural networks with Markov switching and mixed delays. Based on a delay fractionation technique, Rakkiyappan et al. established enhanced stability conditions for Markov jump stochastic neural networks of neutral type in [21].

It is noteworthy that the convergence behavior of many practical systems often exhibits forms slower than exponential decay, such as polynomial or logarithmic decay. In contrast, previous studies on the stability of stochastic delayed neural networks with Markov switching have focused predominantly on exponential stability or moment stability [19, 24–28]. Among these, Yu and Deng investigated the almost sure stability of neutral-type Cohen–Grossberg neural networks with Lévy noise, time-varying delays, and Markovian switching in [27]. Similarly, Cui et al. examined the mean-square exponential stability of neutral-type neural networks subject to Lévy noise and mixed time-varying delays in [28]. Confining the analysis to an exponential decay framework can lead to conservative conclusions, making it difficult to accurately characterize the true asymptotic behavior of such systems. Consequently, researchers have shifted their focus to stability with a general decay rate [29–32], employing a ψ-type function to describe the decay process. For example, Sheng et al. established comprehensive criteria for ψ-type stability and robust stability of stochastic reaction-diffusion neural networks with unbounded time delays in [30], generalizing exponential and polynomial stability.

Classical Lyapunov stability theory conventionally assumes that the origin constitutes a system equilibrium point. However, in numerous practical neural network implementations, external inputs or nontrivial equilibrium configurations render the origin nonequilibrium, invalidating standard asymptotic stability definitions. Caraballo et al. introduced the conceptual framework of practical stability [33], which relaxes the requirement for asymptotic convergence to the origin, instead examining whether the system trajectories become permanently confined to a neighborhood of the origin. Research on practical stability has made preliminary advances in recent years, as exemplified by [34–36]. However, there has been relatively little research on practical stability under a general decay rate, and this insight paves the way for our subsequent research. We aim to integrate practical stability with the decay rate, thus proposing a generalized concept of practical stability with a preset decay rate. This framework can serve as an effective theoretical tool for analyzing systems with nontrivial equilibrium points or those subject to persistently bounded disturbances.

Building upon these foundations, this paper investigates stability properties of neutral stochastic delayed neural networks with Markov switching (NSDNNs-MS) under general decay rate conditions. The principal contributions of this research are summarized as follows: First, by employing the Lyapunov function method and the nonnegative semimartingale convergence theorem, the global existence and uniqueness of the solution are established under weaker local Lipschitz conditions. Subsequently, the stability of the solution with the traditional equilibrium point at zero is studied. Second, for scenarios where the origin is not an equilibrium, the almost sure practical stability of the system is analyzed under weaker conditions on the neural term. Finally, the effectiveness and feasibility of the obtained sufficient conditions are verified through numerical simulations.

2. Model description and preliminaries

Notation. Let $\mathcal{R} = (-\infty, +\infty)$ and $\mathcal{R}^+ = [0, +\infty)$. For integers $m, n \geq 1$, denote by $\mathcal{R}^n$ the n-dimensional Euclidean space and by $\mathcal{R}^{n\times m}$ the set of all $n \times m$ real matrices. The Euclidean norm of a vector $y \in \mathcal{R}^n$ is $|y| = \sqrt{y^T y}$. For $t_0 \geq 0$, let $(Q, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{R}}, P)$ denote a complete probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\in\mathcal{R}}$ satisfying the standard requirement. Let $W_j(t)$ be an m-dimensional $\mathcal{F}_t$-adapted Brownian motion. Let $\mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n]$ be the space of càdlàg functions $\eta : [-\mathcal{D}, 0] \to \mathcal{R}^n$ equipped with the norm $\|\eta\| = \sup_{-\mathcal{D}\leq\theta\leq 0} |\eta(\theta)|$. We denote by $\mathbb{C}^p_{\mathcal{F}_t}([-\mathcal{D}, 0]; \mathcal{R}^n)$ the family of all $\mathcal{F}_t$-measurable, $\mathcal{R}^n$-valued stochastic processes $\eta = \{\eta(\theta) : -\mathcal{D} \leq \theta \leq 0\}$ that are almost surely continuous and satisfy the p-th moment boundedness condition $\|\eta\|_{\mathcal{L}^p} = \left(E\left[\sup_{-\mathcal{D}\leq\theta\leq 0} |\eta(\theta)|^p\right]\right)^{1/p} < \infty$.

Consider a right-continuous Markov switching process $\alpha(t) : \mathcal{R}^+ \to \mathcal{S}$ defined in the aforementioned probability space, where $\mathcal{S} = \{1, 2, \ldots, Z\}$ is a finite state space, and its dynamics are governed by a generator $\Gamma = (\gamma_{ij})_{Z\times Z}$ given by

$$P_{jk}(\Delta) = P\{\alpha(t + \Delta) = k|\alpha(t) = j\} = \delta_{jk} + \gamma_{jk}\Delta + o(\Delta), \tag{2.1}$$

where $\Delta > 0$, $\delta_{jk} = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}$ is the Kronecker delta function and $\gamma_{jk} \geq 0$ is the transition rate from j to k if $j \neq k$ while

$$\gamma_{jj} = -\sum_{j\neq k} \gamma_{jk} \leq 0.$$

2.1. Model description

In this article, we consider the following neutral stochastic delayed neural networks with Markov switching

$$\begin{aligned} &d\left[y_i(t) - G_i(t, y_i(t + \theta), \alpha(t))\right] \\ &= \left[-a_i(\alpha(t))y_i(t) + \sum_{j=1}^{m} b_{ij}(\alpha(t)) f_j(y_j(t + \theta))\right] dt \\ &\quad + \sum_{j=1}^{m} e_{ij} g_j(t, y_j(t), y_j(t + \theta), \alpha(t))\, dW_j(t), t \geq 0, \theta \in [-\mathcal{D}, 0], \end{aligned} \tag{2.2}$$

where the initial value $y_0 = \varphi \in \mathbb{C}^p_{\mathcal{F}_{t_0}}$, $y_i(t) \in \mathbb{C}^P_{\mathcal{F}_t}$ is the i-th neuron state with the Càdlàg trajectories. $f_j : \mathcal{R}^+ \times \mathcal{R}^n \times \mathbb{C}^P_{\mathcal{F}_t} \times \mathcal{S} \to \mathcal{R}^n$, $g_j : \mathcal{R}^+ \times \mathcal{R}^n \times \mathbb{C}^P_{\mathcal{F}_t} \times \mathcal{S} \to \mathcal{R}^{n\times m}$ represent the activation functions of the

j-th neuron. Without loss of generality, we assume $f_j(0) = 0$, $g_j(0) = 0$. θ represents the time delay. The coefficients $a_i > 0$, $b_{ij} \in \mathcal{R}$, and $e_{ij} \in \mathcal{R}$, mean intensity of the i-th neuron on the j-th neuron at current time.

To facilitate the stability analysis of the neutral-type system (2.2), we introduce a key auxiliary variable $Q_i(t) := y_i(t) - G_i(t, y_{i,t}, \alpha(t)), t \geq 0$, for each neuron i, where $G_i : [0, +\infty] \times \mathbb{C}^P_{\mathcal{F}_t} \times \mathcal{S} \to \mathcal{R}^n$ is the Borel measurable function. The variable $Q_i(t)$ captures the deviation between the instantaneous state $y_i(t)$ and its history-dependent neutral term $G_i(\cdot)$. This transformation allows us to rewrite the original system in a form that is more amenable to the application of Itô's formula and Lyapunov techniques. The stability analysis of $Q_i(t)$ will be central to deriving the stability criteria for the original state $y_i(t)$.

For any $(t, y_i, y_i(t + \theta), k) \in \mathcal{R}^+ \times \mathcal{R}^n \times \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n] \times \mathcal{S}$, by the generalized Itô formula,

$$
\begin{aligned}
&\mathcal{L}V(t, Q_i(t), k) \\
&= V_t(t, Q_i(t), k) + \sum_{i=1}^{n} \left[-a_i(\alpha(t)) y_i(t) + \sum_{j=1}^{m} b_{ij}(\alpha(t)) f_j(y_j(t + \theta)) \right] V_{y_i} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \operatorname{tr} \left[\sum_{h=1}^{m} e_{jh} g_h(t, y_h(t), y_h(t + \theta), \alpha(t))^T V_{y_i y_j} \sum_{h=1}^{m} e_{jh} g_h(t, y_h(t), y_h(t + \theta), \alpha(t)) \right] \\
&\quad + \sum_{l=1}^{Z} \gamma_{kl} V(t, Q_i(t), l).
\end{aligned}
$$

Then,

$$
V(t, Q_i(t), \alpha(t)) = V(0, Q_i(0), \alpha_0) + \int_0^t \mathcal{L}V(s, Q_i(s), \alpha(s))\, ds + M_t,
$$

where

$$
M_t = \int_0^t V_y(s, Q_i(s), \alpha(s)) g_i(s, y_i(s), y_i(s + \theta), \alpha(s))\, dW_i(s).
$$

$\{M_t\}_{t \geq 0}$ is a local martingale, and V_t, V_y, and V_{yy} are as given by Mao in [9].

2.2. Basic assumptions and lemma

To study the properties of NSDNN-MS, the following assumptions related to impulses are given below.

