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Research article

General decay for a coupled wave problem with Lord-Shulman thermal heat law and delay

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Abstract: This article study a coupled one-dimensional hyperbolic system with Lord-Shulman thermal heat law on the first equation, and a general weak inherent damping, along with time-dependent coefficient and time-varying delay term on the second equation. Using the multiplier method, through a suitable Lyapunov functional, we prove a general decay stability result that encompasses both exponential and polynomial decay estimates as special cases. This result extends and generalizes existing findings in the literature related to the system under study.

Keywords: general decay; coupled wave; Lord-Shulman thermal law; time varying coefficient and delay

1. Introduction

We investigate a coupled wave problem with Lord-Shulman thermal law, a general weak inherent damping, along with time-dependent coefficient and time-varying delay term of the form

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \lambda\psi + \delta(\theta_x + \tau\theta_{xt}) = 0, & \text{in } [0, L] \times \mathbb{R}_+, \\ \psi_{tt} - \psi_{xx} + \lambda\varphi + b(t)\left[\alpha_1 f_1(\psi_t) + \alpha_2 f_2(\psi_t(t - \sigma(t)))\right] = 0, & \text{in } [0, L] \times \mathbb{R}_+, \\ \rho(\theta_t + \tau\theta_{tt}) - \kappa\theta_{xx} + \beta\varphi_{xt} = 0, & \text{in } [0, L] \times \mathbb{R}_+, \end{cases} \tag{1.1}$$

with initial data

$$\begin{array}{ll} \varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x), & \text{in } [0, L], \\ \psi(x, 0) = \psi_0(x), \ \psi_t(x, 0) = \psi_1(x), & \text{in } [0, L], \\ \theta(x, 0) = \theta_0(x), \ \theta_t(x, 0) = \theta_1(x), & \text{in } [0, L], \\ \psi_t(x, t - \sigma(0)) = g_0(x, t - \sigma(0)), & \text{in } [0, L] \times [0, \sigma(0)] \end{array} \tag{1.2}$$

and Dirichlet-Dirichlet-Neumann boundary conditions given by

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \theta_x(0, t) = \theta_x(L, t) = 0, \quad \text{in } [0, \infty[, \tag{1.3}$$

where $\varphi = \varphi(x, t)$ and $\psi = \psi(x, t)$ represent the displacements of two elastic membranes surfaces under some form of elastic force, which can attracts one membrane towards the other with a coefficient of attraction $\lambda > 0$, such that $(1 - \lambda c_p) > 0$, with c_p the Poincaré's constant. Also, $\theta = \theta(x, t)$ is the temperature deviation from the reference temperature between the two membranes and the positive constants $\delta, \sigma, \rho, \kappa$, and β, represent coupling coefficients which depends on the material properties. The function $b(t)$ is a nonlinear weight function, α_1 and α_2 are real positive damping constants, f_1 and f_2 are nonlinear damping functions, and $\tau(t) > 0$ is the time-varying delay. Due to the boundary conditions imposed on θ, the application of Poincaré's inequality is not achievable directly. Thus integrating equation $(1.1)_3$ over $[0, L]$, and on account of the boundary conditions (1.3), we obtain

$$\tau \frac{d^2}{dt^2}\left[\int_0^L \theta(x, t)dx\right] + \frac{d}{dt}\left[\int_0^L \theta(x, t)dx\right] = 0. \tag{1.4}$$

Solving (1.4) and using the initial data for θ , we obtain

$$\int_0^L \theta(x, t)dx = \left(1 - e^{\frac{-t}{\tau}}\right)\int_0^L \theta_1(x)dx + \int_0^L \theta_0(x)dx.$$

Therefore, if we set

$$\widehat{\theta}(x, t) = \frac{1}{L}\left[\left(1 - e^{\frac{-t}{\tau}}\right)\int_0^L \theta_1(x)dx + \int_0^L \theta_0(x)dx\right], \tag{1.5}$$

then

$$\int_0^L \widehat{\theta}(x, t)dx = 0. \tag{1.6}$$

Thus, we can apply Poincaré's inequality for $\widehat{\theta}$ and it is easy to verify that $(\varphi, \psi, \widehat{\theta})$ satisfies (1.1)–(1.3) with initial data for $\widehat{\theta}$ given by

$$\widehat{\theta}(x, 0) = \theta_0(x) - \frac{1}{L}\int_0^L \theta_0(x)dx \text{ and } \widehat{\theta}_t(x, 0) = \theta_1(x) - \frac{1}{L}\int_0^L \theta_1(x)dx.$$

Going forward, we work with $(\varphi, \psi, \widehat{\theta})$, but, we write (φ, ψ, θ) for simplicity. For $b(t) \equiv 0$ and $\tau \equiv 0$ in system (1.1), Hizia et al. [1] considered

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \lambda\psi + \delta\theta_x = 0, \\ \psi_{tt} - \psi_{xx} + \lambda\varphi = 0, \\ \rho\theta_t - \kappa\theta_{xx} + \delta\varphi_{xt} = 0, \end{cases} \tag{1.7}$$

and established the well-posedness and a polynomial stability result. For recent results with Lord-Shulman thermal heat law, general weak internal damping, time varying coefficient, and time-varying delay: Apalara and Almutairi [2] considered a carbon-nanotube model with Lord-Shulman thermal heat law and established the well-posedness and exponential decay result. Raposo el at. [3] studied a suspension bridge model with weak internal damping and proved an exponential stability result. Mukiawa [4] considered a plate equation with time-varying delay feedback and proved a general stability result. For more related results, we refer the reader to [5–11] and the references therein.

For previous results without thermal damping, Alabau [12] investigated a coupled wave with only linear damping on the first equation, namely

$$\begin{cases} \varphi_{tt} - \Delta\varphi + \lambda\psi + \varphi_t = 0, \\ \psi_{tt} - \Delta\psi + \lambda\varphi = 0, \end{cases} \tag{1.8}$$

for $x \in \Omega$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n, n \geq 1$. The author proved the indirect stabilization and a polynomial decay result of the system. Moreover, Santos et al. [13] improved the polynomial decay result in [12] to the rate t^{-1} and Lobato et al. [14] further improved the result to an optimal rate of $t^{-\frac{1}{2}}$. Recently, Guesmia et al. [15] studied

$$\begin{cases} \varphi_{tt} + A\varphi - \int_0^t g(t-s)A^{\alpha}\varphi(s)ds + \lambda\psi = 0, \\ \psi_{tt} + A\psi + \lambda\varphi = 0, \end{cases} \tag{1.9}$$

where $\alpha \in [0, 1]$ and proved a general decay result. We refer readers to Almeida and Santos [16] for a polynomial decay result on system (1.9) when $\alpha = 1$.

The current work, motivated by the articles in [1, 2, 17], we study the asymptotic behaviour of solutions to the coupled one-dimensional hyperbolic system with Lord-Shulman thermal heat law on the first equation, and a general weak inherent damping, along with time-dependent coefficients and time-varying delay terms on the second equation given in (1.1)–(1.3). Therefore, the result in [1] becomes a particular case of the present paper.

