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Research article

Research on synchronized assets of multiple individuals based on delayed feedback control

Huaiqin Yu[1], **Qiang Fu**[2] **and Xiaohuan Wang**[2,*]

[1] Business Administration, JOONGBU University, South Korea

[2] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

* **Correspondence:** Email: xiaohuanw@126.com.

Abstract: From a theoretical perspective, it is worthwhile to use the idea of control to synchronize multiple individual assets. We regard personal assets as multi-agent systems, i.e., second-order multi-agent systems (SOMAS). The delayed feedback control for consensus and average quasi-consensus of delayed SOMAS is studied. First, two weight matrices are studied, where the interactions are not entirely cooperative into the SOMAS with mixed time-varying delays (the delay varies with time). Second, a delayed feedback control is designed based on two weight matrices where the interactions are cooperative to get the consensus about followers and the average quasi-consensus about leader and followers. The consensus and average quasi-consensus of the delayed SOMAS are established by the graph-theoretic technique and Lyapunov functions. Finally, some numerical simulations is given to verify the theory.

Keywords: delayed feedback control; consensus; average quasi-consensus; SOMAS; graph-theoretic technique; mixed time-varying delays

1. Introduction

The total assets of an individual are influenced by both external and internal factors, and the fluctuations in their rate of change are often influenced by their acceleration (such as consumption acceleration, investment acceleration). The classic Samuelson model was born from this purpose. This paper regards personal assets as multi-agent systems, with the goal of using feedback control to achieve synchronization among multiple individual assets, that is, to achieve common prosperity. We consider two different groups, the first to become wealthy (leader) and the second to become wealthy (follower).

Multi-agent systems (MAS) are a very important aspect of research, and consensus is the hot topic in MAS study. With the contributions of many related researchers, many definitions and problems about MAS consensus have been put forward and studied. **Velocities Consensus:** [1–4] studied a common consensus problem about SOMAS. To achieve the research goal, the authors of the four papers designed this specific type of feedback control which used the weight matrix of the positions and did not use the weight matrix of the velocities. In particular, [1–4] achieved the consensus of the velocities by making the velocities tend to zero. [5–7] achieved the consensus of the velocities by considering the weight matrix of the velocities instead of making the velocities tend to zero. [8] considered the consensus performance of first-order agents. [9–13] studied finite-time consensus about SOMAS and gave some criteria. [14] established the consensus about MAS under asynchronous denial-of-service attacks.

Positions and Velocities Consensus: Though [1–7] solved the consensus problems of SOMAS, it was difficult to make the position state attach a specific state when the consensus was achieved. Many researchers considered the tracking control of SOMAS [15–17], and not only obtained the consensus of the leader and the followers, but also made the positions and the velocities of the followers tend to the position and the velocity of the leader, respectively. In [17], the authors considered the related weight matrix for the non-control part instead of the control part, which was different from most papers, such as [1–7].

Time Delay: In reality, the time delay situation can not be ignored. [18–20] designed a delay feedback control for consensus about SOMAS, where the time delay was a constant. Time-varying delays were considered for the containment control of SOMAS in [21]. [22, 23] studied time-varying delays of the non-control part instead of the control part, and [24] specifically considered the mixed delay times. In [25], authors not only considered the delay time, but also studied the difference of the weight matrix of the positions and the weight matrix of the velocities, when they designed the feedback control. [26] studied the strong consensus of convex SOMAS with time-varying topologies. [27] obtained the asynchronous impulsive consensus of discrete-time nonlinear MAS with time-varying delays. [28] obtained the consensus and average quasi-consensus of first-order delayed multi-agent systems. [29] obtained the containment control of heterogeneous multi-agent systems subject to Markovian randomly switching topologies and unbounded delays. In this paper, we consider the bounded delay, which is different from [29]. From the perspective of control theory, it is reasonable to always use known information to control the future, making time-delay feedback control reasonable. That is to say, it is reasonable to use previous information to control the present. A natural question, how long ago do we need to use the information? That is, how long should the time delay be? [30] studied the impact of time-dealy on stability, where they proved the small time-delay can stabilize the system, but when the time-delay becomes large, the system will become unstable. In this paper, we use a similar idea to consider the consensus issue.

Different Strategy: [31] considered the distributed dynamic event-triggered consensus control of multiagent systems subject to external disturbances, which is different from the strategy "feedback control". [32] established an adaptive dynamic event-triggered bipartite time-varying output formation tracking problem of heterogeneous multi-agent systems. [33] studied the problem of robust adaptive leaderless consensus for heterogeneous uncertain non-minimum phase linear multi-agent systems over directed communication graphs. In our future work, event-triggered strategy and leaderless consensus will be considered.

Inspired by [17] and [25], we consider two different weight matrices where the interactions are not entirely cooperative in SOMAS. Second, we design a delayed feedback control based on two different weight matrices where the interactions are cooperative. Inspired by [22] and [24], we not only consider the mixed time-varying delays for the non-control part, but also study feedback control, which is based on the mixed time-varying delays. Inspired by [15–17], we study consensus about followers and average quasi-consensus about leader and followers, and the definition of the average quasi-consensus between the leader and the follower will be given in the following section. The reason why we consider the average quasi-consensus is the following fact:

$$\int_0^\infty k_1(s)^2 ds \leq \int_0^\infty k_2(s) ds \not\Longrightarrow k_1(t) \leq k_2(t),$$

where $k_1(t), k_2(t)$ are positive functions. But, we can consider the average, so we introduce a new definition "average quasi-consensus". In fact, it is difficult to achieve asset synchronization at all times, but it is possible to achieve synchronization within a certain time interval, or, in the sense of time averaging, assets are synchronized.

In this paper, we face two key problems. First, different from common MAS, SOMAS are controlled only in terms of the velocities, while the positions are not controlled, which may affects the consensus of the positions. Second, we need to consider feedback control with time-varying delays and the non-control with mixed time-varying delays.

[34–36] have given some methods for studying the delay equations. [37] provided an especial useful research method for the second problem. [38] also gave some ideas for the first problems. So we combine [37] and [38], and give a reasonable research method. In a future paper, we will use a similar method to consider the multi-UAV system [39].

Contributions:

(i) A new delayed SOMAS is introduced, and a new definition of "average quasi-consensus" is given.

(ii) A feedback control with mixed time-varying delays for SOMAS is designed. More precisely, a special Lyapunov function based on the state of the followers of SOMAS to study the consensus of the followers is constructed; moreover, the second special Lyapunov function based on the state of the leader and the followers to get the average quasi-consensus between the leader and the follower is designed.

(iii) A new strategy for consensus of personal assets is given.

The rest of this paper is organized as follows. In Section 2, problem formulation and preliminaries are given. Section 3 is concerned with the main results. Some numerical examples are given in Section 4.

2. Problem formulation and preliminaries

2.1. Notation

$\mathcal{N} = \{1, 2, 3, \ldots, M\}$, $\mathbb{N} = 1, 2, 3, \ldots$, $\mathbf{0}_k$ represents the k-dimensional zero matrix, and I_k represents the k-dimensional identity matrix. $\mathbf{1}_{M-1}$ represents a column vector composed of the element 1 with size $M-1$. $\forall x \in R^m$, $|x| = \sqrt{x^\top x}$, $\forall A \in R^{m\times m}$, $\|A\| = \sqrt{\sum_{i=1}^m \sum_{j=1}^m \left|a_{ij}\right|^2}$. Let $\mathcal{G} = (\mathcal{V}, \Upsilon)$ represent a graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and arc set Υ. Denote by $(\mathcal{G}, \mathbb{A})$ the graph $\mathcal{G}$ with weight matrix $\mathbb{A} = (a_{ij})_{M\times M}$, where $a_{ij} \neq 0$ means that there exists an arc $(j, i) \in \Upsilon$, $a_{ij} > 0$ means that the interaction between i and

j is cooperative, and $a_{ij} < 0$ means that the interaction between i and j is competitive. Let $\tau > 0$ and $\delta > 0$. $C([-\tau - \delta, 0]; R^m)$ denotes the family of continuous functions ξ from $[-\tau - \delta, 0]$ to R^m. $\otimes$ is the Kronecker product. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ represent its smallest and largest eigenvalues, respectively.

2.2. Problem formulation

Consider the following delayed SOMAS:

$$\begin{cases} \dot{x}_0(l) &= v_0(l), l > 0, \\ \dot{v}_0(l) &= a_1 x_0(l) + a_2 v_0(l) + b_1 x_0(l - \delta_1(l)) \\ &\quad +b_2 v_0(l - \delta_2(l)) + f(l), l > 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dot{x}_i(l) &= v_i(l), l > 0, \\ \dot{v}_i(l) &= a_1 x_i(l) + a_2 v_i(l) + b_1 x_i(l - \delta_1(l)) \\ &\quad +b_2 v_i(l - \delta_2(l)) + g(l) \\ &\quad + \sum_{j=1}^{M} a_{ij}\Gamma_1[x_j(l - \delta_1(l)) - x_i(l - \delta_1(l))] \\ &\quad + \sum_{j=1}^{M} b_{ij}\Gamma_2[v_j(l - \delta_2(l)) - v_i(l - \delta_2(l))], i \in \mathcal{N}, l > 0, \end{cases} \tag{2.2}$$

where $x_0(l) \in R^m$ and $v_0(l) \in R^m$ are the position state and the velocity state of the leader, $x_i(l) \in R^m$ and $v_i(l) \in R^m$ are the position state and the velocity state of the follower i, $0 \le \delta_1(l) \le \delta$ and $0 \le \delta_2(l) \le \delta$, $x_0(u) = \xi_{01}(u)$ and $v_0(u) = \xi_{02}(u) \in C([-\tau - \delta, 0]; R^m)$, and $x_i(u) = \xi_{i1}(u)$ and $v_i(u) = \xi_{i2}(u) \in C([-\tau - \delta, 0]; R^m)$. a_1, a_2, b_1, and b_2 are known positive constants, $\Gamma_1 \in R^{m\times m}$ and $\Gamma_2 \in R^{m\times m}$ are the inner-coupling matrices, $f(l) \in R^m$ and $g(l) \in R^m$ are the disturbances of the MAS with $\max\{|f(l)|, |g(l)|\} \le d$, and $d > 0$. $\mathbb{A} = (a_{ij})_{M\times M}$ is the outer-coupling matrix of the position state, $i, j \in \mathcal{N}$. When the jth subsystem has influence on the ith subsystem, $a_{ij} \neq 0$, otherwise, a_{ij}=0, and assume a_{ii}=0 for all $i \in \mathcal{N}$. $\mathbb{B} = (b_{ij})_{M\times M}$ is the outer-coupling matrix of the velocity state, $i, j \in \mathcal{N}$. When the jth subsystem has influence on the ith subsystem, $b_{ij} \neq 0$, otherwise, b_{ij}=0, and assume b_{ii}=0 for all $i \in \mathcal{N}$.