$\mathcal{A}_1$ **(Local Lipschitz Condition)** For each $1 \leq h \leq m$, for all $y, \tilde{y} \in \mathcal{R}^n$, $y_t = y(t + \theta), \tilde{y}_t = \tilde{y}(t + \theta) \in \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n]$ with $|y| \vee |\tilde{y}| \vee \|y_t\| \vee \|\tilde{y}_t\| \leq h$, $k \in \mathcal{S}$, $t \geq 0$, there exists a constant $\zeta_h > 0$ such that

$$
\begin{aligned}
&|F_h(t, y, y_t, k) - F_h(t, \tilde{y}, \tilde{y}_t, k)| \vee |g_h(t, y, y_t, k) - g_h(t, \tilde{y}, \tilde{y}_t, k)| \\
&\leq \zeta_h(|y - \tilde{y}| + \|y_t - \tilde{y}_t\|),
\end{aligned} \tag{2.3}
$$

where $F_i(t, y, y_t, k) = \left[-a_i(\alpha(t)) y_i(t) + \sum_{j=1}^{m} b_{ij}(\alpha(t)) f_j(y_j(t + \theta)) \right]$.

$\mathcal{A}_2$ Assume that a function $\mu(\cdot) : [-\mathcal{D}, 0] \to \mathcal{R}^n$ exists with $\int_{-\mathcal{D}}^0 \mu(s)ds = 1$, and a constant $\kappa \in (0,1)$ exists for which the mapping $G_i : \mathcal{R}^+ \times \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n] \times \mathcal{S} \to \mathcal{R}^n$ satisfies

$$|G_i(t,\xi,k)|^p \leq \kappa \int_{-\mathcal{D}}^0 \mu(s)|\xi(s)|^p ds, \quad t \geq 0, k \in S, \tag{2.4}$$

where $\xi \in \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n]$.

Remark 2.1. The neutral term G_i characterizes the intensity of the system's memory effect by relating the p-th moment of the current neutral term to a weighted average of the historical states $|\varphi(s)|^p$ over the delay interval $[-\mathcal{D}, 0]$. The weighting function $\mu(s)$ describes the distribution of influences from different historical time instants. The normalization condition is essentially a mathematical convention; when normalized, $\mu(s)$ can be interpreted as a probability density function defined on $[-\mathcal{D}, 0]$, thereby providing a standardized measure of historical influence. This formulation renders $\kappa < 1$ a clear and dimensionless intensity coefficient, and the boundedness condition ensures that the strength of the system's historical feedback is strictly weaker than the direct effect of the current state. According to the literature [9], we can assume that there exists a constant $\kappa \in (0,1)$ satisfying $|G(t,\xi,k)| \leq \kappa\|\xi\|$, and, thus, $|G(t,\xi,k)|^p \leq \kappa^p\|\xi\|^p$, $k \in \mathcal{S}$, where $G(0) = 0$ for any $\xi \in \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n]$. Our assumption generalizes the uniform boundedness assumption , allowing for more flexible nonuniform historical dependence.

$\mathcal{A}_3$ There exist constants $c_1, c_2 > 0$ such that, for every $p > 0$, the Lyapunov function $V(\cdot,\cdot,\cdot) \in C^{1,2}(\mathcal{R}^+ \times \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n] \times \mathcal{S}; \mathcal{R}^+)$ satisfies

$$c_1|Q_i|^p \leq V(t, Q_i, k) \leq c_2|Q_i|^p. \tag{2.5}$$

For any $(t, y_i, y_i(t+\theta), k) \in \mathcal{R}^+ \times \mathcal{R}^n \times \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n] \times S$, there exist constants $\alpha_1, \alpha_2 > 0$ and a function $n(\cdot) : [-\mathcal{D}, 0] \to \mathcal{R}^n$ such that

$$\mathcal{L}V(t, y_i, y_i(t+\theta), k) \leq -\alpha_1|y_i|^p + \alpha_2 \int_{-\mathcal{D}}^0 n(\theta)|y_i(t+\theta)|^p d\theta, \tag{2.6}$$

where $n(\cdot)$ has the same meaning as the previously defined $\mu(s)$ and satisfies $\int_{-\mathcal{D}}^0 n(s)ds = 1$. Moreover, $\alpha_1 \geq \alpha_2$.

Lemma 2.1. (See [9])

(1) Let $a \in (0, 1]$, and $o, k \in (0, +\infty]$, there is

$$(o+k)^a \leq o^a + k^a.$$

(2) Let $a \in (1, +\infty)$, $\upsilon > 0$ and $o, k \in (0, +\infty]$, there is

$$(o+k)^a \leq (1 + \upsilon^{\frac{1}{a-1}})^{a-1}\left(o^a + \frac{k^a}{\upsilon}\right).$$

The proof of this lemma can be established by employing the sub-additivity property of concave functions and Hölder's inequality.

3. Main results

3.1. The existence and uniqueness of global solution of equation

Theorem 3.1. *Under $\mathcal{A}_{1-3}$, the system exists a unique global solution $y_i(t, \varphi, k_0)$ on $t \in [-\mathcal{D}, \infty)$, given any initial condition $\varphi \in \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n]$ and initial mode $k_0 \in \mathbb{S}$.*

Proof. Given initial value $\varphi \in \mathbb{C}([-\mathcal{D}, 0]; \mathcal{R}^n)$, $\mathcal{A}_1$ guarantees the existence of a unique solution $y_i(\cdot)$ on $[-\mathcal{D}, T)$, where T denotes the explosion time. Then we define the stopping time $\mathcal{D}_\iota = T \wedge \inf\{t_0 \leq t \leq T \| y(t)| \geq \iota\}$, for any $\iota \geq \iota_0, \iota_0 \in \mathcal{R}^+$, where $\|\varphi\| \geq \iota_0$. Clearly, $\mathcal{D}_\iota$ is nondecreasing and $\mathcal{D}_\iota \to T$ almost surely as $\iota \to \infty$. Therefore, we only need to prove $\lim_{\iota\to\infty} \mathcal{D}_\iota = \infty$ a.s., and it can further be shown that $T \to \infty$ a.s.. That is, the explosion time does not exist, which in turn demonstrates that the equation admits a unique global solution $y(t)$.

For any $t \geq 0$, let $I\{\mathcal{D}_\iota \leq t\}$ be the indicator function of the event $\{\mathcal{D}_\iota \leq t\}$. Then, we have $E[I(\mathcal{D}_\iota \leq t)] = P(\mathcal{D}_\iota \leq t)$, and our next objective is to prove the following limit:

$$\lim_{\iota\to\infty} E[I(\mathcal{D}_\iota \leq t)] = 0.$$

This is equivalent to $\lim_{\iota\to\infty} P(\mathcal{D}_\iota > t) = 1$, thereby proving that $\mathcal{D}_\iota \to \infty$ almost surely as $\iota \to \infty$. The proof is divided into two steps based on the range of the exponent p.

Step One: When $p \in (0, 1]$, and applying the Itô formula under $\mathcal{A}_{1-3}$, there is

$$\begin{aligned} & E[V(t \wedge \mathcal{D}_\iota, Q_i(t \wedge \mathcal{D}_\iota), \alpha(t \wedge \mathcal{D}_\iota))] \\ & \leq E[V(0, Q_i(0), \alpha(0))] + E\left[\int_0^{t\wedge\mathcal{D}_\iota} \left[-\alpha_1 |y(s)|^p + \alpha_2 \int_{-\mathcal{D}}^0 |y(s+\theta)|^p n(\theta)\, d\theta\right] ds\right] + E(M_t) \\ & = l_1 + l_2, \end{aligned} \tag{3.1}$$

where $l_1 = E[V(0, Q_i(0), \alpha(0))]$, $l_2 = E\left[\int_0^{t\wedge\mathcal{D}_\iota} \left[-\alpha_1 |y(s)|^p + \alpha_2 \int_{-\mathcal{D}}^0 |y(s+\theta)|^p n(\theta)\, d\theta\right] ds\right]$, $E(M_t) = 0$.

Since $\alpha_1 \geq \alpha_2$, where

$$\begin{aligned} l_2 \leq & E\left(\int_0^{t\wedge\mathcal{D}_\iota} \left[-\alpha_2 |y(s)|^p + \alpha_2 \int_{-\mathcal{D}}^0 |y(s+\theta)|^p n(\theta)\, d\theta\right] ds\right) \\ & - (\alpha_1 - \alpha_2) E\left(\int_0^{t\wedge\mathcal{D}_\iota} |y(s)|^p\, ds\right) \\ \leq & -\alpha_2 E\left(\int_0^{t\wedge\mathcal{D}_\iota} |y(s)|^p\, ds\right) + \alpha_2 E\left(\int_\theta^{(t\wedge\mathcal{D}_\iota)+\theta} |y(s)|^p\, ds \int_{-\mathcal{D}}^0 n(\theta)\, d\theta\right) \\ \leq & -\alpha_2 E\left(\int_0^{t\wedge\mathcal{D}_\iota} |y(s)|^p\, ds\right) + \alpha_2 E\left(\int_{-\mathcal{D}}^{t\wedge\mathcal{D}_\iota} |y(s)|^p\, ds\right) \\ \leq & \alpha_2 E\left(\int_{-\mathcal{D}}^0 |y(s)|^p\, ds\right). \end{aligned}$$

Thus,

$$l_2 \leq \alpha_2 \mathcal{D} E \|\varphi_i\|^p. \tag{3.2}$$

By Lemma 2.1, we know

$$l_1 \le c_2 E|y(0)|^p + c_2\kappa E\left(\int_{-\mathcal{D}}^{0} |y(\theta)|^p \mu(\theta)\, d\theta\right) \\ \le c_2(1+\kappa)E\|\varphi_i\|^p. \tag{3.3}$$

Therefore, from $\mathcal{A}_3$,

$$E|Q_i(t\wedge \mathcal{D}_\iota)|^p \le \frac{1}{c_2}E(V(t\wedge \mathcal{D}_\iota, Q_i(t\wedge \mathcal{D}_\iota), \alpha(t\wedge \mathcal{D}_\iota))) \\ \le \frac{1}{c_2}[c_2(1+\kappa)+\alpha_2\mathcal{D}]E\|\varphi_i\|^p.$$