The paper is arrange as follows. In Section 2, we transform our problem into the correct functional setting and state the needed assumptions. In Section 3, we provide some needed lemmas for the main result. In Section 4, we establish the main stability result. Throughout this paper, c and c_i, $i = 1, 2, \cdots$, are positive constants, which are not necessarily the same from line to line.

2. Problem transformation and conditions

In this section, we define a new function and impose some conditions on the given functions. Using similar definition as in [18], we define a new function as follows:

$$u(x, z, t) = \psi_t(x, t - \sigma(t)z), \quad \text{for } (x, z, t) \in [0, L] \times [0, 1] \times [0, \infty[. \tag{2.1}$$

Simple implicit differentiation gives

$$\sigma(t)u_t(x, z, t) + (1 - \sigma'(t)z)u_z(x, z, t) = 0. \tag{2.2}$$

Therefore, system (1.1) becomes

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \lambda\psi + \delta(\theta_x + \tau\theta_{xt}) = 0, & \text{in } [0, L] \times \mathbb{R}_+, \\ \psi_{tt} - \psi_{xx} + \lambda\varphi + b(t)\left[\alpha_1 f_1(\psi_t) + \alpha_2 f_2(u(x, 1, t))\right] = 0, & \text{in } [0, L] \times \mathbb{R}_+, \\ \rho(\theta_t + \tau\theta_{tt}) - \kappa\theta_{xx} + \beta\varphi_{xt} = 0, & \text{in } [0, L] \times \mathbb{R}_+, \\ \sigma(t)u_t(x, z, t) + (1 - \sigma'(t)z)u_z(x, z, t) = 0, & \text{in } [0, L] \times [0, 1] \times \mathbb{R}_+ \end{cases} \tag{2.3}$$

subjected to the boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \text{in } [0,\infty[, \\ \theta_x(0,t) = \theta_x(L,t) = 0, & \text{in } [0,\infty[, \\ u(x,0,t) = \psi_t(x,t), & \text{in } [0,L]\times[0,\infty[, \end{cases} \tag{2.4}$$

and initial data

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & \text{in } [0,L], \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), & \text{in } [0,L], \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & \text{in } [0,L], \\ u(x,z,0) = \psi_t(x,-\sigma(0)) = g_0(x,-\sigma(0)z), & \text{in } [0,L]\times[0,1]. \end{cases} \tag{2.5}$$

For the nonlinear time-varying damping coefficient b, nonlinear functions f_1 and f_2, and the time-varying feedback σ, we assume the following conditions, see [17,18], where similar assumptions have been used.

(P_1) : $b : [0,+\infty) \longrightarrow (0,+\infty)$ is a decreasing C^1-function and there exists a positive constant c such that

$$|b'(t)| \leq cb(t), \quad \int_0^{+\infty} b(t)dt = +\infty. \tag{2.6}$$

(P_2) : $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing continuous function and there exist positive constants c_1, c_2, ζ and a convex, non-decreasing function $L \in C^1([0,+\infty)) \cap C^2((0,+\infty))$ satisfying $L(0) = 0$ or L is a strictly nonlinear convex C^2-function on $[0,\zeta]$ such that $L'(0), L'' > 0$ and satisfies the following:

$$s^2 + f_1^2(s) \leq\ L^{-1}(sf_1(s)), \quad \text{for all } |s| \leq \zeta, \tag{2.7}$$

$$c_1 s^2 \leq sf_1(s) \leq c_2 s^2, \quad \text{for all } |s| \geq \zeta. \tag{2.8}$$

$f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing and odd C^1- function such that for some positive constants c_3, η_1, η_2,

$$|f_2'(s)| \leq c_3, \tag{2.9}$$

$$\eta_1(sf_2(s)) \leq F(s) \leq \eta_2(sf_1(s)), \tag{2.10}$$

with

$$F(s) = \int_0^s f_2(y)dy. \tag{2.11}$$

(P_3) : There exist $\sigma_0, \sigma_1 > 0$ such that

$$0 < \sigma_0 \leq \sigma(t) \leq \sigma_1,\ \forall\ t > 0, \tag{2.12}$$

$$\sigma \in W^{2,\infty}(0,T),\ \forall T > 0, \tag{2.13}$$

$$\sigma'(t) \leq d < 1,\ \forall\ t > 0. \tag{2.14}$$

(P_4) : The damping coefficients satisfy

$$\alpha_2\eta_2(1 - d\eta_1) < \eta_1(1 - d)\alpha_1. \tag{2.15}$$

Remark 2.1. *On account of the monotonicity of f_2 and thanks to the mean value theorem for integrals, we have*

$$F(s) = \int_0^s f_2(y)dy < sf_2(s). \tag{2.16}$$

Therefore, due to (2.10)*, we get that* $\eta_1 < 1$.

For completeness, we define the following Sobolev spaces:

$$\begin{aligned}
&L^2_\star([0,L]) = \{w \in L^2(0,L)\ :\ \int_0^L w(x)dx = 0\} \text{ and } H^1_*(0,L) = H^1(0,L) \cap L^2_\star([0,L]),\\
&H^2_\star([0,L]) = \{w \in H^2([0,L]) : w_x(0) = w_x(L) = 0\},\\
&H = \left[H^1_0([0,L]) \times L^2([0,L]L)\right]^2 \times H^1_\star([0,L]) \times L^2_\star([0,L]),\\
&H_1 = \left[H^2([0,L]) \cap H^1_0([0,L]) \times H^1_0([0,L])\right]^2 \times H^2_\star([0,L]) \cap H^1_*([0,L]) \times H^1_*([0,L]).
\end{aligned}$$

The existence and uniqueness result for problems (1.1)–(1.3) is given below.

Theorem 2.1. *Let*

$$\begin{aligned}
&(\varphi_0, \psi_0, \theta_0) \in H^2([0,L]) \cap H^1_0([0,L]) \times H^2([0,L]) \cap H^1_0(0,L) \times H^2_\star([0,L]) \cap H^1_*([0,L]),\\
&(\varphi_1, \psi_1, \theta_1) \in H^1_0(0,L) \times H^1_0(0,L) \times H^1_*([0,L]),\ g_0(., -\sigma(0)) \in H^1_0\left([0,L]; H^1([0,L])\right)
\end{aligned}$$

be given such that the compatibility condition $g_0(.,0) = \psi_1$ *holds. Suppose conditions* (P_1)–(P_4) *hold. Then, problems* (1.1)–(1.3) *have a unique global weak solution in the class*

$$\begin{aligned}
&\varphi, \psi \in L^\infty\left([0,+\infty); H^2(0,L) \cap H^1_0([0,L])\right),\ \varphi_t, \psi_t \in L^\infty\left([0,+\infty); H^1_0([0,L])\right),\\
&\varphi_{tt}, \psi_{tt} \in L^\infty\left((0,+\infty); L^2([0,L])\right),\\
&\theta \in L^\infty\left([0,+\infty); H^2_\star([0,L]) \cap H^1_*([0,L])\right),\ \theta_t \in L^\infty\left([0,+\infty); H^1_*([0,L])\right),\\
&\theta_{tt} \in L^\infty\left((0,+\infty); L^2_*([0,L])\right).
\end{aligned}$$

The result in Theorem 2.1 can be proof thanks to the Faedo-Galerkin approximation method, see (Theorem 2.1, [19]). To define the energy functional of our problem, we consider the positive number $\bar{\mu}$ such that

$$\frac{\alpha_2(1-\eta_1)}{\eta_1(1-d)} < \bar{\mu} < \frac{\alpha_1 - \alpha_2\eta_2}{\eta_2} \tag{2.17}$$

and set

$$\mu(t) = \bar{\mu}b(t).$$

The continuous energy functional of systems (2.3)–(2.5) is given by

$$\begin{aligned}
E(t) = &\frac{1}{2}\int_0^1 \left[\varphi_t^2 + \psi_t^2 + \varphi_x^2 + \psi_x^2 + 2\lambda\varphi\psi + \frac{\rho\delta}{\beta}(\theta + \tau\theta_t)^2 + \frac{\kappa\tau\delta}{\beta}\theta_x^2\right]dx\\
&+ \mu(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx,
\end{aligned} \tag{2.18}$$

where F is the function defined in (2.11).