Remark 2.1. *System (2.2) shows that MAS has its own weight matrices. Due to the difference of the position state and the velocity state, the position and the velocity of followers have weight matrix* $\mathbb{A}$ *and weight matrix* $\mathbb{B}$*, respectively.*

Then we give the corresponding delayed feedback control system as follows:

$$\begin{cases} \dot{\hat{x}}_0(l) &= \hat{v}_0(l), l > 0, \\ \dot{\hat{v}}_0(l) &= a_1 \hat{x}_0(l) + a_2 \hat{v}_0(l) + b_1 \hat{x}_0(l - \delta_1(l)) \\ &\quad +b_2 \hat{v}_0(l - \delta_2(l)) + f(l), l > 0, \end{cases} \tag{2.3}$$

and

$$
\begin{cases}
\dot{\hat{x}}_i(l) &= \hat{v}_i(l), l > 0,\\
\dot{\hat{v}}_i(l) &= a_1\hat{x}_i(l) + a_2\hat{v}_i(l) + b_1\hat{x}_i(l-\delta_1(l))\\
& +b_2\hat{v}_i(l-\delta_2(l)) + g(l)\\
& + \sum_{j=1}^{M} a_{ij}\Gamma_1[\hat{x}_j(l-\delta_1(l)) - \hat{x}_i(l-\delta_1(l))]\\
& + \sum_{j=1}^{M} b_{ij}\Gamma_2[\hat{v}_j(l-\delta_2(l)) - \hat{v}_i(l-\delta_2(l))]\\
& + \sum_{j=1}^{M} c_{ij}\Gamma_3[\hat{x}_j(l-\tau_1(l)) - \hat{x}_i(l-\tau_1(l))]\\
& + \sum_{j=1}^{M} d_{ij}\Gamma_4[\hat{v}_j(l-\tau_2(l)) - \hat{v}_i(l-\tau_2(l))]\\
& +\alpha\Gamma_5[\hat{x}_0(l-\tau_1(l)) - \hat{x}_i(l-\tau_1(l))]\\
& +\beta\Gamma_6[\hat{v}_0(l-\tau_2(l)) - \hat{v}_i(l-\tau_2(l))],\ i \in \mathcal{N}, l > 0,
\end{cases}
\tag{2.4}
$$

where $0 \leq \tau_1(l), \tau_2(l) \leq \tau$, $(\hat{x}_0(u), \hat{v}_0(u)) = (\xi_{01}(u), \xi_{02}(u)) \in C([-\tau-\delta, 0]; R^m)^2$, $(\hat{x}_i(u), \hat{v}_i(u)) = (\xi_{i1}(u), \xi_{i2}(u) \in C([-\tau-\delta, 0]; R^m)^2$, and $\Gamma_3 \in R^{m\times m}$, $\Gamma_4 \in R^{m\times m}$, $\Gamma_5 \in R^{m\times m}$, and $\Gamma_6 \in R^{m\times m}$ are the inner-coupling matrices. $\mathbb{C} = (c_{ij})_{M\times M}$ is the outer-coupling matrix, and c_{ij} is a known nonnegative constant, $i, j \in \mathcal{N}$. $c_{ij} > 0$ or c_{ij}=0, and assume c_{ii}=0 for all $i \in \mathcal{N}$. $\mathbb{D} = (d_{ij})_{M\times M}$ is the outer-coupling matrix, and d_{ij} is a known nonnegative constant, $i, j \in \mathcal{N}$. $d_{ij} > 0$ or d_{ij}=0, and assume d_{ii}=0 for all $i \in \mathcal{N}$. $\sum_{j=1}^{M} c_{ij}\Gamma_3[\hat{x}_j(l-\tau_1(l)) - \hat{x}_i(l-\tau_1(l))] + \sum_{j=1}^{M} d_{ij}\Gamma_4[\hat{v}_j(l-\tau_2(l)) - \hat{v}_i(l-\tau_2(l))] + \alpha\Gamma_5[\hat{x}_0(l-\tau_1(l)) - \hat{x}_i(l-\tau_1(l))] + \beta\Gamma_6[\hat{v}_0(l-\tau_2(l)) - \hat{v}_i(l-\tau_2(l))]$ is the delayed feedback control. $\alpha > 0$ and $\beta > 0$ represent that leader has influence on all the followers.

Remark 2.2. *In (2.4), we design a delayed feedback control for SOMAS based on the weight matrix $\mathbb{C}$, weight matrix $\mathbb{D}$, α, and β.*

The roles of Γ_i $(i = 1, \ldots, 6)$ are as follows: Γ_1 makes the individuals x_i and x_j have the same class; Γ_2 makes the individuals x_i and x_j have the same transition rate; Γ_3 makes the individuals x_i and x_j have the same class under the different time-delay; Γ_4 makes the individuals x_i and x_j have the same transition rate under the different time-delay; Γ_5 makes the individuals x_i and x_0 have the same class; and Γ_6 makes the individuals x_i and x_0 have the same transition rate.

Let $p_i(l) = \hat{x}_i(l) - \hat{x}_1(l)$, $q_i(l) = \hat{v}_i(l) - \hat{v}_1(l)$, $i = 1, 2, ..., M$. Using system (2.4) and the special case $i = 1$ (difference between two systems), we can rewrite the error system as

$$
\begin{cases}
\dot{p}_i(l) &= q_i(l)\\
\dot{q}_i(l) &= a_1p_i(l) + a_2q_i(l) + b_1p_i(l-\delta_1(l)) + b_2q_i(l-\delta_2(l))\\
& + \sum_{j=1}^{M} a_{ij}\Gamma_1[p_j(l-\delta_1(l)) - p_i(l-\delta_1(l))]\\
& + \sum_{j=1}^{M} b_{ij}\Gamma_2[q_j(l-\delta_2(l)) - q_i(l-\delta_2(l))]\\
& + \sum_{j=1}^{M} c_{ij}\Gamma_3[p_j(l-\tau_1(l)) - p_i(l-\tau_1(l))]\\
& + \sum_{j=1}^{M} d_{ij}\Gamma_4[q_j(l-\tau_2(l)) - q_i(l-\tau_2(l))]\\
& - \sum_{j=2}^{M} a_{1j}\Gamma_1p_j(l-\delta_1(l)) - \sum_{j=2}^{M} b_{1j}\Gamma_2q_j(l-\delta_2(l))\\
& - \sum_{j=2}^{M} c_{1j}\Gamma_3p_j(l-\tau_1(l)) - \sum_{j=2}^{M} d_{1j}\Gamma_4q_j(l-\tau_2(l))\\
& -\alpha\Gamma_5p_i(l-\tau_1(l)) - \beta\Gamma_6q_i(l-\tau_2(l)), i = 2, ..., M.
\end{cases}
\tag{2.5}
$$

Consequently, we have the compact form

$$
\begin{cases}
\dot{e}(l) = A_1e(l) + B_{11}e(l-\delta_1(l)) + B_{12}e(l-\delta_2(l))\\
-K_{11}e(l-\tau_1(l)) - K_{12}e(l-\tau_2(l)), l > 0,\\
e(l) = \xi_e(l), -\delta-\tau \leq l \leq 0,
\end{cases}
\tag{2.6}
$$

where

$$\begin{aligned} e(l) &= (p_2^\top(l), p_3^\top(l), ..., p_M^\top(l), q_2^\top(l), q_3^\top(l), ..., q_M^\top(l))^\top, \\ \xi_e(l) &= (\xi_{21}^\top(l) - \xi_{11}^\top(l), \cdots, \xi_{M1}^\top(l) - \xi_{11}^\top(l), \xi_{22}^\top(l) - \xi_{12}^\top(l), \cdots, \xi_{M2}^\top(l) - \xi_{12}^\top(l))^\top. \end{aligned}$$

Denote

$$\begin{aligned} A_1 &= \begin{pmatrix} \mathbf{0}_{mM-m} & I_{mM-m} \\ a_1 I_{mM-m} & a_2 I_{mM-m} \end{pmatrix}, \quad B_{11} = \begin{pmatrix} \mathbf{0}_{mM-m} & \mathbf{0}_{mM-m} \\ \Phi_1 & \mathbf{0}_{mM-m} \end{pmatrix}, \quad B_{12} = \begin{pmatrix} \mathbf{0}_{mM-m} & \mathbf{0}_{mM-m} \\ \mathbf{0}_{mM-m} & \Phi_2 \end{pmatrix}, \\ K_{11} &= \begin{pmatrix} \mathbf{0}_{mM-m} & \mathbf{0}_{mM-m} \\ \Phi_3 & \mathbf{0}_{mM-m} \end{pmatrix}, \quad K_{12} = \begin{pmatrix} \mathbf{0}_{mM-m} & \mathbf{0}_{mM-m} \\ \mathbf{0}_{mM-m} & \Phi_4 \end{pmatrix}, \end{aligned}$$

where $\Phi_1 = b_1 I_{mM-m} - L_{1,M-1} \otimes \Gamma_1 - \mathbf{1}_{M-1} \otimes [(a_{12}, a_{13}, ..., a_{1M}) \otimes \Gamma_1]$, $\Phi_2 = b_2 I_{mM-m} - L_{2,M-1} \otimes \Gamma_2 - \mathbf{1}_{M-1} \otimes [(b_{12}, b_{13}, ..., b_{1M}) \otimes \Gamma_2]$, $\Phi_3 = L_{3,M-1} \otimes \Gamma_3 + \mathbf{1}_{M-1} \otimes [(c_{12}, c_{13}, ..., c_{1M}) \otimes \Gamma_3] + \alpha I_{M-1} \otimes \Gamma_5$, $\Phi_4 = L_{4,M-1} \otimes \Gamma_4 + \mathbf{1}_{M-1} \otimes [(d_{12}, d_{13}, ..., d_{1M}) \otimes \Gamma_4] + \beta I_{M-1} \otimes \Gamma_6$, and

$$L_{1,M-1} = \begin{pmatrix} \sum_{j=1}^M a_{2j} & -a_{23} & \cdots & -a_{2M} \\ -a_{32} & \sum_{j=1}^M a_{3j} & \cdots & -a_{3M} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{M2} & -a_{M3} & \cdots & \sum_{j=1}^M a_{Mj} \end{pmatrix},$$

$$L_{2,M-1} = \begin{pmatrix} \sum_{j=1}^M b_{2j} & -b_{23} & \cdots & -b_{2M} \\ -b_{32} & \sum_{j=1}^M b_{3j} & \cdots & -b_{3M} \\ \vdots & \vdots & \cdots & \vdots \\ -b_{M2} & -b_{M3} & \cdots & \sum_{j=1}^M b_{Mj} \end{pmatrix},$$

$$L_{3,M-1} = \begin{pmatrix} \sum_{j=1}^M c_{2j} & -c_{23} & \cdots & -c_{2M} \\ -c_{32} & \sum_{j=1}^M c_{3j} & \cdots & -c_{3M} \\ \vdots & \vdots & \cdots & \vdots \\ -c_{M2} & -c_{M3} & \cdots & \sum_{j=1}^M c_{Mj} \end{pmatrix},$$

$$L_{4,M-1} = \begin{pmatrix} \sum_{j=1}^M d_{2j} & -d_{23} & \cdots & -d_{2M} \\ -d_{32} & \sum_{j=1}^M d_{3j} & \cdots & -d_{3M} \\ \vdots & \vdots & \cdots & \vdots \\ -d_{M2} & -d_{M3} & \cdots & \sum_{j=1}^M d_{Mj} \end{pmatrix}.$$

Let $\hat{p}_i(l) = \hat{x}_i(l) - \hat{x}_0(l)$, $\hat{q}_i(l) = \hat{v}_i(l) - \hat{v}_0(l)$, $i = 1, ..., M$. According to (2.3) and (2.4), we get

$$\left\{ \begin{aligned} \dot{\hat{p}}_i(l) &= \hat{q}_i(l) \\ \dot{\hat{q}}_i(l) &= a_1 \hat{p}_i(l) + a_2 \hat{q}_i(l) + b_1 \hat{p}_i(l - \delta_1(l)) + b_2 \hat{q}_i(l - \delta_2(l)) \\ &\quad + \sum_{j=1}^M a_{ij} \Gamma_1 [\hat{p}_j(l - \delta_1(l)) - \hat{p}_i(l - \delta_1(l))] \\ &\quad + \sum_{j=1}^M b_{ij} \Gamma_2 [\hat{q}_j(l - \delta_2(l)) \\ &\quad - \hat{q}_i(l - \delta_2(l))] + \sum_{j=1}^M c_{ij} \Gamma_3 [\hat{p}_j(l - \tau_1(l)) \\ &\quad - \hat{p}_i(l - \tau_1(l))] + \sum_{j=1}^M d_{ij} \Gamma_4 [\hat{q}_j(l - \tau_2(l)) \\ &\quad - \hat{q}_i(l - \tau_2(l))] - \alpha \Gamma_5 \hat{p}_i(l - \tau_1(l)) \\ &\quad - \beta \Gamma_6 \hat{q}_i(l - \tau_2(l)) + g(l) - f(l), i = 1, ..., M. \end{aligned} \right. \tag{2.7}$$

