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Research article

Existence and uniqueness of solutions to variational inequalities involving a double-phase degenerate parabolic operator with variable exponents arising from American put option valuation analysis

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Abstract: This paper analyzes a class of variational inequalities involving a double-phase degenerate parabolic operator with variable exponents, which arises in the valuation of American put options. By establishing energy estimates for the associated penalty problem and applying a limiting argument, the existence of a weak solution is obtained. The uniqueness of the weak solution is also discussed.

Keywords: variational inequalities; American put option; double-phase degenerate parabolic operator with variable exponents; existence; uniqueness

1. Introduction

Variational inequality problems have good applications in daily life. Consider the example of an American put option. Suppose an investor owns a certain stock, and its price $S(t)$ satisfies

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}w(t),$$

where $S(0)$ is known, μ is the expected return rate of the stock, σ is the stock's volatility, and $w(t)$ is a standard Wiener process representing the background noise in the market. The investor is concerned that the stock price may decline over the future time interval $[0, T]$, potentially leading to substantial losses. At time 0, the investor pays a premium V to acquire an option contract, which grants them the right to sell the stock $S(t)$ at a predetermined price K during the time period $[0, T]$. Since this contract only confers a right and does not impose an obligation, the option effectively enables the investor to lock in a minimum price for the stock $S(t)$ over the time interval $[0, T]$. According to [1,2], the value of this American put option at time t, denoted as $V(S, t)$, satisfies

$$\begin{cases} \max\{L_1V, V_0 - V\} = 0 \text{ in } \Omega_T, \\ V(S,0) = V_0 = \max\{K - S, 0\} \text{ in } \Omega, \\ V(0,t) = K\exp\{-r(T-t)\} \text{ in } (0,T), \\ V_0(B,t) = 0 \text{ in } (0,T), \end{cases} \tag{1}$$

where the parabolic operator L_1V fulfills

$$L_1V = \partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V + rS\partial_S V - rV, \tag{2}$$

r denotes the risk-free interest rate in the financial market, $\Omega_T = (0,B) \times (0,T)$, and B represents the upper barrier level set by the option issuer. The structure of Eq (1) aligns with the variational inequality framework we are investigating, which serves as one motivation for our study. Additionally, when the investor exercises the option and executes a trade, if transaction costs are present, the volatility σ is related to $|\partial_S V|^{p_0-2}\partial_S V$ [3], where p_0 is a constant greater than 2. This provides another motivation for the present work.

Let $\Omega \subset R_N$ be a bounded, connected open domain with a smooth boundary $\partial\Omega$. Define the cylindrical domain $\Omega_T = \Omega \times (0,T)$. On Ω_T, this paper investigates a class of variational inequalities governed by a double-phase degenerate parabolic operator, which arises in the valuation of American put options. The problem is formulated as follows:

$$\begin{cases} \max\{L\pi, \pi_0 - \pi\} = 0 \,\text{in}\, \Omega_T, \\ \pi(\,\cdot\,, 0) = \pi_0 \,\text{in}\, \Omega, \\ \pi = D_i\pi = 0 \,\text{in}\, \partial\Omega \times (0,T), \end{cases} \tag{3}$$

where $L\pi$ is a double-phase degenerate parabolic operator defined on Ω_T, satisfying the structural condition

$$L\pi = \partial_t \pi - \sum_{i=1}^{N} [D_i(|D_i\pi|^{p(x,t)-2}D_i\pi) + D_i(|D_i\pi|^{q(x,t)-2}D_i\pi)]. \tag{4}$$

We assume that the variational inequality (3) satisfies the following linking condition:

$$\sum_{i=1}^{N} [D_i(|D_i\pi_0|^{p(x,t)-2}D_i\pi_0) + D_i(|D_i\pi_0|^{q(x,t)-2}D_i\pi_0)] = 0 \text{ in } \partial\Omega. \tag{5}$$

Here,

$$p^+ = \max\{p(x,t), (x,t) \in \Omega_T\},\ \ p^- = \min\{p(x,t), (x,t) \in \Omega_T\},$$

$$q^+ = \max\{q(x,t), (x,t) \in \Omega_T\},\ \ q^- = \min\{q(x,t), (x,t) \in \Omega_T\},$$

and we further assume $p^- \geq 2$ as well as $q^- \geq 2$. Let $W_0^{1,p}(\Omega)$ denote the set of all functions satisfying $\nabla\omega \in L^p(\Omega)$ that vanish on the boundary $\partial\Omega$. Finally, we assume that the initial value π_0 is nonnegative on Ω and satisfies

$$\pi_0 \in W_0^{1,p^+}(\Omega) \cap W_0^{1,q^+}(\Omega). \tag{6}$$

In the course of our discussion, we also employ additional notations from Banach spaces, specifically

$$L^{p(\cdot,\cdot)}(\Omega_T) = \{u(x,t)|u \text{ is measurable in } \Omega_T,\ A_{p(\cdot,\cdot)}(u) = \int_{\Omega_T} |u|^{p(x,t)} \mathrm{d}x\mathrm{d}t < \infty\},$$

$$\|u\|_{L^{p(\cdot,\cdot)}(\Omega_T)} = \inf\{\lambda |\ A_{p(\cdot,\cdot)}(u/\lambda) \leq 1\},$$
$$W^{1,p(\cdot,\cdot)}(\Omega_T) = \{u(x,t)|u \text{ is measurable in } \Omega_T,\ A_{p(\cdot,\cdot)}(u) < \infty, A_{p(\cdot,\cdot)}(\nabla u) < \infty\}.$$

The existence and uniqueness of solutions to degenerate parabolic initial-boundary value problems have been extensively studied in the literature and form a fundamental basis for research on variational inequalities. Reference [4] investigated a nonlinear parabolic initial-boundary value problem involving the fractional Laplacian, with a focus on the global existence, uniqueness, long-time behavior, and stability of weak solutions. In [5], a degenerate parabolic problem with nonlinear boundary conditions induced by inflow flux on the boundary was studied. The work emphasized the existence and stability of positive solutions bifurcating from the trivial solution, extending the framework established by Crandall and Rabinowitz through a combination of the Lyapunov–Schmidt method and classical local bifurcation theory. Reference [6] examined an initial-boundary value problem for a class of nonlocal semilinear pseudo-parabolic equations. Under different initial energy conditions, the authors addressed the global existence, uniqueness, and asymptotic behavior of solutions, as well as the blow-up phenomenon of weak solutions. Meanwhile, Reference [7] established results on the global existence and uniqueness of weak solutions to a parabolic p-Laplace-type equation with supercritical nonlinear growth.