Remark 2.2. *Using Young's and Poincaré's inequalities, we have*

$$\lambda \int_0^L \varphi\psi dx \geq -\frac{\lambda c_p}{2}\int_0^L \varphi_x^2 dx - \frac{\lambda c_p}{2}\int_0^L \psi_x^2 dx. \tag{2.19}$$

It follows from (2.18) *that*

$$\begin{aligned} E(t) \geq \frac{1}{2}\int_0^L \left[\varphi_t^2 + \psi_t^2 + (1-\lambda c_p)\left(\varphi_x^2 + \psi_x^2\right) + \frac{\rho\delta}{\beta}(\theta+\tau\theta_t)^2 + \frac{\kappa\tau\delta}{\beta}\theta_x^2\right] dx \\ + \mu(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx, \end{aligned} \tag{2.20}$$

for any positive λ satisfying $(1-\lambda c_p) > 0$.

3. Needed lemmas

In this section, we provide needed lemmas that are crucial in establishing the main stability result.

Lemma 3.1. *Under the assumptions of Theorem 2.1, the energy functional* (2.18) *satisfies*

$$\begin{aligned} \frac{dE(t)}{dt} \leq & -\frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx - b(t)\left[\alpha_1 - \bar{\mu}\eta_2 - \alpha_2\eta_2\right]\int_0^L \psi_t f_1(\psi_t)dx \\ & - b(t)\left[\bar{\mu}(1-\sigma'(t))\eta_1 - \alpha_2(1-\eta_1)\right]\int_0^L u(x,1,t)f_2(u(x,1,t))dx \\ \leq & \ 0, \ \forall\, t \geq 0. \end{aligned} \tag{3.1}$$

Proof. We begin by multiplying $(2.3)_1$ by φ_t, $(2.3)_2$ by ψ_t, and $(2.3)_3$ by $\dfrac{\delta}{\beta}(\theta+\tau\theta_t)$, Integrating the resulting outcome over $[0, L]$, using integration by parts and the boundary conditions (2.4), we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left[\varphi_t^2 + \varphi_x^2\right] dx + \lambda\int_0^L \psi\varphi_t dx - \delta\int_0^L (\theta+\tau\theta_t)\,\varphi_{xt} dx = 0, \tag{3.2}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L \left[\,\psi_t^2 + \psi_x^2 + \lambda(\varphi-u)^2\right] dx + \lambda\int_0^L \varphi\psi_t dx \\ + \alpha_1 b(t)\int_0^L \psi_t f_1(\psi_t)dx + \alpha_2 b(t)\int_0^1 \psi_t f_2(u(x,1,t))dx = 0, \end{aligned} \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left[\frac{\rho\delta}{\beta}(\theta+\tau\theta)^2 + \frac{\kappa\tau\delta}{\beta}\theta_x^2\right] dx + \frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx + \delta\int_0^L (\theta+\tau\theta_t)\,\varphi_{xt} dx = 0. \tag{3.4}$$

Adding (3.2)–(3.4), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\varphi_t^2 + \psi_t^2 + \varphi_x^2 + \psi_x^2 + 2\lambda\varphi\psi + \frac{\rho\delta}{\beta}(\theta+\tau\theta_t)^2 + \frac{\kappa\tau\delta}{\beta}\theta_x^2\right] dx \\ = -\alpha_1 b(t)\int_0^L \psi_t f_1(\psi_t)dx - \alpha_2 b(t)\int_0^1 \psi_t f_2(u(x,1,t))dx - \frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx. \end{aligned} \tag{3.5}$$

Next, multiplying equation $(2.3)_4$ by $\mu(t)f_2(u(x,z,t))$ and integrating the outcome over $[0,L]\times[0,1]$, we get

$$\mu(t)\sigma(t)\int_0^L\int_0^1 u_t(x,z,t)f_2(u(x,z,t))dzdx \\ +\mu(t)\int_0^L\int_0^1(1-\sigma'(t)z)u_z(x,z,t)f_2(u(x,z,t))dzdx=0. \tag{3.6}$$

It follows from (2.11) that

$$\frac{\partial}{\partial z}\left[F(u(x,z,t))\right]=u_z(x,z,t)f_2(u(x,z,t)). \tag{3.7}$$

Thus, Eq (3.6) leads to

$$\mu(t)\sigma(t)\int_0^L\int_0^1 u_t(x,z,t)f_2(u(x,z,t))dzdx \\ = -\mu(t)\int_0^L\int_0^1(1-\sigma'(t)z)\frac{\partial}{\partial z}\left[F(u(x,z,t))\right]dzdx \tag{3.8}$$

from which we get

$$\begin{aligned}
&\frac{d}{dt}\left(\mu(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx\right)\\
&= -\mu(t)\int_0^L\int_0^1\frac{\partial}{\partial z}\left[(1-\sigma'(t)z)F(u(x,z,t))\right]dzdx+\mu'(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx\\
&=\mu(t)\int_0^L\left(F(u(x,0,t))-F(u(x,1,t))\right)dx+\mu(t)\sigma'(t)\int_0^L F(u(x,1,t))dx\\
&\quad+\mu'(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx\\
&=\mu(t)\int_0^L F(\psi_t(x,t))dx-\mu(t)(1-\sigma'(t))\int_0^1 F(u(x,1,t)dx\\
&\quad+\mu'(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx.
\end{aligned} \tag{3.9}$$

Recalling (2.18), and summing Eqs (3.5) and (3.9), we arrive at

$$\begin{aligned}
\frac{dE(t)}{dt}=&-\alpha_1 b(t)\int_0^L\psi_t f_1(\psi_t)dx-\alpha_2 b(t)\int_0^1\psi_t f_2(u(x,1,t))dx-\frac{\kappa\delta}{\beta}\int_0^L\theta_x^2dx\\
&+\mu(t)\int_0^L F(\psi_t(x,t))dx-\mu(t)(1-\sigma'(t))\int_0^1 F(u(x,1,t)dx\\
&+\mu'(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx.
\end{aligned} \tag{3.10}$$