Then, we have the compact form

$$\begin{cases} \dot{\hat{e}}(l) &= A_2\hat{e}(l) + B_{21}\hat{e}(l-\delta_1(l)) + B_{22}\hat{e}(l-\delta_2(l)) \\ & \quad -K_{21}\hat{e}(l-\tau_1(l)) - K_{22}\hat{e}(l-\tau_2(l)) \\ & \quad +\begin{pmatrix}\mathbf{1}_{mM}\otimes\mathbf{0}_1 \\ d(l)\end{pmatrix}, l>0, \\ \hat{e}(l) &= \xi_{\hat{e}}(l), -\delta-\tau\le l\le 0, \end{cases} \tag{2.8}$$

where $\hat{e}(l) = (\hat{p}_1^\top(l), \hat{p}_2^\top(l), ..., \hat{p}_M^\top(l), \hat{q}_1^\top(l), \hat{q}_2^\top(l), ..., \hat{q}_M^\top(l))^\top$, $\xi_{\hat{e}}(l) = (\xi_{11}^\top(l)-\xi_{01}^\top(l), \xi_{21}^\top(l)-\xi_{01}^\top(l), ..., \xi_{M1}^\top(l)-\xi_{01}^\top(l), \xi_{12}^\top(l)-\xi_{02}^\top(l), \xi_{22}^\top(l)-\xi_{02}^\top(l), ..., \xi_{M2}^\top(l)-\xi_{02}^\top(l))^\top$. $d(l) = \mathbf{1}_M\otimes(g(l)-f(l))$. Denote

$$\begin{aligned} A_2 &= \begin{pmatrix}\mathbf{0}_{mM} & I_{mM} \\ a_1 I_{mM} & a_2 I_{mM}\end{pmatrix}, \quad B_{21} = \begin{pmatrix}\mathbf{0}_{mM} & \mathbf{0}_{mM} \\ \Phi_5 & \mathbf{0}_{mM}\end{pmatrix}, \quad B_{22} = \begin{pmatrix}\mathbf{0}_{mM} & \mathbf{0}_{mM} \\ \mathbf{0}_{mM} & \Phi_6\end{pmatrix}, \\ K_{21} &= \begin{pmatrix}\mathbf{0}_{mM} & \mathbf{0}_{mM} \\ \Phi_7 & \mathbf{0}_{mM}\end{pmatrix}, \quad K_{22} = \begin{pmatrix}\mathbf{0}_{mM} & \mathbf{0}_{mM} \\ \mathbf{0}_{mM} & \Phi_8\end{pmatrix}, \end{aligned}$$

where $\Phi_5 = b_1 I_{mM} - L_1\otimes\Gamma_1$, $\Phi_6 = b_2 I_{mM} - L_2\otimes\Gamma_2$, $\Phi_7 = L_3\otimes\Gamma_3 + \alpha I_M\otimes\Gamma_5$, $\Phi_8 = L_4\otimes\Gamma_4 + \beta I_M\otimes\Gamma_6$, and

$$L_1 = \begin{pmatrix}\sum_{j=1}^M a_{1j} & -a_{12} & \cdots & -a_{1M} \\ -a_{21} & \sum_{j=1}^M a_{2j} & \cdots & -a_{2M} \\ \vdots & \cdots & \cdots & \vdots \\ -a_{M1} & -a_{M2} & \cdots & \sum_{j=1}^M a_{Mj}\end{pmatrix},$$

$$L_2 = \begin{pmatrix}\sum_{j=1}^M b_{1j} & -b_{12} & \cdots & -b_{1M} \\ -b_{21} & \sum_{j=1}^M b_{2j} & \cdots & -b_{2M} \\ \vdots & \cdots & \cdots & \vdots \\ -b_{M1} & -b_{M2} & \cdots & \sum_{j=1}^M b_{Mj}\end{pmatrix},$$

$$L_3 = \begin{pmatrix}\sum_{j=1}^M c_{1j} & -c_{12} & \cdots & -c_{1M} \\ -c_{21} & \sum_{j=1}^M c_{2j} & \cdots & -c_{2M} \\ \vdots & \cdots & \cdots & \vdots \\ -c_{M1} & -c_{M2} & \cdots & \sum_{j=1}^M c_{Mj}\end{pmatrix},$$

$$L_4 = \begin{pmatrix}\sum_{j=1}^M d_{1j} & -d_{12} & \cdots & -d_{1M} \\ -d_{21} & \sum_{j=1}^M d_{2j} & \cdots & -d_{2M} \\ \vdots & \cdots & \cdots & \vdots \\ -d_{M1} & -d_{M2} & \cdots & \sum_{j=1}^M d_{Mj}\end{pmatrix}.$$

Definition 2.1. *SOMAS (2.4) achieves the consensus among the followers, if*

$$\lim_{l\to\infty} |e(l)|^2 = 0.$$

Definition 2.2. *SOMAS (2.3) and (2.4) achieve the average quasi-consensus between the leader and the follower, if there exists a positive constant η such that*

$$\lim_{l\to\infty}\frac{1}{l}\int_{2\tau+2\delta}^{l}|\hat{e}(s)|^2\,\mathrm{d}s\le\eta.$$

Assumption 2.1. *$\delta_1(l)$, $\delta_2(l)$, $\tau_1(l)$, and $\tau_2(l)$ are bounded and continuously differentiable functions, satisfying $\dot{\delta}_1(l)<\hat{\delta}<1$, $\dot{\delta}_2(l)<\hat{\delta}<1$, $\dot{\tau}_1(l)<\hat{\tau}<1$, and $\dot{\tau}_2(l)<\hat{\tau}<1$.*

Assumption 2.2. *There are positive constants λ_1–λ_7, such that*

$$\lambda_{max}\{\Psi_1\}\le-8\lambda_1-\lambda_2, \tag{2.9}$$
$$\lambda_{max}\{\Psi_2\}\le\lambda_3, \tag{2.10}$$
$$\lambda_{max}\{\Psi_3\}\le\lambda_4, \tag{2.11}$$
$$\lambda_{max}\{\Psi_4\}\le-8\lambda_1-\lambda_5, \tag{2.12}$$
$$\lambda_{max}\{\Psi_5\}\le\lambda_6, \tag{2.13}$$
$$\lambda_{max}\{\Psi_6\}\le\lambda_7, \tag{2.14}$$

where $-\lambda_2<-\frac{\lambda_3+\lambda_4}{1-\hat{\delta}}$ and $-\lambda_5<-\frac{\lambda_6+\lambda_7}{1-\hat{\delta}}$,

$$\begin{aligned}
\Psi_1 &= \begin{pmatrix}(2a_1+2)I_{mM-m} & (a_1+a_2)I_{mM-m}\\ (a_1+a_2)I_{mM-m} & 2(2+a_2)I_{mM-m}\end{pmatrix}-\begin{pmatrix}\Phi_3+\Phi_3^\top & \mathbf{0}_{mM-m}\\ \mathbf{0}_{mM-m} & \Phi_4+\Phi_4^\top\end{pmatrix},\\
\Psi_2 &= \begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix}^\top\begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix},\ \Psi_3=\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix}^\top\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix},\\
\Psi_4 &= \begin{pmatrix}(2a_1+3)I_{mM} & (a_1+a_2)I_{mM}\\ (a_1+a_2)I_{mM} & 2(2.5+a_2)I_{mM}\end{pmatrix}-\begin{pmatrix}\Phi_7+\Phi_7^\top & \mathbf{0}_{mM}\\ \mathbf{0}_{mM} & \Phi_8+\Phi_8^\top\end{pmatrix},
\end{aligned}$$

and

$$\Psi_5=\begin{pmatrix}\Phi_5 & \mathbf{0}_{mM}\\ \Phi_5 & \mathbf{0}_{mM}\end{pmatrix}^\top\begin{pmatrix}\Phi_5 & \mathbf{0}_{mM}\\ \Phi_5 & \mathbf{0}_{mM}\end{pmatrix},\ \Psi_6=\begin{pmatrix}\mathbf{0}_{mM} & \Phi_6\\ \mathbf{0}_{mM} & \Phi_6\end{pmatrix}^\top\begin{pmatrix}\mathbf{0}_{mM} & \Phi_6\\ \mathbf{0}_{mM} & \Phi_6\end{pmatrix}.$$

Assumptions 2.1 show that the delay is bounded, and Assumption 2.2 synchronizes between followers and between followers and leaders. Because we used the Lyapunov method, we imposed restrictions on its main eigenvalues.

3. Main results

In this section, we will give the key lemma and the main theorem.

Lemma 3.1. *Let $e(l)$ and $\hat{e}(l)$ be the solutions of (2.6) and (2.8), respectively. The following estimates hold:*

$$|e(l)| \le [|e(0)|+\sup_{-\tau-\delta\le u\le 0}|\xi_e(u)|(\|B_{11}\|+\|B_{12}\|+\|K_{11}\|+\|K_{12}\|)l]\exp\{g_1 l\}, \tag{3.1}$$
$$|\hat{e}(l)| \le [|\hat{e}(0)|+\sup_{-\tau-\delta\le u\le 0}|\xi_{\hat{e}}(u)|(\|B_{21}\|+\|B_{22}\|+\|K_{21}\|+\|K_{22}\|)l+2Mdl]\exp\{g_2 l\}, \tag{3.2}$$

where $g_1=\|A_1\|+\|B_{11}\|+\|B_{12}\|+\|K_{11}\|+\|K_{12}\|$ and $g_2=\|A_2\|+\|B_{21}\|+\|B_{22}\|+\|K_{21}\|+\|K_{22}\|$.

Proof. First, we prove (3.1). According to (2.6), we obtain

$$\begin{aligned} e(l) &= e(0) + \int_0^l [A_1 e(s) + B_{11} e(s-\delta_1(s)) + B_{12} e(s-\delta_2(s)) \\ &\quad -K_{11} e(s-\tau_1(s)) - K_{12} e(s-\tau_2(s))]\mathrm{d}s, \end{aligned}$$

which implies that

$$\begin{aligned} |e(l)| &= \Big| e(0) + \int_0^l [A_1 e(s) + B_{11} e(s-\delta_1(s)) + B_{12} e(s-\delta_2(s)) \\ &\quad -K_{11} e(s-\tau_1(s)) - K_{12} e(s-\tau_2(s))]\mathrm{d}s \Big| \\ &\le |e(0)| + \|A_1\| \int_0^l |e(s)|\,\mathrm{d}s + \|B_{11}\| \int_0^l |e(s-\delta_1(s))|\,\mathrm{d}s \\ &\quad + \|B_{12}\| \int_0^l |e(s-\delta_2(s))|\,\mathrm{d}s + \|K_{11}\| \int_0^l |e(s-\tau_1(s))|\,\mathrm{d}s \\ &\quad + \|K_{12}\| \int_0^l |e(s-\tau_2(s))|\,\mathrm{d}s. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \sup_{0\le u\le l} |e(u)| &\le |e(0)| + \|A_1\| \int_0^l \sup_{0\le u\le s} |e(u)|\,\mathrm{d}s + \|B_{11}\| \int_0^l \Big(\sup_{-\tau-\delta\le u\le 0} |\xi_e(u)| \\ &\quad + \sup_{0\le u\le s} |e(u)| \Big)\mathrm{d}s + \|B_{12}\| \int_0^l \Big(\sup_{-\tau-\delta\le u\le 0} |\xi_e(u)| + \sup_{0\le u\le s} |e(u)| \Big)\mathrm{d}s \\ &\quad + \|K_{11}\| \int_0^l \Big(\sup_{-\tau-\delta\le u\le 0} |\xi_e(u)| + \sup_{0\le u\le s} |e(u)| \Big)\mathrm{d}s \\ &\quad + \|K_{12}\| \int_0^l \Big(\sup_{-\tau-\delta\le u\le 0} |\xi_e(u)| + \sup_{0\le u\le s} |e(u)| \Big)\mathrm{d}s \\ &= |e(0)| + \sup_{-\tau-\delta\le u\le 0} |\xi_e(u)|\,(\|B_{11}\| + \|B_{12}\| + \|K_{11}\| + \|K_{12}\|)l \\ &\quad +(\|A_1\| + \|B_{11}\| + \|B_{12}\| + \|K_{11}\| + \|K_{12}\|) \int_0^l \sup_{0\le u\le s} |e(u)|\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

The Grönwall's inequality and (3.3) yield that

$$\begin{aligned} |e(l)| &\le \sup_{0\le u\le l} |e(u)| \\ &\le [|e(0)| + \sup_{-\tau-\delta\le u\le 0} |\xi_e(u)|\,(\|B_{11}\| + \|B_{12}\| + \|K_{11}\| + \|K_{12}\|)l] \exp\{g_1 l\}. \end{aligned}$$