In recent years, initial-boundary value problems governed by double-phase degenerate parabolic operators have also attracted considerable attention from researchers. In [8], employing surjectivity theorems for pseudo-monotone operators, the theory of modular function spaces, and embedding theorems in generalized Orlicz spaces, the existence of weak solutions to a class of double-phase quasilinear elliptic equations with logarithmic convection terms was established. The methodology developed in [8] can be naturally extended to a broader class of unbalanced double-phase problems featuring logarithmic perturbations and gradient-dependent source terms. Reference [9] investigated a nonlocal evolution equation incorporating a pseudo-parabolic third-order term, which models an approximately unidirectional two-phase flow in Brinkman-type regimes. The existence of weak solutions was proved, and the behavior of solutions in various asymptotic limits was analyzed. References [10] and [11] further contributed to this line of research by establishing the existence of weak solutions for double-phase parabolic initial-boundary value problems, along with the Hölder regularity and Calderón-Zygmund estimates for such solutions.

Variational inequalities governed by degenerate parabolic operators have also become a prominent research focus. In [12], leveraging properties of variable-exponent Lebesgue and Sobolev spaces, the existence of solutions to a class of parabolic variational inequalities was established. Reference [13] investigated a generalized second-order delayed variational inequality, proving the well-posedness of the system, namely, the existence and uniqueness of solutions, and further analyzing the convergence properties of these solutions. In [14], a virtual element numerical method was developed for two-dimensional parabolic variational inequalities on unstructured polygonal grids, with the existence of solutions demonstrated through numerical analysis. Reference [15], within the framework of parabolic Kirchhoff-type operators, examined a class of variational inequalities arising from financial contract modeling. In addition to deriving gradient supremum norm estimates for the solutions, the work also established the existence of solutions.

This paper studies a class of variational inequalities governed by a double-phase degenerate parabolic operator, which originates from a valuation model for an American put option based on career choice. One of the key contributions of this work is that the exponents of the degenerate parabolic operator depend on both time and space variables. By analyzing the boundedness of solutions to the variational inequality (3) and leveraging its structural properties, we define the notion of a weak solution. Using the penalty method, we further establish the existence and uniqueness of such weak solutions. The main challenge arises from the presence of variable exponents, which complicates the differentiation process and makes it difficult to derive estimates for the solution and its gradient. To overcome this, we introduce an exponential-type upper bound involving logarithmic functions, which enables us to obtain the necessary estimates.

2. Main results

Note that the double-phase degenerate parabolic operator (4) becomes degenerate at $\{(x,t)|D_i\pi = 0\}$, and the inequality structure in (3) imposes significant constraints on deriving energy norm estimates. To address this, we first introduce an auxiliary problem. From (3), it is straightforward to obtain

$$\pi(x,t) \geq \pi_0(x) \geq 0, \forall (x,t) \in \Omega_T. \tag{7}$$

To examine the upper bound of solutions to the variational inequality (3), we choose a test function $\phi = (\pi - k)_+$. Note that, in conjunction with (3), $\phi \geq 0$ leads to $\phi \times L\pi \leq 0$ in Ω_T. Integrating $\phi \times L\pi \leq 0$ over Ω then yields

$$\begin{aligned}&\int_\Omega \partial_t\pi \cdot (\pi-k)_+ \mathrm{d}x + \sum_{i=1}^N \int_\Omega |D_i\pi|^{p(x,t)-2} D_i\pi \cdot D_i(\pi-k)_+ \mathrm{d}x + \sum_{i=1}^N \int_\Omega |D_i\pi|^{q(x,t)-2} D_i\pi \cdot D_i(\pi-k)_+ \mathrm{d}x \\ &\leq \sum_{i=1}^N \int_{\partial\Omega} |D_i\pi|^{p(x,t)-2} D_i\pi \cdot (\pi-k)_+ \cos(x_i,\vec{\upsilon}) \mathrm{d}x + \sum_{i=1}^N \int_{\partial\Omega} |D_i\pi|^{q(x,t)-2} D_i\pi \cdot (\pi-k)_+ \cos(x_i,\vec{\upsilon}) \mathrm{d}x.\end{aligned} \tag{8}$$

Note that k is a constant. It is clear that $D_i\pi = D_i(\pi - k)$ holds. Moreover, when $\pi \leq k$ is satisfied, we have $(\pi - k)_+ = 0$, and under this condition, $D_i(\pi - k)_+ = 0$ also holds, $i = 1, 2, \cdots, N$. Therefore,

$$\sum_{i=1}^N \int_\Omega |D_i\pi|^{p(x,t)-2} D_i\pi \cdot D_i(\pi-k)_+ \mathrm{d}x = \sum_{i=1}^N \int_\Omega |D_i(\pi-k)_+|^{p(x,t)} \mathrm{d}x \geq 0. \tag{9}$$

Similarly, it follows that

$$\sum_{i=1}^N \int_\Omega |D_i\pi|^{q(x,t)-2} D_i\pi \cdot D_i(\pi-k)_+ \mathrm{d}x = \sum_{i=1}^N \int_\Omega |D_i(\pi-k)_+|^{q(x,t)} \mathrm{d}x \geq 0. \tag{10}$$

Observe that $\partial_t\pi = \partial_t(\pi - k)$, and when $\pi \leq k$ holds, it follows that $(\pi - k)_+ = 0$ and $\partial_t(\pi - k)_+ = 0$, which implies

$$\int_\Omega \partial_t\pi \cdot (\pi-k)_+ \mathrm{d}x = \frac{1}{2}\int_\Omega (\pi-k)_+^2 \mathrm{d}x - \frac{1}{2}\int_\Omega (\pi_0-k)_+^2 \mathrm{d}x. \tag{11}$$

On the other hand, from the boundary condition of (3), it is straightforward to conclude that

$$\sum_{i=1}^N \int_{\partial\Omega} |D_i\pi|^{p(x,t)-2} D_i\pi \cdot (\pi-k)_+ \cos(x_i,\vec{\upsilon}) \mathrm{d}x = 0, \sum_{i=1}^N \int_{\partial\Omega} |D_i\pi|^{q(x,t)-2} D_i\pi \cdot (\pi-k)_+ \cos(x_i,\vec{\upsilon}) \mathrm{d}x = 0.$$

Therefore, substituting (9)–(11) into (8), we obtain

$$\int_\Omega (\pi-k)_+^2 \mathrm{d}x \leq \int_\Omega (\pi_0-k)_+^2 \mathrm{d}x. \tag{12}$$