From the definition, we know $|y(\mathcal{D}_\iota)| = \iota$ and $|y(t\wedge \mathcal{D}_\iota)| \le \iota$, hence,

$$|Q_i(t\wedge \mathcal{D}_\iota)|^p \ge |Q_i(t\wedge \mathcal{D}_\iota)|^p \cdot I_{\{\mathcal{D}_\iota \le t\}} \\ \ge |y_i(t\wedge \mathcal{D}_\iota)|^p I_{\{\mathcal{D}_\iota \le t\}} - \kappa\int_{-\mathcal{D}}^{0} |y((t\wedge \mathcal{D}_\iota)+\theta)|^p \mu(\theta)\, d\theta \cdot I_{\{\mathcal{D}_\iota \le t\}} \\ \ge \iota^p(1-\kappa) I_{\{\mathcal{D}_\iota \le t\}}.$$

Thus,

$$P(\mathcal{D}_\iota \le t) = E[I_{\{\mathcal{D}_\iota \le t\}}] \le \frac{1}{c_1(1-\kappa)b^p}[c_1(1+\kappa)+\alpha_2\mathcal{D}]E\|\varphi_i\|^p.$$

Therefore,

$$\lim_{\iota\to\infty} P(\mathcal{D}_\iota \le t) = 0. \tag{3.4}$$

Step Two: When $p > 1$, from inequalities (3.1) and (3.2), we know

$$E[V(t\wedge \mathcal{D}_\iota, Q_i(t\wedge \mathcal{D}_\iota), \alpha(t\wedge \mathcal{D}_\iota))] \le l_1 + \alpha_2\mathcal{D}E\|\varphi_i\|^p.$$

By Lemma 2.1, for $p \in (1, +\infty)$,

$$|y_i(t)|^p \le 2^{p-1}|Q_i(t)|^p + 2^{p-1}\kappa\int_{-\mathcal{D}}^{0} |y_i(t+\theta)|^p n(\theta)\, d\theta,$$

hence,

$$E(V(t\wedge \mathcal{D}_\iota, Q_i(t\wedge \mathcal{D}_\iota), \alpha(t\wedge \mathcal{D}_\iota))) \\ \le 2^{p-1}c_2E\|y_i(0)\|^p + 2^{p-1}c_2\kappa E\left(\int_{-\mathcal{D}}^{0} |y_i(t)|^p n(\theta)\, d\theta\right) + \mathcal{D}\alpha_2 E\|\varphi_i\|^p \\ \le [2^{p-1}c_2(1+\kappa)+\mathcal{D}\alpha_2]E\|\varphi_i\|^p.$$

Therefore,

$$
\begin{aligned}
|Q_i(t \wedge \mathcal{D}_\iota)|^p &\geq |Q_i(t \wedge \mathcal{D}_\iota)|^p I_{\{\mathcal{D}_\iota \leq t\}} \\
&\geq |y_i(t \wedge \mathcal{D}_\iota)|^p I_{\{\mathcal{D}_\iota \leq t\}} \frac{1}{2^{p-1}} - \kappa \int_{-\mathcal{D}}^{0} |y_i((t \wedge \mathcal{D}_\iota) + \theta)|^p \mu(\theta)\, d\theta I_{\{\mathcal{D}_\iota \leq t\}} \\
&\geq \iota^p \left(\frac{1}{2^{p-1}} - \kappa \right) I_{\{\mathcal{D}_\iota \leq t\}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
P(\mathcal{D}_\iota \leq t) = E[I_{\{\mathcal{D}_\iota \leq t\}}] &\leq \frac{1}{b^p \left(\frac{1}{2^{p-1}} - \kappa \right)} E|Q_i(t \wedge \mathcal{D}_\iota)|^p \\
&\leq \frac{1}{c_1 b^p \left(\frac{1}{2^{p-1}} - \kappa \right)} E[V(t \wedge \mathcal{D}_\iota)] \\
&\leq \frac{1}{c_1 b^p \left(\frac{1}{2^{p-1}} - \kappa \right)} [2^{p-1} c_2 (1+\kappa) + \mathcal{D}\alpha_2] E\|\varphi_i\|^p.
\end{aligned}
$$

Therefore,

$$
\lim_{\iota \to \infty} P(\mathcal{D}_\iota \leq t) = 0. \tag{3.5}
$$

In summary, combining Eqs (3.4) and (3.5), the equation indeed possesses a global solution $y_i(t, \varphi, i_0)$.

3.2. Stability at zero equilibrium

To investigate asymptotic stability with general decay rates, we present some fundamental definitions as follows.

Definition 3.1. [(ψ-type Function) [29]] If there exists a function $\psi : \mathcal{R}^+ \to (0, \infty)$ such that we have the following requirements

(i) It is nondecreasing, continuous, and differentiable in $\mathcal{R}^+$ continuously ;
(ii) It satisfies the boundary conditions $\psi(0) = 1$ and $\lim_{t\to\infty} \psi(t) = \infty$;
(iii) Let $\tilde{\psi}(t) = \frac{\psi'(t)}{\psi(t)}$, satisfies

$$
\sup_{t>0} |\tilde{\psi}(t)| = h < \infty;
$$

(iv) We have $\psi(a) \leq \psi(b)\psi(a-b)$, for every $a, b \geq 0$,

then this function is called a ψ-type function.

Definition 3.2. [p-th Moment ψ-type Stability] If there exists an integer $q > 0$ such that for $t \geq 0$ and any initial value φ, the solution of Eq (2.2) satisfies

$$
\limsup_{t\to\infty} \frac{\ln \mathbb{E}|y(t, \varphi, k_0)|^p}{\ln \psi(t)} \leq -q, \tag{3.6}
$$

then the solution is called to be p-th moment ψ-type stable.

Definition 3.3. [Almost Sure ψ-type Stability] If there exists constant $q > 0$ such that for any initial value φ, the solution of Eq (2.2) satisfies

$$\limsup_{t\to\infty} \frac{\ln|y(t,\varphi,k_0)|}{\ln\psi(t)} < -q \quad \text{a.s.,} \tag{3.7}$$

then the solution is called to be almost surely ψ-type stable.

Theorem 3.2. *Suppose $\mathcal{A}_{1-3}$ hold. If there exists a constant $q > 0$ satisfying*

$$\begin{cases} \varphi(p)\kappa\psi^q(\mathcal{D}) < 1 \\ \varphi(p)c_2hq\left[1 + \kappa\psi^q(\mathcal{D})\right] + \alpha_2\psi^q(\mathcal{D}) < \alpha_1, \end{cases} \tag{3.8}$$

where $\varphi(p) = \begin{cases} 2^{p-1}, & p \in (1,+\infty) \\ 1, & p \in (0,1], \end{cases}$ *and κ is given by $\mathcal{A}_2$, for all $(t, y_i, y_i(t+\theta), i)$ on $\mathcal{R}^+ \times \mathcal{R}^n \times \mathbb{C}[[-\mathcal{D},0];\mathcal{R}^n] \times \mathcal{S}$, then the system* (2.2) *is p-th moment ψ-type stable.*

Remark 3.1. To investigate the stability of the system under a general decay rate, we consider a weighted Lyapunov function $\psi^q(t)V(t, Q_i(t), \alpha(t))$, where ψ is a ψ-type function and $q > 0$ controls the decay speed. Applying the generalized Itô formula to this function and utilizing the differential inequality in $\mathcal{A}_3$, we aim to derive an estimate of the form

$$\mathbb{E}[\psi^q(t)|y(t)|^p] \le C\|\varphi\|^p,$$

which then implies $\mathbb{E}|y(t)|^p = O(\psi^{-q}(t))$, i.e., the p-th moment decays at rate $\psi^{-q}(t)$.

Proof. The proof is divided into two steps based on the distribution of the p-value.

Step 1: For any $t \ge 0$, let $p \in (0,1]$ and $q \in (0,1)$. Under $\mathcal{A}_{1-3}$ and the generalized Itô formula, it can be deduced that

$$\begin{aligned} &E\left[\psi^q(t)V(t, Q_i(t), \alpha(t))\right] \\ &= E\left[\psi^q(0)V(0, Q_i(0), \alpha(0))\right] + E\left[\int_0^t \mathcal{L}(\psi^q(s)V(s, Q_i(s), \alpha(s)))\,ds\right] + E\left[\tilde{M}(s)\right] \\ &= l_1 \\ &\quad + E\left(\int_0^t \psi^q(s)\left[q\frac{\psi'(s)}{\psi(s)}V(s, Q_i(s), \alpha(s)) + \mathcal{L}V(s, Q_i(s), \alpha(s))\right]ds\right) \\ &\le l_1 + c_2hqE\left(\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right) \\ &\quad + c_2hq\kappa E\left(\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p\mu(\theta)\,d\theta\,ds\right) \\ &\quad + \alpha_2E\left(\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p n(\theta)d\theta\,ds\right) - \alpha_1E\left(\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right). \end{aligned} \tag{3.9}$$

By Definition 3.1, we have

$$\begin{aligned}
&E\left(\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p\mu(\theta)\,d\theta\,ds\right)\\
&\leq E\left(\int_{-\mathcal{D}}^0 \mu(\theta)\int_{\theta}^{t+\theta} \psi^q(s-\theta)|y_i(s)|^p\,ds\,d\theta\right)\\
&\leq E\left(\int_{-\mathcal{D}}^0 \mu(\theta)\int_{\theta}^{t+\theta} \psi^q(s)\psi^q(-\theta)|y_i(s)|^p\,ds\,d\theta\right)\\
&\leq E\left(\int_{-\mathcal{D}}^t \psi^q(s)\psi^q(\mathcal{D})|y_i(s)|^p\,ds\right)\\
&\leq \psi^q(\mathcal{D})\left[E\left(\int_{-\mathcal{D}}^0 \psi^q(s)|y_i(s)|^p\,ds+\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right)\right]\\
&\leq \psi^q(\mathcal{D})\left[\mathcal{D}E\|\varphi_i\|^p+E\left(\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right)\right]. \qquad (3.10)
\end{aligned}$$