Next, we prove (3.2). According to (2.8), we get

$$
\begin{aligned}
|\hat{e}(l)| &= \Big|\hat{e}(0) + \int_0^l [A_2\hat{e}(s) + B_{21}\hat{e}(s-\delta_1(s)) + B_{22}\hat{e}(s-\delta_2(s)) - K_{21}\hat{e}(s-\tau_1(s)) \\
&\quad -K_{22}\hat{e}(s-\tau_2(s)) + \begin{pmatrix}\mathbf{1}_{mM}\otimes\mathbf{0}_1\\ d(l)\end{pmatrix}]\mathrm{d}s\Big| \\
&\leq |\hat{e}(0)| + \|A_2\|\int_0^l |\hat{e}(s)|\,\mathrm{d}s + \|B_{21}\|\int_0^l |\hat{e}(s-\delta_1(s))|\,\mathrm{d}s \\
&\quad + \|B_{22}\|\int_0^l |\hat{e}(s-\delta_2(s))|\,\mathrm{d}s + \|K_{21}\|\int_0^l |\hat{e}(s-\tau_1(s))|\,\mathrm{d}s \\
&\quad + \|K_{22}\|\int_0^l |\hat{e}(s-\tau_2(s))|\,\mathrm{d}s + 2Mdl.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\sup_{0\leq u\leq l}|\hat{e}(u)| &\leq |\hat{e}(0)| + \|A_2\|\int_0^l \sup_{0\leq u\leq s}|\hat{e}(u)|\,\mathrm{d}s + (\|B_{21}\| + \|B_{22}\| + \|K_{21}\| \\
&\quad + \|K_{22}\|)\int_0^l \Big(\sup_{-\tau-\delta\leq u\leq 0}|\xi_{\hat{e}}(u)| + \sup_{0\leq u\leq s}|\hat{e}(u)|\Big)\mathrm{d}s + 2Mdl \\
&\leq |\hat{e}(0)| + \sup_{-\tau-\delta\leq u\leq 0}|\xi_{\hat{e}}(u)|\,(\|B_{21}\| + \|B_{22}\| + \|K_{21}\| + \|K_{22}\|)l + 2Mdl \\
&\quad +(\|A_2\| + \|B_{21}\| + \|B_{22}\| + \|K_{21}\| + \|K_{22}\|)\int_0^l \sup_{0\leq u\leq s}|\hat{e}(u)|\,\mathrm{d}s,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|\hat{e}(l)| &\leq \sup_{0\leq u\leq l}|\hat{e}(u)| \\
&\leq [|\hat{e}(0)| + \sup_{-\tau-\delta\leq u\leq 0}|\xi_{\hat{e}}(u)|\,(\|B_{21}\| + \|B_{22}\| + \|K_{21}\| + \|K_{22}\|)l + 2Mdl]\exp\{g_2 l\}.
\end{aligned}
$$

Hence the proof is complete.

Theorem 3.1. *Let Assumptions 2.1 and 2.2 hold. Assume that $e(l)$ and $\hat{e}(l)$ are the solutions of (2.6) and (2.8), respectively. Let $\tau_1 > 0$ and $\tau_2 > 0$ be the roots to Eqs (3.4) and (3.5), respectively, and*

$$
-\frac{5\tau^2(\|K_{11}\|^2 + \|K_{12}\|^2)}{\lambda_1}\times\Big(\frac{\|B_{11}\|^2 + \|B_{12}\|^2}{1-\hat{\delta}} + \|A_1 - K_{11} - K_{12}\|^2\Big) = \frac{\lambda_3 + \lambda_4}{1-\hat{\delta}} - \lambda_2, \tag{3.4}
$$

$$
-\frac{6\tau^2(\|K_{21}\|^2 + \|K_{22}\|^2)}{\lambda_1}\times\Big(\frac{\|B_{21}\|^2 + \|B_{22}\|^2}{1-\hat{\delta}} + \|A_2 - K_{21} - K_{22}\|^2\Big) = \frac{\lambda_6 + \lambda_7}{1-\hat{\delta}} - \lambda_5. \tag{3.5}
$$

Let

$$
\Phi_3^\top + \Phi_4 > I_{mM-m}, \tag{3.6}
$$

$$
\Phi_7^\top + \Phi_8 > I_{mM}. \tag{3.7}
$$

Then, for any $\tau \in (0, \tau_*)$, there exists a positive constant η such that

$$
\lim_{l\to\infty}|e(l)|^2 = 0, \tag{3.8}
$$

$$\lim_{l\to\infty}\frac{1}{l}\int_{2\tau+2\delta}^{l}|\hat{e}(s)|^2\,\mathrm{d}s\le\eta,\tag{3.9}$$

where $\tau_*=\min\left\{\tau_1,\tau_2,\ \sqrt{\frac{3}{20(\|K_{11}\|^2+\|K_{12}\|^2)}},\ \sqrt{\frac{1}{8(\|K_{21}\|^2+\|K_{22}\|^2)}}\right\}$.

Proof. First, we prove (3.8). We denote $\Phi=\Phi_3^\top+\Phi_4$ and

$$P_1=\begin{pmatrix}\Phi & I_{mM-m}\\ I_{mM-m} & I_{mM-m}\end{pmatrix}.\tag{3.10}$$

It follows from (3.6) that $P_1>0$. Define $W_1(l)=e^\top(l)P_1e(l)$ and

$$\tilde{W}_1(l)\ =\ 2e^\top(l)P_1[A_1e(l)+B_{11}e(l-\delta_1(l))+B_{12}e(l-\delta_2(l))-K_{11}e(l)-K_{12}e(l)].$$

Then we obtain

$$\begin{aligned}
&\tilde{W}_1(l)+8\lambda_1|e(l)|^2\\
=\ &2e^\top(l)\begin{pmatrix}a_1I_{mM-m} & \Phi+a_2I_{mM-m}\\ a_1I_{mM-m} & (1+a_2)I_{mM-m}\end{pmatrix}e(l)+2e^\top(l)\begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix}e(l-\delta_1(l))\\
&+2e^\top(l)\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix}e(l-\delta_2(l))-2e^\top(l)\begin{pmatrix}\Phi_3 & \Phi_4\\ \Phi_3 & \Phi_4\end{pmatrix}e(l)+8\lambda_1|e(l)|^2\\
=\ &e^\top(l)\begin{pmatrix}2a_1I_{mM-m} & \Phi+(a_1+a_2)I_{mM-m}\\ (a_1+a_2)I_{mM-m}+\Phi^\top & 2(1+a_2)I_{mM-m}\end{pmatrix}e(l)\\
&+2e^\top(l)\begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix}e(l-\delta_1(l))+2e^\top(l)\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix}e(l-\delta_2(l))\\
&-e^\top(l)\begin{pmatrix}\Phi_3+\Phi_3^\top & \Phi_3^\top+\Phi_4\\ \Phi_3+\Phi_4^\top & \Phi_4+\Phi_4^\top\end{pmatrix}e(l)+8\lambda_1|e(l)|^2\\
=\ &e^\top(l)\begin{pmatrix}2a_1I_{mM-m} & (a_1+a_2)I_{mM-m}\\ (a_1+a_2)I_{mM-m} & 2(1+a_2)I_{mM-m}\end{pmatrix}e(l)\\
&-e^\top(l)\begin{pmatrix}\Phi_3+\Phi_3^\top & \mathbf{0}_{mM-m}\\ \mathbf{0}_{mM-m} & \Phi_4+\Phi_4^\top\end{pmatrix}e(l)+2e^\top(l)\begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix}e(l-\delta_1(l))\\
&+2e^\top(l)\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix}e(l-\delta_2(l))+8\lambda_1|e(l)|^2.
\end{aligned}\tag{3.11}$$

By the inequality $X^\top Y+Y^\top X\le X^\top G^{-1}X+Y^\top GY$, where $X,Y\in R^m$ and $0<G\in R^{m\times m}$, we get

$$\begin{aligned}
&2e^\top(l)\begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix}e(l-\delta_1(l))+2e^\top(l)\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix}e(l-\delta_2(l))\\
\le\ &2e^\top(l)e(l)+\left|\begin{pmatrix}\Phi_1 & \mathbf{0}_{mM-m}\\ \Phi_1 & \mathbf{0}_{mM-m}\end{pmatrix}e(l-\delta_1(l))\right|^2+\left|\begin{pmatrix}\mathbf{0}_{mM-m} & \Phi_2\\ \mathbf{0}_{mM-m} & \Phi_2\end{pmatrix}e(l-\delta_2(l))\right|^2.
\end{aligned}\tag{3.12}$$

(3.11) and (3.12) yield that

$$
\begin{aligned}
& \tilde{W}_1(l) + 8\lambda_1 |e(l)|^2 \\
\leq \;& e^\top(l)\begin{pmatrix} (2a_1+2)I_{mM-m} & (a_1+a_2)I_{mM-m} \\ (a_1+a_2)I_{mM-m} & 2(2+a_2)I_{mM-m} \end{pmatrix} e(l) \\
& -e^\top(l)\begin{pmatrix} \Phi_3+\Phi_3^\top & \mathbf{0}_{mM-m} \\ \mathbf{0}_{mM-m} & \Phi_4+\Phi_4^\top \end{pmatrix} e(l) + 8\lambda_1 |e(l)|^2 \\
& + \left| \begin{pmatrix} \Phi_1 & \mathbf{0}_{mM-m} \\ \Phi_1 & \mathbf{0}_{mM-m} \end{pmatrix} e(l-\delta_1(l)) \right|^2 + \left| \begin{pmatrix} \mathbf{0}_{mM-m} & \Phi_2 \\ \mathbf{0}_{mM-m} & \Phi_2 \end{pmatrix} e(l-\delta_2(l)) \right|^2 .
\end{aligned}
$$

By (2.9)–(2.11), we get

$$
\tilde{W}_1(l) + 8\lambda_1 |e(l)|^2 \leq -\lambda_2 |e(l)|^2 + \lambda_3 |e(l-\delta_1(l))|^2 + \lambda_4 |e(l-\delta_2(l))|^2 . \tag{3.13}
$$

When $l \geq 2\delta + 2\tau$, we build a Lyapunov function (see [35])

$$
W_2(l) = W_1(l) + \frac{\|K_{11}\|^2 + \|K_{12}\|^2}{\lambda_1} \int_{-\tau}^{0} \int_{l+s}^{l} \varphi(z)\mathrm{d}z\mathrm{d}s, \tag{3.14}
$$

where

$$
\begin{aligned}
\varphi(l) \;=\;& \tau|A_1 e(l) + B_{11} e(l-\delta_1(l)) + B_{12} e(l-\delta_2(l)) \\
& -K_{11} e(l-\tau_1(l)) - K_{12} e(l-\tau_2(l))|^2.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\dot{W}_2(l) \;=\;& \tilde{W}_1(l) + 2e^\top(l)[K_{11}e(l) - K_{11}e(l-\tau_1(l)) + K_{12}e(l) - K_{12}e(l-\tau_2(l))] \\
& + \frac{\|K_{11}\|^2 + \|K_{12}\|^2}{\lambda_1}\tau\varphi(l) - \frac{\|K_{11}\|^2 + \|K_{12}\|^2}{\lambda_1}\int_{l-\tau}^{l}\varphi(s)\mathrm{d}s \\
\leq\;& \tilde{W}_1(l) + 8\lambda_1 |e(l)|^2 + \frac{\|K_{11}\|^2 |e(l) - e(l-\tau_1(l))|^2}{4\lambda_1} \\
& + \frac{\|K_{12}\|^2 |e(l) - e(l-\tau_2(l))|^2}{4\lambda_1} + \frac{\|K_{11}\|^2 + \|K_{12}\|^2}{\lambda_1}\tau\varphi(l) \\
& - \frac{\|K_{11}\|^2 + \|K_{12}\|^2}{\lambda_1}\int_{l-\tau}^{l}\varphi(s)\mathrm{d}s.
\end{aligned} \tag{3.15}
$$

By the definition of $\varphi(l)$, we obtain

$$
\begin{aligned}
\varphi(l) \;=\;& \tau|(A_1 - K_{11} - K_{12})e(l) + B_{11}e(l-\delta_1(l)) + B_{12}e(l-\delta_2(l)) \\
& + K_{11}(e(l) - e(l-\tau_1(l))) + K_{12}(e(l) - e(l-\tau_2(l)))|^2 \\
\leq\;& 5\tau(\|A_1 - K_{11} - K_{12}\|^2 |e(l)|^2 + \|B_{11}\|^2 |e(l-\delta_1(l))|^2 \\
& + \|B_{12}\|^2 |e(l-\delta_2(l))|^2 + \|K_{11}\|^2 |e(l) - e(l-\tau_1(l))|^2 \\
& + \|K_{12}\|^2 |e(l) - e(l-\tau_2(l))|^2),
\end{aligned} \tag{3.16}
$$

and

$$|e(l)-e(l-\tau_i(l))|^2 = \left|\int_{l-\tau_i(l)}^{l} 1\mathrm{d}e(s)\right|^2 \le \tau \int_{l-\tau_i(l)}^{l} |\dot{e}(s)|^2\,\mathrm{d}s \le \int_{l-\tau}^{l} \varphi(s)\mathrm{d}s,\ i=1,2. \tag{3.17}$$