By choosing $k = |\pi_0|_\infty$, we obtain $\int_\Omega (\pi_0 - k)_+^2 \mathrm{d}x = 0$, which implies that

$$\pi \le |\pi_0|_\infty \text{ in } \Omega_T. \tag{13}$$

Here, $|\pi_0|_\infty$ denotes the supremum norm of π_0 over Ω_T. Based on (7) and (13), we construct a penalty function $\beta_\varepsilon(x)$ satisfying

$$\beta_\varepsilon \in C^1(\mathrm{R}_+),\ \beta_\varepsilon \ge 0 \text{ in } \mathrm{R}_+, \beta_\varepsilon = 0 \text{ in } (\varepsilon, \infty),\ \beta_\varepsilon(0) = M_0, \tag{14}$$

where M_0 is a nonnegative constant. According to (13), we further define $M_0 \le |\pi_0|_\infty$. Following the approach in [15], the variational inequality (3) can be approximated by the following penalty problem:

$$\begin{cases} L_\varepsilon \pi_\varepsilon + \beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) = 0 \,\text{in}\, \Omega_T, \\ \pi_\varepsilon(\,\cdot\,, 0) = \pi_0 + \varepsilon \,\text{in}\, \Omega, \\ \pi_\varepsilon = \varepsilon \,\text{in}\, \partial\Omega \times (0, T). \end{cases} \tag{15}$$

Note that $L\pi_\varepsilon = -\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \le 0$ in Ω_T. In combination with the linking condition (5), we have $L\pi_0 = 0$ in Ω_T, and together with $L\varepsilon = 0$, it follows that $L\pi_\varepsilon \le L(\pi_0 + \varepsilon)$ in Ω_T. Furthermore, since $\pi_\varepsilon(\,\cdot\,, 0) = \pi_0 + \varepsilon \,\text{in}\, \Omega$ and $\pi_\varepsilon = \varepsilon \,\text{in}\, \partial\Omega \times (0, T)$ hold, the comparison principle yields

$$\pi_\varepsilon \ge \pi_0 + \varepsilon \text{ in } \Omega_T. \tag{16}$$

Moreover, since $|\pi_0|_\infty$ is a constant, it is evident that $L(|\pi_0|_\infty + \varepsilon) = 0$ holds in Ω_T, which implies $L\pi_\varepsilon \ge L|\pi_0|_\infty + \varepsilon$ in Ω_T. Additionally, on the boundary, $|\pi_0|_\infty + \varepsilon \ge \pi_0 + \varepsilon \,\text{in}\, \Omega$ and $|\pi_0|_\infty \ge 0 \,\text{in}\, \partial\Omega \times (0, T)$. Therefore, applying the comparison principle again yields

$$\pi_\varepsilon \le |\pi_0|_\infty + \varepsilon \text{ in } \Omega_T. \tag{17}$$

By comparing (7), (13), (16), and (17), it can be observed that the initial-boundary value problem formulated with the penalty function $\beta_\varepsilon(x)$ shares the same structure as the variational inequality (3). Furthermore, according to [15, 16], the penalty problem admits a weak solution in the following sense:

Definition 2.1. *A function π_ε is called a weak solution of the penalty problem (15) if the following conditions are satisfied:*
1) $\pi_\varepsilon \in L^\infty(\Omega_T) \cap L^\infty(0, T; W_0^{1,p(x,t)}(\Omega)) \cap L^\infty(0, T; W_0^{1,q(x,t)}(\Omega))$;
2) for every $\phi \in C_0^\infty(\Omega_T)$, it holds that

$$\begin{aligned} &\int_{\Omega_T} \partial_t \pi_\varepsilon \cdot \phi \mathrm{d}x\mathrm{d}t + \sum_{i=1}^N \int_{\Omega_T} |D_i \pi_\varepsilon|^{p(x,t)-2} D_i \pi_\varepsilon D_i \phi \mathrm{d}x\mathrm{d}t + \sum_{i=1}^N \int_{\Omega_T} |D_i \pi_\varepsilon|^{q(x,t)-2} D_i \pi_\varepsilon D_i \phi \mathrm{d}x\mathrm{d}t \\ &= -\int_0^t \int_\Omega \beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon)\phi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{18}$$

References [15,16] have demonstrated that the variational inequality does not admit an analytical solution. Therefore, this paper focuses on analyzing the existence and uniqueness of solutions to variational inequality (3) in the weak sense. Furthermore, drawing on the results presented in [15], we investigate weak solutions of the following form:

Definition 2.2. *A function (π, ξ) is said to be a weak solution of the variational inequality (3) if and only if the following conditions are satisfied:*
(a) $\pi \in L^\infty(0, T; W_0^{1,p(x,t)}(\Omega)) \cap L^\infty(0, T; W_0^{1,q(x,t)}(\Omega)),\ \partial_t \pi \in L^\infty(0, T, L^2(\Omega))$;
(b) $\xi \in G$ for any $(x, t) \in \Omega_T$;
(c) $\pi(x, t) \geq \pi_0(x),\ \pi(x, 0) = \pi_0(x)$ for any $(x, t) \in \Omega_T$;
(d) for every test function $\varphi \in C^1(\bar{\Omega}_T)$*, the following equality holds:*

$$\int_{\Omega_T} \partial_t \pi \cdot \varphi + \sum_{i=1}^{N} |D_i \pi|^{p(x,t)-2} D_i \pi D_i \varphi + \sum_{i=1}^{N} |D_i \pi|^{q(x,t)-2} D_i \pi D_i \varphi \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \xi \varphi \mathrm{d}x\mathrm{d}t. \tag{19}$$

The main results of this paper are summarized as follows:

Theorem 2.1. *If $|\partial_t p(x, t)| \leq L$ and $|\partial_t q(x, t)| \leq L$ hold, then the variational inequality (3) admits a unique weak solution in the sense of Definition 2.2, and moreover,*

$$\begin{aligned} &\|\partial_t \pi\|^2_{L^2(\Omega_T)} + \sum_{i=1}^{N} \sup_{t \in (0,T)} \int_\Omega \frac{1}{p(x,t)} |D_i \pi|^{p(x,t)} \mathrm{d}x + \sum_{i=1}^{N} \sup_{t \in (0,T)} \int_\Omega \frac{1}{q(x,t)} |D_i \pi|^{q(x,t)} \mathrm{d}x \\ &\leq C \int_\Omega |\pi_0|^2 \mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \frac{1}{p(x,0)} |D_i \pi_0|^{p(x,0)} \mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \frac{1}{q(x,0)} |D_i \pi_0|^{q(x,0)} \mathrm{d}x. \end{aligned} \tag{20}$$

Returning to the case of Eq (1), note that $E(S, t) = V(S, t) - (B - S)K \exp\{-r(T - t)\}$, where $V(S, t)$ represents the value of an American put option. We also introduce the European put option, which grants the investor the right (but not the obligation) to sell stock S at time T, in contrast to the American option, where the holder may exercise the right at any time during the entire life of the option $[0, T]$. According to [15], the value of the European put option $v(S, t)$, satisfies

$$\begin{cases} L_1 v = 0 \text{ in } \Omega_T, \\ v(S, 0) = v_0 = \max\{K - S, 0\} \text{ in } \Omega, \\ v(0, t) = K \exp\{-r(T - t)\} \text{ in } (0, T), \\ v_0(B, t) = 0 \text{ in } (0, T). \end{cases} \tag{21}$$