Using assumptions (P_1) and (2.10), it follows that

$$\begin{aligned}\frac{dE(t)}{dt} \leq &-(\alpha_1 b(t) - \mu(t)\alpha_2)\int_0^L \psi_t f_1(\psi_t)dx - \alpha_2 b(t)\int_0^L \psi_t f_2(u(x,1,t))dx\\ &-\mu(t)(1-\sigma'(t))\int_0^L F(u(x,1,t))dx - \frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx.\end{aligned} \tag{3.11}$$

At this stage, we consider the convex conjugate of the function F defined by

$$F^*(t) = t(F')^{-1}(t) - F((F')^{-1}(t)),\ \forall\, t \geq 0 \tag{3.12}$$

satisfying the generalized Young inequality, see [20]:

$$XY \leq F^*(X) + F(Y),\ \forall X, Y > 0. \tag{3.13}$$

Using the definition of F and (2.10), we obtain

$$F^*(t) = t f_2^{-1}(t) - F(f_2^{-1}(t)),\ \forall\, t \geq 0. \tag{3.14}$$

Thus, on account of (2.10) and (3.14), we have

$$\begin{aligned}F^*(f_2(u(x,1,t))) &= u(x,1,t)f_2(u(x,1,t)) - F(u(x,1,t))\\ &\leq (1-\eta_1)u(x,1,t)f_2(u(x,1,t)).\end{aligned} \tag{3.15}$$

On account of (3.13) and (3.15), it follows from (3.11) that

$$\begin{aligned}\frac{dE(t)}{dt} \leq &-(\alpha_1 b(t) - \mu(t)\alpha_2)\int_0^L \psi_t f_1(\psi_t)dx - \alpha_2 b(t)\int_0^L \psi_t f_2(u(x,1,t))dx\\ &-\mu(t)(1-\sigma'(t))\int_0^L F(u(x,1,t)dx - \frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx\\ \leq &-(\alpha_1 b(t) - \mu(t)\eta_2)\int_0^L \psi_t f_1(\psi_t)dx + \alpha_2 b(t)\eta_2\int_0^L \psi_t f_1(\psi_t)dx\\ &+\alpha_2 b(t)(1-\eta_1)\int_0^L u(x,1,t)f_2(u(x,1,t))dx\\ &-\mu(t)(1-\sigma'(t))\int_0^L F(u(x,1,t)dx - \frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx\\ \leq &-(\alpha_1 b(t) - \mu(t)\eta_2 - \alpha_2 b(t)\eta_2)\int_0^L \psi_t f_1(\psi_t)dx\\ &-(\mu(t)(1-\sigma'(t))\eta_1 - \alpha_2 b(t)(1-\eta_1))\int_0^L u(x,1,t)f_2(u(x,1,t))dx\\ &-\frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx.\end{aligned} \tag{3.16}$$

Since $\mu(t) = \bar{\mu}b(t)$, the inequality in (3.16) becomes

$$\begin{aligned}\frac{dE(t)}{dt} \leq &-\frac{\kappa\delta}{\beta}\int_0^L \theta_x^2 dx - b(t)\left[\alpha_1 - \bar{\mu}\eta_2 - \alpha_2\eta_2\right]\int_0^L \psi_t f_1(\psi_t)dx\\ &-b(t)\left[\bar{\mu}(1-\sigma'(t))\eta_1 - \alpha_2(1-\eta_1)\right]\int_0^L u(x,1,t)f_2(u(x,1,t))dx.\end{aligned} \tag{3.17}$$

On account of (2.14) and (2.17), we obtain (3.1). This completes the proof.

Lemma 3.2. *Let* $W = (\varphi, \psi, \theta, u)$ *be the solution of systems* (2.3)–(2.5) *in Theorem 2.1, then the functional* G_1*, defined by*

$$G_1(t) := \rho \int_0^L \varphi_t \int_0^x (\theta(y,t) + \tau\theta_t(y,t))\, dydx,$$

satisfies, for any positive ϵ_1 *and* ϵ_2*, the estimate*

$$\begin{aligned} G_1'(t) \le & -\frac{\beta}{2}\int_0^L \varphi_t^2 dx + \epsilon_1 \int_0^L \varphi_x^2 dx + \epsilon_2 \int_0^L \psi_x^2 dx + c\int_0^L \theta_x^2 dx \\ & + c\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\int_0^L (\theta + \tau\theta_t)^2 dx, \quad \forall t \ge 0. \end{aligned} \tag{3.18}$$

Proof. Differentiation of G_1 with respect to t, using $(2.3)_1$, $(2.3)_3$, integration by parts, and exploiting the boundary conditions (2.4), we have

$$\begin{aligned} G_1'(t) &= \rho \int_0^L \varphi_{tt} \int_0^x (\theta(y,t) + \tau\theta_t(y,t))\, dydx + \rho \int_0^L \varphi_t \int_0^x (\theta_t(y,t) + \tau\theta_{tt}(y,t))\, dydx \\ &= \rho \int_0^L [\varphi_{xx} - \lambda\psi - \delta(\theta_x + \tau\theta_{xt})] \int_0^x (\theta(y,t) + \tau\theta_t(y,t))\, dydx \\ &\quad + \int_0^L \varphi_t \int_0^x \left(\kappa\theta_{yy}(y,t) - \beta\varphi_{ty}(y,t)\right) dydx \\ &= -\beta \int_0^L \varphi_t^2 dx + \kappa \int_0^L \varphi_t\theta_x dx - \rho \int_0^L \varphi_x(\theta + \tau\theta_t) dx \\ &\quad - \rho\lambda \int_0^L \psi \int_0^x (\theta(y,t) + \tau\theta_t(y,t))\, dydx + \rho\delta \int_0^L (\theta + \tau\theta_t)^2 dx. \end{aligned} \tag{3.19}$$

Applying Cauchy-Schwarz, Young and Poincaré inequalities, we obtain

$$\begin{aligned} & \kappa \int_0^L \varphi_t\theta_x dx \le \frac{\beta}{2}\int_0^L \varphi_t^2 dx + \frac{\kappa^2}{2\beta^2}\int_0^L \theta_x^2 dx, \\ & -\rho \int_0^L \varphi_x(\theta + \tau\theta_t) dx \le \epsilon_1 \int_0^L \varphi_x^2 dx + \frac{\rho^2}{4\epsilon_1}\int_0^L (\theta + \tau\theta_t)^2 dx, \quad \forall \epsilon_1 > 0, \\ & -\rho\lambda \int_0^L \psi \int_0^x (\theta(y,t) + \tau\theta_t(y,t))\, dydx \\ & \qquad \le \epsilon_2 \int_0^L \psi_x^2 dx + \frac{(\rho\lambda)^2 c_p^2}{4\epsilon_2}\int_0^L (\theta + \tau\theta_t)^2 dx, \quad \forall \epsilon_2 > 0. \end{aligned} \tag{3.20}$$

Substituting the estimates in (3.20) into (3.22) leads to (3.18). This completes the proof.