(3.15)–(3.17) imply that

$$\begin{aligned}
&\dot{W}_2(l)\\
\le\ & \tilde{W}_1(l) + 8\lambda_1 |e(l)|^2 + \Big(\frac{\|K_{11}\|^2}{4\lambda_1} + \frac{5\tau^2\|K_{11}\|^4}{\lambda_1} + \frac{5\tau^2\|K_{11}\|^2\|K_{12}\|^2}{\lambda_1}\Big)|e(l)-e(l-\tau_1(l))|^2\\
&+\Big(\frac{\|K_{12}\|^2}{4\lambda_1} + \frac{5\tau^2\|K_{12}\|^4}{\lambda_1} + \frac{5\tau^2\|K_{11}\|^2\|K_{12}\|^2}{\lambda_1}\Big)|e(l)-e(l-\tau_2(l))|^2\\
&+\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)}{\lambda_1}(\|A_1-K_{11}-K_{12}\|^2|e(l)|^2 + \|B_{11}\|^2|e(l-\delta_1(l))|^2\\
&+\|B_{12}\|^2|e(l-\delta_2(l))|^2) - \frac{\|K_{11}\|^2+\|K_{12}\|^2}{\lambda_1}\int_{l-\tau}^{l}\varphi(s)\mathrm{d}s\\
\le\ & \Big(\frac{\|K_{11}\|^2+\|K_{12}\|^2}{4\lambda_1} + \frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)^2}{\lambda_1} - \frac{\|K_{11}\|^2+\|K_{12}\|^2}{\lambda_1}\Big)\int_{l-\tau}^{l}\varphi(s)\mathrm{d}s\\
&+\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)}{\lambda_1}(\|A_1-K_{11}-K_{12}\|^2|e(l)|^2 + \|B_{11}\|^2|e(l-\delta_1(l))|^2\\
&+\|B_{12}\|^2|e(l-\delta_2(l))|^2) + \tilde{W}_1(l) + 8\lambda_1|e(l)|^2.
\end{aligned}$$

According to (3.13) and $\tau \in (0,\tau_*)$, we obtain

$$\begin{aligned}
\dot{W}_2(l) \le\ & -\lambda_2|e(l)|^2 + \lambda_3|e(l-\delta_1(l))|^2 + \lambda_4|e(l-\delta_2(l))|^2\\
&+\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)}{\lambda_1}(\|A_1-K_{11}-K_{12}\|^2|e(l)|^2 + \|B_{11}\|^2|e(l-\delta_1(l))|^2\\
&+\|B_{12}\|^2|e(l-\delta_2(l))|^2).
\end{aligned}$$

Let $\xi = s-\delta_i(s)$, $i=1,2$. Then, $\mathrm{d}s = \frac{\mathrm{d}\xi}{1-\dot{\delta}_i(s)}$, and from Assumption 2.1, we have

$$\begin{aligned}
\int_{2\tau+2\delta}^{l}|e(s-\delta_i(s))|^2\,\mathrm{d}s \le\ & \frac{1}{1-\hat{\delta}}\int_{2\tau+2\delta-\delta_i(2\tau+2\delta)}^{l-\delta_i(l)}|e(\xi)|^2\,\mathrm{d}\xi\\
\le\ & \frac{1}{1-\hat{\delta}}\int_{2\tau+\delta}^{2\tau+2\delta}|e(\xi)|^2\,\mathrm{d}\xi + \frac{1}{1-\hat{\delta}}\int_{2\tau+2\delta}^{l}|e(\xi)|^2\,\mathrm{d}\xi.
\end{aligned}$$

Thus, we get

$$
\begin{aligned}
W_2(l) &\le W_2(2\tau+2\delta)+\Big(\frac{\lambda_3+\lambda_4}{1-\hat{\delta}}+\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)(\|B_{11}\|^2+\|B_{12}\|^2)}{\lambda_1(1-\hat{\delta})}\Big)\\
&\quad\times\int_{2\tau+\delta}^{2\tau+2\delta}|e(s)|^2\,\mathrm{d}s-\Big[\lambda_2-\frac{\lambda_3+\lambda_4}{1-\hat{\delta}}-\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)}{\lambda_1}\\
&\quad\times\Big(\frac{\|B_{11}\|^2+\|B_{12}\|^2}{1-\hat{\delta}}+\|A_1-K_{11}-K_{12}\|^2\Big)\Big]\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s\\
&= W_2(2\tau+2\delta)+Q_1-Q_2\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s,
\end{aligned}
$$

where

$$
Q_1=\Big(\frac{\lambda_3+\lambda_4}{1-\hat{\delta}}+\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)(\|B_{11}\|^2+\|B_{12}\|^2)}{\lambda_1(1-\hat{\delta})}\Big)\times\int_{2\tau+\delta}^{2\tau+2\delta}|e(s)|^2\,\mathrm{d}s,
$$

$$
Q_2=\lambda_2-\frac{\lambda_3+\lambda_4}{1-\hat{\delta}}-\frac{5\tau^2(\|K_{11}\|^2+\|K_{12}\|^2)}{\lambda_1}\times\Big(\frac{\|B_{11}\|^2+\|B_{12}\|^2}{1-\hat{\delta}}+\|A_1-K_{11}-K_{12}\|^2\Big).
$$

By Lemma 3.1 and (3.14), we get the both $W_2(2\tau+2\delta)$ and Q_1 are finite, and, furthermore, we have $W_2(l)\ge 0$. Using $\tau\in(0,\tau_*)$, we have

$$
\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s\le\frac{W_2(2\tau+2\delta)+Q_1}{Q_2}, \tag{3.18}
$$

which implies that

$$
\lim_{l\to\infty}\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s\le\frac{W_2(2\tau+2\delta)+Q_1}{Q_2}. \tag{3.19}
$$

It follows from Lemma 3.1 and (3.18) that there exists a positive constant M_1 such that

$$
\begin{aligned}
|e(l)|^2 &\le |e(2\tau+2\delta)|^2+2(\|A_1\|+2)\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s\\
&\quad+\|B_{11}\|^2\int_{2\tau+2\delta}^{l}|e(s-\delta_1(s))|^2\,\mathrm{d}s+\|B_{12}\|^2\int_{2\tau+2\delta}^{l}|e(s-\delta_2(s))|^2\,\mathrm{d}s\\
&\quad+\|K_{11}\|^2\int_{2\tau+2\delta}^{l}|e(s-\tau_1(s))|^2\,\mathrm{d}s+\|K_{12}\|^2\int_{2\tau+2\delta}^{l}|e(s-\tau_2(s))|^2\,\mathrm{d}s\\
&\le |e(2\tau+2\delta)|^2+2(\|A_1\|+2)\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s\\
&\quad+\frac{\|B_{11}\|^2+\|B_{12}\|^2}{1-\hat{\delta}}\int_{2\tau+\delta}^{2\tau+2\delta}|e(s)|^2\,\mathrm{d}s+\frac{\|B_{11}\|^2+\|B_{12}\|^2}{1-\hat{\delta}}\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s\\
&\quad+\frac{\|K_{11}\|^2+\|K_{12}\|^2}{1-\hat{\tau}}\int_{\tau+2\delta}^{2\tau+2\delta}|e(s)|^2\,\mathrm{d}s+\frac{\|K_{11}\|^2+\|K_{12}\|^2}{1-\hat{\tau}}\int_{2\tau+2\delta}^{l}|e(s)|^2\,\mathrm{d}s
\end{aligned}
$$

$$\leq \quad M_1.$$

Then, there exists a positive constant M_2 such that

$$\begin{aligned}
\left| |e(l_2)|^2 - |e(l_1)|^2 \right| \quad \leq \quad & \Big| \int_{l_1}^{l_2} [2e^\top(s)A_1 e(s) + 4e^\top(s)e(s) \\
& + e^\top(s-\delta_1(s))B_{11}^\top B_{11} e(s-\delta_1(s)) \\
& + e^\top(s-\delta_2(s))B_{12}^\top B_{12} e(s-\delta_2(s)) \\
& + e^\top(s-\tau_1(s))K_{11}^\top K_{11} e(s-\tau_1(s)) \\
& + e^\top(s-\tau_2(s))K_{12}^\top K_{12} e(s-\tau_2(s))]\mathrm{d}s \Big| \\
\leq \quad & 2(\|A_1\|+2) \int_{l_1}^{l_2} |e(s)|^2 \,\mathrm{d}s \\
& + (\|B_{11}\|^2 + \|B_{12}\|^2) \int_{l_1}^{l_2} \left(\sup_{2\tau+2\delta \leq u \leq s} |e(u)|^2 + \sup_{2\tau+\delta \leq u \leq 2\tau+2\delta} |e(u)|^2 \right) \mathrm{d}s \\
& + (\|K_{11}\|^2 + \|K_{12}\|^2) \int_{l_1}^{l_2} \left(\sup_{2\tau+2\delta \leq u \leq s} |e(u)|^2 + \sup_{\tau+2\delta \leq u \leq 2\tau+2\delta} |e(u)|^2 \right) \mathrm{d}s \\
\leq \quad & 2(\|A_1\|+2) \int_{l_1}^{l_2} M_1 \mathrm{d}s \\
& + (\|B_{11}\|^2 + \|B_{12}\|^2) \int_{l_1}^{l_2} (M_1 + \sup_{2\tau+\delta \leq u \leq 2\tau+2\delta} |e(u)|^2) \mathrm{d}s \\
& + (\|K_{11}\|^2 + \|K_{12}\|^2) \int_{l_1}^{l_2} (M_1 + \sup_{\tau+2\delta \leq u \leq 2\tau+2\delta} |e(u)|^2) \mathrm{d}s \\
\leq \quad & M_2 |l_2 - l_1|,
\end{aligned}$$

for $l_1, l_2 > 2\tau + 2\delta$. It is obvious that $|e(l)|^2$ is uniformly continuous in $[2\tau + 2\delta, \infty)$, which implies that

$$\lim_{l\to\infty} |e(l)|^2 = 0.$$

Next, we prove (3.9). By (2.8), we denote $\hat{\Phi} = \Phi_7^\top + \Phi_8$ and

$$P_2 = \begin{pmatrix} \hat{\Phi} & I_{mM} \\ I_{mM} & I_{mM} \end{pmatrix}. \tag{3.20}$$

It follows from (3.7) that $P_2 > 0$. Define $W_3(l) = \hat{e}^\top(l) P_2 \hat{e}(l)$ and

$$\begin{aligned}
\tilde{W}_3(l) \quad = \quad & 2\hat{e}^\top(l) P_2 \Big[A_2 \hat{e}(l) + B_{21} \hat{e}(l - \delta_1(l)) + B_{22} \hat{e}(l - \delta_2(l)) \\
& - K_{21} \hat{e}(l) - K_{22} \hat{e}(l) + \begin{pmatrix} \mathbf{1}_{mM} \otimes \mathbf{0}_1 \\ d(l) \end{pmatrix} \Big].
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\tilde{W}_3(l) + 8\lambda_1 |\hat{e}(l)|^2 \\
= \;& 2\hat{e}^\top(l)\begin{pmatrix} a_1 I_{mM} & \hat{\Phi} + a_2 I_{mM} \\ a_1 I_{mM} & (1+a_2) I_{mM} \end{pmatrix}\hat{e}(l) + 2\hat{e}^\top(l)\begin{pmatrix} \Phi_5 & \mathbf{0}_{mM} \\ \Phi_5 & \mathbf{0}_{mM} \end{pmatrix}\hat{e}(l-\delta_1(l)) \\
& +2\hat{e}^\top(l)\begin{pmatrix} \mathbf{0}_{mM} & \Phi_6 \\ \mathbf{0}_{mM} & \Phi_6 \end{pmatrix}\hat{e}(l-\delta_2(l)) - 2\hat{e}^\top(l)\begin{pmatrix} \Phi_7 & \Phi_8 \\ \Phi_7 & \Phi_8 \end{pmatrix}\hat{e}(l) + 8\lambda_1 |\hat{e}(l)|^2 \\
& +2\hat{e}^\top(l)\begin{pmatrix} d(l) \\ d(l) \end{pmatrix} \\
\leq \;& \hat{e}^\top(l)\begin{pmatrix} (2a_1+3) I_{mM} & (a_1+a_2) I_{mM} \\ (a_1+a_2) I_{mM} & 2(2.5+a_2) I_{mM} \end{pmatrix}\hat{e}(l) \\
& -\hat{e}^\top(l)\begin{pmatrix} \Phi_7 + \Phi_7^\top & \mathbf{0}_{mM} \\ \mathbf{0}_{mM} & \Phi_8 + \Phi_8^\top \end{pmatrix}\hat{e}(l) + 8\lambda_1 |\hat{e}(l)|^2 \\
& +\left|\begin{pmatrix} \Phi_5 & \mathbf{0}_{mM} \\ \Phi_5 & \mathbf{0}_{mM} \end{pmatrix}\hat{e}(l-\delta_1(l))\right|^2 + \left|\begin{pmatrix} \mathbf{0}_{mM} & \Phi_6 \\ \mathbf{0}_{mM} & \Phi_6 \end{pmatrix}\hat{e}(l-\delta_2(l))\right|^2 + \left|\begin{pmatrix} d(l) \\ d(l) \end{pmatrix}\right|^2.
\end{aligned}$$