For condition $\alpha_1 > c_2hq(1+\kappa\psi^q(\mathcal{D}))+\alpha_2\psi^q(\mathcal{D})$ together with (3.9), we then obtain

$$\begin{aligned}
&E\left(\psi^q(t)V(t,Q_i(t),\alpha(t))\right)\\
&\leq l_1+\left[c_2hq(1+\kappa\psi^q(\mathcal{D}))+\alpha_2\psi^q(\mathcal{D})-\alpha_1\right]\\
&\quad E\left(\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right)+\left[\mathcal{D}\psi^q(\mathcal{D})(c_2hq\kappa+\alpha_2)\right]E\|\varphi_i\|^p\\
&\leq l_1+\left[\mathcal{D}\psi^q(\mathcal{D})(c_2hq\kappa+\alpha_2)\right]E\|\varphi_i\|^p.
\end{aligned}$$

By Lemma 2.1,

$$E\left(|y_i(t)|^p\psi^q(t)\right)\leq \kappa E\left(\psi^q(t)\int_{-\mathcal{D}}^0 |y_i(t+\theta)|^p\mu(\theta)\,d\theta\right)+E\left(\psi^q(t)|Q_i(t)|^p\right), \qquad (3.11)$$

where

$$\begin{aligned}
&E\left(\psi^q(t)|Q_i(t)|^p\right)\\
&\leq \frac{1}{c_1}E\left(\psi^q(t)V(t,Q_i(t),\alpha(t))\right)\\
&\leq \frac{c_2}{c_1}E\left(|Q_i(0)|^p\right)+\frac{\mathcal{D}\psi^q(\mathcal{D})}{c_1}(c_2hq\kappa+\alpha_2)E\|\varphi_i\|^p\\
&\leq \frac{c_2}{c_1}E\left(\|\varphi_i\|^p\right)+\kappa\frac{c_2}{c_1}\sup_{\theta\in[\mathcal{D},0]}|y_i(\theta)|^p+\frac{\mathcal{D}\psi^q(\mathcal{D})}{c_1}(c_2hq\kappa+\alpha_2)E\|\psi\|^p\\
&\leq \left(\frac{c_2}{c_1}\left[1+\kappa+hq\kappa\mathcal{D}\psi^q(\mathcal{D})\right]+\frac{\alpha_2}{c_1}\mathcal{D}\psi^q(\mathcal{D})\right)E\|\varphi_i\|^p. \qquad (3.12)
\end{aligned}$$

Additionally, for any $t\in[0,T]$,

$$
\begin{aligned}
&E\left(\psi^q(t)\int_{-\mathcal{D}}^{0}|y_i(t+\theta)|^p\mu(\theta)\,d\theta\right)\\
&\le E\left(\int_{-\mathcal{D}}^{0}\psi^q(t+\theta)\psi^q(-\theta)|y_i(t+\theta)|^p\mu(\theta)\,d\theta\right)\\
&\le \psi^q(\mathcal{D})\sup_{t\in[-\mathcal{D},T]}E(\psi^q(t)|y_i(t)|^p)\\
&\le \psi^q(\mathcal{D})\left(\sup_{-t\in[-\mathcal{D},0]}E(\psi^q(t)|y_i(t)|^p)+\sup_{t\in[0,T]}E(\psi^q(t)|y_i(t)|^p)\right)\\
&\le \psi^q(\mathcal{D})\left(E\|\varphi_i\|^p+\sup_{t\in[0,T]}E\left(\psi^q(t)|y_i(t)|^p\right)\right).
\end{aligned}\tag{3.13}
$$

Hence, combining (3.12) and (3.13), inequality (3.11) can be further bounded as follows

$$
\begin{aligned}
&\sup_{t\in[0,T]}E(\psi^q(t)|y_i(t)|^p)\\
&\le \sup_{t\in[0,T]}E(\psi^q(t)|Q_i(t)|^p)+\kappa\psi^q(\mathcal{D})E\|\varphi_i\|^p+\sup_{t\in[0,T]}\kappa\psi^q(\mathcal{D})E(\psi^q(t)|y_i(t)|^p)\\
&\le \frac{E\|\varphi_i\|^p}{1-\kappa\psi^q(\mathcal{D})}\left[\frac{c_2}{c_1}(1+\kappa+hq\kappa\mathcal{D}\psi^q(\mathcal{D}))+\frac{\alpha_2}{c_1}\mathcal{D}\psi^q(\mathcal{D})+\kappa\psi^q(\mathcal{D})\right]\\
&\le A_1E\|\varphi_i\|^p,
\end{aligned}
$$

where $A_1=\frac{1}{1-\kappa\psi^q(\mathcal{D})}\left[\frac{c_2}{c_1}(1+\kappa+hq\kappa\mathcal{D}\psi^q(\mathcal{D}))+\frac{\alpha_2}{c_1}\mathcal{D}\psi^q(\mathcal{D})+\kappa\psi^q(\mathcal{D})\right]$. Thus, when $T\to\infty$, we have

$$
\sup_{t\in[0,\infty)}E(\psi^q(t)|y_i(t)|^p)\le A_1E\|\varphi_i\|^p,
$$

that is, $E|y_i(t)|^p\le A_1E\|\varphi_i\|^p\psi^{-q}(t), t\ge 0$.

Hence, for all $t\ge 0$,

$$
\limsup_{t\to\infty}\frac{\ln E|y_i(t)|^p}{\ln\psi(t)}\le\limsup_{t\to\infty}\left(-q+\frac{\ln(A_1E\|\varphi_i\|^p)}{\ln\psi(t)}\right)=-q.\tag{3.14}
$$

Step 2: When $p\in(1,+\infty)$. Similarly to (3.9), obtain the following

$$
\begin{aligned}
&E\left[\psi^q(t)V(t,Q_i(t),\alpha(t))\right]\\
&\le l_1+2^{p-1}c_2hqE\left(\int_0^t\psi^q(s)|y_i(s)|^p\,ds\right)\\
&+2^{p-1}c_2hq\kappa E\left(\int_0^t\psi^q(s)\int_{-\mathcal{D}}^{0}|y_i(s+\theta)|^p\mu(\theta)\,d\theta\,ds\right)\\
&+\alpha_2E\left(\int_0^t\psi^q(s)\int_{-\mathcal{D}}^{0}|y_i(s+\theta)|^pn(\theta)d\theta\,ds\right)-\alpha_1E\left(\int_0^t\psi^q(s)|y_i(s)|^p\,ds\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&E\left[\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p \mu(\theta)\,d\theta\,ds\right] \\
&\le \psi^q(\mathcal{D})\left(\mathcal{D}E\|\varphi_i\|^p + E\left[\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right]\right).
\end{aligned}
$$

By assumption $\alpha_1 > 2^{p-1}c_2hq\left[1+\kappa\psi^q(\mathcal{D})\right] + \alpha_2\psi^q(\mathcal{D})$, then

$$
\begin{aligned}
&E\left[V(t,Q_i(t),\alpha(t))\psi^q(t)\right] \\
&\le l_1 + \left[2^{p-1}c_2hq(1+\kappa\psi^q(\mathcal{D})) + \alpha_2\psi^q(\mathcal{D}) - \alpha_1\right] \\
&\quad E\left(\int_0^t \psi^q(s)|y_i(s)|^p\,ds\right) + \left[\mathcal{D}\psi^q(\mathcal{D})(2^{p-1}c_2hq\kappa + \alpha_2)\right]E\|\varphi_i\|^p \\
&\le l_1 + \left[\mathcal{D}\psi^q(\mathcal{D})(2^{p-1}c_2hq\kappa + \alpha_2)\right]E\|\varphi_i\|^p.
\end{aligned}
$$

By Lemma 2.1,

$$
E\left(\psi^q(t)|y_i(t)|^p\right) \le 2^{p-1}E\left(\psi^q(t)|Q_i(t)|^p\right) + 2^{p-1}\kappa E\left(\psi^q(t)\int_{-\mathcal{D}}^0 |y_i(t+\theta)|^p\mu(\theta)\,d\theta\right), \tag{3.15}
$$

where

$$
\begin{aligned}
&E\left(\psi^q(t)|Q_i(t)|^p\right) \\
&\le \frac{1}{c_1}E\left(\psi^q(t)V(t,Q_i(t),\alpha(t))\right) \\
&\le \left(\frac{2^{p-1}c_2}{c_1}\left[1+\kappa+hq\kappa\mathcal{D}\psi^q(\mathcal{D})\right] + \frac{\alpha_2}{c_1}\mathcal{D}\psi^q(\mathcal{D})\right)E\|\varphi_i\|^p.
\end{aligned}
$$

Applying the methodology outlined in Step 1, for every $t \in [0,T]$, yields

$$
\begin{aligned}
&E\left(\psi^q(t)\int_{-\mathcal{D}}^0 |y_i(t+\theta)|^p\mu(\theta)\,d\theta\right) \\
&\le \psi^q(\mathcal{D})\left(E\|\varphi_i\|^p + \sup_{t\in[0,T]} E\left(\psi^q(t)|y_i(t)|^p\right)\right).
\end{aligned}
$$