Note that $B > K$. According to [15], the solution for $v(S, t)$ is given by

$$v(S, t) = K \exp\{-r(T - t)\} N(-d_2) - S N(-d_1), \tag{22}$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and

$$d_1 = \frac{\ln S - \ln K + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}},\ d_2 = \frac{\ln S - \ln K + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}}.$$

Straightforward calculation then yields

$$\partial_t v(S, t) = rK \exp\{-r(T - t)\} N(-d_2) - \frac{\sigma S N'(d_1)}{2 \sqrt{T - t}}, \tag{23}$$

$$\partial_S v(S, t) = -N(-d_1). \tag{24}$$

On the other hand, since an American option can be exercised at any time during its lifespan $[0, T]$, while a European option can only be exercised at maturity T, the value of the European option, $v(S, t)$, is more sensitive to changes in t and S compared to the American option $V(S, t)$ (since the American option retains time value if not exercised, reducing the investor's need to closely track movements in the underlying asset price S) [15]. Therefore,

$$|\partial_S v(S, t)| \leq |\partial_S v(S, t)|, \quad |\partial_t v(S, t)| \leq |\partial_t v(S, t)| \ \text{ in } (0, B) \times (0, T).$$

Furthermore, we have

$$|\partial_S E(S, t)| \leq 1 + rBK \text{ in } (0, B) \times (0, T),$$
$$|\partial_t E(S, t)| \leq rK(1 + B)\exp\{-r(T - t)\} \text{ in } (0, B) \times (0, T).$$

Next, we verify the result of Theorem 2.1. First, we compute the left-hand side of Eq (20). We have

$$\|\partial_t E\|^2_{L^2(\Omega_T)} + \frac{1}{2} \sup_{t\in(0,T)} \int_\Omega |D_S E|^2 \mathrm{d}x \leq rK(1 + B)BT + \frac{1}{2}(1 + rBK)B, \tag{25}$$

$$\begin{aligned} &C \int_\Omega |E_0|^2 \mathrm{d}x + \tfrac{1}{2} \sup_{t\in(0,T)} \int_\Omega |D_S E_0|^2 \mathrm{d}x \\ &= C \int_0^B |(K - S)_+ - (B - S)K|^2 \mathrm{d}S + \tfrac{1}{2} \int_0^B | - I_{S<K} + K|^2 \mathrm{d}S. \end{aligned} \tag{26}$$

To compute the integral on the right-hand side of (26), we have

$$C \int_\Omega |E_0|^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |D_S E_0|^2 \mathrm{d}x = \frac{1}{2}C(B - K)^2 + \frac{1}{5}CK^4B^5 + \frac{1}{2}K + \frac{1}{2}K^2 + \frac{1}{2}K^2B. \tag{27}$$

Comparing (25) and (27), we can choose a suitable constant C (which depends only on r, B, and K) such that

$$\|\partial_t E\|^2_{L^2(\Omega_T)} + \frac{1}{2} \sup_{t\in(0,T)} \int_\Omega |D_S E|^2 \mathrm{d}x \leq C \int_\Omega |E_0|^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |D_S E_0|^2 \mathrm{d}x.$$

Thus, the result of Theorem 2.1 also holds for the case of the American put option described in Eq (1).

3. Prior estimates

Since $\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \geq 0$ in Ω_T, it follows that $L_\varepsilon \pi_\varepsilon = -\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \leq 0$ in Ω_T. Moreover, because $\pi_\varepsilon \geq \pi_0 + \varepsilon \geq 0$ in Ω_T holds, we conclude that for any $t \in (0, T)$, the following inequality is satisfied:

$$\int_\Omega \pi_\varepsilon \partial_t \pi_\varepsilon \mathrm{d}x + \sum_{i=1}^N \int_\Omega D_i(|D_i\pi_\varepsilon|^{p(x,t)-2} D_i\pi_\varepsilon)\pi_\varepsilon \mathrm{d}x + \sum_{i=1}^N \int_\Omega D_i(|D_i\pi_\varepsilon|^{q(x,t)-2} D_i\pi_\varepsilon)\pi_\varepsilon \mathrm{d}x \leq 0.$$

By further applying integration by parts, we obtain

$$\begin{aligned} &\int_\Omega \pi_\varepsilon \partial_t \pi_\varepsilon \mathrm{d}x + \sum_{i=1}^N \int_\Omega |D_i\pi_\varepsilon|^{p(x,t)-2} |D_i\pi_\varepsilon|^2 \mathrm{d}x + \sum_{i=1}^N \int_\Omega |D_i\pi_\varepsilon|^{q(x,t)-2} |D_i\pi_\varepsilon|^2 \mathrm{d}x \\ &\leq \sum_{i=1}^N \int_{\partial\Omega} |D_i\pi_\varepsilon|^{p(x,t)-2} D_i\pi_\varepsilon \pi_\varepsilon \cos(x_i, \vec{\upsilon}) \mathrm{d}x + \sum_{i=1}^N \int_{\partial\Omega} |D_i\pi_\varepsilon|^{q(x,t)-2} D_i\pi_\varepsilon \pi_\varepsilon \cos(x_i, \vec{\upsilon}) \mathrm{d}x. \end{aligned} \tag{28}$$

Using the boundary conditions of the penalty problem (15), it is straightforward to verify that

$$\sum_{i=1}^{N}\int_{\partial\Omega}|D_i\pi_\varepsilon|^{p(x,t)-2}D_i\pi_\varepsilon\pi_\varepsilon\cos(x_i,\vec{\upsilon})\mathrm{d}x=\sum_{i=1}^{N}\int_{\partial\Omega}|D_i\pi_\varepsilon|^{q(x,t)-2}D_i\pi_\varepsilon\pi_\varepsilon\cos(x_i,\vec{\upsilon})\mathrm{d}x=0.$$

Note that $\pi_\varepsilon\partial_t\pi_\varepsilon=\frac{1}{2}\partial_t\pi_\varepsilon^2$. Integrating (28) over $(0,t)$ then yields, for any $t\in(0,T)$, the following estimate:

$$\frac{1}{2}\int_{\Omega}\pi_\varepsilon^2\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega_t}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega_t}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\le\frac{1}{2}\int_{\Omega}\pi_0^2\mathrm{d}x. \tag{29}$$

Lemma 3.1. *Suppose that $|\partial_t p(x,t)|\le L$ and $|\partial_t q(x,t)|\le L$ hold in Ω_T. Then the solution π_ε of the penalty problem (15) satisfies*

$$\sup_{t\in(0,T)}\|\pi_\varepsilon(\cdot,t)\|_{2,\Omega}^2+2\sup_{t\in(0,T)}\sum_{i=1}^{N}\int_{\Omega_T}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x+2\sup_{t\in(0,T)}\sum_{i=1}^{N}\int_{\Omega_T}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\le\int_{\Omega}\pi_0^2\mathrm{d}x. \tag{30}$$