Lemma 3.3. *Let* $W = (\varphi, \psi, \theta, u)$ *be the solution of systems* (2.3)–(2.5) *in Theorem 2.1, then the functional* G_2*, defined by*

$$G_2(t) := -\tau\rho \int_0^L \theta\,(\theta + \tau\theta_t)\, dx,$$

satisfies, for any positive ϵ_3*, the estimate*

$$G_2'(t) \le -\frac{\rho}{2}\int_0^L (\theta + \tau\theta_t)^2\, dx + \epsilon_3 \int_0^L \varphi_t^2 dx + c\left(1 + \frac{1}{\epsilon_3}\right)\int_0^L \theta_x^2 dx, \quad \forall t \ge 0. \tag{3.21}$$

Proof. Differentiating G_2 with respect to t, using $(2.3)_3$ and integrating by parts, and exploiting the boundary conditions (2.4), we have

$$\begin{aligned} G_2'(t) =& -\tau\rho\int_0^L \theta_t\,(\theta+\tau\theta_t)\,dx - \tau\rho\int_0^L \theta\,(\theta_t+\tau\theta_{tt})\,dx \\ =& -\rho\int_0^L (\theta+\tau\theta_t-\theta)\,(\theta+\tau\theta_t)\,dx + \tau\int_0^L \theta\,(-\kappa\theta_{xx}+\beta\varphi_{xt})\,dx \\ =& -\rho\int_0^L (\theta+\tau\theta_t)^2\,dx + \rho\int_0^L \theta(\theta+\tau\theta_t)dx + \kappa\tau\int_0^L \theta_x^2dx - \beta\tau\int_0^L \varphi_t\theta_x dx. \end{aligned} \tag{3.22}$$

Making use of Young and Poincaré inequalities, we get, for any $\epsilon_3 > 0$,

$$\begin{aligned} G_2'(t) \le& -\frac{\rho}{2}\int_0^L (\theta+\tau\theta_t)^2\,dx + \frac{\rho c_p}{2}\int_0^L \theta_x^2dx + \kappa\tau\int_0^L \theta_x^2dx \\ &+ \epsilon_3\int_0^L \varphi_t^2dx + \frac{\beta^2\tau^2}{4\epsilon_3}\int_0^L \theta_x^2 \\ =& -\frac{\rho}{2}\int_0^L (\theta+\tau\theta_t)^2\,dx + \epsilon_3\int_0^L \varphi_t^2dx + c\left(1+\frac{1}{\epsilon_3}\right)\int_0^L \theta_x^2dx. \end{aligned}$$

This completes the proof.

Lemma 3.4. *Let* $W = (\varphi,\psi,\theta,u)$ *be the solution of systems* (2.3)–(2.5) *in Theorem 2.1, then the functional* G_3, *defined by*

$$G_3(t) := \int_0^L (\varphi\varphi_t + \psi\psi_t)\,dx,$$

satisfies the estimate

$$\begin{aligned} G_3'(t) \le& -\int_0^L \left(\frac{1}{2}\varphi_x^2 + \frac{1}{2}\psi_x^2 + 2\lambda\varphi\psi\right)dx + \int_0^L \left(\varphi_t^2+\psi_t^2\right)dx \\ &+ c\int_0^L (\theta+\tau\theta_t)^2\,dx + c\int_0^L |f_1(\psi_t)|^2dx + c\int_0^L |f_2(u(x,1,t))|^2dx, \quad \forall\, t\ge 0. \end{aligned} \tag{3.23}$$

Proof. Differentiating G_3 with respect to t, using $(2.3)_1$, $(2.3)_2$, then integration by parts, and boundary conditions (2.4), we have

$$\begin{aligned} G_3'(t) =& \int_0^L \left(\varphi_t^2+\psi_t^2\right)dx + \int_0^L (\varphi\varphi_{tt}+\psi\psi_{tt})\,dx \\ =& \int_0^L \left(\varphi_t^2+\psi_t^2\right)dx + \int_0^L \varphi\,[\varphi_{xx} - \lambda\psi - \delta(\theta_x+\tau\theta_{xt})]\,dx \\ &+ \int_0^L \psi\,[\psi_{xx} - \lambda\varphi - \alpha_1 b(t) f_1(\psi_t) - \alpha_2 b(t) f_2(u(x,1,t))]\,dx \\ =& -\int_0^L \left(\varphi_x^2 + 2\lambda\varphi\psi + \psi_x^2\right)dx + \int_0^L \left(\varphi_t^2+\psi_t^2\right)dx + \delta\int_0^L \varphi_x(\theta+\tau\theta_t)dx \\ &- \alpha_1 b(t)\int_0^L \psi f_1(\psi_t)dx - \alpha_2 b(t)\int_0^L \psi f_2(u(x,1,t))dx. \end{aligned} \tag{3.24}$$

Now, using condition (P_1), and Young's and Poincaré's inequalities, we obtain

$$
\begin{aligned}
&\delta \int_0^L \varphi_x(\theta+\tau\theta_t)dx \le \frac{1}{2}\int_0^L \varphi_x^2 dx + \frac{\delta^2}{2}\int_0^L (\theta+\tau\theta_t)^2\,dx,\\
&-\alpha_1 b(t)\int_0^L \psi f_1(\psi_t)dx \le \frac{1}{4}\int_0^L \psi_x^2 dx + \frac{3b^2(0)c_p^2\alpha_1^2}{4}\int_0^L f_1(\psi_t)|^2 dx,\\
&-\alpha_2 b(t)\int_0^L \psi f_2(u(x,1,t))dx \le \frac{1}{4}\int_0^L \psi_x^2 dx + \frac{3b^2(0)c_p^2\alpha_2^2}{4}\int_0^L |f_2(u(x,1,t))|^2 dx.
\end{aligned}
\tag{3.25}
$$

Substituting (3.25) into (3.24) leads to (3.23). This completes the proof.

Lemma 3.5. *Let $W = (\varphi,\psi,\theta,u)$ be the solution of systems* (2.3)–(2.5) *in Theorem 2.1, then the functional*

$$
G_4(t) := \bar{\mu}\sigma(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}F(u(x,z,t))dzdx,
$$

satisfies the estimate

$$
G_4'(t) \le -2G_4(t) + \frac{\bar{\mu}\eta_2}{2}\int_0^L \left(\psi_t^2 + |f_1(\psi_t)|^2\right)dx,\ \forall\, t\ge 0.
\tag{3.26}
$$

Proof. Differentiating G_4, we get

$$
\begin{aligned}
G_4'(t) =\ & \bar{\mu}\sigma'(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}F(u(x,z,t))dzdx\\
&-2\bar{\mu}\sigma(t)\sigma'(t)\int_0^L\int_0^1 ze^{-2\sigma(t)z}F(u(x,z,t))dzdx\\
&+\bar{\mu}\sigma(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}u_t(x,z,t)f_2(u(x,z,t))dzdx.
\end{aligned}
\tag{3.27}
$$