Therefore, inequality (3.15) can be further bounded as follows

$$\begin{aligned}
&\sup_{t\in[0,T]} E(\psi^q(t)|y_i(t)|^p) \\
&\le 2^{p-1}[\sup_{t\in[0,T]} E(\psi^q(t)|Q_i(t)|^p) + \kappa\psi^q(\mathcal{D})E\|\varphi_i\|^p \\
&\quad + \kappa\psi^q(\mathcal{D}) \sup_{t\in[0,T]} E(\psi^q(t)|y_i(t)|^p)] \\
&\le \frac{2^{p-1}E\|\varphi_i\|^p}{1-2^{p-1}\kappa\psi^q(\mathcal{D})}\left[\frac{2^{p-1}c_2}{c_1}(1+\kappa+hq\kappa\mathcal{D}\psi^q(\mathcal{D})) + \frac{\alpha_2}{c_1}\mathcal{D}\psi^q(\mathcal{D}) + \kappa\psi^q(\mathcal{D})\right] \\
&\le A_2E\|\varphi_i\|^p,
\end{aligned}$$

where

$$A_2 = \frac{2^{p-1}}{1-2^{p-1}\kappa\psi^q(\mathcal{D})}\left[\frac{2^{p-1}c_2}{c_1}(1+\kappa+hq\kappa\mathcal{D}\psi^q(\mathcal{D})) + \frac{\alpha_2}{c_1}\mathcal{D}\psi^q(\mathcal{D}) + \kappa\psi^q(\mathcal{D})\right].$$

Hence, similarly to (3.15), for any $t \in [0, +\infty]$, we get

$$\limsup_{t\to\infty} \frac{\ln E|y_i(t)|^p}{\ln\psi(t)} \le -q. \tag{3.16}$$

In conclusion, taking $A = \max(A_1, A_2)$, we have

$$\limsup_{t\to\infty} \frac{\ln E|y_i(t)|^p}{\ln\psi(t)} \le -q.$$

Therefore, Theorem 3.2 holds.

Theorem 3.3. *Under $\mathcal{A}_{1-3}$, if a constant $q > 0$ exists for which inequality* (3.8) *is satisfied, then the solution of Eq* (2.2) *is almost surely ψ-type stable, and the decay occurs at the rate q/p.*

Remark 3.2. To prove the conclusion in Theorem 3.3 using Definition 3.3, it suffices to show

$$\limsup_{t\to\infty} \frac{\ln|y_i(t)|}{\ln\psi(t)} \le -\frac{q}{p} \quad \text{a.s..} \tag{3.17}$$

Proof. ***Step 1:*** For any $t \in \mathcal{R}^+$, assume $p \in (0, 1]$, and let $q \in (0, 1)$. Under $\mathcal{A}_{1-3}$ and the generalized Itô formula, we can derive

$$\begin{aligned}
&\psi^q(t)V(t, Q_i(t), \alpha(t)) \\
&= \psi^q(0)V(0, Q_i(0), \alpha(0)) + \int_0^t \mathcal{L}(\psi^q(s)V(s, Q_i(s), \alpha(s)))\,ds + \int_0^t \psi^q(s)dM_s \\
&= V(0, Q_i(0), \alpha(0)) \\
&\quad + \int_0^t \psi^q(s)\left[q\frac{\psi'(s)}{\psi(s)}V(s, Q_i(s), \alpha(s)) + \mathcal{L}V(s, Q_i(s), \alpha(s))\right] ds + \tilde{M}_t \\
&= I_1 + I_2 + \tilde{M}_t,
\end{aligned} \tag{3.18}$$

where $\tilde{M}_t = \int_0^t \psi^q(s)dM_s$ is a certain continuous local martingale with $\tilde{M}_0 = 0$, and $I_1 = V(0, Q_i(0), \alpha(0))$, $I_2 = \int_0^t \psi^q(s)\left[q\frac{\psi'(s)}{\psi(s)}V(s, Q_i(s), \alpha(s)) + \mathcal{L}V(s, Q_i(s), \alpha(s))\right] ds$. From the property of function $\psi(t)$, we obtain

$$\begin{aligned}I_2 &\le c_2hq\int_0^t \psi^q(s)|y_i(s)|^p\,ds + c_2hq\kappa\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p\mu(\theta)\,d\theta\,ds\\ &+ \alpha_2\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 n(\theta)|y_i(s+\theta)|^p d\theta\,ds - \alpha_1\int_0^t \psi^q(s)|y_i(s)|^p\,ds.\end{aligned}$$

From inequality (3.10), we have

$$\begin{aligned}&\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p\mu(\theta)\,d\theta\,ds\\ &\le \psi^q(\mathcal{D})\left[\mathcal{D}||\varphi_i||^p + \int_0^t \psi^q(s)|y_i(s)|^p\,ds\right].\end{aligned}$$

Analogously, we can prove

$$\begin{aligned}&\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p n(\theta)\,d\theta\,ds\\ &\le \psi^q(\mathcal{D})\left[\mathcal{D}||\varphi_i||^p + \int_0^t \psi^q(s)|y_i(s)|^p\,ds\right].\end{aligned}$$

Then

$$\begin{aligned}I_2 \le &\left[c_2hq(1+\kappa\psi^q(\mathcal{D})) + \alpha_2\psi^q(\mathcal{D}) - \alpha_1\right]\int_0^t \psi^q(s)|y_i(s)|^p\,ds\\ &+ \left[\mathcal{D}\psi^q(\mathcal{D})(c_2hq\kappa + \alpha_2)\right]||\varphi_i||^p.\end{aligned}$$

For the condition $\alpha_1 > c_2hq(1+\kappa\psi^q(\mathcal{D})) + \alpha_2\psi^q(\mathcal{D})$, then we obtain

$$\psi^q(t)V(t, Q_i(t), \alpha(t)) \le I_1 + \mathcal{D}\psi^q(\mathcal{D})(c_2hq\kappa + \alpha_2)||\varphi_i||^p + \tilde{M}_t.$$

By the nonnegative semimartingale convergence theorem [37], it can be inferred that a random value $l_1 > 0$ exists for which

$$\psi^q(t)V(t, Q_i(t), \alpha(t)) \le l_1 \quad \text{a.s..}$$

Similarly to inequalities (3.11)–(3.13), we can obtain

$$\psi^q(t)|y_i(t)|^p \le \psi^q(t)|Q_i(t)|^p + \kappa\psi^q(\mathcal{D})||\varphi_i||^p + \kappa\psi^q(\mathcal{D})\sup_{t\ge 0}\psi^q(t)|y_i(t)|^p.$$

For all nonnegative value T, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p \\
&\le \sup_{t\in[0,T]} \psi^q(t)|Q_i(t)|^p + \kappa\psi^q(\mathcal{D})\|\varphi_i\|^p + \kappa\psi^q(\mathcal{D}) \sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p \\
&\le \frac{1}{1-\kappa\psi^q(\mathcal{D})} \sup_{t\in[0,T]} \psi^q(t)|Q_i(t)|^p + \frac{\kappa\psi^q(\mathcal{D})}{1-\kappa\psi^q(\mathcal{D})}\|\varphi_i\|^p \\
&\le \frac{l_1}{c_1[1-\kappa\psi^q(\mathcal{D})]} + \frac{\kappa\psi^q(\mathcal{D})}{1-\kappa\psi^q(\mathcal{D})}\|\varphi_i\|^p.
\end{aligned}
$$

That is, for all $t \in \mathcal{R}^+$, a constant $M > 0$ exists for which $\psi^q(t)|y_i(t)|^p \le M$, thus,

$$
\frac{\ln|y_i(t)|}{\ln\psi(t)} < \frac{\ln M}{p\ln\psi(t)} - \frac{q}{p}. \tag{3.19}
$$

Hence, from the property of function $\psi(t)$, we deduce

$$
\limsup_{t\to\infty} \frac{\ln|y_i(t)|}{\ln\psi(t)} \le -\frac{q}{p}.
$$

Step 2: When $p \in (1, +\infty)$, Eq (3.18) still holds, and by Lemma 2.1, we have

$$
\begin{aligned}
I_2 \le\ & 2^{p-1}c_2hq\int_0^t \psi^q(s)|y_i(s)|^p ds + 2^{p-1}c_2hq\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p\mu(\theta)d\theta ds \\
&+ \alpha_2\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p n(\theta)d\theta ds - \alpha_1\int_0^t \psi^q(s)|y_i(s)|^p ds \\
\le\ & [2^{p-1}c_2hq(1+\kappa\psi^q(\mathcal{D})) - \alpha_1 + \alpha_2\psi^q(\mathcal{D})]\int_0^t \psi^q(s)|y_i(s)|^p ds \\
&+ (2^{p-1}c_2hq\kappa + \alpha_2)\psi^q(\mathcal{D})\mathcal{D}\|\varphi_i\|^p.
\end{aligned}
$$

For the condition $\alpha_1 > [2^{p-1}c_2hq(1+\kappa\psi^q(\mathcal{D}))] + \alpha_2\psi^q(\mathcal{D})$, then we obtain

$$
\psi^q(t)V(t, Q_i(t), \alpha(t)) \le I_1 + \mathcal{D}\psi^q(\mathcal{D})(2^{p-1}c_2hq\kappa + \alpha_2)\|\varphi_i\|^p + \tilde{M}_t.
$$

Similarly, a random constant $l_2 > 0$ exists for which

$$
\psi^q(t)V(t, Q_i(t), \alpha(t)) \le l_2.
$$

Similar to Step 1 and by applying Lemma 2.1, for all $T \in \mathcal{R}^+$, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p \\
&\le 2^{p-1}[\sup_{t\in[0,T]} \psi^q(t)|Q_i(t)|^p + \kappa\psi^q(\mathcal{D})\|\varphi_i\|^p + \kappa\psi^q(\mathcal{D}) \sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p] \\
&\le \frac{2^{p-1}l_2}{c_1[1-2^{p-1}\kappa\psi^q(\mathcal{D})]} + \frac{2^{p-1}\kappa\psi^q(\mathcal{D})}{1-2^{p-1}\kappa\psi^q(\mathcal{D})}\|\varphi_i\|^p.
\end{aligned}
$$

Following inequality (3.19), we have

$$
\limsup_{t\to\infty} \frac{\ln|y_i(t)|}{\ln\psi(t)} \le -\frac{q}{p}.
$$

In conclusion, Theorem 3.3 holds.

