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Research article

Markovian switching induced dynamics of a predator-prey system with prey refuge and habitat selection

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Abstract: In natural ecosystems, population dynamics are influenced by white and colored noise, which drive the population system to transition between multiple states. Therefore, it is essential to investigate predator-prey models that incorporate the concurrent impacts of white and colored noise. In this paper, we examined a three-dimensional stochastic predator-prey model involving dual prey species and a single predator by incorporating the refuge effect and habitat selection behavior under Markovian switching. By leveraging the theoretical framework of stochastic differential equations, its stochastic persistence was analyzed. Additionally, by constructing suitable Lyapunov functions, the conditions for an ergodic stationary distribution were deduced. Additionally, the extinction of populations and the asymptotic properties of solutions were explored. The findings revealed that habitat selection behavior has a significant and detrimental impact on the corresponding prey, while the refuge effect positively influences the prey and the predator. Ultimately, numerical simulations were performed to validate the theoretical outcomes. The results of these simulations strongly confirm the accuracy of the theoretical deductions.

Keywords: glipin-ayala model; Markov switching; stationary distribution; persistence

1. Introduction

The predator-prey relationship is an important mechanism of interaction among species in natural ecosystems. The functional response is a key element in understanding the predator-prey dynamics, which delineates the relationship between the predation rate of predators and the population density of prey. The Holling functional response, which effectively describes how the predation rate changes with prey density, is widely utilized in ecological modeling [1]. However, an increasing body of evidence indicates that predator interference, a phenomenon where predators engage in competition or impede one another during the hunting process, can exert a substantial impact on these dynamics [2, 3]. To address this, Arditi and Ginzburg put forward the ratio-dependent model [4], which incorporates the effects of predator interference and is more consistent with reality. In recent decades, the ratio-dependent predator-prey model has garnered significant attention from researchers worldwide. These studies have provided valuable insights into the mechanisms driving population interactions and ecosystem stability [5–7]. In ecological research, scholars commonly employ the Logistic growth model: $\frac{dx}{dt} = x(r - kx)$, primarily proposed by Verhulst [8], to depict the growth mechanisms of prey populations. Nevertheless, this model supposes a linear relationship between population growth and density, which may not fully capture the complexities of real-world ecosystems. To address this limitation, Gilpin and Ayala [9] introduced an improved version of the Logistic model, formulated as: $\frac{dx}{dt} = x(r - kx^{\theta})$, where θ is a scaling factor that introduces nonlinearity into the model. This modification bolsters the model's capacity to mirror realistic population dynamics, thereby providing a more accurate representation of real-world scenarios. While the inclusion of θ increases the analytical complexity, it overcomes the restrictive assumption of the traditional Logistic model, which presumes a constant reduction in growth rate per unit increase in population. The Gilpin-Ayala model has since gained significant attention and is being actively investigated by researchers for its improved accuracy in describing population behavior [10, 11]. Therefore, we incorporate ratio-dependent response and Gilpin-Ayala type growth into the model to enhance its realism.

The predator-prey relationship is a complex dynamical system with profound interacting ecological effects and environmental factors, such as cooperative hunting [12, 13], the fear effect [14, 15], Allee effect [16], additional food supplement [13], predator-taxis or prey-taxis [17], the refuge effect [18], and anti-predation behavior [19]. Among these mechanisms, habitat selection [20, 21] stands out as a common anti-predation strategy, where many animals choose suitable habitats to minimize predation risk. However, habitat selection imposes costs on prey reproductive success. For instance, Werner et al. discovered that when bass are present, bluegills will select habitats not rich in resources but more. Moreover, bluegills reared in the presence of bass attain only approximately 80% of the mass compared to those raised without the bass present [22]. To better understand these dynamics, Ives et al. put forward a model tailored for habitat selection [23], providing a theoretical framework to explore the spatial distribution and interactions between predators and prey:

$$\begin{cases} \dfrac{dx}{dt} = x\left[\lambda\left(1 - \dfrac{x}{k}\right) - v - e^{-\varepsilon v}\dfrac{qy}{1 + ax}\right], \\ \dfrac{dy}{dt} = \dfrac{ce^{-\varepsilon v}qxy}{1 + ax} - my. \end{cases} \tag{1.1}$$

It is evident that habitat selection behavior significantly influences the reproductive capacity of prey populations, leading to decreased fecundity or increased mortality from non-predatory factors: vx, while imposing costs on predators as they adapt to such behavioral strategies: $e^{-\varepsilon v}$. To mitigate predation risk, prey species actively seek out, create, or utilize specific areas as refuges, a phenomenon widely recognized as the refuge effect. This ecological mechanism exerts a vital influence on forming predator-prey dynamics and maintaining ecosystem balance. In ecological modeling, the refuge effect is frequently incorporated to better capture the spatial and behavioral interactions between predators and prey, providing a more realistic representation of their complex relationships. Numerous researchers have examined predator-prey models that incorporate refuge effects [24–26]. Thus, we aim to investigate populations that exhibit habitat selection behavior and refuge effects.

Natural population systems are frequently subject to diverse random perturbations, which can exert significant influences on their dynamics. As May pointed out [27], environmental noise impacts key parameters such as growth rates, carrying capacity, and competition coefficients to varying degrees. To more objectively characterize and predict population ecosystems, it is essential to investigate population models incorporating stochastic disturbances. Environmental noise is commonly categorized into white noise and colored noise. White noise represents minor, random fluctuations in the environment, and its impact on population systems has been thoroughly examined by numerous scholars [28, 29]. In contrast, colored noise encompasses more structured and periodic disturbances, such as seasonal variations, disease outbreaks, rainfall patterns, droughts, cold snaps, and other climatic fluctuations. For instance, the growth rates of certain species can vary dramatically between dry and rainy seasons [30]. Colored noise induces transitions between environmental states within the system [31]. Wang et al. carried out a comprehensive and meticulous investigation into the persistence and extinction dynamics of a stochastic predator-prey model that incorporates the Beddington-DeAngelis functional response within the context of state switching [32]. Their research conclusively demonstrated the existence of a unique stationary distribution. Moreover, the study of predator-prey models with state-switching remains a vibrant research area. Numerous academics, from fields like ecology and mathematics, persistently explore the intricate complexities these models entail. They combine theoretical formulation with numerical analysis and seek to elucidate how state transitions affect predator-prey dynamics and ecological stability, thereby contributing to the advancement of ecological theory and its applications [33].

This paper is structured in the following manner: In Section 2, a single-predator-dual-prey model that accounts for white and colored noise is formulated. During the model-building process, several crucial ecological factors are incorporated, namely the habitat-selection behavior of species, the prey-refuge effect, Gilpin-Ayala type growth, and a ratio-dependent functional response. Moreover, concepts from Markov chain theory are introduced to enhance the model's comprehensiveness and accuracy in representing real-world ecological dynamics. In Section 3, sufficient conditions for stochastic persistence are deduced. These conditions expound on the scenarios in which populations are capable of maintaining themselves within a stochastic environment as time progresses. In Section 4, the existence of a solitary ergodic stationary distribution is ascertained. This discovery significantly contributes to our comprehension of the model's long-term behavior, shedding light on how the system evolves and stabilizes over extended periods. In Section 5, the extinction and persistence conditions of the populations are investigated. By conducting an in-depth analysis, several crucial asymptotic properties are derived in Section 6. These properties play a pivotal role in forecasting the future trends of the population, providing valuable insights into its long-term viability and potential trajectories. In Section 7, numerical simulations are conducted to validate the principal conclusions derived from the foregoing theoretical analysis. The numerical outcomes vividly illustrate the accuracy and efficacy of our theoretical findings. In the concluding section, we systematically summarize the principal results derived from the entire study and expound upon its practical significance.

2. Model formulation and basic knowledge

2.1. Model formulation

Recognizing the classical Logistic model's limitations, especially its unrealistic linear growth assumptions, and factoring in the ecological importance of habitat selection and the refuge effect, we present the following refined model:

$$
\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}t} = x(r_1 - k_1 x^{\theta_1}) - \dfrac{b_1(1-f)xz}{1+a(1-f)x}, \\
\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(r_2 - k_2 y^{\theta_2} - v) - \dfrac{e^{-vm} b_2 yz}{y+\rho z}, \\
\dfrac{\mathrm{d}z}{\mathrm{d}t} = z\left[r_3 + \dfrac{c_1(1-f)x}{1+a(1-f)x} + \dfrac{c_2 e^{-vm} y}{y+\rho z} - d - k_3 z\right],
\end{cases}
\tag{2.1}
$$

where x and y signify the population densities of the two prey species, and z represents the density of the predator. The intrinsic growth rates are given by r_i $(i = 1, 2, 3)$, and the coefficients of intra-specific competition are denoted by k_i $(i = 1, 2, 3)$. The processing time of the predator is a. The scaling factors for the prey species are θ_i, and the maximum predation rates are b_i, $i = 1, 2$. The food conversion rates, defined as $c_i = \epsilon_i b_i$, where $\epsilon_i \in (0, 1)$, reflect the efficiency of energy transfer from prey to predator. The intensity of habitat selection behavior is represented by v, and f denotes the proportion of prey in refuge. The half-saturation constant is ρ, while d signifies the natural mortality rate. Moreover, m determines the efficiency of habitat selection behavior. $\frac{b_1 x}{1+ax}$ and $\frac{b_2 y}{y+\rho z}$ represent the Holling type II functional response and the ratio-dependent functional response, respectively. All the parameters present in the model are constants with positive values.

Given the ubiquity of random environmental disturbances, we incorporate white noise into model (2.1) to account for the impact of environmental fluctuations. White noise refers to ubiquitous minor disturbances in the environment, such as temperature changes, humidity variations, and air flow. Mathematically, it is often described by the formal derivative of Brownian motion. Furthermore, stochastic noise can affect numerous parameters of an ecosystem; in the modeling process, we typically assume that certain key parameters are subject to noise interference. For instance, Liu et al. assumed that white noise mostly affects the growth rate of prey and the mortality rate of predators, expressed as: $r_1 \to r_1 + \sigma_1 \dot{B}_1(t)$, $-r_2 \to -r_2 + \sigma_2 \dot{B}_2(t)$ [34]. Similarly, Zhang et al. assumed that white noise influences the growth rate of prey, the internal competition coefficient, and the mortality rate, given by: $r \to r + \sigma_{11} \dot{B}_1(t)$, $-\frac{r}{K} \to -\frac{r}{K} + \sigma_{12} \dot{B}_1(t)$, $-d \to -d + (\sigma_{21} + \sigma_{22} y)\dot{B}_2(t)$ [35]. Specifically, we postulate that the parameters r_i are subject to the impact of environmental noise. This assumption is made to mirror the actual fluctuations inherent in natural conditions. In other words,

$$
r_i \to r_i + \sigma_i \dot{B}_i(t), \quad i = 1, 2, 3,
$$

where σ_i represent the intensity values of white noise, $B_i(t)$ denote the standard Brownian motions. Therefore, a stochastic version of model (2.1) is developed to incorporate environmental variability, enhancing its ability to simulate realistic predator-prey dynamics under random disturbances.

$$
\begin{cases}
\mathrm{d}x = x\left(r_1 - k_1 x^{\theta_1} - \frac{b_1(1-f)z}{1+a(1-f)x}\right)\mathrm{d}t + \sigma_1 x \mathrm{d}B_1(t), \\
\mathrm{d}y = y\left(r_2 - k_2 y^{\theta_2} - v - \frac{e^{-vm} b_2 z}{y+\rho z}\right)\mathrm{d}t + \sigma_2 y \mathrm{d}B_2(t), \\
\mathrm{d}z = z\left(r_3 + \frac{c_1(1-f)x}{1+a(1-f)x} + \frac{c_2 e^{-vm} y}{y+\rho z} - d - k_3 z\right)\mathrm{d}t + \sigma_3 z \mathrm{d}B_3(t).
\end{cases}
\tag{2.2}
$$

In reality, population systems are also affected by colored noise, which causes system parameters to fluctuate as environmental states change. Specifically, a population dynamical system has multiple states, and it switches between these states over time. These transitions are memoryless, with the duration of each state following an exponential distribution. In most cases, this process is modeled using a continuous-time Markov chain with a finite number of discrete states [36]. For instance, scholars such as Wang and Jiang have used continuous-time Markov chains taking values in a finite state space to characterize the influence of colored noise [32, 33]. To better simulate the impacts of real-world environments, we have developed a stochastic model that incorporates white noise and colored noise. The latter is represented by a Markov switching mechanism, enabling the model to account for random shifts in environmental states and their effects on population dynamics:

$$\begin{cases} \mathrm{d}x = x\Big(r_1(\gamma(t)) - k_1(\gamma(t))x^{\theta_1(\gamma(t))} - \frac{b_1(\gamma(t))(1-f(\gamma(t)))z}{1+a(\gamma(t))(1-f(\gamma(t)))x}\Big)\mathrm{d}t + \sigma_1(\gamma(t))x\mathrm{d}B_1(t), \\ \mathrm{d}y = y\left(r_2(\gamma(t)) - k_2(\gamma(t))y^{\theta_2(\gamma(t))} - v(\gamma(t)) - \dfrac{e^{-v(\gamma(t))m(\gamma(t))}b_2(\gamma(t))z}{y+\rho(\gamma(t))z}\right)\mathrm{d}t + \sigma_2(\gamma(t))y\mathrm{d}B_2(t), \\ \mathrm{d}z = z\left(r_3(\gamma(t)) + \dfrac{c_1(\gamma(t))(1-f(\gamma(t)))x}{1+a(\gamma(t))(1-f(\gamma(t)))x}\right. \\ \qquad \left. + \dfrac{c_2(\gamma(t))e^{-v(\gamma(t))m(\gamma(t))}y}{y+\rho(\gamma(t))z} - d(\gamma(t)) - k_3(\gamma(t))z\right)\mathrm{d}t + \sigma_3(\gamma(t))z\mathrm{d}B_3(t), \end{cases} \tag{2.3}$$

where $\gamma(t)$ represents a right-continuous Markov chain taking values within a finite state space $S = \{1, 2, ..., N\}$, and $B_i(t)$ and $\gamma(t)$ are independent of one another. Model (2.3) can switch between N states, where each parameter in the model is affected and may take different values under different states. The sub-model of model (2.3) corresponding to a specific state is formulated as follows

$$\begin{cases} \mathrm{d}x = x\Big(r_1(i) - k_1(i)x^{\theta_1(i)} - \frac{b_1(i)(1-f(i))z}{1+a(i)(1-f(i))x}\Big)\mathrm{d}t + \sigma_1(i)x\mathrm{d}B_1(t), \\ \mathrm{d}y = y\Big(r_2(i) - k_2(i)y^{\theta_2(i)} - v(i) - \frac{e^{-v(i)m(i)}b_2(i)z}{y+\rho(i)z}\Big)\mathrm{d}t + \sigma_2(i)y\mathrm{d}B_2(t), \\ \mathrm{d}z = z\Big(r_3(i) + \frac{c_1(i)(1-f(i))x}{1+a(i)(1-f(i))x} + \frac{c_2(i)e^{-v(i)m(i)}y}{y+\rho(i)z} - d(i) - k_3(i)z\Big)\mathrm{d}t + \sigma_3(i)z\mathrm{d}B_3(t). \end{cases} \tag{2.4}$$

Remark 1. *White noise and colored noise are ubiquitous in ecosystems, exerting impacts on numerous ecological dynamical systems. Through the study of this model, we aim to examine the influence of the Markov switching mechanism on the survival status of populations in model* (2.3)*, as well as its connection with the scenario where only white noise interference is present in the sub-states.*

2.2. Basic knowledge

Consider a complete probability space equipped with a filtration $\{F_t\}_{t\geq 0}$ that satisfies the standard conditions: F_0 contains all P-zero subsets and the filtration is right-continuous. The elements of the resulting matrix $\mathbf{Q} = (q_{ij})_{N\times N}$ satisfy

$$q_{ij} = \begin{cases} q_{ij}\Delta t + o(\Delta t), j \neq i \\ 1 + q_{ii}\Delta t + o(\Delta t), j = i \end{cases}$$

where $\Delta t > 0$. $q_{ij} \geq 0 (i \neq j)$ represents the transfer rate from state i to state j with $\sum_{j=1}^{N} q_{ij} = 0$. It is customarily assumed that $q_{ij} > 0 (i \neq j)$ and Markov chains are irreducible, with irreducibility signifying that Markov chains can move from any state to any other state, which implies that there is a transition path with positive probability between any two states. It is ergodic and admits a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in \mathbb{R}^{1\times \mathrm{N}}$. This stationary distribution satisfies $\pi\mathbf{Q} = 0$ along with $\sum\limits_{i=1}^{N} \pi_i = 1, \pi_i > 0$ for all $i \in S$.