From Eq $(2.3)_4$, the last term on the right hand-side of (3.26) can be express as follows:

$$
\begin{aligned}
&\bar{\mu}\sigma(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}u_t(x,z,t)f_2(u(x,z,t))dzdx\\
&\quad=\bar{\mu}\int_0^L\int_0^1 e^{-2\sigma(t)z}\left(\sigma'(t)z-1\right)u_z(x,z,t)f_2(u(x,z,t))dzdx\\
&\quad=\bar{\mu}\int_0^L\int_0^1 \frac{\partial}{\partial z}\left[e^{-2\sigma(t)z}\left(\sigma'(t)z-1\right)F(u(x,z,t))\right]dzdx\\
&\qquad+2\bar{\mu}\sigma(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}\left(\sigma'(t)z-1\right)F(u(x,z,t))dzdx\\
&\qquad-\bar{\mu}\sigma'(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}F(u(x,z,t))dzdx\\
&\quad=-\bar{\mu}(1-\sigma'(t))e^{-2\sigma(t)}\int_0^L F(u(x,1,t))dx+\int_0^L F(\psi_t)dx\\
&\qquad+2\bar{\mu}\sigma(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}\left(\sigma'(t)z-1\right)F(u(x,z,t))dzdx\\
&\qquad-\bar{\mu}\sigma'(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}F(u(x,z,t))dzdx.
\end{aligned}
\tag{3.28}
$$

The substitution of (3.28) into (3.27) leads to

$$
\begin{aligned}
G_4'(t)=&-2\bar{\mu}\sigma(t)\int_0^L\int_0^1 e^{-2\sigma(t)z}F(u(x,z,t))dzdx+\bar{\mu}\int_0^L F(\psi_t)dx\\
&-\bar{\mu}(1-\sigma'(t))e^{-2\sigma(t)}\int_0^L F(u(x,1,t))dx.
\end{aligned}
\tag{3.29}
$$

On account of assumptions (2.10), (2.14) and Young's inequality, we obtain (3.26). This completes the proof.

Lemma 3.6. Let $W=(\varphi,\psi,\theta,u)$ be the solution of system (2.3)–(2.5) in Theorem 2.1. Then, for some positive numbers n, n_1, n_2 to be specified, the Lyapunov functional Q, defined by

$$
Q(t):=nE(t)+n_1G_1(t)+n_2G_2(t)+G_3(t)+G_4(t),
\tag{3.30}
$$

satisfies

$$
a_1E(t)\le Q(t)\le a_2E(t),\qquad \forall t\ge 0
\tag{3.31}
$$

and

$$
Q'(t)\le-\gamma E(t)+c\int_0^L(\psi_t^2+|f_1(\psi_t)|^2)dx+c\int_0^L|f_2(u(x,1,t))|^2dx,\quad \forall t\ge 0,
\tag{3.32}
$$

for some positive constants a_1, a_2, γ.

Proof. Thanks to Cauchy-Schwarz's, Young's and Poincaré's inequalities, we obtain

$$
\begin{aligned}
|Q(t)-nE(t)| \leq\ & n_1\rho\int_0^L\left|\varphi_t\int_0^x(\theta(y,t)+\tau\theta_t(y,t))\,dy\right|dx+n_2\tau\rho\int_0^L|\theta\,(\theta+\tau\theta_t)|\,dx\\
&+\int_0^L|(\varphi\varphi_t+\psi\psi_t)|\,dx+\bar{\mu}\sigma(t)\int_0^L\left|\int_0^1 e^{-2\sigma(t)z}F(u(x,z,t))dz\right|dx\\
&+\lambda\int_0^L\varphi\psi dx-\lambda\int_0^L\varphi\psi dx\\
\leq\ & \frac{n_1\rho+1}{2}\int_0^L\varphi_t^2dx+\frac{1}{2}\int_0^L\psi_t^2dx+c_p\int_0^L\varphi_x^2dx+c_p\int_0^L\psi_x^2dx\\
&\lambda\int_0^L\varphi\psi dx+\frac{n_1\rho+n_2\tau\rho}{2}\int_0^L(\theta+\tau\theta_t)^2\,dx+\frac{n_2\tau\rho c_p}{2}\int_0^L\theta_x^2dx\\
&+\bar{\mu}\sigma(t)\int_0^L\int_0^1F(u(x,z,t))dzdx.
\end{aligned}
\tag{3.33}
$$

From (3.33), we get

$$|Q(t)-nE(t)|\leq n_0E(t), \tag{3.34}$$

for some positive consatant n_o. Thus for n large enough such that

$$a_1=n-n_0>0,\ a_2=n+n_0>0, \tag{3.35}$$

we arrive at (4.15). Next on account of Lemmas 3.1–3.5, we have

$$
\begin{aligned}
Q'(t)\leq &-\left[n_1\frac{\beta}{2}-n_2\epsilon_3-1\right]\int_0^L\varphi_t^2dx-\int_0^L\psi_t^2dx-\left[\frac{1}{2}-n_1\epsilon_1\right]\int_0^L\varphi_x^2dx\\
&-\left[\frac{1}{2}-n_1\epsilon_2\right]\int_0^L\psi_x^2dx-2\lambda\int_0^L\varphi\psi dx\\
&-\left[n_2\frac{\rho}{2}-cn_1\left(1+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)-c\right]\int_0^L(\theta+\tau\theta_t)^2\,dx\\
&-\left[\frac{\kappa\delta}{\beta}n-cn_1-cn_2\left(1+\frac{1}{\epsilon_3}\right)\right]\int_0^L\theta_x^2dx\\
&-\frac{2e^{-2\sigma_1}}{b(0)}\mu(t)\sigma(t)\int_0^L\int_0^1F(u(x,z,t))dzdx+\left[2+\frac{\bar{\mu}\eta_2}{2}\right]\int_0^L\psi_t^2dx\\
&+\left[c+\frac{\bar{\mu}\eta_2}{2}\right]\int_0^L|f_1(\psi_t)|^2dx+c\int_0^1|f_2(u(x,1,t))|^2dx.
\end{aligned}
$$

Choosing

$$\epsilon_1=\epsilon_2=\frac{1}{4n_1},\epsilon_3=\frac{n_1\beta}{4n_2},$$

we arrive at

$$
\begin{aligned}
Q'(t) \leq & -\left[n_1\frac{\beta}{4}-1\right]\int_0^L \varphi_t^2 dx - \int_0^L \psi_t^2 dx - \frac{1}{4}\int_0^L \varphi_x^2 dx - \frac{1}{4}\int_0^L \psi_x^2 dx \\
& - 2\lambda \int_0^L \varphi\psi dx - \left[n_2\frac{\rho}{2} - cn_1\left(1+8n_1\right) - c\right]\int_0^L \left(\theta+\tau\theta_t\right)^2 dx \\
& - \left[\frac{\kappa\delta}{\beta}n - cn_1 - cn_2\left(1+\frac{4n_2}{n_1\beta}\right)\right]\int_0^L \theta_x^2 dx \\
& - \frac{2e^{-2\sigma_1}}{b(0)}\mu(t)\sigma(t)\int_0^L\int_0^1 F(u(x,z,t))dzdx + \left[2+\frac{\bar{\mu}\eta_2}{2}\right]\int_0^L \psi_t^2 dx \\
& + \left[c+\frac{\bar{\mu}\eta_2}{2}\right]\int_0^L |f_1(\psi_t)|^2 dx + c\int_0^1 |f_2(u(x,1,t))|^2 dx.
\end{aligned} \tag{3.36}
$$

Next, we select n_1 large such that

$$
\left[n_1\frac{\beta}{4}-1\right] > 0,
$$

follow by choosing n_2 very large such that

$$
\left[n_2\frac{\rho}{2} - cn_1\left(1+8n_1\right) - c\right] > 0.
$$

Finally, we choose n even larger such that (3.31) remains true and

$$
\left[\frac{\kappa\delta}{\beta}n - cn_1 - cn_2\left(1+\frac{4n_2}{n_1\beta}\right)\right] > 0.
$$

Thus, recalling (2.18), it follows from (3.36) that

$$
Q'(t) \leq -\gamma E(t) + c\int_0^L (\psi_t^2 + |f_1(\psi_t)|^2)dx + c\int_0^L |f_2(u(x,1,t))|^2 dx, \quad \forall t \geq 0, \tag{3.37}
$$

for some constant $\gamma > 0$. This completes the proof.