By the (2.12)–(2.14), we get

$$\tilde{W}_3(l) + 8\lambda_1 |\hat{e}(l)|^2 \leq -\lambda_5 |\hat{e}(l)|^2 + \lambda_6 |\hat{e}(l-\delta_1(l))|^2 + \lambda_7 |\hat{e}(l-\delta_2(l))|^2 + 8Md^2. \tag{3.21}$$

When $l \geq 2\delta + 2\tau$, we build a Lyapunov function

$$W_4(l) = W_3(l) + \frac{\|K_{21}\|^2 + \|K_{22}\|^2}{\lambda_1}\int_{-\tau}^{0}\int_{l+s}^{l}\hat{\varphi}(z)\mathrm{d}z\mathrm{d}s, \tag{3.22}$$

where

$$\begin{aligned}
\hat{\varphi}(l) \;=\; & \tau\Big|A_2\hat{e}(l) + B_{21}\hat{e}(l-\delta_1(l)) + B_{22}\hat{e}(l-\delta_2(l)) - K_{21}\hat{e}(l-\tau_1(l)) \\
& -K_{22}\hat{e}(l-\tau_2(l)) + \begin{pmatrix} \mathbf{1}_{mM}\otimes\mathbf{0}_1 \\ d(l) \end{pmatrix}\Big|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\dot{W}_4(l) \;\leq\; & \tilde{W}_3(l) + 8\lambda_1 |\hat{e}(l)|^2 + \frac{\|K_{21}\|^2 |\hat{e}(l) - \hat{e}(l-\tau_1(l))|^2}{4\lambda_1} \\
& +\frac{\|K_{22}\|^2 |\hat{e}(l) - \hat{e}(l-\tau_2(l))|^2}{4\lambda_1} + \frac{\|K_{21}\|^2 + \|K_{22}\|^2}{\lambda_1}\tau\hat{\varphi}(l) \\
& -\frac{\|K_{21}\|^2 + \|K_{22}\|^2}{\lambda_1}\int_{l-\tau}^{l}\hat{\varphi}(s)\mathrm{d}s.
\end{aligned} \tag{3.23}$$

By the definition of $\hat{\varphi}(l)$, we get

$$\begin{aligned}
\hat{\varphi}(l) \;\leq\; & 6\tau(\|A_2 - K_{21} - K_{22}\|^2 |\hat{e}(l)|^2 + \|B_{21}\|^2 |\hat{e}(l-\delta_1(l))|^2 + \|B_{22}\|^2 |\hat{e}(l-\delta_2(l))|^2 \\
& + \|K_{21}\|^2 |\hat{e}(l) - \hat{e}(l-\tau_1(l))|^2 + \|K_{22}\|^2 |\hat{e}(l) - \hat{e}(l-\tau_2(l))|^2 + 4Md^2),
\end{aligned} \tag{3.24}$$

and

$$|\hat{e}(l)-\hat{e}(l-\tau_i(l))|^2=\left|\int_{l-\tau_i(l)}^{l}1\mathrm{d}\hat{e}(s)\right|^2\le\tau\int_{l-\tau_i(l)}^{l}\left|\dot{\hat{e}}(s)\right|^2\mathrm{d}s\le\int_{l-\tau}^{l}\hat{\varphi}(s)\mathrm{d}s,\, i=1,2. \tag{3.25}$$

(3.23)–(3.25) yield that

$$\begin{aligned}\dot{W}_4(l)\ \le\ & \tilde{W}_3(l)+8\lambda_1|\hat{e}(l)|^2\\ &+\Big(\frac{\|K_{21}\|^2+\|K_{22}\|^2}{4\lambda_1}+\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)^2}{\lambda_1}-\frac{\|K_{21}\|^2+\|K_{22}\|^2}{\lambda_1}\Big)\int_{l-\tau}^{l}\hat{\varphi}(s)\mathrm{d}s\\ &+\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)}{\lambda_1}(\|A_2-K_{21}-K_{22}\|^2|\hat{e}(l)|^2+\|B_{21}\|^2|\hat{e}(l-\delta_1(l))|^2\\ &+\|B_{22}\|^2|\hat{e}(l-\delta_2(l))|^2+4Md^2).\end{aligned}$$

According to (3.21) and $\tau\in(0,\tau_*)$, we obtain

$$\begin{aligned}\dot{W}_4(l)\ \le\ & -\lambda_5|\hat{e}(l)|^2+\lambda_6|\hat{e}(l-\delta_1(l))|^2+\lambda_7|\hat{e}(l-\delta_2(l))|^2+8Md^2\\ &+\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)}{\lambda_1}(\|A_2-K_{21}-K_{22}\|^2|\hat{e}(l)|^2+\|B_{21}\|^2|\hat{e}(l-\delta_1(l))|^2\\ &+\|B_{22}\|^2|\hat{e}(l-\delta_2(l))|^2+4Md^2).\end{aligned}$$

It follows from Assumption 2.1 that

$$\int_{2\tau+2\delta}^{l}|\hat{e}(s-\delta_i(s))|^2\mathrm{d}s\le\frac{1}{1-\hat{\delta}}\int_{2\tau+\delta}^{2\tau+2\delta}|\hat{e}(s)|^2\mathrm{d}s+\frac{1}{1-\hat{\delta}}\int_{2\tau+2\delta}^{l}|\hat{e}(s)|^2\mathrm{d}s,\, i=1,2.$$

Consequently, we get

$$\begin{aligned}&W_4(l)\\ \le\ & W_4(2\tau+2\delta)+\Big(\frac{\lambda_6+\lambda_7}{1-\hat{\delta}}+\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)(\|B_{21}\|^2+\|B_{22}\|^2)}{\lambda_1(1-\hat{\delta})}\Big)\times\int_{2\tau+\delta}^{2\tau+2\delta}|\hat{e}(s)|^2\mathrm{d}s\\ &-\Big[\lambda_5-\frac{\lambda_6+\lambda_7}{1-\hat{\delta}}-\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)}{\lambda_1}\times\Big(\frac{\|B_{21}\|^2+\|B_{22}\|^2}{1-\hat{\delta}}\\ &+\|A_2-K_{21}-K_{22}\|^2\Big)\Big]\int_{2\tau+2\delta}^{l}|\hat{e}(s)|^2\mathrm{d}s+8Md^2l+\frac{24\tau^2Md^2(\|K_{21}\|^2+\|K_{22}\|^2)l}{\lambda_1}\\ =\ & W_4(2\tau+2\delta)+Q_3-Q_4\int_{2\tau+2\delta}^{l}|\hat{e}(s)|^2\mathrm{d}s+8Md^2l+\frac{24\tau^2Md^2(\|K_{21}\|^2+\|K_{22}\|^2)l}{\lambda_1},\end{aligned}$$

where

$$Q_3=\Big(\frac{\lambda_6+\lambda_7}{1-\hat{\delta}}+\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)(\|B_{21}\|^2+\|B_{22}\|^2)}{\lambda_1(1-\hat{\delta})}\Big)\times\int_{2\tau+\delta}^{2\tau+2\delta}|\hat{e}(s)|^2\mathrm{d}s,$$

$$Q_4=\lambda_5-\frac{\lambda_6+\lambda_7}{1-\hat{\delta}}-\frac{6\tau^2(\|K_{21}\|^2+\|K_{22}\|^2)}{\lambda_1}\times\Big(\frac{\|B_{21}\|^2+\|B_{22}\|^2}{1-\hat{\delta}}+\|A_2-K_{21}-K_{22}\|^2\Big).$$

By Lemma 3.1 and (3.22), we get not only that $W_4(2\tau + 2\delta)$ and Q_3 are finite, but also that $W_4(l) \geq 0$. Together with $\tau \in (0, \tau_*)$, we obtain

$$\int_{2\tau+2\delta}^{l} |\hat{e}(s)|^2 \, \mathrm{d}s \leq \frac{W_4(2\tau + 2\delta) + Q_3}{Q_4} + \frac{8Md^2 l}{Q_4} + \frac{24\tau^2 Md^2(\|K_{21}\|^2 + \|K_{22}\|^2)l}{\lambda_1 Q_4},$$

which implies that

$$\lim_{l\to\infty} \frac{1}{l} \int_{2\tau+2\delta}^{l} |\hat{e}(s)|^2 \, \mathrm{d}s \leq \frac{8Md^2}{Q_4} + \frac{24\tau^2 Md^2(\|K_{21}\|^2 + \|K_{22}\|^2)}{\lambda_1 Q_4}.$$

Hence the proof is complete.

Remark 3.1. *When Assumptions 2.1 and 2.2, and (3.4)–(3.7) hold simultaneously, the multi-agent systems (2.3) and (2.4) can achieve consensus about followers and achieve the average quasi-consensus about leader and followers.*

4. Numerical example

Example 1. Considering (2.1) and (2.2), let $\delta_1(l) = 0.1 \arctan(l)$, $\delta_2(l) = 0.09 \arctan(l)$, $a_{11} = a_{22} = 0$, $a_{12} = a_{21} = -0.1$, $b_{11} = b_{22} = 0$, $b_{12} = b_{21} = -0.2$, $\Gamma_1 = 0.1I_2$, $\Gamma_2 = 0.2I_2$, $\lambda_1 = 0.1$, $M = 2$, and $m = 2$. Let $a_1 = a_2 = b_1 = b_2 = 0.5$, $x_0(l) = (x_{01}(l), x_{02}(l))^\top$, $x_1(l) = (x_{11}(l), x_{12}(l))^\top$, $x_2(l) = (x_{21}(l), x_{22}(l))^\top$, $\xi_{01}(l) = (-11 + \cos(l), 13 + \cos(l))^\top$, $\xi_{11}(l) = (31 + 2l, 24 + \sin(l))^\top$, $\xi_{21}(l) = (47 + 2\sin(l), 57 + l)^\top$, $v_0(l) = (v_{01}(l), v_{02}(l))^\top$, $v_1(l) = (v_{11}(l), v_{12}(l))^\top$, $v_2(l) = (v_{21}(l), v_{22}(l))^\top$, $\xi_{02}(l) = (-3 + \cos(l), 4 + \sin(l))^\top$, $\xi_{12}(l) = (11 + \cos(l), 3 + \frac{\sin(l)}{3})^\top$, $\xi_{22}(l) = (21 + \sin(l), -6 + \frac{\sin(l)}{3})^\top$, $f(l) = (\cos(l) + 1, 2\cos(l) + 1)^\top$, and $g(l) = (2 + 0.5\cos(l), 3\cos(l))^\top$. (2.1) and (2.2) become (4.1) and (4.2) respectively, the corresponding numerical simulation results are shown in Figure 1(a),(b).

$$\begin{cases} \dot{x}_0(l) = v_0(l), l > 0, \\ \dot{v}_0(l) = 0.5x_0(l) + 0.5v_0(l) + 0.5x_0(l - 0.1\arctan(l)) \\ \qquad +0.5v_0(l - 0.09\arctan(l)) + \begin{pmatrix} \cos(l) + 1 \\ 2\cos(l) + 1 \end{pmatrix}, l > 0, \end{cases} \tag{4.1}$$

$$\begin{cases} \dot{x}_1(l) = v_1(l), l > 0, \\ \dot{v}_1(l) = 0.5x_1(l) + 0.5v_1(l) + 0.5x_1(l - 0.1\arctan(l)) \\ \qquad +0.5v_1(l - 0.09\arctan(l)) + \begin{pmatrix} 0.5\cos(l) + 2 \\ 3\cos(l) \end{pmatrix} \\ \qquad -0.01[x_2(l - 0.1\arctan(l)) - x_1(l - 0.1\arctan(l))] \\ \qquad -0.04[v_2(l - 0.09\arctan(l)) - v_1(l - 0.09\arctan(l))], l > 0, \\ \dot{x}_2(l) = v_2(l), l > 0, \\ \dot{v}_2(l) = 0.5x_2(l) + 0.5v_2(l) + 0.5x_2(l - 0.1\arctan(l)) \\ \qquad +0.5v_2(l - 0.09\arctan(l)) + \begin{pmatrix} 0.5\cos(l) + 2 \\ 3\cos(l) \end{pmatrix} \\ \qquad -0.01[x_1(l - 0.1\arctan(l)) - x_2(l - 0.1\arctan(l))] \\ \qquad -0.04[v_1(l - 0.09\arctan(l)) - v_2(l - 0.09\arctan(l))], l > 0. \end{cases} \tag{4.2}$$

Numerical simulation shows that under the condition of no feedback control, Figure 1(a) indicates that the consensus among followers cannot be achieved, and Figure 1(b) indicates that the average quasi-consensus between the leader and the follower cannot be achieved either.