Consider a random differential equation with a Markov chain for $X_0 \in \mathbb{R}^n, t \geq 0$:

$$\mathrm{d}X(t) = \mathbf{F}(X(t), t, \gamma(t))\mathrm{d}t + \mathbf{G}(X(t), t, \gamma(t))\mathrm{d}B(t),$$

where $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$, $\mathbf{G} : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}$. The space $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S, \mathbb{R}_+)$ consists of non-negative functions $V(X(t), t, i)$ that are twice continuously differentiable with respect to X and once continuously differentiable with respect to t. For a function $V \in C^{2,1}$, the operator $\mathcal{L}V$ is defined as follows:

$$\mathcal{L}V = V_t + V_X\mathbf{F} + \frac{1}{2}trace[\mathbf{G}^T V_{XX}\mathbf{G}] + \sum_{j=1}^{N} q_{ij}V,$$

where

$$\sum_{j=1}^{N} q_{ij}V = \sum_{j\neq i\in S} q_{ij}(V(X, t, j) - V(X, t, i)).$$

3. Stochastic persistence

In this study, we adopt the following notations:

$$G(i) = \min\left\{r_1(i) - k_1(i), r_2(i) - k_2(i) - v(i), r_3(i) - d(i) - \frac{b_1(i)}{a(i)} - e^{-v(i)m(i)}b_2(i)\right\},$$

$$R(i) = \max\left\{\sigma_1^2(i), \sigma_2^2(i), \sigma_3^2(i)\right\}, \ \ D(i) = \max\left\{k_1(i), k_2(i), k_3(i)\right\}, \ \ \beta_i = G(i) - \frac{1}{2}R(i),$$

$$B = \sum_{i=1}^{N} \pi_i\beta_i, \ \ |X(t)| = \sqrt{x^2 + y^2 + z^2}, \ \ \hat{f} = \min_{i\in S} f(\cdot), \ \ \check{f} = \max_{i\in S} f(\cdot).$$

Theorem 1. *For any initial values* $(x_0, y_0, z_0) \in \mathbb{R}^3_+$, *model* (2.3) *admits a unique global positive solution* $(x(t), y(t), z(t))$ *that stays in* $\mathbb{R}^3_+$ *almost surely.*

Proof. We adopt the same method as in Theorem 3 of [37]. The functions on the right-hand side of model (2.3) satisfy the local Lipschitz condition. Thus, for a given initial value $(x_0, y_0, z_0) \in \mathbb{R}^3_+$, there is a unique local solution on $[0, \tau_e)$, where τ_e denotes the explosion time. To prove that this solution is global, we need only show $\tau_e = \infty$. Let n_0 be a sufficiently large positive constant such that x_0, y_0, z_0 all lie in $\left[\frac{1}{n_0}, n_0\right]$. For $n \geq n_0$, define

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\left\{x(t), y(t), z(t)\right\} \leq \frac{1}{n} \text{ or } \max\left\{x(t), y(t), z(t)\right\} \geq n\right\}.$$

Define $\inf \emptyset = \infty$. Thus, τ_n is increasing as $n \to \infty$. Define $\tau_\infty = \lim_{n\to\infty} \tau_n$, so $\tau_\infty \leq \tau_e$. If we can prove $\tau_\infty = \infty$, then for any $t \geq 0$, we have $\tau_e = \infty$.

Suppose that $\tau_\infty = \infty$ does not hold. Then, there are constants $T_\infty > 0$ and $\varepsilon_\infty \in (0, 1)$, satisfying $P\left\{\tau_\infty \leq T_\infty\right\} > \varepsilon_\infty$. Thus, there exists an integer $n_\infty \geq n_0$, satisfying

$$P\left\{\tau_n \leq T_\infty\right\} > \varepsilon_\infty, \quad n \geq n_\infty.$$

Defining

$$V(x,y,z) = (x-1-\ln x) + (y-1-\ln y) + (z-1-\ln z).$$

$V(x,y,z)$ is nonnegative and

$$\begin{aligned}
\mathcal{L}V(x,y,z,i) =&(x-1)\left(r_1(i) - k_1(i)x^{\theta_1(i)} - \frac{b_1(i)(1-f(i))z}{1+a(i)(1-f(i))x}\right) + \frac{1}{2}\sigma_1^2(i)\\
&+ (y-1)\left(r_2(i) - v(i) - k_2(i)y^{\theta_2(i)} - \frac{e^{-v(i)m(i)}b_2(i)z}{y+\rho(i)z}\right) + \frac{1}{2}\sigma_2^2(i)\\
&+ (z-1)\left(r_3(i) + \frac{c_1(i)(1-f(i))x}{1+a(i)(1-f(i))x} + \frac{e^{-v(i)m(i)}c_2(i)y}{y+\rho(i)z} - d(i) - k_3(i)z\right) + \frac{1}{2}\sigma_3^2(i)\\
\leq& r_1(i)x - k_1(i)x^{1+\theta_1(i)} + k_1(i)x^{\theta_1(i)} + r_2(i)y - v(i)y - k_2(i)y^{1+\theta_2(i)} + k_2(i)y^{\theta_2(i)}\\
&+ r_3(i)z - d(i)z + k_3(i)z + b_1(i)(1-f(i))z - k_3(i)z^2\\
&- r_1(i) - r_2(i) - r_3(i) + v(i) + \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} + d(i) + \frac{1}{2}\sigma_1^2(i) + \frac{1}{2}\sigma_2^2(i) + \frac{1}{2}\sigma_3^2(i)\\
\leq& \tilde{K},
\end{aligned}$$

where $\tilde{K}$ is a positive constant. Thus, we can deduce that

$$\mathrm{d}V \leq \tilde{K}\mathrm{d}t + (x-1)\sigma_1(i)\mathrm{d}B_1(t) + (y-1)\sigma_2(i)\mathrm{d}B_2(t) + (z-1)\sigma_3(i)\mathrm{d}B_3(t).$$

Moreover, integrating over the interval $(0, \tau_n \wedge T_\infty)$ and taking expectations, we have

$$\mathbb{E}V\left(x(\tau_n \wedge T_\infty), y(\tau_n \wedge T_\infty), z(\tau_n \wedge T_\infty)\right) \leq V(x_0,y_0,z_0,i_0) + \tilde{K}\mathbb{E}(\tau_n \wedge T_\infty) \leq V(x_0,y_0,z_0,i_0) + \tilde{K}T_\infty.$$

Let $\Omega_n = \{\omega \in \Omega \mid \tau_n = \tau_n(\omega) \leq T_\infty\}$, then $P(\Omega_n) > \varepsilon_\infty$. Thus, for each $\omega \in \Omega_n$, at least one of $x(\tau_n,\omega)$, $y(\tau_n,\omega)$, or $z(\tau_n,\omega)$ equals n or $\frac{1}{n}$, so

$$V(x_0,y_0,z_0,i_0) + \tilde{K}T_\infty \geq \varepsilon_\infty \min\left\{n - 1 - \ln n,\ \frac{1}{n} - 1 - \ln\frac{1}{n}\right\}.$$

Let $n \to \infty$, then

$$\infty > V(x_0,y_0,z_0,i_0) + \tilde{K}T_\infty = \infty,$$

which is a contradiction. Therefore, the assumption is false, and $\tau_\infty = \infty$. □

Definition 1. *(Boundedness and stochastic persistence* [38]*) The solution of model* (2.3) *is said to exhibit stochastic ultimate boundedness, if* $\forall\varepsilon \in (0,1)$ *and* $\exists\iota = \iota(\varepsilon) > 0$*, such that*

$$\limsup_{t\to+\infty} \mathbb{P}\{|X(t)| > \iota\} < \varepsilon.$$

The model is said to be stochastic persistent if $\exists\delta = \delta(\varepsilon) > 0$ *such that*

$$\liminf_{t\to+\infty} \mathbb{P}\{|X(t)| \leq \iota\} \geq 1-\varepsilon,\quad \liminf_{t\to+\infty} \mathbb{P}\{|X(t)| \geq \delta\} \geq 1-\varepsilon.$$

Suppose $Z^{n\times n} = \{A = (a_{ij})_{n\times n} : a_{ij} < 0, i \neq j\}$. If the sum of every row element of $A = (a_{ij}) \in Z^{n\times n}$ is positive, that is $\sum_{j=1}^{n} a_{ij} > 0$ for all $1 \leq i \leq n$, then $\det(A) > 0$.

Lemma 1. [39] *For $A = (a_{ij}) \in Z^{n\times n}$, the following statements are mutually equivalent: i) A is a non-singular M-matrix; ii) all sequential principal subexpressions of A are positive; and iii) A is semi-positive, meaning $A\vec{p} \gg 0$ for some vector $\vec{p} \gg 0$ in $\mathbb{R}^n$.*

Lemma 2. *For any $(x_0, y_0, z_0, r_0) \in \mathbb{R}^3_+ \times S$, $\flat > 0$, there is a constant $\ell(\flat) > 0$, such that for any solution of model (2.3), $\limsup_{t\to+\infty} \mathbb{E}(x^\flat + y^\flat + z^\flat) \le \ell(\flat)$ holds.*

Proof. Define $V(x, y, z, i) = x^\flat + y^\flat + z^\flat$. Employing the Itô formula, it has

$$\begin{aligned}\mathcal{L}V(x, y, z, i) + V(x, y, z, i) \le& x^\flat\left(1 + \flat\left(r_1(i) - k_1(i)x^{\theta_1(i)}\right) + \frac{1}{2}\flat(\flat - 1)\sigma_1^2(i)\right)\\ &+ y^\flat\left(1 + \flat\left(r_2(i) - k_2(i)y^{\theta_2(i)} - \nu(i)\right) + \frac{1}{2}\flat(\flat - 1)\sigma_2^2(i)\right)\\ &+ z^\flat\left(1 + \flat\left(r_3(i) + \frac{c_1(i)}{a(i)} + e^{-\nu(i)m(i)}c_2(i) - d(i) - k_3(i)z\right) + \frac{1}{2}\flat(\flat - 1)\sigma_3^2(i)\right)\\ \le& \ell(\flat),\end{aligned}$$

then we have

$$\mathcal{L}(e^t V) = e^t\mathcal{L}V + e^t V \le e^t\ell(\flat).$$

By integrating over interval $[0, t]$ and subsequently finding the expectation, we deduce the following result:

$$\limsup_{t\to+\infty} \mathbb{E}(V(x, y, z, i)) \le \ell(\flat).$$

□

Theorem 2. *The solutions of model (2.3) possess the property of stochastic ultimate boundedness.*

Proof. It is known that $|X(t)|^\flat \le 3^{\frac{\flat}{2}} \max\{x^\flat, y^\flat, z^\flat\} \le 3^{\frac{\flat}{2}} V(x, y, z, i)$, so

$$\limsup_{t\to+\infty} \mathbb{E}(|X(t)|^\flat) \le 3^{\frac{\flat}{2}} \limsup_{t\to+\infty} \mathbb{E}(V(x, y, z, i)) \le 3^{\frac{\flat}{2}}\ell(\flat) = \kappa(\flat).$$

For any $\varepsilon > 0$, we set $\iota = \left(\frac{\kappa(\flat)}{\varepsilon}\right)^{\frac{1}{\flat}}$. Utilizing Chebyshev's inequality [39], there is

$$\mathbb{P}\{|X(t)| > \iota\} \le \frac{\mathbb{E}|X(t)|^\flat}{\iota^\flat},$$

and then

$$\limsup_{t\to+\infty} \mathbb{P}\{|X(t)| > \iota\} \le \varepsilon.$$

□

Although population sizes fluctuate under the influence of stochastic factors, these populations will not grow indefinitely in the long run, reflecting the self-regulating capacity of natural ecosystems.

Lemma 3. *There exists a constant* $\theta > 0$ *such that the matrix*

$$\widetilde{A}(\theta) = \mathrm{diag}(\xi_1(\theta), \xi_2(\theta), \cdots, \xi_N(\theta)) - \mathbf{Q}$$

is a non-singular M-matrix when $B > 0$*, where*

$$\xi_i(\theta) = \theta\beta_i - \frac{1}{2}\theta^2 R(i), \quad i \in S.$$

The proof of this lemma follows a methodology analogous to that employed in Lemma 3.4 of [38], utilizing similar techniques and reasoning. Furthermore, $\widetilde{A}(\theta)$ is a non-singular M-matrix when $\beta_i > 0$, for all $i \in S$.

Theorem 3. *The model* (2.3) *is stochastic persistent when* $B > 0$ *and* $0 < \theta_j(i) \leq 1$, $j = 1, 2$.

Proof. By Lemmas 1 and 3, there is $\vec{\eta} = (\eta_1, \eta_2, \cdots \eta_N)^T \gg 0$ satisfying $\widetilde{A}(\theta)\vec{\eta} \gg 0$. Thus,

$$\xi_i(\theta)\eta_i - \sum_{j=1}^{N} q_{ij}\eta_j > 0, \quad i \in S,$$

which means that there is a sufficiently small $l > 0$, satisfying

$$\xi_i(\theta)\eta_i - \sum_{j=1}^{N} q_{ij}\eta_j - l\eta_i > 0, \quad i \in S.$$

Let $\tilde{w} = x + y + z$ and $\mathrm{u} = \frac{1}{\tilde{w}}$. Then we have

$$\mathcal{L}\left[e^{lt}\eta_i(1+\mathrm{u})^{\theta}\right] = e^{lt}\left[l\eta_i(1+\mathrm{u})^{\theta} + \mathcal{L}(\eta_i(1+\mathrm{u})^{\theta})\right]$$

with

$$\begin{aligned}
\mathcal{L}(\eta_i(1+\mathrm{u})^{\theta}) \leq & (1+\mathrm{u})^{\theta-2}\Big\{\eta_i\theta\Big[-(\mathrm{u}^2+\mathrm{u}^3)\Big((r_1(i)-k_1(i))x - k_1(i)x^2 + (r_2(i)-k_2(i)-v(i))y \\
& - k_2(i)y^2 + \left(r_3(i) - d(i) - \frac{b_1(i)}{a(i)} - e^{-v(i)m(i)}b_2(i)\right)z \\
& - k_3(i)z^2\Big) + \left(\mathrm{u} + \frac{\theta+1}{2}\mathrm{u}^2\right)R(i)\Big] + (1+\mathrm{u}^2+2\mathrm{u})\sum_{j=1}^{N} q_{ij}\eta_j\Big\} \\
\leq & (1+\mathrm{u})^{\theta-2}\Big\{\eta_i\theta\Big[-(\mathrm{u}^2+\mathrm{u}^3)\Big(G(i)(x+y+z) - D(i)\left(x^2+y^2+z^2\right)\Big) \\
& + \left(\mathrm{u} + \frac{\theta+1}{2}\mathrm{u}^2\right)R(i)\Big] + (1+\mathrm{u}^2+2\mathrm{u})\sum_{j=1}^{N} q_{ij}\eta_j\Big\} \\
\leq & (1+\mathrm{u})^{\theta-2}\Big\{\eta_i\theta\Big[-(\mathrm{u}+\mathrm{u}^2)G(i) + D(i)(1+\mathrm{u}) \\
& + \left(\mathrm{u} + \frac{\theta+1}{2}\mathrm{u}^2\right)R(i)\Big] + (1+\mathrm{u}^2+2\mathrm{u})\sum_{j=1}^{N} q_{ij}\eta_j\Big\} \\
= & (1+\mathrm{u})^{\theta-2}\Bigg\{-\left(\eta_i\xi_i(\theta) - \sum_{j=1}^{N} q_{ij}\eta_j\right)\mathrm{u}^2 \\
& + \left(\eta_i\theta\left(D(i) - G(i) + R(i)\right) + 2\sum_{j=1}^{N} q_{ij}\eta_j\right)\mathrm{u} + \eta_i\theta D(i) + \sum_{j=1}^{N} q_{ij}\eta_j\Bigg\}.
\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}\left[e^{lt}\eta_i(1+\mathrm{u})^{\theta}\right] \leq & e^{lt}(1+\mathrm{u})^{\theta-2}\left\{-\left(\eta_i\xi_i(\theta)-\sum_{j=1}^{N}q_{ij}\eta_j-l\eta_i\right)\mathrm{u}^2\right.\\ &\left.+\left(\eta_i\theta\left(D(i)-G(i)+R(i)\right)+2\sum_{j=1}^{N}q_{ij}\eta_j+2l\eta_i\right)\mathrm{u}+\eta_i\theta D(i)+\sum_{j=1}^{N}q_{ij}\eta_j+l\eta_i\right\}\\ \leq & Me^{lt},\end{aligned}$$

where

$$\begin{aligned}M=\max&\left\{\sup\left\{(1+\mathrm{u})^{\theta-2}\left[-\left(\eta_i\xi_i(\theta)-\sum_{j=1}^{N}q_{ij}\eta_j-l\eta_i\right)\mathrm{u}^2\right.\right.\right.\\ &\left.\left.\left.+\left(\eta_i\theta(D(i)-G(i)+R(i))+2\sum_{j=1}^{N}q_{ij}\eta_j+2l\eta_i\right)\mathrm{u}+\eta_i\theta D(i)+\sum_{j=1}^{N}q_{ij}\eta_j+l\eta_i\right]\right\},1\right\}.\end{aligned}$$