4. Stability result

Now, we are ready to establish the main stability result of this paper.

Theorem 4.1. *Let $W = (\varphi, \psi, \theta, u)$ be the solution of systems* (2.3)–(2.5) *in Theorem 2.1 and suppose conditions (P_1)–(P_4) hold. Then, the energy functional* (2.18) *satisfies*

$$
E(t) \leq \gamma_1 L_1^{-1}\left(\gamma_2\int_0^t b(s)ds + \gamma_3\right), \quad t \geq 0, \tag{4.1}
$$

for some positive constants $\gamma_1, \gamma_2, \gamma_3$, and ζ_0, with

$$
L_1(t) = \int_t^1 \frac{1}{L_0(s)}ds \;\; \text{and} \;\; L_0(t) = tL'(\zeta_0 t).
$$

Proof. The proof is divide into two parts:

Part I: L is linear. On account of condition (P_2), we have

$$c_1|s| \le |f_1(s)| \le c_2|s|, \quad \forall s \in \mathbb{R}.$$

It follows that

$$f_1^2(s) \le c_2 s f_1(s), \quad \forall s \in \mathbb{R}. \tag{4.2}$$

Thus, by multiplying (3.32) with $b(t)$ and recalling (3.1) and (4.2), we obtain

$$\begin{aligned} b(t)Q'(t) \le & -\gamma b(t)E(t) + cb(t)\int_0^L \psi_t f_1(\psi_t)dx + cb(t)\int_0^L u(x,1,t)f_2(u(x,1,t))dx \\ \le & -\gamma b(t)E(t) - cE'(t), \quad \forall t \ge 0. \end{aligned}$$

By exploiting condition (P_2) again and the equivalent relation (3.31), it follows that

$$Q_0(t) := b(t)Q(t) + cE(t) \sim E(t), \tag{4.3}$$

and Q_0 satisfies

$$Q_0'(t) \le -\delta_1 b(t) Q_0(t), \quad \forall t \ge 0, \tag{4.4}$$

for some positive constant δ_1. By integrating the estimate (4.4) over $(0, t)$, using (4.3), we get

$$E(t) \le \gamma_1 \exp\left(-\gamma_2 \int_0^t b(s)ds\right) = \gamma_1 L_1^{-1}\left(\gamma_2 \int_0^t b(s)ds\right), \ \forall t \ge 0. \tag{4.5}$$

Part II: L is nonlinear on $[0, \zeta]$. Using similar ideas as in [21], we choose $0 < \zeta_1 \le \zeta$ such that

$$s f_1(s) \le \min\{\zeta, L(\zeta)\}, \quad \forall |s| \le \zeta_1. \tag{4.6}$$

Thus using assumption (P_2) and the continuity of f_1 such that $|f_1(s)| > 0$, for $s \neq 0$, we obtain

$$\begin{cases} s^2 + f_1^2(s) \le \ L^{-1}(s f_1(s)), & \forall |s| \le \zeta_1, \\ c_1'|s| \le |f_1(s)| \le c_2'|s|, & \forall |s| \ge \zeta_1. \end{cases} \tag{4.7}$$

Next, we define the following partitions:

$$\begin{aligned} J_1 &= \{x \in [0, L] : \ |\psi_t| \le \zeta_1\}, \qquad J_2 = \{x \in [0, L] : \ |\psi_t| > \zeta_1\}, \\ \tilde{J}_1 &= \{x \in [0, L] : \ |u(x,1,t)| \le \zeta_1\}, \qquad \tilde{J}_2 = \{x \in][0, L] : \ |u(x,1,t)| > \zeta_1\} \end{aligned}$$

and the function g, defined by

$$g(t) = \int_{J_1} \psi_t f_1(\psi_t)dx.$$

On account that L^{-1} is concave (as L is convex) and using Jensen's inequality for nonlinear functions, it follows that

$$L^{-1}(g(t)) \ge c \int_{J_1} L^{-1}\left(\psi_t f_1(\psi_t)\right) dx. \tag{4.8}$$

Now, multiplying the inequality (3.32) by $b(t)$, we get

$$b(t)Q'(t) \le -\gamma b(t)E(t) + cb(t)\int_0^L (\psi_t^2 + |f_1(\psi_t)|^2)dx + cb(t)\int_0^L |f_2(u(x,1,t))|^2 dx, \quad \forall t \ge 0. \tag{4.9}$$

Using (4.7) and (4.8), we have

$$\begin{aligned} b(t)\int_0^L (\psi_t^2 + |f_1(\psi_t)|^2)dx &= b(t)\int_{J_1}\left(\psi_t^2 + f_1^2(\psi_t)\right)dx + b(t)\int_{J_2}\left(\psi_t^2 + f_1^2(\psi_t)\right)dx \\ &\le b(t)\int_{J_1} L^{-1}\left(\psi_t f_1(\psi_t)\right)dx + cb(t)\int_{J_2}\psi_t f_1(\psi_t)dx \\ &\le cb(t)L^{-1}(g(t)) - cE'(t), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &cb(t)\int_0^L |f_2(u(x,1,t))|^2 dx \\ &\quad= b(t)\int_{\bar{J}_1}|f_2(u(x,1,t))|^2dx + b(t)\int_{\bar{J}_2}|f_2(u(x,1,t))|^2dx \\ &\quad\le cb(t)\int_{\bar{J}_1}u(x,1,t)f_2(u(x,1,t))dx + cb(t)\int_{\bar{J}_2}u(x,1,t)f_2(u(x,1,t))dx \\ &\quad\le -cE'(t). \end{aligned} \tag{4.11}$$