Example 2. Based on example 1, let $\Gamma_3 = I_2$, $\Gamma_4 = 1.1I_2$, $\Gamma_5 = 1.2I_2$, $\Gamma_6 = 1.3I_2$, $c_{11} = c_{22} = 0$, $c_{12} = c_{21} = 0.5$, $d_{11} = d_{22} = 0$, $d_{12} = d_{21} = 0.6$,

$$\begin{aligned} e(l) &= (e_1(l), e_2(l), e_3(l), e_4(l))^\top, \\ \hat{e}(l) &= y(l) = (y_1(l), y_2(l), y_3(l), y_4(l), y_5(l), y_6(l), y_7(l), y_8(l))^\top, \end{aligned}$$

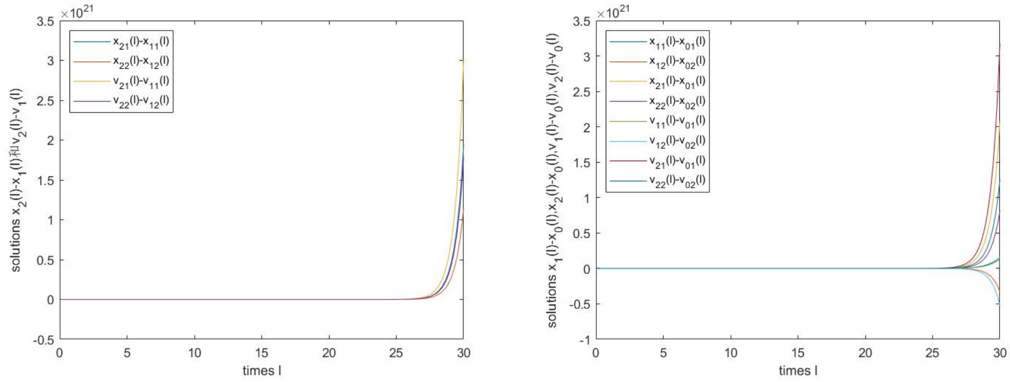
$\alpha = 4$, $\beta = 4.5$. Then, $\lambda_2 = 7.1$, $\lambda_3 = 0.55$, $\lambda_4 = 0.7$, $\lambda_5 = 3.8$, $\lambda_6 = 0.55$, and $\lambda_7 = 0.68$, so we can get an effective upper bound 0.00076 on $\tau_1(l)$ and $\tau_2(l)$. First, we consider the delay time within the effective range(e.g., $\tau_1(l) = 0.0005 - 0.00026\cos(l) \leq 0.00076$ and $\tau_2(l) = 0.0005 - 0.0002\cos(l) \leq 0.00076$), (2.5) and (2.7) become (4.3) and (4.4), respectively, and the corresponding numerical simulation results are shown in Figure 2(a) and Figure 3(a). From Figure 3(a), we can get that $y_1(l)$, $y_2(l)$, $y_3(l)$, $y_4(l)$, $y_5(l)$, $y_6(l)$, $y_7(l)$, and $y_8(l)$ are finite when $l \geq 2\tau + 2\delta$, so it is easy to get that $\lim_{l\to\infty} \frac{1}{l} \int_{2\tau+2\delta}^{l} |\hat{e}(s)|^2 \, ds$ is finite. Meanwhile, Figure 2(a) indicates that the consensus among followers can be achieved, Figure 3(a) indicates that the average quasi-consensus between the leader and the followers can be achieved.

$$\left\{ \begin{aligned} \dot{p}_2(l) &= q_2(l), l > 0, \\ \dot{q}_2(l) &= 0.5p_2(l) + 0.5q_2(l) + 0.5p_2(l - 0.1\arctan(l)) + 0.5q_2(l - 0.09\arctan(l)) \\ &\quad -0.01[p_1(l - 0.1\arctan(l)) - p_2(l - 0.1\arctan(l))] \\ &\quad -0.04[q_1(l - 0.09\arctan(l)) - q_2(l - 0.09\arctan(l))] \\ &\quad +0.5[p_1(l - 0.0005 + 0.00026\cos(l)) - p_2(l - 0.0005 + 0.00026\cos(l))] \\ &\quad +0.66[q_1(l - 0.0005 + 0.0002\cos(l)) - q_2(l - 0.0005 + 0.0002\cos(l))] \\ &\quad +0.01p_2(l - 0.1\arctan(l)) + 0.04q_2(l - 0.09\arctan(l)) \\ &\quad -0.5p_2(l - 0.0005 + 0.00026\cos(l)) - 0.66q_2(l - 0.0005 + 0.0002\cos(l)) \\ &\quad -4.8p_2(l - 0.0005 + 0.00026\cos(l)) - 5.85q_2(l - 0.0005 + 0.0002\cos(l)), l > 0. \end{aligned} \right. \tag{4.3}$$

$$
\begin{cases}
\dot{\hat{p}}_1(l) &=& \hat{q}_1(l), l > 0,\\
\dot{\hat{q}}_1(l) &=& 0.5\hat{p}_1(l) + 0.5\hat{q}_1(l) + 0.5\hat{p}_1(l - 0.1\arctan(l)) + 0.5\hat{q}_1(l - 0.09\arctan(l))\\
&& -0.01[\hat{p}_2(l - 0.1\arctan(l)) - \hat{p}_1(l - 0.1\arctan(l))]\\
&& -0.04[\hat{q}_2(l - 0.09\arctan(l)) - \hat{q}_1(l - 0.09\arctan(l))]\\
&& +0.5[\hat{p}_2(l - 0.0005 + 0.00026\cos(l)) - \hat{p}_1(l - 0.0005 + 0.00026\cos(l))]\\
&& +0.66[\hat{q}_2(l - 0.0005 + 0.0002\cos(l)) - \hat{q}_1(l - 0.0005 + 0.0002\cos(l))]\\
&& -4.8\hat{p}_1(l - 0.0005 + 0.00026\cos(l)) - 5.85\hat{q}_1(l - 0.0005 + 0.0002\cos(l))\\
&& + \begin{pmatrix} 1 - 0.5\cos(l) \\ \cos(l) - 1 \end{pmatrix}, l > 0,\\
\dot{\hat{p}}_2(l) &=& \hat{q}_2(l), l > 0,\\
\dot{\hat{q}}_2(l) &=& 0.5\hat{p}_2(l) + 0.5\hat{q}_2(l) + 0.5\hat{p}_2(l - 0.1\arctan(l)) + 0.5\hat{q}_2(l - 0.09\arctan(l))\\
&& -0.01[\hat{p}_1(l - 0.1\arctan(l)) - \hat{p}_2(l - 0.1\arctan(l))]\\
&& -0.04[\hat{q}_1(l - 0.09\arctan(l)) - \hat{q}_2(l - 0.09\arctan(l))]\\
&& +0.5[\hat{p}_1(l - 0.0005 + 0.00026\cos(l)) - \hat{p}_2(l - 0.0005 + 0.00026\cos(l))]\\
&& +0.66[\hat{q}_1(l - 0.0005 + 0.0002\cos(l)) - \hat{q}_2(l - 0.0005 + 0.0002\cos(l))]\\
&& -4.8\hat{p}_2(l - 0.0005 + 0.00026\cos(l)) - 5.85\hat{q}_2(l - 0.0005 + 0.0002\cos(l))\\
&& + \begin{pmatrix} 1 - 0.5\cos(l) \\ \cos(l) - 1 \end{pmatrix}, l > 0.
\end{cases}
\tag{4.4}
$$

Furthermore, we consider that the delay control time exceeds the effective range (e.g., $\tau_1(l) = 2 > 0.00076$ and $\tau_2(l) = 2 > 0.00076$), (2.5) and (2.7) become (4.5) and (4.6), respectively, and the corresponding numerical simulation results are shown in Figure 4(a) and Figure 5(a). In the case where delayed feedback control exists but the control delay time exceeds the effective range, Figure 4(a) indicates that the consensus among followers cannot be achieved, and Figure 5(a) indicates that the average quasi-consensus between the leader and the followers cannot be achieved either. To further verify the effectiveness of the delay time $\tau_1(l) = \tau_2(l) = 0.00076$, we assign some other values to $\tau_1(l)$ and $\tau_2(l)$ again (e.g., $\tau_1(l) = \tau_2(l) = 0.0001$, $\tau_1(l) = \tau_2(l) = 0.0003$, $\tau_1(l) = \tau_2(l) = 0.7$ and $\tau_1(l) = \tau_2(l) = 1.5$), and the corresponding numerical simulation results are shown in Figure 2(b),(c), Figure 4(b),(c), Figure 3(b),(c), and Figure 5(b),(c). Figure 2(b),(c) and Figure 3(b),(c) indicate that the delay time feedback control takes effect and the system reaches the consensus and the average quasi-consensus under the condition $\tau_1(l) = \tau_2(l) \leq 0.00076$. Figure 4(b),(c) and Figure 5(b),(c) indicate that the delay time feedback control cannot take effect and the system cannot reach the consensus and the average quasi-consensus under the condition that the control delay time is too large.

$$
\begin{cases}
\dot{p}_2(l) &=& q_2(l), l > 0,\\
\dot{q}_2(l) &=& 0.5p_2(l) + 0.5q_2(l) + 0.5p_2(l - 0.1\arctan(l)) + 0.5q_2(l - 0.09\arctan(l))\\
&& -0.01[p_1(l - 0.1\arctan(l)) - p_2(l - 0.1\arctan(l))]\\
&& -0.04[q_1(l - 0.09\arctan(l)) - q_2(l - 0.09\arctan(l))]\\
&& +0.5[p_1(l - 2) - p_2(l - 2)] + 0.66[q_1(l - 2) - q_2(l - 2)]\\
&& +0.01p_2(l - 0.1\arctan(l)) + 0.04q_2(l - 0.09\arctan(l))\\
&& -0.5p_2(l - 2) - 0.66q_2(l - 2)\\
&& -4.8p_2(l - 2) - 5.85q_2(l - 2), l > 0.
\end{cases}
\tag{4.5}
$$

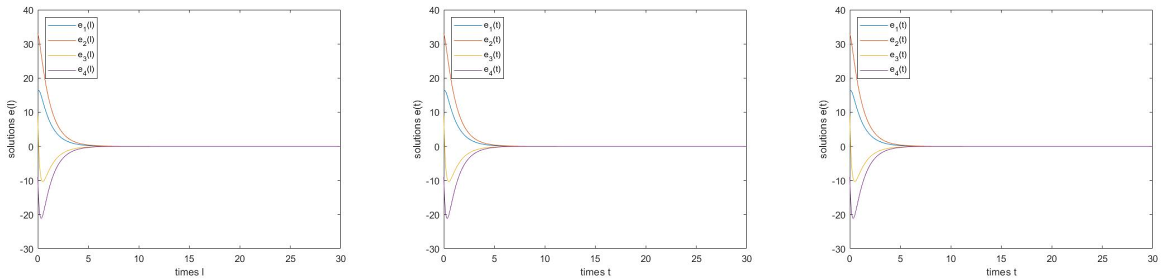


(a) (b)

Figure 1. (a) and (b) respectively represent the non-consensus and non-average quasi-consensus results of (4.1) and (4.2).