We can obtain

$$\mathbb{E}\left(e^{lt}\eta_i(1+\mathrm{u})^{\theta}\right)=\eta_i(1+\mathrm{u}(x_0,y_0,z_0,r_0))^{\theta}+\mathbb{E}\int_0^t\mathcal{L}\left(e^{ls}\eta_i(1+\mathrm{u})^{\theta}\right)\mathrm{d}s,$$

and

$$\mathbb{E}\left(e^{lt}\eta_i(1+\mathrm{u})^{\theta}\right)\leq\eta_i(1+\mathrm{u}(x_0,y_0,z_0,r_0))^{\theta}+\frac{M}{l}(e^{lt}-1),$$

thus

$$\limsup_{t\to+\infty}\mathbb{E}\left(\eta_i(1+\mathrm{u})^{\theta}\right)\leq\frac{M}{l},$$

so

$$\limsup_{t\to+\infty}\mathbb{E}\left(\frac{1}{|X(t)|^{\theta}}\right)\leq 3^{\frac{\theta}{2}}\limsup_{t\to+\infty}\mathbb{E}(\mathrm{u}^{\theta})\leq 3^{\frac{\theta}{2}}\limsup_{t\to+\infty}\mathbb{E}\left((1+\mathrm{u})^{\theta}\right)\leq 3^{\frac{\theta}{2}}\frac{M}{l\hat{\eta}}=H.$$

For any $\varepsilon>0$, set $\delta=(\frac{\varepsilon}{H})^{\frac{1}{\theta}}$, there is

$$\mathbb{P}(|X(t)|<\delta)=\mathbb{P}\left(\frac{1}{|X(t)|}>\frac{1}{\delta}\right)\leq\delta^{\theta}\mathbb{E}\left(\frac{1}{|X(t)|^{\theta}}\right)\leq\varepsilon,$$

and

$$\limsup_{t\to+\infty}\mathbb{P}\{|X(t)|<\delta\}\leq\varepsilon\Rightarrow\liminf_{t\to+\infty}\mathbb{P}\{|X(t)|\geq\delta\}\geq 1-\varepsilon.$$

Given that the model is stochastically ultimately bounded, it follows that

$$\liminf_{t\to+\infty}\mathbb{P}\{|X(t)|\leq\iota\}\geq 1-\varepsilon.$$

□

So far, we derive a sufficient condition for the stochastic persistence of model (2.3), which reflects the resilience of the ecosystem under stochastic perturbations. Even in the presence of random disturbances, the system can avoid the risk of extinction and sustain the possibility of long-term survival by regulating parameters such as its own reproductive capacity and interspecific interactions to satisfy this condition.

Corollary 1. *Model* (2.3) *is stochastic persistent when* $\beta_i > 0$ *for all* $i \in S$ *and* $0 < \theta_j(i) \le 1$, $j = 1, 2$.

Corollary 2. *For some state i, the sub-model* (2.4) *is stochastic persistent when* $\beta_i > 0$ *and* $0 < \theta_j(i) \le 1$, $j = 1, 2$.

Sub-model (2.4) is stochastic persistent for some state i, $\beta_i > 0$, $0 < \theta_j(i) \le 1$, $j = 1, 2$, and model (2.3) is stochastic persistent when every state is stochastic persistent. However, Theorem 3 reveals an interesting fact: When individual states in model (2.3) are extinct, the overall behavior of the model may exhibit stochastic persistence due to Markov switching. The persistence displayed by model (2.3) results from the combined superposition of multiple sub-states. For certain sub-states within the state space, the model may show a tendency toward extinction; however, as long as the condition in Theorem 3 is satisfied, the extinction of these sub-states will not affect the eventual persistence of model (2.3).

4. Stationary distribution

Lemma 4. [39] *The solution of model* (2.3) *exhibits normality and ergodicity. Moreover, a unique ergodic stationary distribution exists, provided that the following conditions are satisfied:*

(i) $q_{ij} > 0, i \ne j$;

(ii) $\widetilde{\mathbf{G}}(X,k) = \mathbf{G}(X,k)\mathbf{G}^T(X,k)$ *is a symmetric matrix for any* $k \in S$ *and*

$$\lambda|\varsigma|^2 \le \varsigma^T \widetilde{\mathbf{G}}(X,k)\varsigma \le \lambda^{-1}|\varsigma|^2, \quad \lambda \in (0,1], \quad \varsigma \in \mathbb{R}^n, \quad X \in \mathbb{R}^n.$$

(iii) There is a nonempty open set $\mathcal{D}$ *containing closed packets, with a nonnegative function* $V(\cdot, k) : \mathcal{D}^c \to \mathbb{R}$ *for* $k \in S$ *such that* $V(\cdot, k) \in C^2$, *and* $\alpha > 0$ *such that*

$$\mathcal{L}V(X,k) \le -\alpha, \quad (X,k) \in \mathcal{D}^c \times S.$$

Define $\mu = \sum_{i \in S} \pi_i \varphi_i$, where

$$\begin{aligned}\varphi_i =& r_1(i) + r_2(i) + r_3(i) - k_1(i) - k_2(i) - v(i) - d(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} \\ &- \frac{1}{2}\sigma_1^2(i) - \frac{1}{2}\sigma_2^2(i) - \frac{1}{2}\sigma_3^2(i) - \frac{(k_3(i) + r_3(i) + b_1(i)(1 - f(i)) - d(i))^2}{4k_3(i)}.\end{aligned}$$

Theorem 4. *For* $\forall (x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$, *model* (2.3) *admits a unique stationary distribution, which is ergodic when* $\mu > 0$, $\sum_{i \in S} \pi_i \left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i) \right) > 0$ *and* $\sum_{i \in S} \pi_i \left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i) \right) > 0$.

Proof. Define a bounded compact subset $U = [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1/\varepsilon]$, where $\varepsilon > 0$ is sufficiently small. We know the diffusion matrix $\widetilde{\mathbf{G}}(x, y, z, i)$ is positive definite,

$$\widetilde{\mathbf{G}}(x,y,z,i) = \begin{pmatrix} \sigma_1^2(i)x^2 & 0 & 0 \\ 0 & \sigma_2^2(i)y^2 & 0 \\ 0 & 0 & \sigma_3^2(i)z^2 \end{pmatrix}.$$

Therefore,

$$\lambda_{\max}(\widetilde{\mathbf{G}}) \geq \lambda_{\min}(\widetilde{\mathbf{G}}) > 0, \tag{4.1}$$

$\lambda_{\max}(\widetilde{\mathbf{G}})$ and $\lambda_{\min}(\widetilde{\mathbf{G}})$ are the largest and smallest eigenvalues of the matrix $\widetilde{\mathbf{G}}$, respectively. Thus,

$$\lambda_{\min}(\widetilde{\mathbf{G}})|\varsigma|^2 \leq \varsigma^T(\widetilde{\mathbf{G}})\varsigma \leq \lambda_{\max}(\widetilde{\mathbf{G}})|\varsigma|^2, \quad \forall \varsigma \in R^3. \tag{4.2}$$

Thus, $\lambda_{\max}(\widetilde{\mathbf{G}})$ and $\lambda_{\min}(\widetilde{\mathbf{G}})$ are continuous functions, and formula (4.1) can be derived as

$$\hat{\lambda} = \min_{(x,y,z,i)\in U^c\times S} \lambda_{\min}(\widetilde{\mathbf{G}}) > 0, \quad \check{\lambda} = \max_{(x,y,z,i)\in U^c\times S} \lambda_{\max}(\widetilde{\mathbf{G}}) > 0.$$

It can be obtained from formula (4.2) that

$$\lambda|\varsigma|^2 \leq \varsigma^T(\widetilde{\mathbf{G}})\varsigma \leq \lambda^{-1}|\varsigma|^2,$$

where $\lambda = \min\left\{\hat{\lambda}, (\check{\lambda})^{-1}, 1\right\}$, condition (ii) in the Lemma 4 is proved, and condition (i) is clear. Now, we prove condition (iii).

Let

$$\begin{aligned} h(x,y,z,i) =&(1-\upsilon\phi_i)y^{-\upsilon} + (1-\upsilon\psi_i)z^{-\upsilon} + x + y + z \\ &- \bar{M}(\ln x - x + \ln y - y + \ln z - z), \end{aligned}$$

where υ is a small enough normal number, $\upsilon < \min\{1/\phi_i, 1/\psi_i, \theta_1(i), \theta_2(i), 1\}$, $\bar{M}$ is a sufficiently large number, and $-\bar{M}\mu + \bar{N} \leq -2$, $\bar{N} = \sup\{-\hat{d}_3 y^{-\upsilon} + \check{d}_4 y^{\theta_2(i)-\upsilon} + \check{d}_5 y - \hat{d}_6 y^{\theta_2(i)+1} - \hat{d}_7 z^{-\upsilon} + \check{d}_8 z^{1-\upsilon} + \check{d}_9 z - \hat{d}_{10} z^2\}$. Since $h(x,y,z,i)$ is continuous, denote $h(\bar{x},\bar{y},\bar{z},i)$ in $\mathbb{R}^3_+ \times S$ as the minimum point, and define the Lyapunov function $W : \mathbb{R}^3_+ \times S \to \mathbb{R}_+$ by

$$W(x,y,z,i) = h(x,y,z,i) - h(\bar{x},\bar{y},\bar{z},i) + \bar{M}(|w| + w_i),$$

and $V_1(x,y,z,i) = (1-\upsilon\phi_i)y^{-\upsilon} + (1-\upsilon\psi_i)z^{-\upsilon}$, $V_2(x,y,z) = x+y+z$, $V_3(x,y,z) = -\bar{M}(\ln x - x + \ln y - y + \ln z - z)$, $V_4(x,y,z) = \bar{M}(|w| + w_i)$, where $w = (w_1, w_2, \cdots, w_N)$, $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, $\psi = (\psi_1, \psi_2, \cdots, \psi_N)$ and $|w| = (w_1^2 + w_2^2 + \cdots + w_N^2)^{\frac{1}{2}}$, which will be given in subsequent proofs.

Utilizing the Itô formula, it has

$$\begin{aligned} \mathcal{L}V_1(x,y,z,i) \leq& \upsilon(1-\upsilon\phi_i)y^{-\upsilon}\Bigg[-\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right) \\ &- \sum_{k\neq i} q_{ik}(\phi_k - \phi_i) + \frac{1}{2}\upsilon\sigma_2^2(i) - \frac{\upsilon\phi_i}{1-\upsilon\phi_i}\sum_{k\neq i} q_{ik}(\phi_k - \phi_i)\Bigg] \\ &+ \upsilon(1-\upsilon\psi_i)z^{-\upsilon}\Bigg[-\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right) \\ &- \sum_{k\neq i} q_{ik}(\psi_k - \psi_i) + \frac{1}{2}\upsilon\sigma_3^2(i) - \frac{\upsilon\psi_i}{1-\upsilon\psi_i}\sum_{k\neq i} q_{ik}(\psi_k - \psi_i)\Bigg] \\ &+ \upsilon(1-\upsilon\phi_i)k_2(i)y^{\theta_2(i)-\upsilon} + \upsilon(1-\upsilon\psi_i)k_3(i)z^{1-\upsilon}, \end{aligned}$$

Define $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_N)^T$, $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N)^T$, where $\vartheta_i = -\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right)$, $\zeta_i = -\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right)$. Since the generating matrix $\mathbf{Q}$ is irreducible, vectors ϕ and ψ are the solutions of the following Poisson systems for ϑ and ζ:

$$\mathbf{Q}\phi - \vartheta = -(\pi\vartheta)e, \quad \mathbf{Q}\psi - \zeta = -(\pi\zeta)e,$$

where $e = (1, 1, \cdots, 1)^T$. Thus,

$$-\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right) - \sum_{k\neq i} q_{ik}(\phi_k - \phi_i)$$
$$= -\sum_{i\in S}\pi_i\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right),$$

$$-\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right) - \sum_{k\neq i} q_{ik}(\psi_k - \psi_i) = -\sum_{i\in S}\pi_i\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right).$$

Therefore,

$$\begin{aligned}\mathcal{L}V_1(x,y,z,i) \leq & \upsilon(1-\upsilon\phi_i)y^{-\upsilon}\left[-\sum_{i\in S}\pi_i\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right)\right.\\ & \left.+\frac{1}{2}\upsilon\sigma_2^2(i) - \frac{\upsilon\phi_i}{1-\upsilon\phi_i}\sum_{k\neq i} q_{ik}(\phi_k - \phi_i)\right]\\ & + \upsilon(1-\upsilon\psi_i)z^{-\upsilon}\left[-\sum_{i\in S}\pi_i\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right)\right.\\ & \left.+\frac{1}{2}\upsilon\sigma_3^2(i) - \frac{\upsilon\psi_i}{1-\upsilon\psi_i}\sum_{k\neq i} q_{ik}(\psi_k - \psi_i)\right]\\ & + \upsilon(1-\upsilon\phi_i)k_2(i)y^{\theta_2(i)-\upsilon} + \upsilon(1-\upsilon\psi_i)k_3(i)z^{1-\upsilon}.\end{aligned}$$

By the Itô formula, it has

$$\mathcal{L}V_2(x,y,z,i) \leq r_1(i)x - k_1(i)x^{\theta_1(i)+1} + r_2(i)y - k_2(i)y^{\theta_2(i)+1} - v(i)y + r_3(i)z - d(i)z - k_3(i)z^2,$$

$$\begin{aligned}\mathcal{L}V_3(x,y,z,i) \leq & \bar{M}\left(-r_1(i) + k_1(i) + b_1(i)(1-f(i))z + \frac{1}{2}\sigma_2^2(i) + r_1(i)x\right.\\ & - r_2(i) + k_2(i) + v(i) + \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} + \frac{1}{2}\sigma_2^2(i) + r_2(i)y - v(i)y\\ & \left.-r_3(i) + d(i) + k_3(i)z + \frac{1}{2}\sigma_3^2(i) + r_3(i)z - d(i)z - k_3(i)z^2\right)\\ \leq & \bar{M}\left(r_1(i)x + r_2(i)y - v(i)y\right) - \bar{M}\left(r_1(i) + r_2(i) + r_3(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)}\right.\\ & - k_1(i) - k_2(i) - v(i) - d(i) - \frac{1}{2}\left(\sigma_1^2(i) + \sigma_2^2(i) + \sigma_3^2(i)\right)\\ & \left.-\frac{(k_3(i) + r_3(i) + b_1(i)(1-f(i)) - d(i))^2}{4k_3(i)}\right)\\ = & \bar{M}(r_1(i)x + r_2(i)y - v(i)y) - \bar{M}\varphi_i,\end{aligned}$$

and

$$\mathcal{L}V_4(i) = \bar{M}\sum_{k\neq i} q_{ik}(w_k - w_i).$$

Since $\mathbf{Q}$ is irreducible, there is a vector $w = (w_1, w_2, \cdots, w_N)$, satisfying the Poisson system:

$$\mathbf{Q}w - \varphi = -(\pi\varphi)e,$$

so

$$-\varphi_i + \sum_{k\neq i} q_{ik}(w_k - w_i) = -\sum_{i\in S}\pi_i\varphi_i = -\mu.$$