From Lemma 3.1, we obtain the following boundedness result for the gradient of π_ε:

$$D_i\pi_\varepsilon\in L^\infty(0,T;L^{p(\cdot,\cdot)}(\Omega))\cap L^\infty(0,T;L^{q(\cdot,\cdot)}(\Omega)),i=1,2,\cdots,N. \tag{31}$$

Furthermore, an estimate for the solution π_ε itself is also derived:

$$\pi_\varepsilon\in L^\infty(0,T;L^2(\Omega)). \tag{32}$$

We now proceed to prove a new result, which implies an L^2 estimate for $\partial_t\pi_\varepsilon$. Note that from (3), we have

$$\partial_t\pi_\varepsilon\times L_\varepsilon\pi_\varepsilon+\partial_t\pi_\varepsilon\times\beta_\varepsilon(\pi_\varepsilon-\pi_0-\varepsilon)=0 \text{ in } \Omega_T. \tag{33}$$

Integrating this expression over Ω, it is straightforward to obtain

$$\begin{aligned}&\int_\Omega|\partial_t\pi_\varepsilon|^2\mathrm{d}x+\int_\Omega\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{p(x,t)-2}D_i\pi_\varepsilon\partial_tD_i\pi_\varepsilon\mathrm{d}x+\int_\Omega|D_i\pi_\varepsilon|^{q(x,t)-2}D_i\pi_\varepsilon\partial_tD_i\pi_\varepsilon\mathrm{d}x\\&=\int_{\partial\Omega}\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{p(x,t)-2}D_i\pi_\varepsilon\partial_t\pi_\varepsilon\cos(x_i,\vec{\upsilon})\mathrm{d}x+\int_{\partial\Omega}\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{q(x,t)-2}D_i\pi_\varepsilon\partial_t\pi_\varepsilon\cos(x_i,\vec{\upsilon})\mathrm{d}x\\&-\int_\Omega\partial_t\pi_\varepsilon\times\beta_\varepsilon(\pi_\varepsilon-\pi_0-\varepsilon)\mathrm{d}x.\end{aligned} \tag{34}$$

We first analyze the right-hand side of Eq (34). From the boundary condition of (3), it is straightforward to conclude that

$$\int_{\partial\Omega}\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{p(x,t)-2}D_i\pi_\varepsilon\partial_t\pi_\varepsilon\cos(x_i,\vec{\upsilon})\mathrm{d}x=\int_{\partial\Omega}\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{q(x,t)-2}D_i\pi_\varepsilon\partial_t\pi_\varepsilon\cos(x_i,\vec{\upsilon})\mathrm{d}x=0. \tag{35}$$

Next, we analyze $\int_\Omega\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{p(x,t)-2}D_i\pi_\varepsilon\partial_tD_i\pi_\varepsilon\mathrm{d}x$ and $\int_\Omega\sum_{i=1}^{N}|D_i\pi_\varepsilon|^{q(x,t)-2}D_i\pi_\varepsilon\partial_tD_i\pi_\varepsilon\mathrm{d}x$. Note that from (15), it is readily obtained that $\pi_\varepsilon\ge\pi_0+\varepsilon>0$ in Ω_T. Applying the rules of differentiation then yields

$$\begin{aligned}&|D_i\pi_\varepsilon|^{p(x,t)-2}D_i\pi_\varepsilon D_i\partial_t\pi_\varepsilon=\tfrac{1}{2}|D_i\pi_\varepsilon|^{p(x,t)-2}\partial_t\left(|D_i\pi_\varepsilon|^2\right)\\&=\partial_t\left(\tfrac{1}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\right)+\tfrac{\partial_tp(x,t)}{p(x,t)^2}|D_i\pi_\varepsilon|^{p(x,t)}-\tfrac{\partial_tp(x,t)}{2p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\ln(|D_i\pi_\varepsilon|^2)\end{aligned} \tag{36}$$

and

$$\begin{aligned}&|D_i\pi_\varepsilon|^{q(x,t)-2}D_i\pi_\varepsilon D_i\partial_t\pi_\varepsilon = \tfrac{1}{2}|D_i\pi_\varepsilon|^{q(x,t)-2}\partial_t\left(|D_i\pi_\varepsilon|^2\right)\\ &= \partial_t\left(\tfrac{1}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\right) + \tfrac{\partial_t q(x,t)}{q(x,t)^2}|D_i\pi_\varepsilon|^{q(x,t)} - \tfrac{\partial_t q(x,t)}{2q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\ln(|D_i\pi_\varepsilon|^2).\end{aligned} \tag{37}$$

Substituting (34)–(37) into (33) and integrating over $[0, t]$, we obtain

$$\begin{aligned}&\tfrac{3}{4}\int_{\Omega_T}|\partial_t\pi_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \int_\Omega\sum_{i=1}^N\tfrac{1}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x + \int_\Omega\sum_{i=1}^N\tfrac{1}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\\ &\le \int_\Omega\sum_{i=1}^N\tfrac{1}{p(x,0)}|D_i\pi_0|^{p(x,0)}\mathrm{d}x + \int_\Omega\sum_{i=1}^N\tfrac{1}{q(x,0)}|D_i\pi_0|^{q(x,0)}\mathrm{d}x - \int_\Omega\partial_t\pi_\varepsilon\times\beta_\varepsilon(\pi_\varepsilon-\pi_0)\mathrm{d}x\\ &\quad-\sum_{i=1}^N\int_{\Omega_T}\tfrac{\partial_t p(x,t)}{p(x,t)^2}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x\mathrm{d}t - \int_{\Omega_T}\tfrac{\partial_t p(x,t)}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\ln(|D_i\pi_\varepsilon|^2)\mathrm{d}x\mathrm{d}t\\ &\quad-\sum_{i=1}^N\int_{\Omega_T}\tfrac{\partial_t q(x,t)}{q(x,t)^2}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\mathrm{d}t - \int_{\Omega_T}\tfrac{\partial_t q(x,t)}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\ln(|D_i\pi_\varepsilon|^2)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{38}$$