3.3. Practical stability at a nonzero equilibrium

In this part, if the intensity measure $\nu(dz)$ is nonzero, then the zero solution $y(0) \equiv 0$ no longer exists for the system (2.2). This work investigates the practical stability properties of the system when the origin fails to be an equilibrium point. Specifically, we examine the convergence behavior of the system trajectories to a small neighborhood of zero, characterized by the ball $B_\delta := \{y \in \mathcal{R}^n : \|y\| < \delta\}$ with radius $\delta > 0$.

Based on the definitions of globally uniform boundedness for the system mentioned by [35] and the convergence of its solutions established in [33], we summarize the definition of almost surely practical stability uniformly globally for the solution of the system.

Definition 3.4. [33] If constants $\delta, \gamma^* > 0$ exist for which the solution fulfills

$$\limsup_{t\to\infty} \frac{\ln|y(t,\varphi,t_0) - \delta|}{\ln\psi(t)} < -\gamma^*, \tag{3.20}$$

where $\psi(t) \in C(\mathcal{R}^+;\mathcal{R}^+)$ is a ψ-type function, then the solution of the equation is called to be almost surely practically stable uniformly globally with ψ-type decay.

Theorem 3.4. *Suppose $\mathcal{A}_{1-2}$ hold, and let $p \in \mathbb{N}^+$. If there exists a function $V(t, y_i, k) \in C^{1,2}(\mathcal{R}^+ \times \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n] \times \mathcal{S}; \mathcal{R}^+)$, with $\psi^q(t)e(t) \in \mathbb{L}^1(\mathcal{R}^+)$, $\psi(t) \in C(\mathcal{R}^+;\mathcal{R}^+)$ is a ψ-type function, and there exist constants $\alpha_1, \alpha_2, c_1, c_2 > 0$ with $\alpha_1 > \alpha_2$, satisfying*

($\mathcal{H}_1$): $c_1|Q_i|^p \le V(t, Q_i, k) \le c_2|Q_i|^p$, *for all* $(t, Q_i, k) \in \mathcal{R}^+ \times \mathbb{C}[[-\mathcal{D}, 0]; \mathcal{R}^n] \times \mathcal{S}$;

($\mathcal{H}_2$): $\mathcal{L}V(t, y_i, y_i(t+\theta), k) \le \alpha_2 \int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p n(\theta)\, d\theta - \alpha_1|y_i(t)|^p + e_i(t)$;

($\mathcal{H}_3$): $\begin{cases} c_1\left(1 - 2^{p-1}\kappa\psi^q(\mathcal{D})\right) > 2^{p-1} \\ \alpha_1 > c_2 hq\,(1 + \kappa\psi^q(\mathcal{D})) + \alpha_2\psi^q(\mathcal{D}); \end{cases}$

($\mathcal{H}_4$): $\lim_{t\to\infty} \frac{e_i(t)}{\psi^q(t)} = 0^+$, *and* $|y_i(t, \varphi, k_0)| > \left(\frac{e_i(t)}{\psi^q(t)}\right)^{1/p}$,

then the solution of Eq (2.2) is almost surely practically ψ-type stable uniformly globally, and the decay occurs at the rate q.

Remark 3.3 Consider the equilibrium point in a small neighborhood $B_\delta := \{y \in \mathcal{R}^n : \|y\| < r\}$, with $\delta > 0$. Let $e(t) = \frac{\delta^p}{\psi^q(t)}$, then it suffices to verify

$$\limsup_{t\to\infty} \frac{\ln\left|y_i(t,\varphi,k_0) - \left(\frac{e_i(t)}{\psi^q(t)}\right)^{1/p}\right|}{\ln\psi(t)} \le -q \quad \text{a.s..} \tag{3.21}$$

Proof. By Assumptions ($\mathcal{H}_1$)–($\mathcal{H}_3$) and using the similar procedure to the proof of Theorem 3.3,

$$
\begin{aligned}
&\psi^q(t)V(t, Q_i(t), \alpha(t)) \\
&\leq I_1 + 2^{p-1}c_2hq\int_0^t \psi^q(s)|y_i(s)|^p ds \\
&\quad + 2^{p-1}c_2hq\kappa\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p\mu(\theta)d\theta ds \\
&\quad + \alpha_2\int_0^t \psi^q(s)\int_{-\mathcal{D}}^0 |y_i(s+\theta)|^p n(\theta)d\theta ds - \alpha_1\int_0^t \psi^q(s)|y_i(s)|^p ds \\
&\quad + \int_0^t \psi^q(s)e_i(s)ds + \tilde{M}_t \\
&\leq I_1 + [2^{p-1}c_2hq(1+\kappa\psi^q(\mathcal{D})) + \alpha_2\psi^q(\mathcal{D}) - \alpha_1]\int_0^t \psi^q(s)|y_i(s)|^p ds \\
&\quad + [2^{p-1}c_2hq\kappa + \alpha_2]\mathcal{D}\psi^q(\mathcal{D})||\varphi_i||^p + \int_0^t \psi^q(s)e_i(s)ds + \tilde{M}_t \\
&\leq I_1 + [2^{p-1}c_2hq\kappa + \alpha_2]\mathcal{D}\psi^q(\mathcal{D})||\varphi_i||^p + \int_0^t \psi^q(s)e_i(s)ds + \tilde{M}_t.
\end{aligned}
$$

By the nonnegative semimartingale convergence theorem, it can be inferred that a random value $l_3 > 0$ exists for which

$$
\psi^q(t)V(t, Q_i(t), \alpha(t)) \leq l_3, \quad t \in \mathcal{R}^+.
$$

Similar to procedure of Theorem 3.3 and by applying Lemma 2.1, for any $T \in \mathcal{R}^+$, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p \\
&\leq 2^{p-1}[\sup_{t\in[0,T]} \psi^q(t)|Q_i(t)|^p + \kappa\psi^q(\mathcal{D})||\varphi_i||^p + \kappa\psi^q(\mathcal{D})\sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p] \\
&\leq \frac{2^{p-1}[l_3 + e_i(t)]}{c_1[1-2^{p-1}\kappa\psi^q(\mathcal{D})]} + \frac{2^{p-1}\kappa\psi^q(\mathcal{D})}{1-2^{p-1}\kappa\psi^q(\mathcal{D})}||\varphi_i||^p.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sup_{t\in[0,T]} \psi^q(t)|y_i(t)|^p - e_i(t) &\leq \sup_{t\in[0,T]} (\psi^q(t)|y_i(t)|^p - \frac{2^{p-1}e_i(t)}{c_1[1-2^{p-1}\kappa\psi^q(\mathcal{D})]} \\
&\leq \frac{2^{p-1}l_3}{c_1[1-2^{p-1}\kappa\psi(\mathcal{D})]} + \frac{2^{p-1}\psi(\mathcal{D})}{1-2^{p-1}\kappa\psi(\mathcal{D})}||\varphi_i||^p.
\end{aligned}
$$

Observe that

$$
\psi^q(t)|y_i(t)|^p - e_i(t) = \psi^q(t)\left(|y_i(t)|^p - \left(\left[\frac{e_i(t)}{\psi^q(t)}\right]^{\frac{1}{p}}\right)^p\right).
$$

Using the identity

$$\begin{aligned}\zeta^p - \theta^p &= (\zeta - \theta)(\zeta^{p-1} + \zeta^{p-2}\theta + \cdots + \zeta\theta^{p-2} + \theta^{p-1}) \\ &= (\zeta - \theta)\sum_{i=1}^{p} \zeta^{p-i}\theta^{i-1}.\end{aligned}$$

Therefore,

$$\psi^q(t)|y_i(t)|^p - e_i(t) = \psi^q(t)\left(|y_i(t)| - \left[\frac{e_i(t)}{\psi^q(t)}\right]^{\frac{1}{p}}\right) \cdot \sum_{k=1}^{p} |y_i(t)|^{p-k}\left[\frac{e_i(t)}{\psi^q(t)}\right]^{\frac{k-1}{p}}.$$

By Assumption ($\mathcal{H}_4$), $\lim_{t\to\infty} \frac{e_i(t)}{\psi^q(t)} = 0^+$, thus, for some $\omega_0 > 0$, there exists $t \geq 0$ such that $\frac{e_i(t)}{\psi^q(t)} = \omega_0$. Since $|y_i(t, \varphi, i_0)| > \left(\frac{e_i(t)}{\psi^q(t)}\right)^{\frac{1}{p}}$, for all $t \geq 0$, we deduce that

$$\sum_{k=1}^{p} |y_i(t)|^{p-k}\left(\frac{e_i(t)}{\psi^q(t)}\right)^{\frac{k-1}{p}} \geq \sum_{k=1}^{p}\left(\frac{e_i(t)}{\psi^q(t)}\right)^{\frac{p-1}{p}} = p\omega_o^{\frac{p-1}{p}} =: \omega.$$

Then, it yields that

$$\psi^q(t)\left(|y_i(t)| - \left[\frac{e_i(t)}{\psi^q(t)}\right]^{\frac{1}{p}}\right) \leq \frac{1}{\omega}(\psi^q(t)|y_i(t)|^p - e_i(t)).$$

Therefore, for all $T \in \mathcal{R}^+$,

$$\begin{aligned}&\sup_{t\in[0,T]} \psi^q(t)\left(|y_i(t)| - \left[\frac{e_i(t)}{\psi^q(t)}\right]^{\frac{1}{p}}\right) \\ &\leq \frac{1}{\omega}\left(\frac{2^{p-1}l_3}{c_1[1 - 2^{p-1}\kappa\psi(\mathcal{D})]} + \frac{2^{p-1}\psi(\mathcal{D})}{1 - 2^{p-1}\kappa\psi(\mathcal{D})}\right)||\varphi_i||^p.\end{aligned}$$

Similar to inequality (3.19), we obtain

$$\limsup_{t\to\infty} \frac{\ln|y_i(t, \varphi, i_0) - (\frac{e_i(t)}{\psi^q(t)})^{\frac{1}{p}}|}{\ln\psi(t)} \leq -q \quad \text{a.s.}.$$

Therefore, Theorem 3.4 follows.