Therefore,

$$\begin{aligned}
\mathcal{L}W(x,y,z,i) \leq& -\bar{M}\mu + r_1(i)x - k_1(i)x^{\theta_1(i)+1} + \bar{M}r_1(i)x \\
&+ \upsilon(1-\upsilon\phi_i)y^{-\upsilon}\Bigg[-\sum_{i\in S}\pi_i\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right) \\
&+\frac{1}{2}\upsilon\sigma_2^2(i) - \frac{\upsilon\phi_i}{1-\upsilon\phi_i}\sum_{k\neq i} q_{ik}(\phi_k - \phi_i)\Bigg] + \upsilon(1-\upsilon\phi_i)k_2(i)y^{\theta_2(i)-\upsilon} \\
&+ r_2(i)y - k_2(i)y^{\theta_2(i)+1} - v(i)y + \bar{M}(r_2(i) - v(i))y \\
&+ \upsilon(1-\upsilon\psi_i)z^{-\upsilon}\Bigg[-\sum_{i\in S}\pi_i\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right) \\
&+\frac{1}{2}\upsilon\sigma_3^2(i) - \frac{\upsilon\psi_i}{1-\upsilon\psi_i}\sum_{k\neq i} q_{ik}(\psi_k - \psi_i)\Bigg] \\
&+ \upsilon(1-\upsilon\psi_i)k_3(i)z^{1-\upsilon} + r_3(i)z - d(i)z - k_3(i)z^2 \\
=& -\bar{M}\mu + d_1x - d_2x^{\theta_1(i)+1} - d_3y^{-\upsilon} + d_4y^{\theta_2(i)-\upsilon} + d_5y - d_6y^{\theta_2(i)+1} \\
&- d_7z^{-\upsilon} + d_8z^{1-\upsilon} + d_9z - d_{10}z^2,
\end{aligned}$$

where

$$d_1 = (\bar{M}+1)r_1(i), d_2 = k_1(i),$$

$$d_3 = \upsilon(1-\upsilon\phi_i)\left[\sum_{i\in S}\pi_i\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right) - \frac{1}{2}\upsilon\sigma_2^2(i) + \frac{\upsilon\phi_i}{1-\upsilon\phi_i}\sum_{k\neq i} q_{ik}(\phi_k - \phi_i)\right],$$

$$d_4 = \upsilon(1-\upsilon\phi_i)k_2(i), d_5 = (\bar{M}+1)(r_2(i) - v(i)), d_6 = k_2(i),$$

$$d_7 = \upsilon(1-\upsilon\psi_i)\left[\sum_{i\in S}\pi_i\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right) - \frac{1}{2}\upsilon\sigma_3^2(i) + \frac{\upsilon\psi_i}{1-\upsilon\psi_i}\sum_{k\neq i} q_{ik}(\psi_k - \psi_i)\right],$$

$$d_8 = \upsilon(1-\upsilon\psi_i)k_3(i), d_9 = r_3(i) - d(i), d_{10} = k_3(i).$$

We choose υ to be sufficiently small, so that

$$\sum_{i\in S}\pi_i\left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)}b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right) - \frac{1}{2}\upsilon\sigma_2^2(i) + \frac{\upsilon\phi_i}{1-\upsilon\phi_i}\sum_{k\neq i} q_{ik}(\phi_k - \phi_i) > 0,$$

$$\sum_{i\in S}\pi_i\left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right) - \frac{1}{2}\upsilon\sigma_3^2(i) + \frac{\upsilon\psi_i}{1-\upsilon\psi_i}\sum_{k\neq i} q_{ik}(\psi_k - \psi_i) > 0.$$

Thus, $\mathcal{L}W \to -\infty$, $(x, y, z, r) \in (U^c \setminus \bar{\Omega}) \times S$ for small enough ε, where

$$\bar{\Omega} = \{(x, y, z) \in R_+^3 | 0 < x < \varepsilon\}.$$

When $x \to 0^+$,

$$\begin{aligned}\mathcal{L}W(x, y, z, i) \leq & -\bar{M}\mu - d_3 y^{-\upsilon} + d_4 y^{\theta_2(i)-\upsilon} + d_5 y - d_6 y^{\theta_2(i)+1} - d_7 z^{-\upsilon} + d_8 z^{1-\upsilon} + d_9 z - d_{10} z^2 \\ \leq & -\bar{M}\mu + \bar{N} \\ \leq & -2.\end{aligned}$$

Therefore, $\mathcal{L}W \leq -1$, $(x, y, z, r) \in U^c \times S$. □

It can be observed that regardless of the initial population sizes and initial state, after a sufficiently long period of time, the overall distribution of population sizes will converge to the same stable pattern. This phenomenon reflects the inherent tendency of the ecosystem to move toward stability in a stochastic environment.

Corollary 3. *For some state i and any* $(x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$, *sub-model* (2.4) *has a unique stationary distribution and is ergodic when* $\varphi_i > 0$, $r_2(i) - v(i) - \frac{e^{-v(i)m(i)} b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i) > 0$ *and* $r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i) > 0$.

When sub-model (2.4) has an ergodic stationary distribution in every state, model (2.3) must also possess such a distribution. Similarly, if sub-model (2.4) lacks an ergodic stationary distribution in certain states but satisfies the sufficient conditions in Theorem 4, model (2.3) will nonetheless have an ergodic stationary distribution.

5. Extinction and permanence

In this section, we aim to derive the sufficient conditions for the extinction or persistence of different populations in model (2.3), thus providing a theoretical foundation for predicting the population growth dynamics.

Definition 2. *(Extinction and Persistence* [40]*) A species x is defined as extinct when* $\lim_{t\to+\infty} x(t) = 0$*; as strongly persistent (in the mean) when* $\liminf_{t\to+\infty} \frac{1}{t}\int_0^t x^\theta \,\mathrm{d}s > 0$*; and as non-persistent (in the mean) when* $\limsup_{t\to+\infty} \frac{1}{t}\int_0^t x^\theta \,\mathrm{d}s = 0$.

Lemma 5. [41] *Suppose* $x(t) \in C[\Omega \times [0, +\infty), \mathbb{R}^+]$ *and* $\lim_{t\to+\infty} \frac{L(t)}{t} = 0$.
(1) If there exist $\lambda_0, T, \lambda > 0$ *with*

$$\ln x(t) \leq \lambda t - \lambda_0 \int_0^t x(s)\mathrm{d}s + L(t)$$

for $t \geq T$*, then there is*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\mathrm{d}s \leq \frac{\lambda}{\lambda_0}.$$

(2) If there exist λ_0, T, $\lambda > 0$ *with*

$$\ln x(t) \geq \lambda t - \lambda_0 \int_0^t x(s)\mathrm{d}s + L(t)$$

for $t \geq T$*, then* $\liminf_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\mathrm{d}s \geq \frac{\lambda}{\lambda_0}$.

Define $J_1(i) = r_1(i) - \frac{1}{2}\sigma_1^2(i)$, $J_{11} = \sum\limits_{i=1}^{N} \pi_i \left(r_1(i) - \frac{1}{2}\sigma_1^2(i)\right)$,

$J_{12} = \sum\limits_{i=1}^{N} \pi_i \left(r_1(i) - \frac{1}{2}\sigma_1^2(i) - k_1(i) + \frac{k_1(i)\theta_1(i)}{\hat{\theta}_1}\right)$.

Theorem 5. *For any* $(x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$ *in model* (2.3)*,*

(1) population x becomes extinct in case of $J_{11} < 0$.

(2) $\limsup\limits_{t\to+\infty} \frac{1}{t} \int_0^t x^{\hat{\theta}_1} \mathrm{d}s \le \frac{\hat{\theta}_1}{\widehat{k_1\theta_1}} J_{12}$ *in case of* $J_{12} > 0$.

(3) $\limsup\limits_{t\to+\infty} \frac{1}{t} \int_0^t x^{\hat{\theta}_1} \mathrm{d}s = 0$ *and x is non-persistent (in the mean) in case of* $J_{12} = 0$.

Proof. (1) Applying the Itô formula gives

$$\mathrm{d}\ln x = \left(r_1(\gamma(t)) - k_1(\gamma(t))x^{\theta_1(\gamma(t))} - \frac{b_1(\gamma(t))(1 - f(\gamma(t)))z}{1 + a(\gamma(t))(1 - f(\gamma(t)))x} - \frac{1}{2}\sigma_1^2(\gamma(t))\right)\mathrm{d}t + \sigma_1(\gamma(t))\mathrm{d}B_1(t).$$

By integrating both sides of the above equation over $[0, t]$, we have

$$\begin{aligned}\ln x(t) - \ln x_0 = \int_0^t \Bigg(& r_1(\gamma(s)) - k_1(\gamma(s))x^{\theta_1(\gamma(s))} - \frac{b_1(\gamma(s))(1 - f(\gamma(s)))z}{1 + a(\gamma(s))(1 - f(\gamma(s)))x} \\ & - \frac{1}{2}\sigma_1^2(\gamma(s))\Bigg)\mathrm{d}s + M_1(t),\end{aligned}$$

where $M_1(t) = \int_0^t \sigma_1(\gamma(s))\mathrm{d}B_1(s)$ is a local martingale and

$$\langle M_1(t), M_1(t)\rangle = \int_0^t \sigma_1^2(\gamma(s))\mathrm{d}s \le \check{\sigma}_1^2 t.$$

By the strong law of numbers [39], we have $\lim\limits_{t\to+\infty} \frac{M_1(t)}{t} = 0$, and then

$$\frac{\ln x(t) - \ln x_0}{t} \le \frac{1}{t}\int_0^t \left(r_1(\gamma(s)) - \frac{1}{2}\sigma_1^2(\gamma(s))\right)\mathrm{d}s + \frac{M_1(t)}{t},$$

Taking upper limits on both sides and applying the ergodic property of Markov chains, we get

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{t} \le \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \left(r_1(\gamma(s)) - \frac{1}{2}\sigma_1^2(\gamma(s))\right)\mathrm{d}s = J_{11} < 0,$$

so $\lim\limits_{t\to+\infty} x(t) = 0$.

(2) Set $x' = x(t)^{\hat{\theta}_1} \ge 0$, $\rho_1 = \frac{\theta_1(\gamma(t))}{\hat{\theta}_1} \ge 1$ and $1 + \rho_1(x' - 1) < x'^{\rho_1}$. We can get

$$\begin{aligned}\mathrm{d}\ln x &= \left(r_1(\gamma(t)) - k_1(\gamma(t))(x^{\hat{\theta}_1})^{\rho_1} - \frac{b_1(\gamma(t))(1 - f(\gamma(t)))z}{1 + a(\gamma(t))(1 - f(\gamma(t)))x} - \frac{1}{2}\sigma_1^2(\gamma(t))\right)\mathrm{d}t + \sigma_1(\gamma(t))\mathrm{d}B_1(t) \\ &\le \left(r_1(\gamma(t)) - \frac{1}{2}\sigma_1^2(\gamma(t)) - k_1(\gamma(t)) - \frac{\widehat{k_1\theta_1}}{\hat{\theta}_1}x^{\hat{\theta}_1} + \frac{k_1(\gamma(t))\theta_1(\gamma(t))}{\hat{\theta}_1}\right)\mathrm{d}t + \sigma_1(\gamma(t))\mathrm{d}B_1(t).\end{aligned}$$

For any $\varepsilon > 0$, there exists $T > 0$, such that

$$\int_0^t \left(r_1(\gamma(s)) - \frac{1}{2}\sigma_1^2(\gamma(s)) - k_1(\gamma(s)) + \frac{k_1(\gamma(s))\theta_1(\gamma(s))}{\hat{\theta}_1} \right) \mathrm{d}s$$
$$\leq \left(\sum_{i=1}^N \pi_i \left(r_1(i) - \frac{1}{2}\sigma_1^2(i) - k_1(i) + \frac{k_1(i)\theta_1(i)}{\hat{\theta}_1} \right) + \varepsilon \right) t$$

when $t > T$. We calculate

$$\mathrm{d}\ln x^{\hat{\theta}_1} \leq \hat{\theta}_1 \left(r_1(\gamma(t)) - \frac{1}{2}\sigma_1^2(\gamma(t)) - k_1(\gamma(t)) - \frac{\widehat{k_1\theta_1}}{\hat{\theta}_1} x^{\hat{\theta}_1} + \frac{k_1(\gamma(t))\theta_1(\gamma(t))}{\hat{\theta}_1} \right) \mathrm{d}t + \hat{\theta}_1 \sigma_1(\gamma(t))\mathrm{d}B_1(t),$$

and

$$\ln x^{\hat{\theta}_1} \leq \hat{\theta}_1 (J_{12} + \varepsilon)t - \widehat{k_1\theta_1} \int_0^t x^{\hat{\theta}_1} \mathrm{d}s + \hat{\theta}_1 M_1(t) + \hat{\theta}_1 \ln x_0.$$

From Lemma 5 and the indeterminacy of ε, we get

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t x^{\hat{\theta}_1} \mathrm{d}s \leq \frac{\hat{\theta}_1}{\widehat{k_1\theta_1}} J_{12}.$$

(3) From case (2), it has $\limsup_{t\to+\infty} \frac{1}{t} \int_0^t x^{\hat{\theta}_1} \mathrm{d}s \leq \frac{\hat{\theta}_1}{\widehat{k_1\theta_1}} J_{12} = 0$, so $\limsup_{t\to+\infty} \frac{1}{t} \int_0^t x^{\hat{\theta}_1} \mathrm{d}s = 0$. □

Define

$$J_2(i) = r_2(i) - v(i) - \tfrac{1}{2}\sigma_2^2(i),\ J_{21} = \sum_{i=1}^N \pi_i \left(r_2(i) - v(i) - \tfrac{1}{2}\sigma_2^2(i) \right),$$
$$J_{22} = \sum_{i=1}^N \pi_i \left(r_2(i) - v(i) - \tfrac{1}{2}\sigma_2^2(i) - k_2(i) + \tfrac{k_2(i)\theta_2(i)}{\check{\theta}_2} - \tfrac{e^{-v(i)m(i)} b_2(i)}{\rho(i)} \right),$$
$$J_{23} = \sum_{i=1}^N \pi_i \left(r_2(i) - v(i) - \tfrac{1}{2}\sigma_2^2(i) - k_2(i) + \tfrac{k_2(i)\theta_2(i)}{\hat{\theta}_2} \right).$$

Theorem 6. *For model* (2.3) *with* $(x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$, *it has*

(1) population y *becomes extinct when* $J_{21} < 0$;

(2) $\liminf_{t\to+\infty} \frac{1}{t} \int_0^t y_2^{\check{\theta}} \mathrm{d}s \geq \frac{\check{\theta}_2}{\widecheck{k_2\theta_2}} J_{22}$ *and* y *is strongly persistent in the mean when* $J_{22} > 0$;

(3) $\limsup_{t\to+\infty} \frac{1}{t} \int_0^t y^{\hat{\theta}_2} \mathrm{d}s \leq \frac{\hat{\theta}_2}{\widehat{k_2\theta_2}} J_{23}$ *when* $J_{23} > 0$;

(4) $\limsup_{t\to+\infty} \frac{1}{t} \int_0^t y^{\hat{\theta}_2} \mathrm{d}s = 0$ *and* y *is non-persistent (in the mean) when* $J_{23} = 0$.

Proof. (1) Applying formula Itô to $\ln y$, we can get

$$\mathrm{d}\ln y = \left(r_2(\gamma(t)) - k_2(\gamma(t)) y^{\theta_2(\gamma(t))} - v(\gamma(t)) - \frac{e^{-v(\gamma(t))m(\gamma(t))} b_2(\gamma(t))z}{y + \rho(\gamma(t))z} \right.$$
$$\left. - \frac{1}{2}\sigma_2^2(\gamma(t)) \right) \mathrm{d}t + \sigma_2(\gamma(t))\mathrm{d}B_2(t).$$

Integrating both sides over the interval $[0, t]$, we have

$$\ln y(t) - \ln y_0 = \int_0^t \Bigg(r_2(\gamma(s)) - k_2(\gamma(s)) y^{\theta_2(\gamma(s))} - v(\gamma(s)) - \frac{e^{-v(\gamma(s))m(\gamma(s))} b_2(\gamma(s)) z}{y + \rho(\gamma(s)) z} - \frac{1}{2}\sigma_2^2(\gamma(s)) \Bigg) \mathrm{d}s + M_2(t),$$

where $M_2(t) = \int_0^t \sigma_2(\gamma(s)) \mathrm{d}B_2(s)$ and

$$\langle M_2(t), M_2(t) \rangle = \int_0^t \sigma_2^2(\gamma(s)) \mathrm{d}s \leq \check{\sigma}_2^2 t.$$

Since $\lim\limits_{t \to +\infty} \frac{M_2(t)}{t} = 0$, then

$$\frac{\ln y(t) - \ln y_0}{t} \leq \frac{1}{t} \int_0^t \left(r_2(\gamma(s)) - v(\gamma(s)) - \frac{1}{2}\sigma_2^2(\gamma(s)) \right) \mathrm{d}s + \frac{M_2(t)}{t}.$$

Taking the upper limit on both sides and using the ergodic property of Markov chains, we get

$$\limsup_{t \to +\infty} \frac{\ln y(t)}{t} \leq \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \left(r_2(\gamma(s)) - v(\gamma(s)) - \frac{1}{2}\sigma_2^2(\gamma(s)) \right) \mathrm{d}s = J_{21} < 0,$$

so $\lim\limits_{t \to +\infty} y(t) = 0$.