Note that on Ω_T, both $|\partial_t p(x,t)| \le L$ and $|\partial_t q(x,t)| \le L$ hold. It then follows from Lemma 2.1 that

$$\begin{aligned}&\left|\sum_{i=1}^N\int_{\Omega_T}\tfrac{\partial_t p(x,t)}{p(x,t)^2}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x\mathrm{d}t\right| + \left|\int_{\Omega_T}\tfrac{\partial_t p(x,t)}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\ln(|D_i\pi|^2)\mathrm{d}x\mathrm{d}t\right|\\ &\left|\sum_{i=1}^N\int_{\Omega_T}\tfrac{\partial_t q(x,t)}{q(x,t)^2}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\mathrm{d}t\right| + \left|\int_{\Omega_T}\tfrac{\partial_t q(x,t)}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\ln(|D_i\pi|^2)\mathrm{d}x\mathrm{d}t\right| \le C\int_\Omega\pi_0^2\mathrm{d}x.\end{aligned} \tag{39}$$

The integrals of $\frac{\partial_t p(x,t)}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\ln(|D_i\pi|^2)$ and $\frac{\partial_t q(x,t)}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\ln(|D_i\pi|^2)$ can be estimated as follows:

$$\int_{\Omega_T}\frac{\partial_t p(x,t)}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\ln(|D_i\pi|^2)\mathrm{d}x\mathrm{d}t \le C_\rho\sum_{i=1}^N\int_\Omega|D_i\pi_\varepsilon|^{p(x,t)+\rho}\mathrm{d}x\mathrm{d}t \tag{40}$$

and

$$\int_{\Omega_T}\frac{\partial_t q(x,t)}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\ln(|D_i\pi|^2)\mathrm{d}x\mathrm{d}t \le C_\rho\sum_{i=1}^N\int_\Omega|D_i\pi_\varepsilon|^{q(x,t)+\rho}\mathrm{d}x\mathrm{d}t, \tag{41}$$

where ρ is an arbitrarily given nonnegative constant. In this procedure, we utilize the following result:

$$\ln(s)^2 \le C_\rho s^\rho,\ \text{ if } s \ge 1;\ln(s)^2 \le C_\rho s^{-\rho},\ \text{ if } s \in (0,\ 1). \tag{42}$$

Lemma 3.2. *If $|\partial_t p(x,t)| \le L$ and $|\partial_t q(x,t)| \le L$ hold in Ω_T, then for any $\rho \in (0,\ 1)$, the following inequality is satisfied:*

$$\begin{aligned}&\|\partial_t\pi_\varepsilon\|^2_{L^2(\Omega_T)} + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\\ &\le C\int_\Omega|\pi_0|^2\mathrm{d}x + C_\rho\int_{\Omega_T}|D_i\pi_\varepsilon|^{p(x,t)+\rho}\mathrm{d}x\mathrm{d}t + C_\rho\int_{\Omega_T}|D_i\pi_\varepsilon|^{q(x,t)+\rho}\mathrm{d}x\mathrm{d}t\\ &\quad+\sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{p(x,0)}|D_i\pi_0|^{p(x,0)}\mathrm{d}x + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{q(x,0)}|D_i\pi_0|^{q(x,0)}\mathrm{d}x.\end{aligned} \tag{43}$$

Note that (43) holds for any $\rho \in (0,\ 1)$. By the right-continuity of the $L^\rho(\Omega_T)$-norm with respect to ρ, we obtain

$$\begin{aligned}&\|\partial_t\pi_\varepsilon\|^2_{L^2(\Omega_T)} + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{p(x,t)}|D_i\pi_\varepsilon|^{p(x,t)}\mathrm{d}x + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{q(x,t)}|D_i\pi_\varepsilon|^{q(x,t)}\mathrm{d}x\\ &\le C\int_\Omega|\pi_0|^2\mathrm{d}x + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{p(x,0)}|D_i\pi_0|^{p(x,0)}\mathrm{d}x + \sum_{i=1}^N\sup_{t\in(0,T)}\int_\Omega\tfrac{1}{q(x,0)}|D_i\pi_0|^{q(x,0)}\mathrm{d}x.\end{aligned} \tag{44}$$

From (44), we obtain the following boundedness result for the gradient of π_ε:

$$D_i\pi_\varepsilon \in L^\infty(0,T;L^{p(\cdot,\cdot)}(\Omega)), D_i\pi_\varepsilon \in L^\infty(0,T;L^{q(\cdot,\cdot)}(\Omega)), i=1,2,\cdots,N. \tag{45}$$

Moreover, an estimate for $\partial_t\pi_\varepsilon$ is also derived:

$$\partial_t\pi_\varepsilon \in L^2(\Omega_T). \tag{46}$$

4. Proof of Theorem 2.1

Prior to establishing the existence of solutions via norm continuity, the boundedness of solutions to the penalty problem (15) is required. Selecting the test function $(\pi_\varepsilon - k)_+$ and combining it with (3) yields

$$(\pi_\varepsilon - k)_+ \times L\pi_\varepsilon \le -(\pi_\varepsilon - k)_+ \times \beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \le 0 \text{ in } \Omega_T. \tag{47}$$

By repeating the proof procedure from (7) to (13), we obtain

$$0 \le \pi_0 + \varepsilon \le \pi_\varepsilon \le |\pi_0|_\infty + \varepsilon \text{ in } \Omega_T. \tag{48}$$

Based on the norm bounds (31), (46), and (48), we obtain the following weak convergence results:

$$\pi_\varepsilon \to \pi \ \text{ a.e. in } \Omega_T \text{ as } \varepsilon \to 0, \tag{49}$$

$$\pi_\varepsilon \overset{weak}{\to} \pi \ \text{ in } L^\infty(0,T;W_0^{1,p(\cdot,\cdot)}(\Omega)) \text{ as } \varepsilon \to 0, \tag{50}$$

$$\pi_\varepsilon \overset{weak}{\to} \pi \ \text{ in } L^\infty(0,T;W_0^{1,q(\cdot,\cdot)}(\Omega)) \text{ as } \varepsilon \to 0, \tag{51}$$

$$\partial_t\pi_\varepsilon \overset{weak}{\to} \partial_t\pi \text{ in } L^2(\Omega_T) \text{ as } \varepsilon \to 0. \tag{52}$$

Lemma 4.1. *For any $\varepsilon > 0$, the following holds:*

$$\pi_\varepsilon \ge \pi \text{ in } \Omega_T. \tag{53}$$

Proof. From (7), we decompose Ω_T into two parts:

$$\mathrm{O}_1 = \{(x,t) \in \Omega_T | \pi = \pi_0 \text{ in } \Omega_T\},\ \mathrm{O}_2 = \{(x,t) \in \Omega_T | \pi > \pi_0 \text{ in } \Omega_T\}. \tag{54}$$