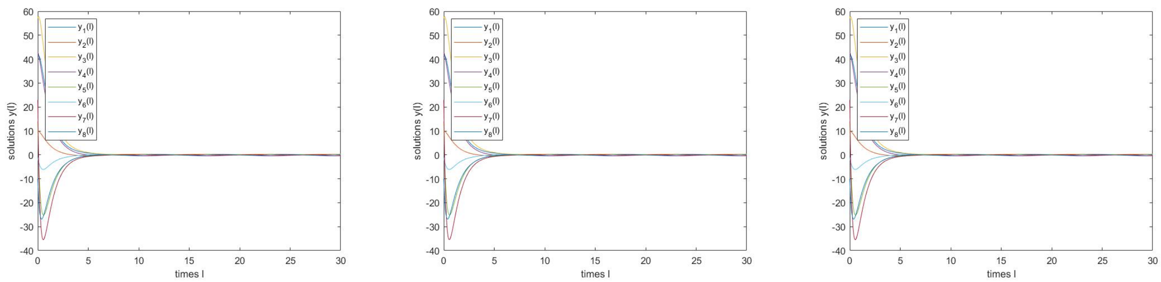
$$\begin{cases} \dot{\hat{p}}_1(l) &= \hat{q}_1(l), l > 0, \\ \dot{\hat{q}}_1(l) &= 0.5\hat{p}_1(l) + 0.5\hat{q}_1(l) + 0.5\hat{p}_1(l-0.1\arctan(l)) + 0.5\hat{q}_1(l-0.09\arctan(l)) \\ & \quad -0.01[\hat{p}_2(l-0.1\arctan(l)) - \hat{p}_1(l-0.1\arctan(l))] \\ & \quad -0.04[\hat{q}_2(l-0.09\arctan(l)) - \hat{q}_1(l-0.09\arctan(l))] \\ & \quad +0.5[\hat{p}_2(l-2) - \hat{p}_1(l-2)] + 0.66[\hat{q}_2(l-2) - \hat{q}_1(l-2)] \\ & \quad -4.8\hat{p}_1(l-2) - 5.85\hat{q}_1(l-2) + \begin{pmatrix} 1-0.5\cos(l) \\ \cos(l)-1 \end{pmatrix}, l > 0, \\ \dot{\hat{p}}_2(l) &= \hat{q}_2(l), l > 0, \\ \dot{\hat{q}}_2(l) &= 0.5\hat{p}_2(l) + 0.5\hat{q}_2(l) + 0.5\hat{p}_2(l-0.1\arctan(l)) + 0.5\hat{q}_2(l-0.09\arctan(l)) \\ & \quad -0.01[\hat{p}_1(l-0.1\arctan(l)) - \hat{p}_2(l-0.1\arctan(l))] \\ & \quad -0.04[\hat{q}_1(l-0.09\arctan(l)) - \hat{q}_2(l-0.09\arctan(l))] \\ & \quad +0.5[\hat{p}_1(l-2) - \hat{p}_2(l-2)] + 0.66[\hat{q}_1(l-2) - \hat{q}_2(l-2)] \\ & \quad -4.8\hat{p}_2(l-2) - 5.85\hat{q}_2(l-2) + \begin{pmatrix} 1-0.5\cos(l) \\ \cos(l)-1 \end{pmatrix}, l > 0. \end{cases} \tag{4.6}$$

Example 3. $\delta_1(l)$ and $\delta_2(l)$ affect the speed of the consensus and the average quasi-consensus of SOMAS. Replace $\delta_1(l) = 0.1arctan(l)$ and $\delta_2(l) = 0.09arctan(l)$ with $\delta_1(l) = \delta_2(l) = 0.1$, $\delta_1(l) = \delta_2(l) = 5$, and $\delta_1(l) = \delta_2(l) = 10$; the corresponding numerical simulation results are shown in Figure 6(a)–(c) and Figure 7(a)–(c). Figure 6(a) shows SOMAS with $\delta(l) = 0.1$ can achieve an effective consensus on $l = 5$, Figure 6(b) shows SOMAS with $\delta(l) = 5$ can achieve an effective consensus on $l = 8$, and Figure 6(c) shows SOMAS with $\delta(l) = 10$ can achieve an effective consensus on $l = 14$. Figure 7(a) shows the SOMAS with $\delta(l) = 0.1$ can achieve an effective average quasi-consensus on $l = 5$, Figure 7(b) shows SOMAS with $\delta(l) = 5$ can achieve an effective average quasi-consensus on $l = 7$, and Figure 7(c) shows SOMAS with $\delta(l) = 10$ can achieve an effective average quasi-consensus on $l = 14$.



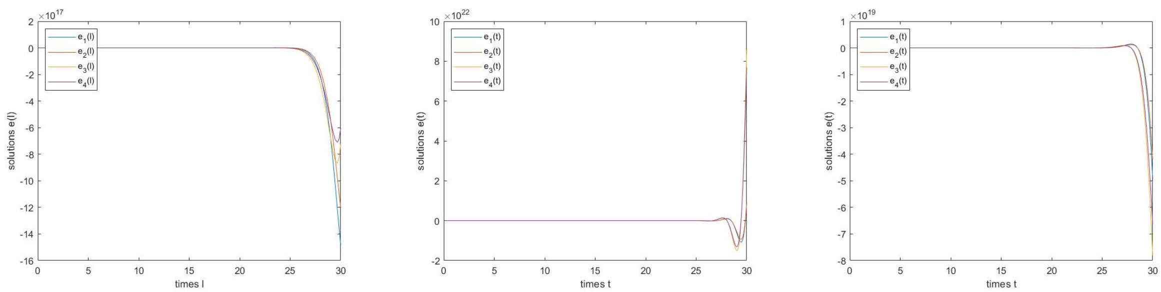
(a) (b) (c)

Figure 2. (a)–(c) respectively represent the consensus results of error to (2.5) under conditions $\tau_1(l) = 0.0005 - 0.00026\cos(l)$ and $\tau_2(l) = 0.0005 - 0.0002\cos(l)$, $\tau_1(l) = 0.0001$ and $\tau_2(l) = 0.0001$, and $\tau_1(1) = 0.0003$ and $\tau_2(l) = 0.0003$ when $\delta_1(l) = 0.1\arctan(l)$ and $\delta_2(l) = 0.09\arctan(l)$.



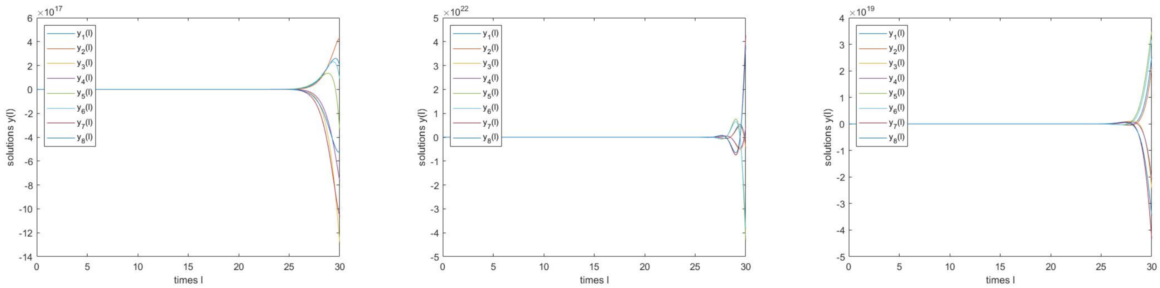
(a) (b) (c)

Figure 3. (a)–(c) respectively represent the average quasi-consensus results of error to (2.7) under conditions $\tau_1(l) = 0.0005 - 0.00026\cos(l)$ and $\tau_2(l) = 0.0005 - 0.0002\cos(l)$, $\tau_1(l) = 0.0001$ and $\tau_2(l) = 0.0001$, and $\tau_1(l) = 0.0003$ and $\tau_2(l) = 0.0003$ when $\delta_1(l) = 0.1\arctan(l)$ and $\delta_2(l) = 0.09\arctan(l)$.



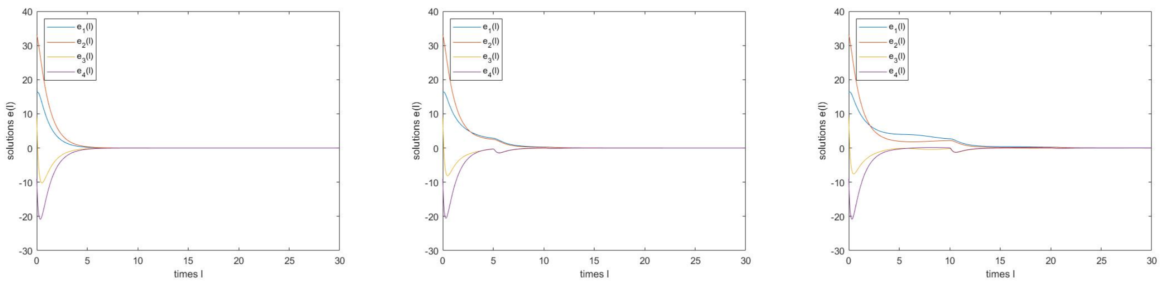
(a) (b) (c)

Figure 4. (a)–(c) respectively represent the non-consensus results of error to (2.5) under conditions $\tau_1(l) = \tau_2(l) = 2$, $\tau_1(l) = \tau_2(l) = 0.7$, and $\tau_1(l) = \tau_2(l) = 1.5$ when $\delta_1(l) = 0.1\arctan(l)$ and $\delta_2(l) = 0.09\arctan(l)$.



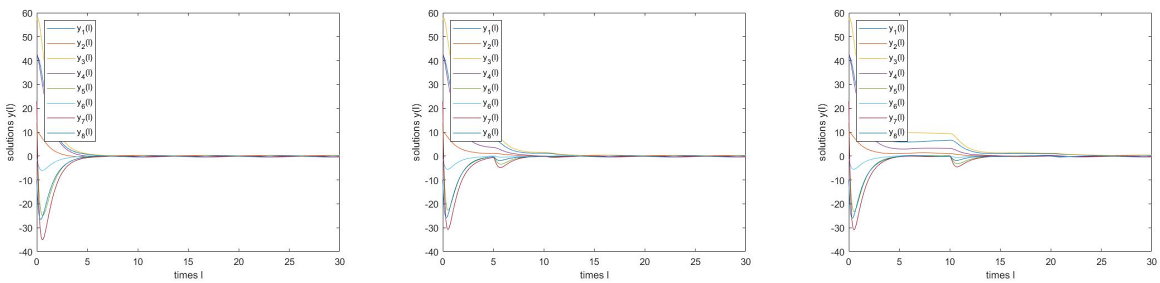
(a) (b) (c)

Figure 5. (a)–(c) respectively represent the non-average quasi-consensus results of (2.7) under conditions $\tau_1(l) = \tau_2(l) = 2$, $\tau_1(l) = \tau_2(l) = 0.7$, and $\tau_1(l) = \tau_2(l) = 1.5$ when $\delta_1(l) = 0.1 \arctan(l)$ and $\delta_2(l) = 0.09 \arctan(l)$.



(a) (b) (c)

Figure 6. (a)–(c) respectively represent the consensus results of error to (2.7) under conditions $\delta_1(l) = \delta_2(l) = 0.1$, $\delta_1(l) = \delta_2(l) = 5$, and $\delta_1(l) = \delta_2(l) = 10$ when $\tau_1(l) = 0.0005 - 0.00026 \cos(l)$ and $\tau_2(l) = 0.0005 - 0.0002 \cos(l)$.



(a) (b) (c)

Figure 7. (a)–(c) respectively represent the average quasi-consensus results of (2.7) under conditions $\delta_1(l) = \delta_2(l) = 0.1$, $\delta_1(l) = \delta_2(l) = 5$, and $\delta_1(l) = \delta_2(l) = 10$ when $\tau_1(l) = 0.0005 - 0.00026 \cos(l)$ and $\tau_2(l) = 0.0005 - 0.0002 \cos(l)$.

5. Conclusions

In this paper, we regard personal assets as second-order multi-agent systems (SOMAS). The delayed feedback control for consensus and average quasi-consensus of delayed SOMAS is studied. This conclusion provides theoretical support for synchronizing multiple individual assets and offers a pathway for common prosperity.

Meanwhile, from the examples, it is easy to find the delay is very small. In other words, for the large time-delay, the method used here is not suitable. In addition, the environment is random, and so the stochastic delayed differential equations should be considered. In future work, we will consider the stochastic environment.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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