(2) We set $y' = y(t)^{\check{\theta}_2}$, $\rho_2 = \frac{\theta_2(\gamma(t))}{\check{\theta}_2} \leq 1$ and $y'^{\rho_2} < 1 + \rho_2(y' - 1)$. We can get

$$\mathrm{d} \ln y \geq \Bigg(r_2(\gamma(t)) - k_2(\gamma(t)) - v(\gamma(t)) - \frac{1}{2}\sigma_2^2(\gamma(t)) + \frac{k_2(\gamma(t))\theta_2(\gamma(t))}{\check{\theta}_2} - \frac{\check{k_2\theta_2}}{\check{\theta}_2} y^{\check{\theta}_2} - \frac{e^{-v(\gamma(t))m(\gamma(t))} b_2(\gamma(t))}{\rho(\gamma(t))} \Bigg) \mathrm{d}t + \sigma_2(\gamma(t)) \mathrm{d}B_2(t).$$

For any $\varepsilon > 0$, there exists $T > 0$, such that

$$\int_0^t \left(r_2(\gamma(s)) - k_2(\gamma(s)) - v(\gamma(s)) - \frac{1}{2}\sigma_2^2(\gamma(s)) + \frac{k_2(\gamma(s))\theta_2(\gamma(s))}{\check{\theta}_2} - \frac{e^{-v(\gamma(s))m(\gamma(s))} b_2(\gamma(s))}{\rho(\gamma(s))} \right) \mathrm{d}s$$
$$\geq \left(\sum_{i=1}^N \pi_i \left(r_2(i) - v(i) - \frac{1}{2}\sigma_2^2(i) - k_2(i) + \frac{k_2(i)\theta_2(i)}{\check{\theta}_2} - \frac{e^{-v(i)m(i)} b_2(i)}{\rho(i)} \right) - \varepsilon \right) t$$

when $t > T$. Then

$$\mathrm{d} \ln y^{\check{\theta}_2} \geq \check{\theta}_2 \Bigg(r_2(\gamma(t)) - k_2(\gamma(t)) - v(\gamma(t)) - \frac{1}{2}\sigma_2^2(\gamma(t)) + \frac{k_2(\gamma(t))\theta_2(\gamma(t))}{\check{\theta}_2} - \frac{\check{k_2\theta_2}}{\check{\theta}_2} y^{\check{\theta}_2} - \frac{e^{-v(\gamma(t))m(\gamma(t))} b_2(\gamma(t))}{\rho(\gamma(t))} \Bigg) \mathrm{d}t + \check{\theta}_2 \sigma_2(\gamma(t)) \mathrm{d}B_2(t)$$

and

$$\ln y^{\check{\theta}_2} \geq \check{\theta}_2 (J_{22} - \varepsilon) t - \check{k_2\theta_2} \int_0^t y^{\check{\theta}_2} \mathrm{d}s + \check{\theta}_2 M_2(t) + \check{\theta}_2 \ln y_0.$$

By Lemma 5 and the arbitrariness of ε, we obtain

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t y^{\check{\theta}}\mathrm{d}s \geq \frac{\check{\theta}_2}{\check{k_2\theta_2}}J_{22}$$

and y is strongly persistent in the mean.

(3) We set $y'' = y(t)^{\hat{\theta}_2} \geq 0, \rho_3 = \frac{\theta_2(\gamma(t))}{\hat{\theta}_2} \geq 1$ and $1+\rho_3(y''-1) < y''^{\rho_3}$. We can get

$$\mathrm{d}\ln y \leq \left(r_2(\gamma(t)) - k_2(\gamma(t)) - v(\gamma(t)) - \frac{1}{2}\sigma_2^2(\gamma(t)) + \frac{k_2(\gamma(t))\theta_2(\gamma(t))}{\hat{\theta}_2} - \frac{\widehat{k_2\theta_2}}{\hat{\theta}_2}y^{\hat{\theta}_2}\right)\mathrm{d}t + \sigma_2(\gamma(t))\mathrm{d}B_2(t).$$

For any $\varepsilon > 0$, there exists $T > 0$, such that

$$\begin{aligned}&\int_0^t \left(r_2(\gamma(s)) - k_2(\gamma(s)) - v(\gamma(s)) - \frac{1}{2}\sigma_2^2(\gamma(s)) + \frac{k_2(\gamma(s))\theta_2(\gamma(s))}{\hat{\theta}_2}\right)\mathrm{d}s\\ \leq& \left(\sum_{i=1}^N \pi_i\left(r_2(i) - v(i) - \frac{1}{2}\sigma_2^2(i) - k_2(i) + \frac{k_2(i)\theta_2(i)}{\hat{\theta}_2}\right) + \varepsilon\right)t\end{aligned}$$

when $t > T$. Then

$$\begin{aligned}\mathrm{d}\ln y^{\hat{\theta}_2} \leq& \hat{\theta}_2\left(r_2(\gamma(t)) - k_2(\gamma(t)) - v(\gamma(t)) - \frac{1}{2}\sigma_2^2(\gamma(t))\right.\\ &\left.+\frac{k_2(\gamma(t))\theta_2(\gamma(t))}{\hat{\theta}_2} - \frac{\widehat{k_2\theta_2}}{\hat{\theta}_2}y^{\hat{\theta}_2}\right)\mathrm{d}t + \hat{\theta}_2\sigma_2(\gamma(t))\mathrm{d}B_2(t)\end{aligned}$$

and

$$\ln y^{\hat{\theta}_2} \leq \hat{\theta}_2(J_{23}+\varepsilon)t - \widehat{k_2\theta_2}\int_0^t y^{\hat{\theta}_2}\mathrm{d}s + \hat{\theta}_2 M_2(t) + \hat{\theta}_2\ln y_0.$$

Thus,

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t y^{\hat{\theta}_2}\mathrm{d}s \leq \frac{\hat{\theta}_2}{\widehat{k_2\theta_2}}J_{23}.$$

(4) It is known from (3) that $\limsup_{t\to+\infty}\frac{1}{t}\int_0^t y^{\hat{\theta}_2}\mathrm{d}s \leq \frac{\hat{\theta}_2}{\widehat{k_2\theta_2}}J_{23} = 0$, so $\limsup_{t\to+\infty}\frac{1}{t}\int_0^t y^{\hat{\theta}_2}\mathrm{d}s = 0$. □

Define$J(i) = r_3(i) + \frac{c_1(i)}{a(i)} + e^{-v(i)m(i)}c_2(i) - d(i) - \frac{1}{2}\sigma_3^2(i)$,

$J_3 = \sum_{i=1}^N \pi_i\left(r_3(i) + \frac{c_1(i)}{a(i)} + e^{-v(i)m(i)}c_2(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right)$.

Theorem 7. *For model* (2.3) *with* $(x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$, *it has*

(1) population z *becomes extinct when* $J_3 < 0$*;*

(2) $\limsup_{t\to+\infty}\frac{1}{t}\int_0^t z\mathrm{d}s \leq \frac{J_3}{\widehat{k_3}}$, *when* $J_3 > 0$*;*

(3) $\limsup_{t\to+\infty}\frac{1}{t}\int_0^t z\mathrm{d}s = 0$ *and* z *is non-persistent (in the mean) when* $J_3 = 0$.

Proof. (1) Applying the Itô formula to $\ln z$, we have

$$\mathrm{d}\ln z = \left(r_3(\gamma(t)) + \frac{c_1(\gamma(t))(1 - f(\gamma(t)))x}{1 + a(\gamma(t))(1 - f(\gamma(t)))x} + \frac{e^{-\nu(\gamma(t))m(\gamma(t))}c_2(\gamma(t))y}{y + \rho(\gamma(t))z} \right.$$
$$\left. - d(\gamma(t)) - k_3(\gamma(t))z - \frac{1}{2}\sigma_3^2(\gamma(t)) \right)\mathrm{d}t + \sigma_3(\gamma(t))\mathrm{d}B_3(t).$$

Integrating both sides over the interval $[0, t]$, we have

$$\ln z - \ln z_0 = \int_0^t \left(r_3(\gamma(s)) + \frac{c_1(\gamma(s))(1 - f(\gamma(s)))x}{1 + a(\gamma(s))(1 - f(\gamma(s)))x} + \frac{e^{-\nu(\gamma(s))m(\gamma(s))}c_2(\gamma(s))y}{y + \rho(\gamma(s))z} - d(\gamma(s)) \right.$$
$$\left. - k_3(\gamma(s))z - \frac{1}{2}\sigma_3^2(\gamma(s)) \right)\mathrm{d}s + M_3(t),$$

where $M_3(t) = \int_0^t \sigma_3(\gamma(s))\mathrm{d}B_3(s)$ and

$$\langle M_3(t), M_3(t) \rangle = \int_0^t \sigma_3^2(\gamma(s))\mathrm{d}s \le \check{\sigma}_3^2 t.$$

Since $\lim_{t\to+\infty} \frac{M_3(t)}{t} = 0$, then we have

$$\frac{\ln z(t) - \ln z_0}{t} \le \frac{1}{t}\int_0^t \left(r_3(\gamma(s)) + \frac{c_1(\gamma(s))}{a(\gamma(s))} + e^{-\nu(\gamma(s))m(\gamma(s))}c_2(\gamma(s)) \right.$$
$$\left. - d(\gamma(s)) - \frac{1}{2}\sigma_3^2(\gamma(s)) \right)\mathrm{d}s + \frac{M_3(t)}{t}.$$

Taking the upper limit on both sides and using the ergodic property of Markov chains, we deduce

$$\limsup_{t\to+\infty} \frac{\ln z(t)}{t} \le \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \left(r_3(\gamma(s)) + \frac{c_1(\gamma(s))}{a(\gamma(s))} + e^{-\nu(\gamma(s))m(\gamma(s))}c_2(\gamma(s)) \right.$$
$$\left. - d(\gamma(s)) - \frac{1}{2}\sigma_3^2(\gamma(s)) \right)\mathrm{d}s = J_3 < 0,$$

so $\lim_{t\to+\infty} z(t) = 0$.

(2) Since

$$\mathrm{d}\ln z \le \left(r_3(\gamma(t)) + \frac{c_1(\gamma(t))}{a(\gamma(t))} + e^{-\nu(\gamma(t))m(\gamma(t))}c_2(\gamma(t)) \right.$$
$$\left. - d(\gamma(t)) - k_3(\gamma(t))z - \frac{1}{2}\sigma_3^2(\gamma(t)) \right)\mathrm{d}t + \sigma_3(\gamma(t))\mathrm{d}B_3(t),$$

then for any $\varepsilon > 0$, there exists $T > 0$, such that

$$\int_0^t \left(r_3(\gamma(s)) + \frac{c_1(\gamma(s))}{a(\gamma(s))} + e^{-\nu(\gamma(s))m(\gamma(s))}c_2(\gamma(s)) - d(\gamma(s)) - \frac{1}{2}\sigma_3^2(\gamma(s)) \right)\mathrm{d}s$$
$$\le \left(\sum_{i=1}^N \pi_i \left(r_3(i) + \frac{c_1(i)}{a(i)} + e^{-\nu(i)m(i)}c_2(i) - d(i) - \frac{1}{2}\sigma_3^2(i) \right) + \varepsilon \right) t$$

when $t > T$. The integral yields

$$\ln z \le (J_3 + \varepsilon)t - \hat{k}_3 \int_0^t z\mathrm{d}s + M_3(t) + \ln z_0.$$

From Lemma 5 and the randomness of ε, we deduce

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t z\mathrm{d}s \le \frac{J_3}{\hat{k}_3}.$$

(3) It is known from case (2) $\limsup_{t\to+\infty} \frac{1}{t} \int_0^t z\mathrm{d}s \le \frac{J_3}{\hat{k}_3} = 0$, so $\limsup_{t\to+\infty} \frac{1}{t} \int_0^t z\mathrm{d}s = 0$. □

Based on Theorems 5–7, we directly present a corollary regarding the extinction of sub-model (2.4).

Corollary 4. *For some state i and any $(x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$, 1) population x becomes extinct when $J_1(i) < 0$; 2) population y becomes extinct when $J_2(i) < 0$; and 3) population z becomes extinct when $J(i) < 0$.*

The impact of Markov transformation on model (2.3) is obvious. The populations corresponding to the sub-model are extinct when $J_1(i) < 0$, $J_2(i) < 0$, and $J(i) < 0$ for certain states $i \in S$. Model (2.3) must be extinct when each sub-model population is extinct. The general behavior may result in extinction due to Markov switching, whereby certain states in model (2.3) persist while others go extinct. In other words, the extinction of populations in model (2.3) results from the combined superposition of multiple sub-states. In certain sub-states, populations may persist; however, as long as the sufficient conditions for population extinction specified in Theorems 5–7 are satisfied, the persistence of populations in these sub-states will not affect the nature of population extinction.

6. Asymptotic boundedness of solutions

Theorem 8. *For any $(x_0, y_0, z_0, r_0) \in \mathbb{R}_+^3 \times S$, the solution of model* (2.3) *satisfies the following properties when $\hat{\theta}_1 \ge 1$, $\hat{\theta}_2 \ge 1$:*

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{\ln t} \le 1, \quad \limsup_{t\to+\infty} \frac{\ln y(t)}{\ln t} \le 1, \quad \limsup_{t\to+\infty} \frac{\ln z(t)}{\ln t} \le 1.$$

Proof. Applying the Itô formula to $e^t \ln x$, $e^t \ln y$ and $e^t \ln z$, we get

$$\begin{aligned}\mathrm{d}(e^t \ln x) = &\Bigg[e^t \ln x + e^t \Bigg(r_1(\gamma(t)) - k_1(\gamma(t))x^{\theta_1(\gamma(t))} - \frac{b_1(\gamma(t))(1 - f(\gamma(t)))z}{1 + a(\gamma(t))(1 - f(\gamma(t)))x} \\ &- \frac{1}{2}\sigma_1^2(\gamma(t))\Bigg)\Bigg] \mathrm{d}t + e^t \sigma_1(\gamma(t))\mathrm{d}B_1(t),\end{aligned}$$

$$\begin{aligned}\mathrm{d}(e^t \ln y) = &\Bigg[e^t \ln y + e^t \Bigg(r_2(\gamma(t)) - k_2(\gamma(t))y^{\theta_2(\gamma(t))} - v(\gamma(t)) - \frac{e^{-v(\gamma(t))m(\gamma(t))} b_2(\gamma(t))z}{y + \rho(\gamma(t))z} \\ &- \frac{1}{2}\sigma_2^2(\gamma(t))\Bigg)\Bigg] \mathrm{d}t + e^t \sigma_2(\gamma(t))\mathrm{d}B_2(t),\end{aligned}$$

$$\mathrm{d}(e^t\ln z)=\Bigg[e^t\ln z+e^t\Bigg(r_3(\gamma(t))+\frac{c_1(\gamma(t))(1-f(\gamma(t)))x}{1+a(\gamma(t))(1-f(\gamma(t)))x}+\frac{e^{-\nu(\gamma(t))m(\gamma(t))}c_2(\gamma(t))y}{y+\rho(\gamma(t))z}$$
$$-d(\gamma(t))-k_3(\gamma(t))z-\frac{1}{2}\sigma_3^2(\gamma(t))\Bigg)\Bigg]\mathrm{d}t+e^t\sigma_3(\gamma(t))\mathrm{d}B_3(t).$$

Integrate both sides over the interval $[0,t]$, we have

$$\begin{aligned}e^t\ln x-\ln x_0=\int_0^t e^s\Bigg[&\ln x(s)+r_1(\gamma(s))-k_1(\gamma(s))x^{\theta_1(\gamma(s))}-\frac{b_1(\gamma(s))(1-f(\gamma(s)))z}{1+a(\gamma(s))(1-f(\gamma(s)))x}\\&-\frac{1}{2}\sigma_1^2(\gamma(s))\Bigg]\mathrm{d}s+N_1(t),\end{aligned}\tag{6.1}$$

$$\begin{aligned}e^t\ln y(t)-\ln y_0=\int_0^t e^s\Bigg[&\ln y(s)+r_2(\gamma(s))-k_2(\gamma(s))y^{\theta_2(\gamma(s))}-\nu(\gamma(s))-\frac{1}{2}\sigma_2^2(\gamma(s))\\&-\frac{e^{-\nu(\gamma(s))m(\gamma(s))}b_2(\gamma(s))z}{y+\rho(\gamma(s))z}\Bigg]\mathrm{d}s+N_2(t),\end{aligned}\tag{6.2}$$

$$\begin{aligned}e^t\ln z-\ln z_0=\int_0^t e^s\Bigg[&\frac{c_1(\gamma(s))(1-f(\gamma(s)))x}{1+a(\gamma(s))(1-f(\gamma(s)))x}+\frac{e^{-\nu(\gamma(s))m(\gamma(s))}c_2(\gamma(s))y}{y+\rho(\gamma(s))z}\\&+\ln z(s)+r_3(\gamma(s))-d(\gamma(s))-k_3(\gamma(s))z-\frac{1}{2}\sigma_3^2(\gamma(s))\Bigg]\mathrm{d}s+N_3(t),\end{aligned}\tag{6.3}$$

where $N_i(t)=\int_0^t e^s\sigma_i(\gamma(s))\mathrm{d}B_i(s)$ are local martingales with the following quadratic variation

$$\langle N_i,N_i\rangle=\int_0^t e^{2s}\sigma_i^2(\gamma(s))\mathrm{d}s,\quad i=1,2,3.$$