When $\pi = \pi_0$, it follows from (16) that

$$\pi_\varepsilon \ge \pi_0 = \pi \text{ in } \mathrm{O}_1, \tag{55}$$

which implies $\pi_\varepsilon \ge \pi$ in $\partial\mathrm{O}_2$. When $\pi > \pi_0$, we obtain from (3) that $L\pi = 0$ in O_2. Moreover, since $L\pi_\varepsilon \le -\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \le 0$ in Ω_T, it follows that $L\pi_\varepsilon \le L\pi = 0$ in O_2. Applying the comparison principle again yields

$$\pi_\varepsilon \ge \pi \text{ in } \mathrm{O}_2. \tag{56}$$

Combining (55) and (56), we conclude that the result of the proposition holds. □

Lemma 4.2. *Let π_ε be a solution to the penalty problem (15) and π be a solution to the variational inequality (3). Then, letting $\varepsilon \to 0$, the following inequality holds:*

$$\sum_{i=1}^{N} \int_{\Omega_T} |D_i(\pi_\varepsilon - \pi)|^{p(x,t)} \mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i(\pi_\varepsilon - \pi)|^{q(x,t)} \mathrm{d}x\mathrm{d}t \to 0.$$

Proof. Choosing $\phi = \pi_\varepsilon - \pi$ in (18) and noting that Lemma 4.1 implies $\phi \geq 0$ in Ω_T, we obtain

$$\begin{aligned}&\int_{\Omega_T} \partial_t \pi_\varepsilon (\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi_\varepsilon|^{p(x,t)-2} D_i\pi_\varepsilon D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi_\varepsilon|^{q(x,t)-2} D_i\pi_\varepsilon D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t \\ &= -\int_0^t \int_\Omega \beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon)(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t \leq 0.\end{aligned}$$

Subtracting $\int_{\Omega_T} |D_i\pi|^{p(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t$ and $\int_{\Omega_T} |D_i\pi|^{q(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t$ from both sides of the above equation yields

$$\begin{aligned}&\int_{\Omega_T} \partial_t \pi_\varepsilon (\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi_\varepsilon|^{p(x,t)-2} D_i\pi_\varepsilon D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t \\ &+ \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi_\varepsilon|^{q(x,t)-2} D_i\pi_\varepsilon D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t \\ &\leq \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi|^{p(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi|^{q(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\Omega_T} \partial_t \pi_\varepsilon (\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i(\pi_\varepsilon - \pi)|^{p(x,t)}\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i(\pi_\varepsilon - \pi)|^{q(x,t)}\mathrm{d}x\mathrm{d}t \\ &\leq \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi|^{p(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi|^{q(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{57}$$

From (46) and (49), combined with Hölder's inequality, we have

$$\left|\int_{\Omega_T} \partial_t \pi_\varepsilon (\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t\right| \leq \int_{\Omega_T} |\partial_t \pi_\varepsilon|^2 \mathrm{d}x\mathrm{d}t \int_{\Omega_T} |\pi_\varepsilon - \pi|^2 \mathrm{d}x\mathrm{d}t \to 0 \text{ as } \varepsilon \to 0.$$

Note that from (50) and (51), it follows that

$$D_i\pi \in L^\infty(0, T; L^{p(\cdot,\cdot)}(\Omega)) \cap L^\infty(0, T; L^{q(\cdot,\cdot)}(\Omega)), i = 1, 2, \cdots, N, \tag{58}$$

which implies

$$\sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi|^{p(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t \to 0 \ as \ \varepsilon \to 0 \tag{59}$$

and

$$\sum_{i=1}^{N} \int_{\Omega_T} |D_i\pi|^{q(x,t)-2} D_i\pi D_i(\pi_\varepsilon - \pi)\mathrm{d}x\mathrm{d}t \to 0 \ as \ \varepsilon \to 0. \tag{60}$$

Substituting (58)–(60) into (57), we conclude that the proposition holds. □

Lemma 4.3. *Letting $\varepsilon \to 0$, we have*

$$\sum_{i=1}^{N} \int_{\Omega_T} \left| |D_i\pi_\varepsilon|^{p(x,t)-2} D_i\pi_\varepsilon - |D_i\pi|^{p(x,t)-2} D_i\pi \right| \mathrm{d}x\mathrm{d}t \to 0, \tag{61}$$

$$\sum_{i=1}^{N} \int_{\Omega_T} \left| |D_i\pi_\varepsilon|^{q(x,t)-2} D_i\pi_\varepsilon - |D_i\pi|^{q(x,t)-2} D_i\pi \right| \mathrm{d}x\mathrm{d}t \to 0. \tag{62}$$

Proof. By the triangle inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_T} \left| |D_i\pi_\varepsilon|^{p(x,t)-2} D_i\pi_\varepsilon - |D_i\pi|^{p(x,t)-2} D_i\pi \right| \mathrm{d}x\mathrm{d}t \\
&\leq \int_{\Omega_T} \left| |D_i\pi_\varepsilon|^{p(x,t)-2} - |D_i\pi|^{p(x,t)-2} \right| |D_i\pi| \mathrm{d}x\mathrm{d}t + \int_{\Omega_T} |D_i\pi_\varepsilon|^{p(x,t)-2} \left| D_i\pi_\varepsilon - D_i\pi \right| \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Applying Minkowsk's inequality and Hölder's inequality yields

$$\begin{aligned}
&\int_{\Omega} \left| |\nabla\pi_\varepsilon|^{p(x,t)-2} - |\nabla\pi|^{p(x,t)-2} \right| |\nabla\pi| \mathrm{d}x \\
&\leq \int_{\Omega} |\nabla\pi_\varepsilon - \nabla\pi|^{p(x,t)-2} |\nabla\pi| \mathrm{d}x \leq \left(\int_{\Omega} |\nabla\pi_\varepsilon - \nabla\pi|^{p(x,t)} \mathrm{d}x \right)^{\frac{p^+-2}{p^-}} \times \left(\int_{\Omega} |\nabla\pi|^2 \mathrm{d}x \right)^{\frac{2}{p^-}}.
\end{aligned}$$

From Lemma 4.2, it follows that the result (61) holds. Replacing $p(x,t)$ with $q(x,t)$, we conclude that (62) also holds. □

Existence. We analyze the limit of $-\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon)$. Note that from (14), $\beta_\varepsilon(\cdot)$ is a monotonically non-increasing function. It then follows from (17) that

$$-\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \to \xi \text{ as } \varepsilon \to 0. \tag{63}$$

On the other hand, by (16), we know that $\pi_\varepsilon \geq \pi_0 + \varepsilon$ in Ω_T. When $\pi_\varepsilon \geq \pi_0 + 2\varepsilon$,

$$-\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) = 0 \text{ in } \Omega_T,$$

which implies that when $\pi \geq \pi_0$ in Ω_T, $\xi = 0$. When $\pi_\varepsilon \in [\pi_0, \pi_0 + \varepsilon)$,

$$-\beta_\varepsilon(\pi_\varepsilon - \pi_0 - \varepsilon) \in [-M_0, 0] \text{ in } \Omega_T,$$

so we conclude that when $\pi = \pi_0$, $\xi \leq 0$ in Ω_T. In summary, we have $\xi \in G(\pi - \pi_0)$.