4. Examples

In this section, we will examine three instances simulating Theorems 3.2–3.4, respectively. Using the mathematical software MATLAB, we will visualize the evolution of the system state over time to further demonstrate the validity of these theorems. Then, consider these neural networks with Markov switching

$$\begin{aligned}d\left[y_i(t) - \kappa_i \int_{-\mathcal{D}}^{0} y_i(t+\theta)\mu(\theta)d\theta\right] &= F_i(t, y(t), y_t, \alpha(t))dt \\ &\quad G_i(t, y_i(t), y_i(t+\theta), \alpha(t))\, dW_i(t), t \geq 0. \qquad (4.1)\end{aligned}$$

Example 4.1 Considering system (4.1) in the one-dimensional case, let

$$\begin{aligned} F_1 &= -2[y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)d\theta], \\ F_2 &= -2.5[y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)d\theta], \\ G_1 &= 0.8[y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)d\theta], \\ G_2 &= 0.6[y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)d\theta], \end{aligned}$$

where $\kappa = \frac{3}{10}, \mathcal{D} = 1$, let $\mathcal{S} = \{1, 2\}$, and let the generator Γ be defined by

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

Taking distinct Lyapunov functions corresponding to each of the two-modes Markov

$$V_1(Q(t)) = |Q(t)|^2, V_2(Q(t)) = \frac{6}{5}|Q(t)|^2,$$

then $|Q|^2 \leq V \leq \frac{6}{5}|Q|^2$. We can compute the generator by its definition

$$\begin{aligned} \mathcal{L}V_1(Q(t)) &= -4Q^2 + 0.64Q^2 - 2Q^2 + 2.4Q^2 \\ &\leq -2\left(y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)\, d\theta\right)^2. \end{aligned}$$

When $k = 2$,

$$\begin{aligned} \mathcal{L}V_2(Q(t)) &= -6Q^2 + 1.2 * 0.36Q^2 + Q^2 - 1.2Q^2 \\ &\leq -5\left(y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)\, d\theta\right)^2. \end{aligned}$$

By using general inequality and Hölder's inequality, we have

$$
\begin{aligned}
\mathcal{L}V_1(Q(t)) &\le -2\left(y(t) - \frac{3}{10}\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&= -2\left[y^2(t) + \frac{9}{100}\left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2\right] \\
&\quad + \frac{6}{5}y(t)\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta \\
&\le -2y^2(t) + 0.36y^2(t) + \left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\quad - 0.18\left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\le -1.64y^2(t) + 0.82\left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\le -1.64y^2(t) + 0.82\int_{-1}^{0} y^2(t+\theta)\mu(\theta)\,d\theta.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}V_2(Q(t)) &\le -5\left[y^2(t) + 0.09\left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2\right] \\
&\quad + 3y(t)\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta \\
&\le -5y^2(t) + 1.5y^2(t) + \left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\quad - 0.45\left(\int_{-1}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\le -2.75y^2(t) + 0.55\int_{-1}^{0} y^2(t+\theta)\mu(\theta)\,d\theta.
\end{aligned}
$$

Thus, for every $k \in \mathcal{S}$, the following holds:

$$
\mathcal{L}V_k(Q(t)) \le 0.82\int_{-1}^{0} y^2(t+\theta)\mu(\theta)\,d\theta - 1.64y^2(t).
$$

Therefore, the coefficients of the system satisfy $\mathcal{A}_{1-3}$ under the following parameter choices $c_1 = 1, c_2 = 1.2, \alpha_1 = 1.64, \alpha_2 = 0.82, \kappa = \frac{3}{10}, \psi(t) = e^t, p = 2$, and $q = 0.1$. Having verified these assumptions, we directly apply Theorems 3.2 and 3.3 to establish that the equation is almost surely ψ-stable.

Example 4.2 Considering system (4.1) in the two-dimensional case, let

$$\begin{aligned} F_1 &= A_1 Q(t), \\ F_2 &= A_2 Q(t), \\ G_1 &= B_1 Q(t), \\ G_2 &= B_2 Q(t), \end{aligned}$$

where

$$Q(t) = \begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta \\ y_2(t) - \frac{3}{10}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -2.5 & 0.3 \\ 0.2 & -3 \end{pmatrix}, B_1 = \begin{pmatrix} 0.6 & 0.1 \\ 0.05 & 0.7 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 0.2 \\ 0.1 & -3.5 \end{pmatrix}, B_2 = \begin{pmatrix} 0.5 & 0.05 \\ 0.1 & 0.6 \end{pmatrix}.$$

Let $\mathcal{S} = \{1, 2\}$ and the generator Γ be defined by

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

We utilize distinct Lyapunov functions corresponding to each of the two Markov subsystems

$$V_1(Q(t)) = \|Q(t)\|^2, V_2(Q(t)) = \frac{6}{5}\|Q(t)\|^2,$$

then $\|Q(t)\|^2 \le V \le \frac{6}{5}\|Q(t)\|^2$. The generator can be computed, when $k = 1$,

$$\begin{aligned} \mathcal{L}V_1(Q(t)) &= 2Q^T A_1 Q + tr(B_1^T B_1 Q Q^T) - 2\|Q\|^2 + 2.4\|Q\|^2 \\ &\le -4.68\|Q\|^2 + 0.72\|Q\|^2 + 0.4\|Q\|^2 \\ &\le -3.56\left(y_1(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\ &\quad - 3.56\left(y_2(t) - \frac{3}{10}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2. \end{aligned}$$

Also, when $k = 2$,

$$\begin{aligned} \mathcal{L}V_2(Q(t)) &\le -6.96\|Q\|^2 + 0.72\|Q\|^2 - 0.2\|Q\|^2 \\ &\le -6.44\left(y_1(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\ &\quad - 6.44\left(y_2(t) - \frac{3}{10}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2. \end{aligned}$$

By using general inequality and Hölder's inequality, we have

$$
\begin{aligned}
\mathcal{L}V_1(Q(t)) &\leq -3.56\left(y_1(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\quad - 3.56\left(y_2(t) - \frac{3}{10}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\leq -3.56|y(t)|^2 - 0.54\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\quad + 0.79|y(t)|^2 + 2.44\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\leq -2.76|y(t)|^2 + 1.9\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\leq 1.9\int_{-0.5}^{0} y^2(t+\theta)\mu(\theta)\,d\theta - 2.76|y(t)|^2.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}V_2(Q(t)) &\leq -6.44\left(y_1(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\quad - 6.44\left(y_2(t) - \frac{3}{10}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\leq -6.44|y(t)|^2 - 0.98\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\quad + 2.59|y(t)|^2 + 2.44\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 \\
&\leq 1.46\int_{-0.5}^{0} y^2(t+\theta)\mu(\theta)\,d\theta - 3.85|y(t)|^2.
\end{aligned}
$$

Thus, for every $k \in \mathcal{S}$, the following holds:

$$
\mathcal{L}V_k(Q(t)) \leq 1.9\int_{-0.5}^{0} y^2(t+\theta)\mu(\theta)\,d\theta - 2.76|y(t)|^2.
$$

Therefore, the coefficients of the system satisfy $\mathcal{A}_{1-3}$ under the following parameter choices $c_1 = 1, c_2 = 1.2, \alpha_1 = 2.76, \alpha_2 = 1.9, \kappa_1 = \frac{1}{4}, \kappa_2 = \frac{3}{10}, q = 0.1, p = 2$, and $\psi(t) = e^t$. Having verified these assumptions, we directly apply Theorems 3.2 and 3.3 to establish that the solution of Eq (4.1) is almost surely ψ-stable.

The numerical simulations to compute the mean-square trajectory $\|y(t)\|^2$ were conducted using the Euler–Maruyama method. Key implementation details include a time step of $\Delta t = 0.01$ and averaging over $N = 100$ independent Monte Carlo trials. Figures 1 and 2 depict the resulting mean-square trajectories for Examples 4.1 and 4.2, respectively. The clear convergence of $\|y(t)\|^2$ to zero in both cases verifies the system's ψ-type stability in the mean-square sense.

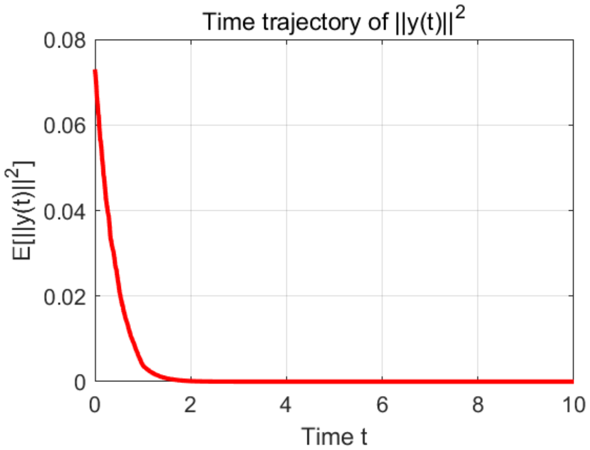

Figure 1. State Trajectories of $||y(t)||^2$ in Example 4.1.