Take $\tilde{\mu}>0$, $\tilde{\xi}>1$, and $0<\varepsilon<1$. For $n_i\in N$, denote $\widetilde{\alpha}=\varepsilon e^{-\tilde{\mu}n_i}$, $\widetilde{\beta}=\frac{\tilde{\xi}e^{\tilde{\mu}n_i}}{\varepsilon}\ln n_i$, $T=\tilde{\mu}n_i$. Then, according to Exponential Martingale Inequality [39], it has

$$\mathbb{P}\left\{\sup_{0\le t\le\tilde{\mu}n_i}\left[N_i(t)-\frac{\varepsilon e^{-\tilde{\mu}n_i}}{2}\langle N_i,N_i\rangle\right]>\frac{\tilde{\xi}e^{\tilde{\mu}n_i}}{\varepsilon}\ln n_i\right\}\le\frac{1}{n_i^{\tilde{\xi}}},\quad i=1,2,3.$$

By Borel-Cantelli lemma [39], it has $\Omega_0\in F$ with $\mathbb{P}(\Omega_0)=1$ and random integer $n_0(\omega)$ such that for any $\omega\in\Omega_0$ and $n_i\ge n_0(\omega)$, there is

$$\sup_{0\le t\le\tilde{\mu}n_i}\left[N_i(t)-\frac{\varepsilon e^{-\tilde{\mu}n_i}}{2}\langle N_i,N_i\rangle\right]\le\frac{\tilde{\xi}e^{\tilde{\mu}n_i}}{\varepsilon}\ln n_i,\quad i=1,2,3$$

and

$$N_i(t)\le\frac{\varepsilon e^{-\tilde{\mu}n_i}}{2}\langle N_i,N_i\rangle+\frac{\tilde{\xi}e^{\tilde{\mu}n_i}}{\varepsilon}\ln n_i,\quad 0\le t\le\tilde{\mu}n_i,\tag{6.4}$$

Substituting (6.4) into (6.1)–(6.3), respectively, we can get

$$e^t\ln x-\ln x_0\le\int_0^t e^s\left[\ln x(s)+r_1(\gamma(s))-k_1(\gamma(s))x^{\theta_1(\gamma(s))}-\frac{1}{2}\sigma_1^2(\gamma(s))(1-\varepsilon e^{s-\tilde{\mu}n_1})\right]\mathrm{d}s+\frac{\tilde{\xi}e^{\tilde{\mu}n_1}}{\varepsilon}\ln n_1,$$

$$e^t \ln y(t) - \ln y_0 \le \int_0^t e^s \left[\ln y(s) + r_2(\gamma(s)) - v(\gamma(s)) - k_2(\gamma(s)) y^{\theta_2(\gamma(s))} - \frac{1}{2}\sigma_2^2(\gamma(s))(1 - \varepsilon e^{s-\tilde{\mu} n_2}) \right] \mathrm{d}s$$
$$+ \frac{\tilde{\xi} e^{\tilde{\mu} n_2}}{\varepsilon} \ln n_2,$$
$$e^t \ln z - \ln z_0 \le \int_0^t e^s \left[\ln z(s) + r_3(\gamma(s)) + \frac{c_1(\gamma(s))}{a(\gamma(s))} + e^{-v(\gamma(s))m(\gamma(s))} c_2(\gamma(s)) - d(\gamma(s)) \right.$$
$$\left. -k_3(\gamma(s)) z - \frac{1}{2}\sigma_3^2(\gamma(s))(1 - \varepsilon e^{s-\tilde{\mu} n_3}) \right] \mathrm{d}s + \frac{\tilde{\xi} e^{\tilde{\mu} n_3}}{\varepsilon} \ln n_3.$$

For any s with $0 \le s \le \tilde{\mu} n_i$, there exists $C_i (i = 1, 2, 3)$ independent of n, and for $(n_i - 1)\tilde{\mu} \le t \le n_i \tilde{\mu}$ and $n_i > n_0$, we have

$$\frac{\ln x(t)}{\ln t} \le e^{-t} \frac{\ln x_0}{\ln t} + C_1 \frac{1 - e^{-t}}{\ln t} + \frac{\tilde{\xi} e^{\tilde{\mu}}}{\varepsilon} \frac{\ln n_1}{\ln t},$$
$$\frac{\ln y(t)}{\ln t} \le e^{-t} \frac{\ln y_0}{\ln t} + C_2 \frac{1 - e^{-t}}{\ln t} + \frac{\tilde{\xi} e^{\tilde{\mu}}}{\varepsilon} \frac{\ln n_2}{\ln t},$$
$$\frac{\ln z(t)}{\ln t} \le e^{-t} \frac{\ln z_0}{\ln t} + C_3 \frac{1 - e^{-t}}{\ln t} + \frac{\tilde{\xi} e^{\tilde{\mu}}}{\varepsilon} \frac{\ln n_3}{\ln t}.$$

We know that $t \to +\infty$ as $n_i \to +\infty$, so

$$\frac{\ln x(t)}{\ln t} \le \frac{\tilde{\xi} e^{\tilde{\mu}}}{\varepsilon}, \quad \frac{\ln y(t)}{\ln t} \le \frac{\tilde{\xi} e^{\tilde{\mu}}}{\varepsilon}, \quad \frac{\ln z(t)}{\ln t} \le \frac{\tilde{\xi} e^{\tilde{\mu}}}{\varepsilon}.$$

Let $\tilde{\mu} \to 0, \tilde{\xi} \to 1, \varepsilon \to 1$, so that

$$\limsup_{t \to +\infty} \frac{\ln x(t)}{\ln t} \le 1, \quad \limsup_{t \to +\infty} \frac{\ln y(t)}{\ln t} \le 1, \quad \limsup_{t \to +\infty} \frac{\ln z(t)}{\ln t} \le 1.$$

□

7. Simulation verifications

For stochastic differential equations, the Milstein's method offers higher accuracy than the EM method [42], with its numerical solutions closer to the true values. We adopt Milstein's method to numerically simulate each state of the model. Taking the i-th state as an example, the discretization method is as follows:

$$\begin{cases} x_{n+1} = x_n + x_n \left(r_1(i) - k_1(i) x_n^{\theta_1(i)} - \dfrac{b_1(i)(1 - f(i)) z_n}{1 + a(i)(1 - f(i)) x_n} \right) \Delta t \\ \qquad + \sigma_1(i) x_n \sqrt{\Delta t} \varepsilon_{1,n} + \dfrac{1}{2}\sigma_1^2(i) x_n (\varepsilon_{1,n}^2 - 1) \Delta t, \\ y_{n+1} = y_n + y_n \left(r_2(i) - k_2(i) y_n^{\theta_2(i)} - v(i) - \dfrac{e^{-v(i)m(i)} b_2(i) z_n}{y_n + \rho(i) z_n} \right) \Delta t \\ \qquad + \sigma_2(i) y_n \sqrt{\Delta t} \varepsilon_{2,n} + \dfrac{1}{2}\sigma_2^2(i) y_n (\varepsilon_{2,n}^2 - 1) \Delta t, \\ z_{n+1} = z_n + z_n \left(r_3(i) + \dfrac{c_1(i)(1 - f(i)) x_n}{1 + a(i)(1 - f(i)) x_n} + \dfrac{e^{-v(i)m(i)} c_2(i) y_n}{y_n + \rho(i) z_n} - d(i) - k_3(i) z_n \right) \Delta t \\ \qquad + \sigma_3(i) z_n \sqrt{\Delta t} \varepsilon_{3,n} + \dfrac{1}{2}\sigma_3^2(i) z_n (\varepsilon_{3,n}^2 - 1) \Delta t, \end{cases}$$

where $\varepsilon_{1,n}$, $\varepsilon_{2,n}$, and $\varepsilon_{3,n}$ are compliant Gauss random variables that follow $N(0,1)$. Consider $S = \{1,2,3\}$ and the initial value $(x_0, y_0, z_0) = (1,1,2)$. States $i = 1,2,3$ correspond to Cases 1–3, respectively. Model (2.3) is Markov switched in three sub-models. Let the generation matrix of Markov chain be

$$\mathbf{Q} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$

The only stationary distribution determined by computation is $\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{1}{3}, \frac{1}{4}, \frac{5}{12}\right)$. We select the following parameters:

$r_1 = (0.6, 0.7, 0.5)$, $r_2 = (0.7, 0.5, 0.6)$, $r_3 = (0.6, 0.6, 0.65)$, $k_1 = (0.15, 0.1, 0.15)$,
$k_2 = (0.2, 0.1, 0.05)$, $k_3 = (0.15, 0.25, 0.2)$, $\theta_1 = (0.5, 0.6, 0.5)$, $\theta_2 = (0.6, 0.7, 0.6)$,
$a = (4, 3, 5)$, $v = (0.4, 0.2, 0.25)$, $m = (3, 4, 2)$, $b_1 = (0.6, 0.5, 0.4)$,
$b_2 = (0.5, 0.6, 0.45)$, $c_1 = (0.45, 0.4, 0.3)$, $c_2 = (0.4, 0.5, 0.4)$, $d = (0.1, 0.05, 0.15)$,
$\rho = (5, 4, 3)$, $f = (0.65, 0.5, 0.6)$, $\sigma_1 = (0.02, 0.01, 0.1)$, $\sigma_2 = (0.03, 0.01, 0.2)$,
$\sigma_3 = (0.01, 0.01, 0.15)$.

Then we have
$\beta_1 = 0.09955, \beta_2 = 0.113686, \beta_3 = 0.127, B = 0.1145 > 0,$
$\varphi_1 = -0.213486, \varphi_2 = 0.179950655, \varphi_3 = 0.09827, \mu = 0.01477 > 0,$
$\sum_{i\in S} \pi_i \left(r_2(i) - v(i) - \frac{e^{-v(i)m(i)} b_2(i)}{\rho(i)} - \frac{1}{2}\sigma_2^2(i)\right) > 0$, $\sum_{i\in S} \pi_i \left(r_3(i) - d(i) - \frac{1}{2}\sigma_3^2(i)\right) > 0.$
The conditions of Theorems 3 and 4 are met, and we can conclude that the Markov switching model (2.3) is stochastic persistent and has a unique ergodic stationary distribution. The Markov switching diagram is shown in Figure 1, which illustrates that the model can switch between three states. The sample orbits and stationary distribution of model (2.3) are shown in Figure 2. For small noise parameters and population parameters satisfying the sufficient conditions, system (2.3) will neither go extinct nor grow indefinitely. Instead, it will fluctuate within a certain range and converge to a stable state. This reflects populations' adaptability to stochastic environments, the supporting role of interspecific interactions in sustaining the system, and the resilience of the population system in maintaining stability amid randomness. The conditions of Corollary 2 are also met, and sub-model (2.4) is stochastic persistent in three cases. The sample orbits of sub-models (2.4) are shown in Figure 3.

Under this set of parameters, we obtain $J_{12} = 0.586$, $J_{22} = 0.235$, $J_{23} = 0.3165$, and $J_3 = 0.801$, where

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t x^{\hat{\theta}_1} ds \le \frac{\hat{\theta}_1}{\widehat{k_1\theta_1}} J_{12} = 4.88, \quad \liminf_{t\to+\infty} \frac{1}{t}\int_0^t y^{\check{\theta}_2} ds \ge \frac{\check{\theta}_2}{\widecheck{k_2\theta_2}} J_{22} = 1.37,$$

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t y^{\hat{\theta}_2} ds \le \frac{\hat{\theta}_2}{\widehat{k_2\theta_2}} J_{23} = 6.33, \quad \limsup_{t\to+\infty} \frac{1}{t}\int_0^t z ds \le \frac{J_3}{\widehat{k_3}} = 5.34.$$

The conditions for the second conclusion of Theorem 5, the second and third conclusions of Theorem 6, and the second conclusion of Theorem 7 are satisfied, and y is strongly persistent in the mean. The simulations are presented in Figure 4.

Some parameters in Cases 1 and 2 are modified as follows: $r_1(1) = 0.4$, $r_1(2) = 0.6$, $\sigma_1(1) = 1$, and $\sigma_1(2) = 2$. We calculate $J_1(1) = -0.1$, $J_1(2) = -1.4$, $J_1(3) = 0.495$, and $J_{11} = -0.177 < 0$. The

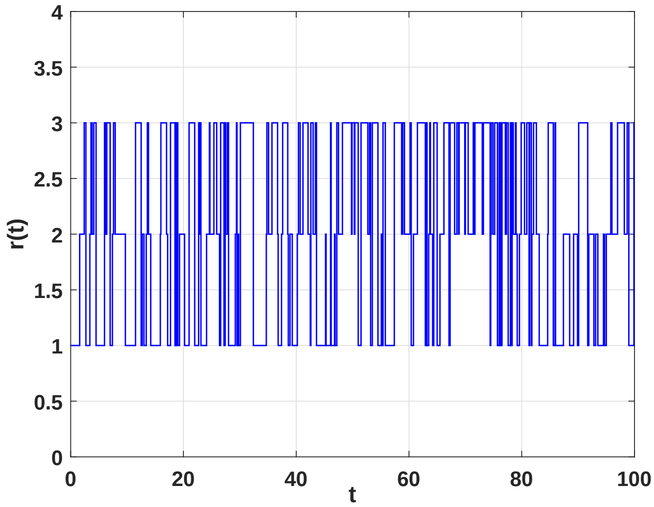

Figure 1. Markov chain with $\pi = (\frac{1}{3}, \frac{1}{4}, \frac{5}{12})$.

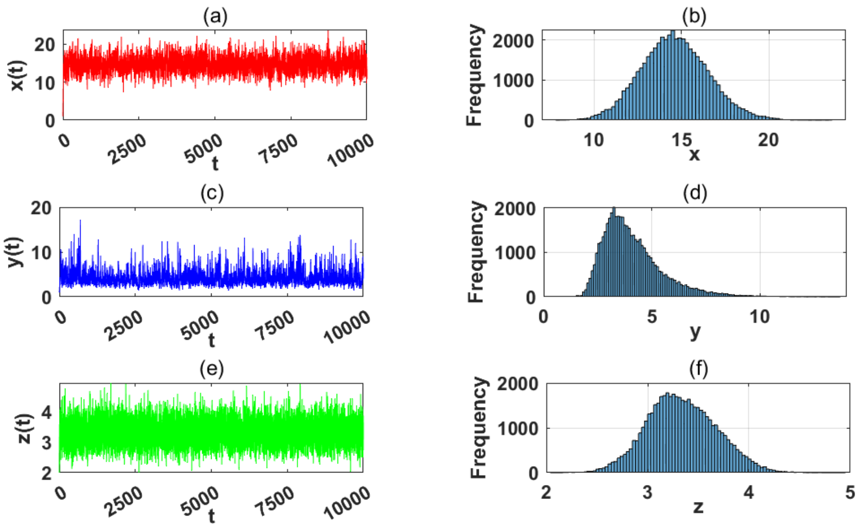

Figure 2. The densities ((a), (c) , (e)) and distribution ((b), (d) , (f)) of x, y, and z, respectively.

condition for the first conclusion of Theorem 5 is satisfied. In model (2.3), the population x is extinct, as depicted in Figure 5. Specifically, x is extinct in Cases 1 and 2, while it is not in Case 3, as illustrated in Figure 6. However, this does not impact the extinction of population x within the Markov system. In Cases 1 and 2, with relatively large noise interference and small intrinsic growth rate of x, x tends to go extinct. In Case 3, as the noise disturbance is very minor, x avoids extinction. At this time, the parameters of model (2.3) meet the condition for the extinction of x. Population x has an inherent propensity for extinction and will not perpetually stay in the state of Case 3; ultimately, x will become extinct.