On account of (4.9)–(4.11), it follows that

$$b(t)Q'(t) + cE'(t) \le -\gamma b(t)E(t) + cb(t)L^{-1}(g(t)). \tag{4.12}$$

Let $Q_1(t) = b(t)L(t) + cE(t)$, using assumption (P_1), it follows that

$$Q_1'(t) \le -\gamma b(t)E(t) + cb(t)L^{-1}(g(t)), \tag{4.13}$$

and

$$Q_1 \sim E \;\text{ by virtue of }\; (3.31). \tag{4.14}$$

Also, let $\zeta_0 < r$ and $\gamma_0 > 0$ be constants to be chosen later and set

$$Q_2(t) := L'\left(\zeta_0\frac{E(t)}{E(0)}\right)Q_1(t) + \gamma_0 E(t).$$

Then, using (4.13), the fact that E is decreasing ($E' \le 0$), and $L' > 0,\ L'' > 0$ on $(0, \zeta]$, the functional Q_2, satisfies

$$\varpi_1 Q_2(t) \le E(t) \le \varpi_2 Q_2(t) \tag{4.15}$$

for some positive constants ϖ_1, ϖ_2 and, we have

$$\begin{aligned} Q_2'(t) &= \zeta_0\frac{E'(t)}{E(0)}L''\left(\zeta_0\frac{E(t)}{E(0)}\right)Q_1(t) + L'\left(\zeta_0\frac{E(t)}{E(0)}\right)Q_1'(t) + \gamma_0E'(t) \\ &\le -\gamma b(t)E(t)L'\left(\zeta_0\frac{E(t)}{E(0)}\right) + \underbrace{cb(t)L'\left(\zeta_0\frac{E(t)}{E(0)}\right)L^{-1}(g(t))}_{G_5} + \gamma_0E'(t). \end{aligned} \tag{4.16}$$

Next, we estimate the term G_5 in (4.16) above. For this, we define the convex conjugate of L denoted by L^* as follows:

$$L^*(s) = s(L')^{-1}(s) - L[(L')^{-1}(s)] \leq s(L')^{-1}(s), \quad \text{if } s \in (0, L'(\zeta)] \tag{4.17}$$

and satisfies the generalized Young inequality

$$AB \leq L^*(A) + L(B), \quad \text{if } A \in (0, L'(\zeta)], \ B \in (0, \zeta]. \tag{4.18}$$

We set $A = L'\left(\zeta_0 \dfrac{E(t)}{E(0)}\right)$ and $B = L^{-1}(g(t))$, and keeping in mind Lemma 3.1 and condition (4.6), then (4.16)–(4.18) lead to

$$\begin{aligned} Q_2'(t) \leq & -\gamma b(t)E(t)L'\left(\zeta_0 \frac{E(t)}{E(0)}\right) + cb(t)\left[L^*\left(L'\left(\zeta_0 \frac{E(t)}{E(0)}\right)\right) + L\left(L^{-1}(g(t))\right)\right] + \gamma_0 E'(t) \\ = & -\gamma b(t)E(t)L'\left(\zeta_0 \frac{E(t)}{E(0)}\right) + cb(t)L^*\left(L'\left(\zeta_0 \frac{E(t)}{E(0)}\right)\right) + cb(t)g(t) + \gamma_0 E'(t) \\ \leq & -\gamma b(t)E(t)L'\left(\zeta_0 \frac{E(t)}{E(0)}\right) + c\zeta_0 b(t)\left(\frac{E(t)}{E(0)}\right)L'\left(\zeta_0 \frac{E(t)}{E(0)}\right) - cE'(t) + \gamma_0 E'(t) \\ \leq & -(\gamma E(0) - c\zeta_0)b(t)\left(\frac{E(t)}{E(0)}\right)L'\left(\zeta_0 \frac{E(t)}{E(0)}\right) + (\gamma_0 - c)E'(t). \end{aligned} \tag{4.19}$$

Now, we choose $\zeta_0 = \dfrac{\gamma E(0)}{2c}$, $\gamma_0 = 2c$, and using the fact that $E'(t) \leq 0$, we obtain, for some $\gamma_1 > 0$

$$Q_2'(t) \leq -\gamma_1 b(t)\frac{E(t)}{E(0)}L'\left(\zeta_0 \frac{E(t)}{E(0)}\right) = -\gamma_1 b(t)L_0\left(\frac{E(t)}{E(0)}\right), \tag{4.20}$$

where $L_0(t) = tL'(\zeta_0 t)$. Base on the fact that L is strictly convex on $]0, \zeta]$, we conclude that $L_0(t) > 0$, $L_0'(t) > 0$ on $]0, 1]$. It follows from (4.15) and (4.20) that the functional Q_3, defined by

$$Q_3(t) = \frac{\varpi_1 Q_2(t)}{E(0)},$$

satisfies

$$Q_3(t) \sim E(t) \tag{4.21}$$

and

$$Q_3'(t) \leq -\delta_2 b(t)L_0(Q_3(t)), \tag{4.22}$$

for some $\gamma_2 > 0$. Thus, we obtain

$$[L_1(Q_3(t))]' \geq \gamma_2 b(t), \tag{4.23}$$

where

$$L_1(t) = \int_t^1 \frac{1}{L_0(s)} ds, \quad t \in (0, 1].$$

By integrating (4.23) over $]0, t]$, recalling the properties of L_0, and the fact that L_1 is strictly decreasing on $]0, 1]$, we arrive at

$$Q_3(t) \leq L_1^{-1}\left(\gamma_2 \int_0^t a(s)ds + \gamma_3\right), \quad \forall t \in \mathbb{R}^+, \tag{4.24}$$

for some $\gamma_3 > 0$. On account of (4.21) and (4.24), we obtain the result in (4.1). This completes the proof.

Examples

In this section, we give some examples to illustrate the result in (4.1).
Let

$$f_0 \in C^2\left([0, +\infty)\right)$$

be a strictly increasing function satisfying

$$\begin{aligned} & f_0(|s|) \leq |f_1(s)| \leq f_0^{-1}(|s|), \ \forall \ |s| \leq \zeta, \\ & c_1 s^2 \leq s f_1(s) \leq c_2 s^2, \ \forall \ |s| \geq \zeta, \end{aligned} \tag{4.25}$$

for some positive constants c_1, c_2, and ζ. Let

$$L(s) = \left(\sqrt{\frac{s}{2}}\right) f_0\left(\sqrt{\frac{s}{2}}\right), \tag{4.26}$$

then L is a C^2-strictly convex function on $(0, \zeta]$ and satisfies condition (P_2). We give some examples of f_0 such that f_1 satisfies (4.25) near 0.

i. Let $f_0(s) = \delta s$, where $\delta > 0$ is a constant. Then $L(s) = \bar{\delta} s$, with $\bar{\delta} = \dfrac{\delta}{2}$ satisfies (P_2) near 0. Using (4.1), we obtain

$$E(t) \leq \bar{\gamma} \exp\left(-\gamma_2 \int_0^t b(s) ds\right), \ \forall \ t \geq 0.$$

ii. Let $f_0(s) = e^{-\frac{1}{s}}$. Then $L(s) = \sqrt{\frac{s}{2}} e^{-\sqrt{\frac{2}{s}}}$ satisfies (P_2) near 0. On account of (4.1), we obtain

$$E(t) \leq \gamma_1 \left(\ln\left(\gamma_2 \int_0^t a(s) ds + \gamma_3\right)\right)^{-2}, \ \forall \ t \geq 0. \tag{4.27}$$

iii. Let $f_0(s) = \dfrac{1}{s} e^{-\frac{1}{s^2}}$. Then $L(s) = e^{-\frac{2}{s}}$ satisfies (P_2) near 0. It follows from (4.1) that

$$E(t) \leq \gamma_1 \left(\ln\left(\gamma_2 \int_0^t b(s) ds + \gamma_3\right)\right)^{-1}, \ \forall \ t \geq 0. \tag{4.28}$$

Use of AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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