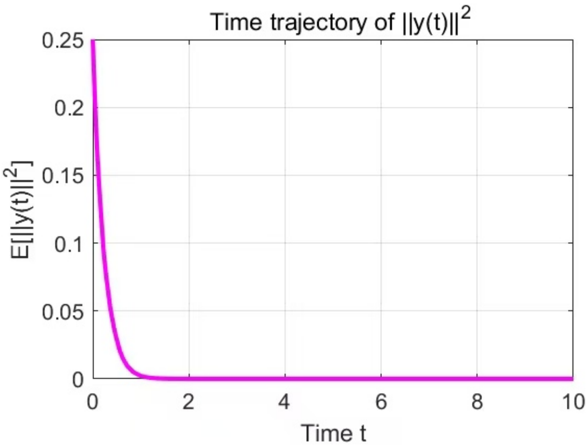

Figure 2. State Trajectories of $||y(t)||^2$ in Example 4.2.

Example 4.3 We apply the theoretical framework to an autonomous vehicle trajectory tracking control system. In this system, the acquisition and processing of sensor data are subject to random time delays; the vehicle may switch randomly between different driving modes; and the system is continuously subjected to bounded random disturbances, such as road unevenness. The control problem can be modeled using system (4.1). The specific parameters are expressed as follows:

$$\begin{aligned} F_1 &= -2.2[y(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)d\theta], \\ F_2 &= -2.8[y(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)d\theta], \\ G_1 &= 0.7e^{-t}tanhy(t), \\ G_2 &= 0.5e^{-t}tanhy(t), \end{aligned}$$

where $\kappa = \frac{1}{4}$. Let $\mathcal{S} = \{1, 2\}$ and the generator Γ be defined by

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

Here, the system state $y(t)$ represents the lateral tracking error between the vehicle and the desired trajectory at time t; the Markov process $\alpha(t)$ describes the random switching of driving modes; and the neutral-type delay term $\kappa_i \int_{-\mathcal{D}}^{0} y_i(t+\theta)\mu(\theta)d\theta$ reflects the history of tracking errors over a past period.

For the vehicle control model established above, we define the function V for the two-modes Markov

$$V_1(Q(t)) = |Q(t)|^2, V_2(Q(t)) = \frac{6}{5}|Q(t)|^2,$$

then $|Q|^2 \leq V \leq \frac{6}{5}|Q|^2$. We can compute the generator by its definition when $k = 1$,

$$\mathcal{L}V_1(Q(t)) = -4.4Q^2 + 0.49e^{-2t}tanh^2y(t) - 2Q^2 + 2.4Q^2$$
$$= -4\left(y(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)d\theta\right)^2 + 0.49e^{-2t}tanh^2y(t).$$

When $k = 2$,

$$\mathcal{L}V_2(Q(t)) = -6.72Q^2 + 0.3e^{-2t}tanh^2y(t) + Q^2 - 1.2Q^2$$
$$= -6.92\left(y(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)d\theta\right)^2 + 0.3e^{-2t}tanh^2y(t).$$

By using general inequality and Hölder's inequality, we have

$$\begin{aligned}\mathcal{L}V_1(Q(t)) &= -4\left(y(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)d\theta\right)^2 + 0.49e^{-2t}tanh^2y(t)\\ &\le -4y^2(t) + y^2(t) - \frac{1}{4}\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2\\ &\quad + \left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 + 0.49e^{-2t}tanh^2y(t)\\ &\le -3y^2(t) + \frac{3}{4}\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 + 0.49e^{-2t}tanh^2y(t)\\ &\le -3y^2(t) + \frac{3}{4}\int_{-0.5}^{0} y^2(t+\theta)\mu(\theta)\,d\theta + 0.49e^{-2t}tanh^2y(t).\end{aligned}$$

$$\begin{aligned}\mathcal{L}V_2(Q(t)) &= -6.92\left(y(t) - \frac{1}{4}\int_{-0.5}^{0} y(t+\theta)\mu(\theta)d\theta\right)^2 + 0.3e^{-2t}tanh^2y(t)\\ &\le -6.92y^2(t) + 2.473y^2(t) - 0.43\left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2\\ &\quad + \left(\int_{-0.5}^{0} y(t+\theta)\mu(\theta)\,d\theta\right)^2 + 0.3e^{-2t}tanh^2y(t)\\ &\le -4.4y^2(t) + 0.57\int_{-0.5}^{0} y^2(t+\theta)\mu(\theta)\,d\theta + 0.3e^{-2t}tanh^2y(t).\end{aligned}$$

Thus, for every $k \in \mathcal{S}$, the following holds:

$$\mathcal{L}V_k(Q(t)) \le -3y^2(t) + \frac{3}{4}\int_{-0.5}^{0} y^2(t+\theta)\mu(\theta)\,d\theta + 0.49e^{-2t}tanh^2y(t).$$

Therefore, the model with parameters $c_1 = 1, c_2 = 1.2, \alpha_1 = 3, \alpha_2 = \frac{3}{4}, \kappa = \frac{1}{4}, \psi(t) = e^t, p = 2, q = 0.1, e(t) = 0.49e^{-2t}$, and $r = (\frac{e(t)}{\psi^q(t)})^{\frac{1}{p}} = 0.7e^{-1.05t}$ meets all the requirements of Theorem 3.4. This theoretically proves that, under the given delays, mode switching, and disturbances, the tracking error $y(t)$ will converge almost surely with an exponential decay rate $q = 0.1$ to a time-varying neighborhood of radius $r(t) = 0.7e^{-1.05t}$.

Numerical simulation of this vehicle control system is conducted using MATLAB and the Euler-Maruyama method. Key implementation parameters are set as follows: a total simulation time of $T = 10$ seconds with a fixed step size of $\Delta t = 0.01$, and the results are statistically evaluated over $N = 1000$ independent Monte Carlo trials.

The simulation results are presented in two figures. Figure 3 depicts the mean-square trajectory $\|y(t)\|^2$, computed by averaging over all 1000 trials. Its decay clearly validates the theoretical stability bound. Figure 4 visually demonstrates the sample path behavior: the light-colored curves represent 100 randomly selected individual sample paths, while the dark-colored curve traces their sample mean. Collectively, they show that under the influence of random delays, mode switching, and external disturbances, the lateral tracking error $y(t)$ is effectively suppressed and converges to a boundary layer near zero that decays over time. This strongly validates Theorem 3.4 and indicates that the designed control law ensures the tracking error remains within an engineering acceptable range.

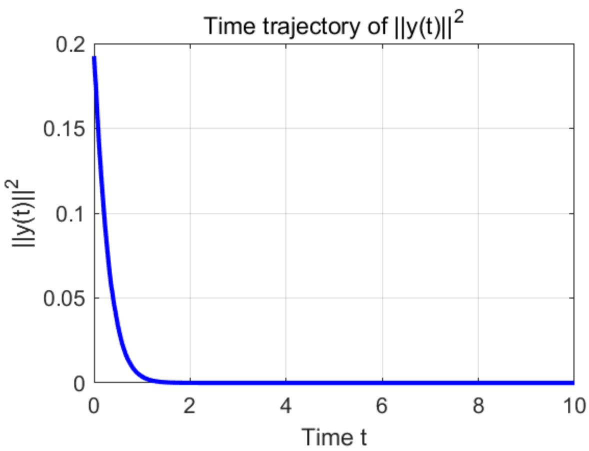

Figure 3. State Trajectories of $\|y(t)\|^2$ in Example 4.3.

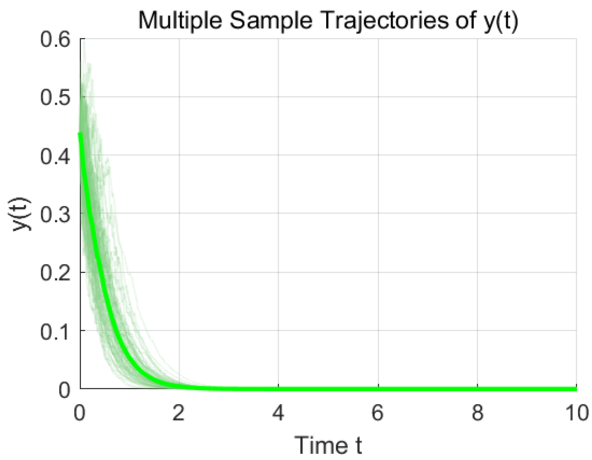

Figure 4. State Trajectories in Example 4.3.

5. Conclusions

This paper investigates the stability of NSDNNs-MS under a general decay rate. By constructing appropriate Lyapunov functions and employing the nonnegative semimartingale convergence theorem, we establish sufficient conditions for p-th moment ψ-type stability, almost sure ψ-type stability, and almost sure practical ψ-type stability under weaker assumptions on the neutral term. This study not only extends the traditional exponential stability framework, but also provides a theoretical foundation for the engineering applications of systems with nonzero equilibrium points or persistent disturbances. The effectiveness and practical applicability of the proposed methods are validated through numerical simulations and an application example in an autonomous vehicle control system, demonstrating the potential of this theory in the analysis of complex dynamical systems.

Although the present work has analyzed stability conditions under relatively weak time-delay assumptions and neutral terms, the obtained sufficient conditions remain rather restrictive with respect to the system parameters. Furthermore, in recent years, increasing attention has been paid to models with discontinuous disturbances. For instance, in reference [38], Caraballo et al. investigated the p-th moment h-stability of neutral stochastic differential equations with Lévy noise and Markov switching. In light of these considerations, our future work will focus on establishing sufficient conditions for the practical stability with a general decay rate of NSDNNs-MS subject to discontinuous disturbances under even milder assumptions, thereby extending the scope of our research to encompass both practical and theoretical dimensions of such networks.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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