Some parameters in Cases 1 and 2 are modified as follows: $\sigma_2(1) = 1$ and $\sigma_1(2) = 2$. We calculate $J_2(1) = -0.2$, $J_2(2) = -1.7$, $J_2(3) = 0.33$, and $J_{21} = -0.354 < 0$. Additionally, condition for the first

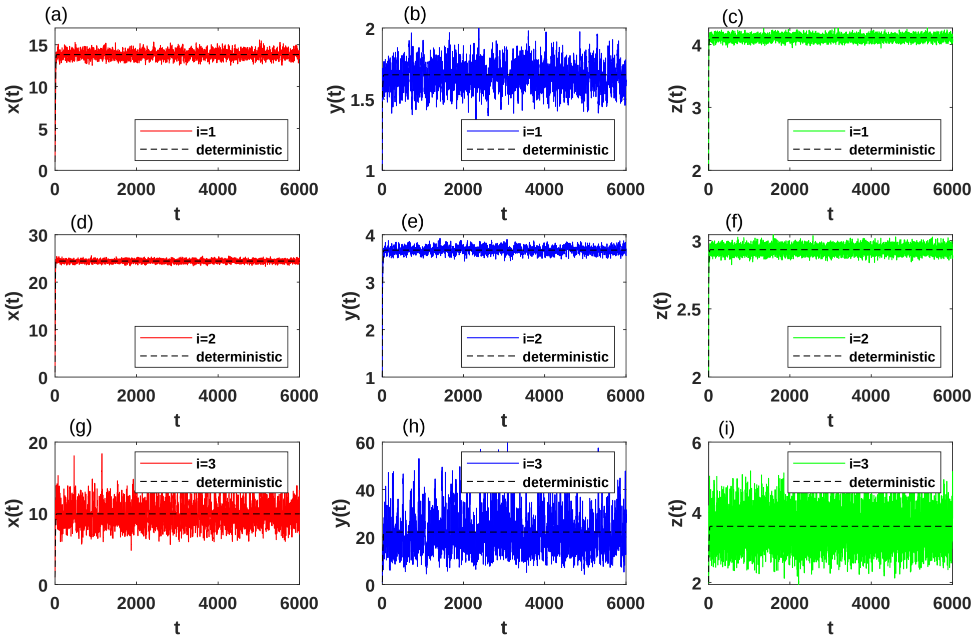

Figure 3. The probability densities of the variables x, y, and z within sub-model (2.4) for its three cases.

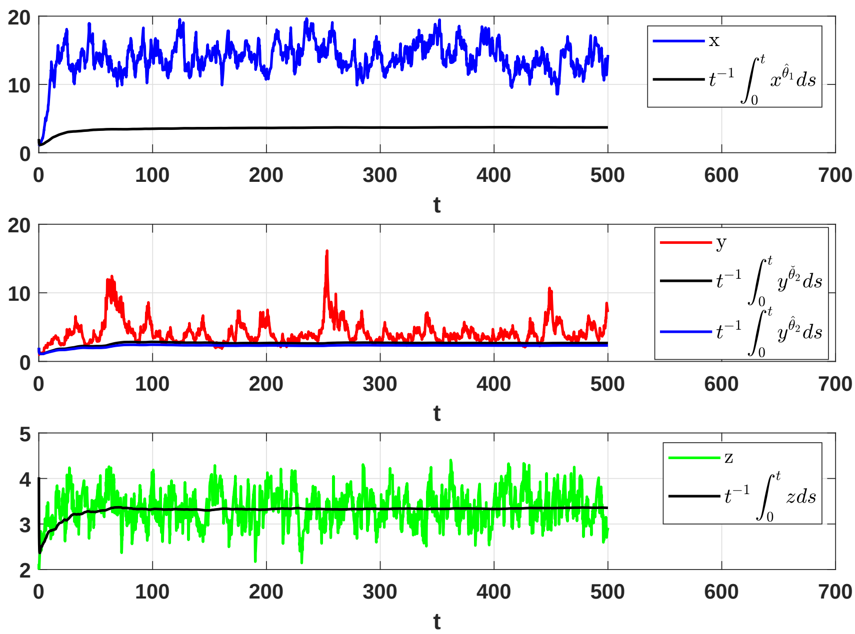

Figure 4. Sample paths for populations and corresponding mean value functions to model (2.3).

conclusion of Theorem 6 is satisfied. In model (2.3), population y becomes extinct, as demonstrated

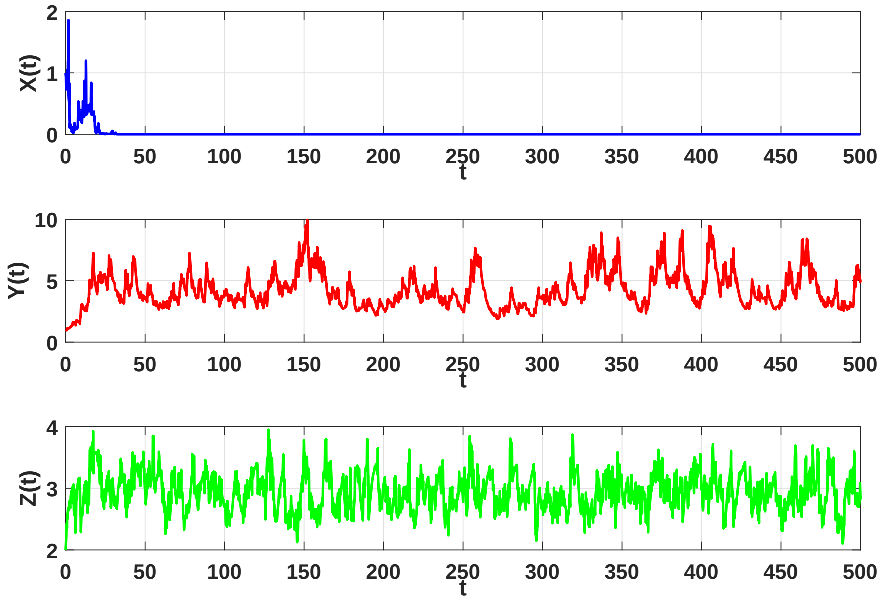

Figure 5. The survival status of species x, y, and z in model (2.3), and x is extinct.

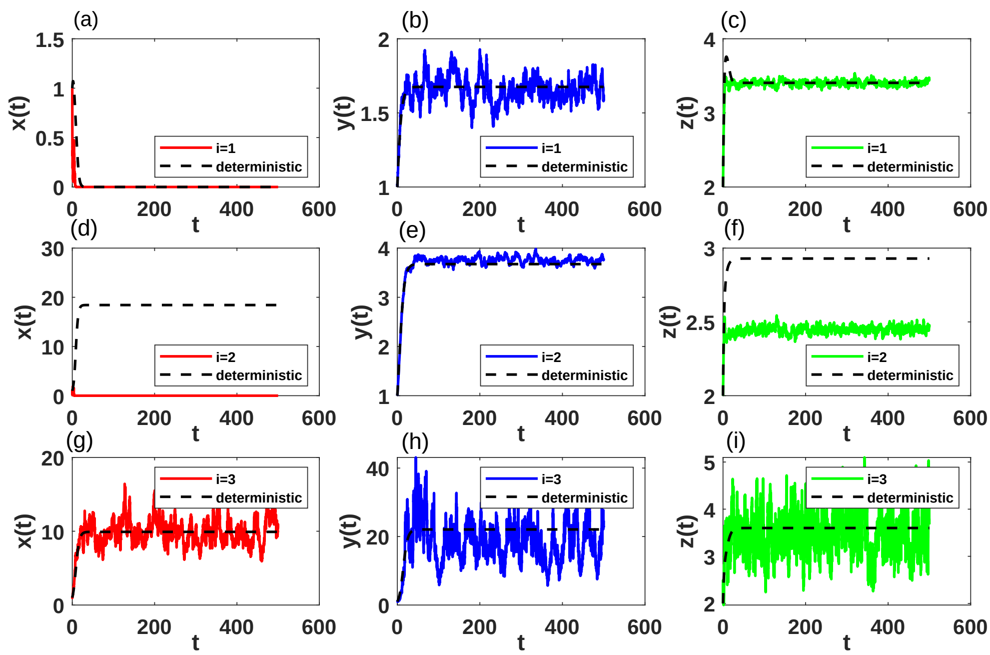

Figure 6. The survival status of species x, y, and z in sub-model (2.4) for three cases with $r_1(1) = 0.4$, $r_1(2) = 0.6$, $\sigma_1(1) = 1$, and $\sigma_1(2) = 2$.

in Figure 7. Specifically, y is extinct in Cases 1 and 2, but not in Case 3, as evidenced in Figure 8. Nevertheless, this does not alter the fact that population y goes extinct in the Markov system. In Cases 1 and 2, given larger noise interference, y tends to go extinct. In Case 3, with very small noise disturbance, y will not go extinct. At this time, the parameters of model (2.3) satisfy the condition for the extinction of y. y has an inherent propensity to go extinct, and y will become extinct.

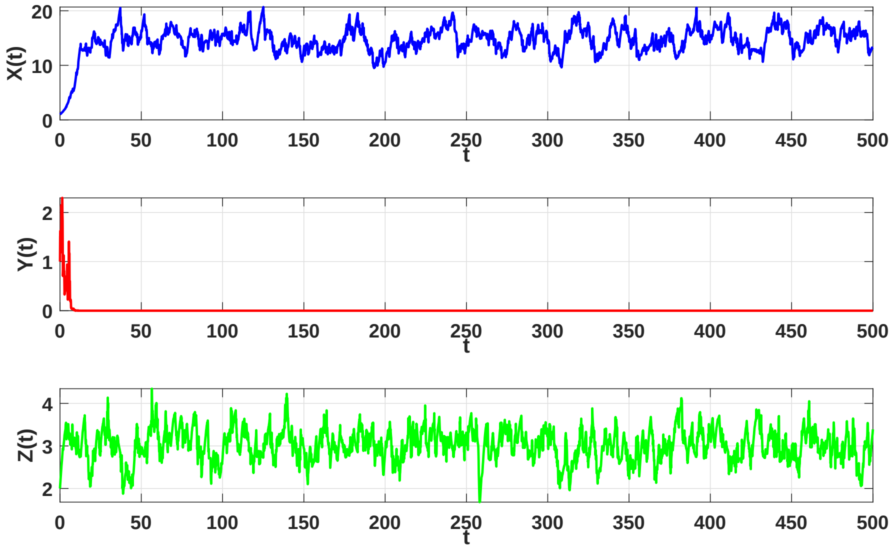

Figure 7. The population densities of x, y, and z in model (2.3), and y is extinct.

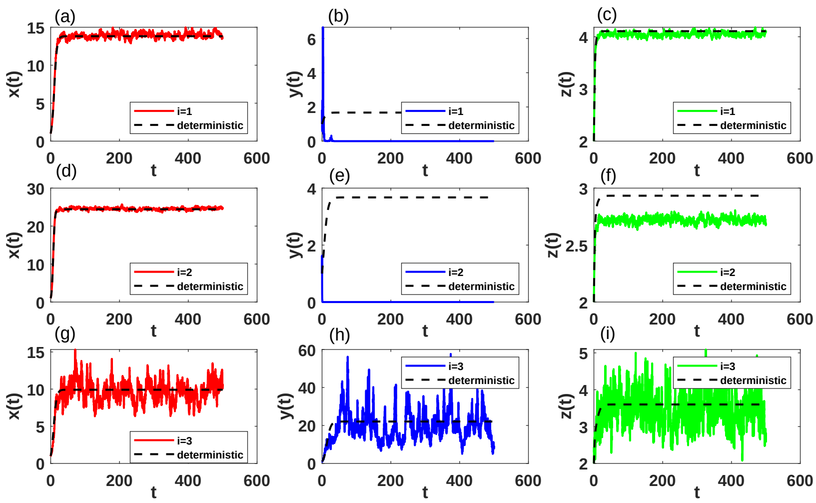

Figure 8. The population densities of x, y, and z in sub-model (2.4) for three cases with $\sigma_2(1) = 1$ and $\sigma_1(2) = 2$.

Some parameters are modified as follows: $\sigma_3(1) = 1.6$, $\sigma_3(2) = 1.5$, and $\sigma_3(3) = 1$. We calculate

$J(1) = -0.547$, $J(2) = -0.217$, $J(3) = 0.3026$, and $J_3 = -0.1105 < 0$. Thus, condition for the first conclusion of Theorem 7 is satisfied. In model (2.3), population z becomes extinct, as exhibited in Figure 9. Specifically, z is extinct for Cases 1 and 2, but not for Case 3, as demonstrated in Figure 10. Nonetheless, this discrepancy does not impede the extinction of population z within the Markov system. In Cases 1 and 2, when the noise interference is relatively large, z will tend to go extinct. In Case 3, although the noise disturbance has increased, it does not meet the condition for extinction, so z will not go extinct. At this time, the parameters of model (2.3) satisfy the condition for the extinction of z. z has an inherent tendency to go extinct, and z will become extinct.

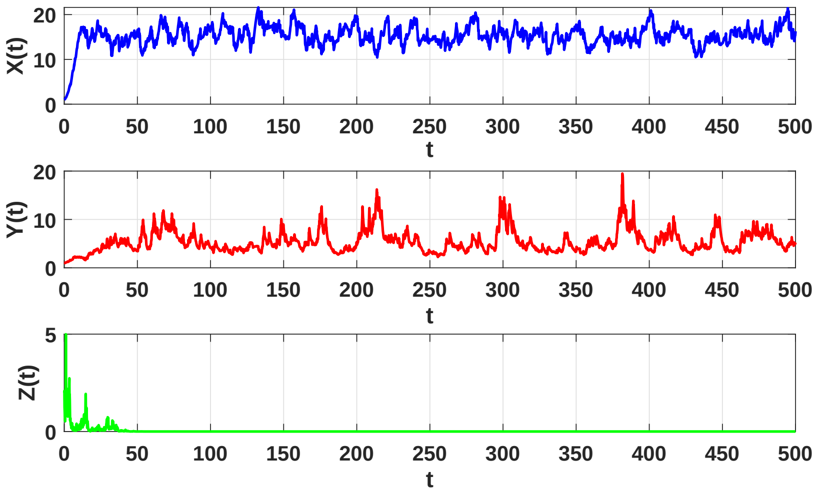

Figure 9. The survival of species x, y and z in model (2.3) and z is extinct.

In Case 1, we consider the impact of habitat selection behavior and select $v = 0.4, 0.6$, and 0.8. As shown in Figure 11, the value of v significantly influences the population of y, whereas it has negligible effects on the populations of x and z. That is to say, as the level of y's habitat selection behavior increases, its own growth will decrease. When such behavior becomes excessively strong, y will tend to go extinct, meaning that the cost of habitat selection behavior to y's own growth far outweighs the benefits derived from weakening predation capacity. Habitat selection behavior has almost no impact on other populations.

In Case 2, we consider the influence of the refuge effect and select $r_1(2) = 0.4$, $f = 0.2, 0.3$, and 0.7. The value of f significantly influences the populations of x and z, while having negligible effects on the population of y, as illustrated in Figure 12. When the value of f is small, the population of x becomes extinct, and the value of z is correspondingly low. As f increases, the populations of x and z also increase. The refuge effect plays a positive role in the prey x and the predator z.

8. Conclusions

In this study, we construct a stochastic two-prey-one-predator model (2.3). This model incorporates white noise and colored noise as disturbance factors while considering habitat selection behavior and the refuge effect. Our analysis reveals that the species in model (2.3) exhibit stochastic ultimate

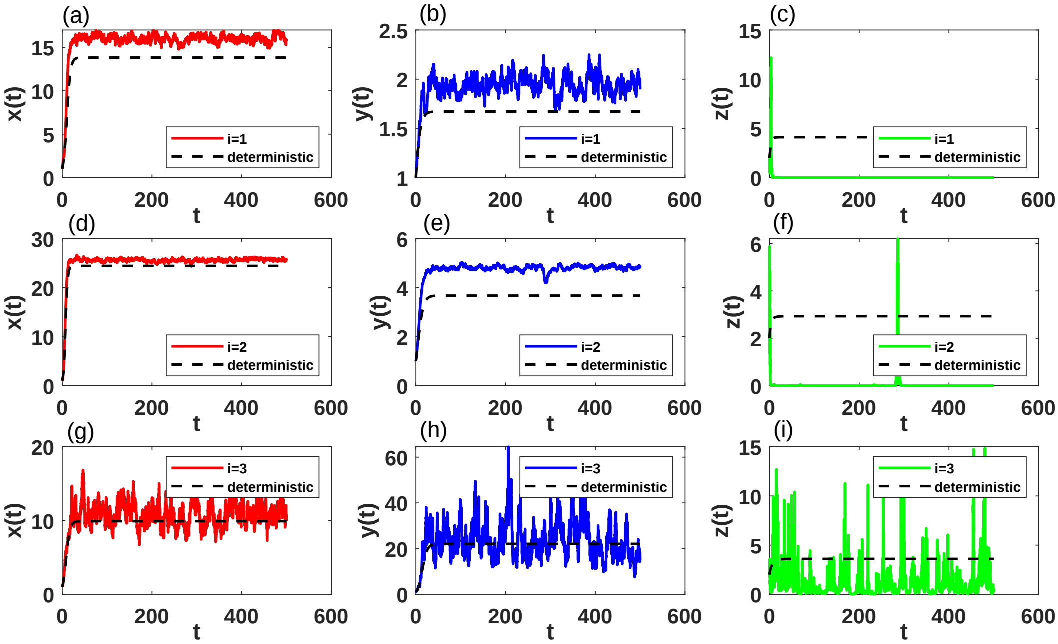

Figure 10. The survival of species x, y, and z in sub-model (2.4) for three cases with $\sigma_3(1) = 1.6$, $\sigma_3(2) = 1.5$, and $\sigma_3(3) = 1$.

boundedness and stochastic persistence when $B > 0$ and $0 < \theta_j(i) \leq 1$ ($j = 1, 2$). By constructing appropriate Lyapunov functions, we derive the conditions for the existence of an ergodic stationary distribution of the model. Additionally, we discuss the scenarios of population extinction and the asymptotic behavior of solutions. It can be shown that population x goes extinct when $J_{11} < 0$, population y goes extinct when $J_{21} < 0$, and population z goes extinct when $J_3 < 0$.