From (49)–(52), Lemma 4.3, and $\xi \in G(\pi - \pi_0)$, it is straightforward to verify that the dual quantity (π, ξ) satisfies all conditions of Definition 2.1. Therefore, the variational inequality (3) admits a weak solution in the sense of Definition 2.1. Finally, using (45) and Lemma 4.2, it is readily obtained that inequality (20) also holds.

Uniqueness. Suppose (π_1, ξ_1) and (π_2, ξ_2) are weak solutions of the variational inequality (3). Then,

$$\int_{\Omega_T} \partial_t\pi_1 \cdot \varphi + \sum_{i=1}^{N} |D_i\pi_1|^{p(x,t)-2} \nabla\pi_1 \nabla\varphi + \sum_{i=1}^{N} |D_i\pi_1|^{q(x,t)-2} \nabla\pi_1 \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \xi_1 \varphi \mathrm{d}x\mathrm{d}t, \tag{64}$$

$$\int_{\Omega_T} \partial_t\pi_2 \cdot \varphi + \sum_{i=1}^{N} |D_i\pi_2|^{p(x,t)-2} \nabla\pi_2 \nabla\varphi + \sum_{i=1}^{N} |D_i\pi_2|^{q(x,t)-2} \nabla\pi_2 \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \xi_2 \varphi \mathrm{d}x\mathrm{d}t. \tag{65}$$

Subtracting (65) from (64) and choosing $\varphi = \pi_1 - \pi_2$, we readily obtain

$$\begin{aligned} &\int_{\Omega_T} \sum_{i=1}^{N} (|D_i\pi_1|^{p(x,t)-2} D_i\pi_1 - |D_i\pi_2|^{p(x,t)-2} D_i\pi_2)(D_i\pi_1 - D_i\pi_2)\, \mathrm{d}x\mathrm{d}t \\ &+ \int_{\Omega_T} \sum_{i=1}^{N} (|D_i\pi_1|^{q(x,t)-2} \nabla\pi_1 - |D_i\pi_2|^{q(x,t)-2} D_i\pi_2)(D_i\pi_1 - D_i\pi_2)\, \mathrm{d}x\mathrm{d}t \\ &+ \int_{\Omega_T} (\partial_t\pi_1 - \partial_t\pi_2) \times (\pi_1 - \pi_2)\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} (\xi_1 - \xi_2)(\pi_1 - \pi_2)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{66}$$

Moreover, according to [17], we know that

$$(|D_i\pi_1|^{p(x,t)-2} \nabla\pi_1 - |D_i\pi_2|^{p(x,t)-2} D_i\pi_2)(D_i\pi_1 - D_i\pi_2) \geq |D_i\pi_1 - D_i\pi_2|^{p(x,t)} \geq 0 \tag{67}$$

and

$$(|D_i\pi_1|^{q(x,t)-2} \nabla\pi_1 - |D_i\pi_2|^{q(x,t)-2} D_i\pi_2)(D_i\pi_1 - D_i\pi_2) \geq |D_i\pi_1 - D_i\pi_2|^{q(x,t)} \geq 0. \tag{68}$$

We now analyze $\int_{\Omega_T} \xi_1(\pi_1 - \pi_2)\mathrm{d}x\mathrm{d}t$. When $\pi_1 > \pi_2$ holds, it necessarily follows that $\xi_1 = 0$ in Ω_T. Note that $\xi_2 \leq 0$ in Ω_T, which implies

$$\int_{\Omega_T} (\xi_1 - \xi_2)(\pi_1 - \pi_2)\mathrm{d}x\mathrm{d}t \leq 0. \tag{69}$$

When $\pi_1 < \pi_2$, we have $\xi_2 = 0$ in Ω_T. Observing that $\xi_1 \leq 0$ in Ω_T, it follows that $\xi_1 - \xi_2 \leq 0$ and $\pi_1 - \pi_2 \leq 0$; under this condition, (69) still holds. Substituting (67)–(69) into (66), we obtain

$$\int_{\Omega_T} (\partial_t\pi_1 - \partial_t\pi_2) \times (\pi_1 - \pi_2)\mathrm{d}x\mathrm{d}t = \int_{\Omega} (\pi_1(,T) - \pi_2(,T))^2\mathrm{d}x - \int_{\Omega} (\pi_1(,0) - \pi_2(,0))^2\mathrm{d}x \leq 0. \tag{70}$$

Note that both (π_1, ξ_1) and (π_2, ξ_2) are weak solutions of the variational inequality (3) and share the same initial-boundary conditions. Therefore,

$$\int_{\Omega} (\pi_1(,T) - \pi_2(,T))^2\mathrm{d}x \leq \int_{\Omega} (\pi_1(,0) - \pi_2(,0))^2\mathrm{d}x = 0,$$

which implies that (π_1, ξ_1) and (π_2, ξ_2) coincide in Ω_T.

5. Conclusions and discussion

This paper, from the perspective of American put options, formulates a class of variational inequality problems governed by a double-phase parabolic operator with variable exponents:

$$L\pi = \partial_t\pi - \sum_{i=1}^{N} [D_i(|D_i\pi|^{p(x,t)-2} D_i\pi) + D_i(|D_i\pi|^{q(x,t)-2} D_i\pi)].$$

The primary focus of this work is to study the existence and uniqueness of weak solutions to the variational inequality problem. First, we analyzed the boundedness of the variational inequality (3), namely,

$$|\pi_0|_\infty \geq \pi(x,t) \geq \pi_0(x) \geq 0, \forall (x,t) \in \Omega_T$$

and constructed a penalty problem to approximate its solution. Second, by applying integration by parts, Hölder's inequality, Young's inequality, and other techniques, we derived several energy estimates for the penalty initial-boundary value problem, including gradient energy estimates. Based on these, the existence of weak solutions to the variational inequality was established via a limiting procedure. The key challenge lay in handling structural changes in the gradient due to variable exponents and the resulting difficulties in deriving energy estimates. Finally, using a proof by contradiction, we assumed the existence of two distinct weak solutions and demonstrated that if they share the same initial-boundary conditions, they coincide in the L^2-norm, thereby completing the proof of uniqueness.

Due to the introduction of the variable-exponent parabolic operator in this work, certain challenges arose in deriving energy estimates for the solution and its gradient, particularly in establishing upper bounds, as detailed in Eqs (36)–(42). To address this issue, we introduced Eq (42); however, this approach still requires the additional condition

$$|\partial_t p(x,t)| \leq L,\ |\partial_t q(x,t)| \leq L \text{ in } \Omega_T.$$

Furthermore, throughout this paper, we assumed that $p(x,t) \geq 2$ and $q(x,t) \geq 2$ hold in Ω_T. In future studies, we aim to explore ways to relax these constraints.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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