The study revealed that the survival status of individual sub-models does not influence the overall survival status of the model. Specifically, populations may be extinct within certain sub-models, yet they may persist within the overarching model. Conversely, a population may survive within a particular sub-state but be extinct in the context of the model. Moreover, it is evident that the value of v significantly influences the population of y, whereas it has negligible effects on the populations of x and z. Additionally, it is evident that the value of f significantly influences the populations of x and z, while having negligible effects on the population of y. In natural systems, the survival status of populations can be influenced through artificial intervention in the magnitude of environmental random noise. By increasing or decreasing such noise, we can maintain or suppress the populations' growth.

By incorporating habitat selection behavior, the refuge effect, white-noise interference, and colored-noise interference into the model, we can more realistically reflect the survival strategies of prey in natural environments and make more accurate predictions of predator-prey interactions, thereby fostering a deeper understanding of the complexities inherent in population dynamics. Via theoretical research to derive the properties of the model's solutions and analysis of population survival status, we can optimize conservation strategies to reduce the risk of species extinction. Practical applications involve the artificial construction of suitable habitats for endangered prey under intense predation pressure,

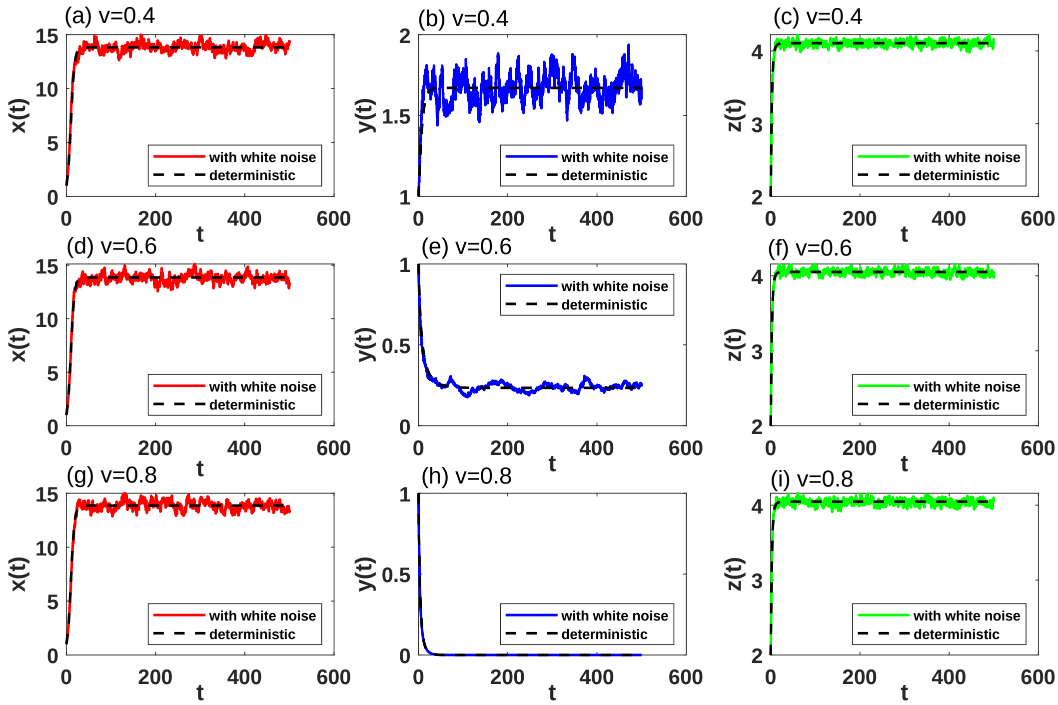

Figure 11. Time series of the sub-model (2.4) with different level v in Case 1.

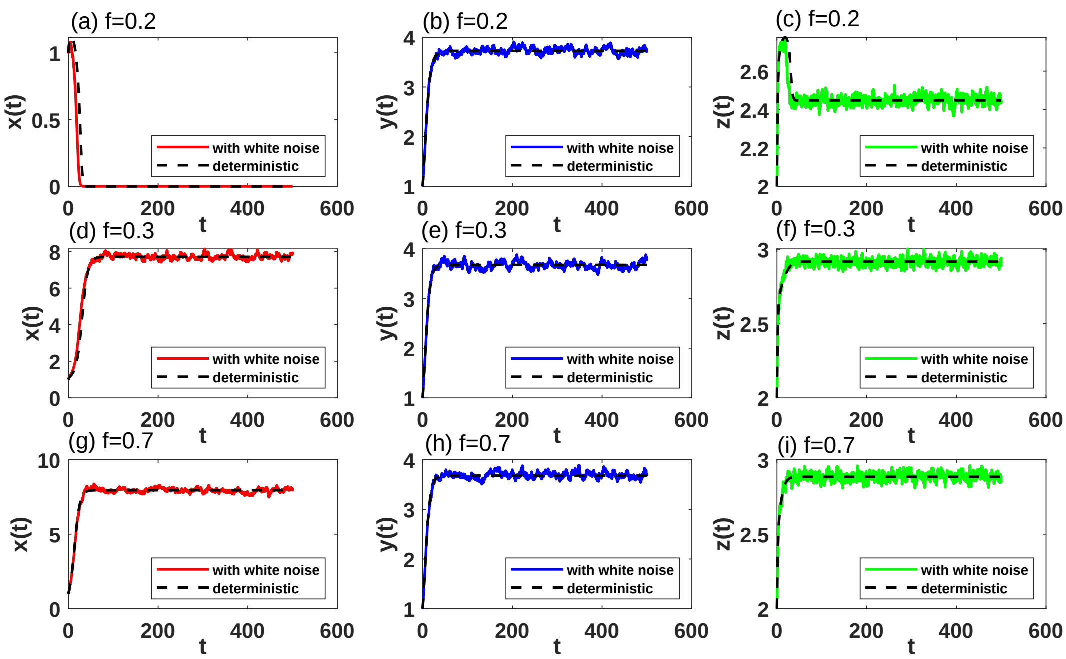

Figure 12. Time series of the sub-model (2.4) with different level f in Case 2.

which can act as natural refuges and effectively mitigate predation pressure. Thus, exploring the intricate dynamical behaviors associated with these ecological regulation processes represents a valuable and promising direction for future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. C. S. Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entomol. Soc. Can.*, **97** (1965), 5–60. https://doi.org/10.4039/entm9745fv
2. R. Arditi, H. Saiah, Empirical evidence of the role of heterogeneity in ratio-dependent consumption, *Ecology*, **73** (1992), 1544–1551. https://doi.org/10.2307/1940007
3. A. P. Gutierrez, Physiological basis of ratio-dependent predator-prey theory: The metabolic pool model as a paradigm, *Ecology*, **73** (1992), 1552–1563. https://doi.org/10.2307/1940008
4. R. Arditi, L. R. Ginzburg, Coupling in predator-prey dynamics: Ratio-dependence, *J. Theor. Biol.*, **139** (1989), 311–326. https://doi.org/10.1016/S0022-5193(89)80211-5
5. M. Chen, R. Wu, B. Liu, L. Chen, Spatiotemporal dynamics in a ratio-dependent predator-prey model with time delay near the Turing-Hopf bifurcation point, *Commun. Nonlinear Sci. Numer. Simul.*, **77** (2019), 141–167. https://doi.org/10.1016/j.cnsns.2019.04.024
6. X. Jiang, Z. She, S. Ruan, Global dynamics of a predator-prey system with density-dependent mortality and ratio-dependent functional response, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 1967–1990. https://doi.org/10.3934/dcdsb.2020041
7. M. Chen, X. Z. Li, C. Tian, Spatiotemporal dynamics in a three-component predator-prey model, *Appl. Math. Lett.*, **163** (2025), 109424. https://doi.org/10.1016/j.aml.2024.109424
8. P. F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Corr. Math. Phys.*, **10** (1838), 113–129.
9. F. J. Ayala, M. E. Gilpin, J. G. Ehrenfeld, Competition between species: Theoretical models and experimental tests, *Theor. Popul. Biol.*, **4** (1973), 331–356. https://doi.org/10.1016/0040-5809(73)90014-2

10. Y. Jiang, Z. Liu, J. Yang, Y. Tan, Dynamics of a stochastic Gilpin-Ayala population model with Markovian switching and impulsive perturbations, *Adv. Differ. Equations*, **2020** (2020), 530. https://doi.org/10.1186/s13662-020-02900-w

11. X. Yu, M. Huang, Dynamics of a Gilpin-Ayala predator-prey system with state feedback weighted harvest strategy, *AIMS Math.*, **8** (2023), 26968–26990. https://doi.org/10.3934/math.20231380

12. D. Song, C. Li, Y. Song, Stability and cross-diffusion-driven instability in a diffusive predator-prey system with hunting cooperation functional response, *Nonlinear Anal. Real World Appl.*, **54** (2020), 103106. https://doi.org/10.1016/j.nonrwa.2020.103106

13. X. Yan, Y. Tian, K. Sun, Hybrid effects of cooperative hunting and inner fear on the dynamics of a fishery model with additional food supplement, *Math. Methods Appl. Sci.*, **48** (2025), 9389–9403. https://doi.org/10.1002/mma.10805

14. D. Mukherjee, Role of fear in predator-prey system with intraspecific competition, *Math. Comput. Simul.*, **177** (2020), 263–275. https://doi.org/10.1016/j.matcom.2020.04.025

15. X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator-prey interactions, *J. Math. Biol.*, **73** (2016), 1179–1204. https://doi.org/10.1007/s00285-016-0989-1

16. D. Sen, S. Ghorai, S. Sharma, M. Banerjee, Allee effect in prey's growth reduces the dynamical complexity in prey-predator model with generalist predator, *Appl. Math. Model.*, **91** (2021), 768–790. https://doi.org/10.1016/j.apm.2020.09.046

17. M. Chen, C. Tian, S. Ham, H. Kim, J. Kim, Impact of prey-taxis on a harvested intraguild predation predator-prey model, *Eur. J. Appl. Math.*, **36** (2025), 1183–1220. https://doi.org/10.1017/S0956792525000087

18. H. Molla, S. Sarwardi, M. Haque, Dynamics of adding variable prey refuge and an Allee effect to a predator-prey model, *Alex. Eng. J.*, **61** (2022), 4175–4188. https://doi.org/10.1016/j.aej.2021.09.039

19. T. Wen, Y. Xu, M. He, L. Rong, Modelling the dynamics in a predator-prey system with Allee effects and anti-predator behavior, *Qual. Theory Dyn. Syst.*, **22** (2023), 116. https://doi.org/10.1007/s12346-023-00821-z

20. E. Gazagne, T. Savini, D. Ngoprasert, P. Poncin, M. C. Huynen, F. Brotcorne, When northern pigtailed macaques (Macaca leonina) cannot select for ideal sleeping sites in a degraded habitat, *Int. J. Primatol.*, **41** (2020), 614–633. https://doi.org/10.1007/s10764-020-00173-4

21. D. Li, Q. Zhou, X. Tang, H. Huang, C. Huang, Sleeping site use of the white-headed langur Trachypithecus leucocephalus: The role of predation risk, territorial defense, and proximity to feeding sites, *Curr. Zool.*, **57** (2011), 260–268. https://doi.org/10.1093/czoolo/57.3.260

22. E. E. Werner, J. F. Gilliam, D. J. Hall, G. G. Mittelbach, An experimental test of the effects of predation risk on habitat use in fish, *Ecology*, **64** (1983), 1540–1548. https://doi.org/10.2307/1937508

23. A. R. Ives, A. P. Dobson, Antipredator behavior and the population dynamics of simple predator-prey systems, *Am. Nat.*, **130** (1987), 431–447. https://doi.org/10.1086/284719

24. Y. Wang, X. Y. Meng, Bifurcation and control of a delayed Leslie-Gower fractional order predator-prey model with fear effect and prey refuge, *Adv. Cont. Discr. Mod.*, **2025** (2025), 103. https://doi.org/10.1186/s13662-025-03959-z

25. X. Y. Meng, X. Z. Sun, Dynamics of a stage-structure diffusive predator-prey model with nonlinear prey refuge, delay and anti-predator behavior, *Int. J. Appl. Comput. Math.*, **12** (2026), 2. https://doi.org/10.1007/s40819-025-02077-4

26. Z. Liang, X. Meng, Stability and Hopf bifurcation of a multiple delayed predator-prey system with fear effect, prey refuge and Crowley-Martin function, *Chaos, Solitons Fractals*, **175** (2023), 113955. https://doi.org/10.1016/j.chaos.2023.113955

27. R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, 2001. https://doi.org/10.1515/9780691206912

28. C. Qin, J. Du, Y. Hui, Dynamical behavior of a stochastic predator-prey model with Holling-type III functional response and infectious predator, *AIMS Math.*, **7** (2022), 7403–7418. https://doi.org/10.3934/math.2022413

29. H. Qi, X. Meng, Dynamics of a stochastic predator-prey model with fear effect and hunting cooperation, *J. Appl. Math. Comput.*, **69** (2023), 2077–2103. https://doi.org/10.1007/s12190-022-01746-7

30. S. Hu, J. Li, L. Zhu, J. Akeodi, The artificial propagation effects comparison of African catfish in rainy season and dry season, *Guizhou Agric. Sci.*, **3** (2009), 108–110.

31. Q. Luo, X. Mao, Stochastic population dynamics under regime switching, *J. Math. Anal. Appl.*, **334** (2007), 69–84. https://doi.org/10.1016/j.jmaa.2006.12.032

32. Q. Wang, L. Zu, D. Jiang, D. O'Regan, Study on dynamic behavior of a stochastic predator-prey system with Beddington-DeAngelis functional response and regime switching, *Mathematics*, **11** (2023), 2735. https://doi.org/10.3390/math11122735

33. X. Jiang, L. Zu, D. Jiang, D. O'Regan, Analysis of a stochastic Holling type II predator-prey model under regime switching, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 2171–2197. https://doi.org/10.1007/s40840-019-00798-6

34. Q. Liu, L. Zu, D. Jiang, Dynamics of stochastic predator-prey models with Holling II functional response, *Commun. Nonlinear Sci. Numer. Simul.*, **37** (2016), 62–76. https://doi.org/10.1016/j.cnsns.2016.01.005

35. S. Zhang, T. Zhang, S. Yuan, Dynamics of a stochastic predator-prey model with habitat complexity and prey aggregation, *Ecol. Complexity*, **45** (2021), 100889. https://doi.org/10.1016/j.ecocom.2020.100889

36. C. Jeffries, Stability of predation ecosystem models, *Ecology*, **57** (1976), 1321–1325. https://doi.org/10.2307/1935058

37. Y. Tian, J. Zhu, J. Zheng, K. Sun, Modeling and analysis of a prey-predator system with prey habitat selection in an environment subject to stochastic disturbances, *Electron. Res. Arch.*, **33** (2025), 744–767. https://doi.org/10.3934/era.2025034

38. X. Li, A. Gray, D. Jiang, X. Mao, Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching, *J. Math. Anal. Appl.*, **376** (2011), 11–28. https://doi.org/10.1016/j.jmaa.2010.10.053

39. X. Mao, *Stochastic Differential Equations and Applications*, 2^{nd} edition, Woodhead Publishing, Cambridge, 2007.

40. A. Settati, A. Lahrouz, On stochastic Gilpin-Ayala population model with Markovian switching, *Biosystems*, **130** (2015), 17–27. https://doi.org/10.1016/j.biosystems.2015.01.004

41. M. Liu, K. Wang, Q. Wu, Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle, *Bull. Math. Biol.*, **73** (2011), 1969–2012. https://doi.org/10.1007/s11538-010-9569-5

42. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302