



---

*Research article*

## Constructing abundant exact stochastic solutions to the Newell-Whitehead-Segel equation with multiplicative advection noise

Abeer H. Alblowy<sup>1</sup>, Wael W. Mohammed<sup>1,\*</sup>, Hessa W. Alshammari<sup>1</sup> and Elsayed M. Elsayed<sup>2,3</sup>

<sup>1</sup> Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia

<sup>2</sup> Mathematics Department, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

<sup>3</sup> Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

\* **Correspondence:** Email: w.mohammed@uoh.edu.sa.

**Abstract:** In this work, we consider the stochastic Newell-Whitehead-Segel (SNWS) equation, which is forced in the Itô sense by multiplicative advection noise. We demonstrate that the solution of the SNWS equation can be discovered by solving certain deterministic counterparts of the Newell-Whitehead-Segel equation with an extra diffusion term (DNWS), and then the results are combined with a solution for stochastic ordinary differential equations. We use two techniques to solve the DNWS equation: the  $(\frac{G'}{G})$ -expansion method and the  $\exp(-\psi(\eta))$ -expansion method. The purpose of studying the Newell-Whitehead-Segel equation with multiplicative noise is to explore how stochastic fluctuations influence the stability and evolution of patterns near a bifurcation. Incorporating noise provides a more realistic description of physical systems and reveals impacts that are omitted from the deterministic model, such as changing thresholds, different pattern structures, and increased variability. Therefore, we use a MATLAB tool to exhibit numerous 3D and 2D graphs that show how multiplicative advection noise impacts the solutions of the SNWS equation.

**Keywords:** noise strength; simulation; stochastic solutions; analytical methods; mathematical model; exact solutions

---

### 1. Introduction

The Newell-Whitehead-Segel (NWS) equation is significant in pattern formation theory and can be applied to a variety of physical, biological, and chemical phenomena. In fluid dynamics, it describes the amplitude of convection rolls in Rayleigh-Bénard convection and the beginning of oscillatory or stationary flow patterns. In nonlinear optics, the equation characterizes the evolution of slowly changing light-field envelopes in laser cavities and optical media on the verge of instability. It

is also commonly employed in chemical reaction-diffusion systems to characterize Turing patterns, such as stripes, spots, and front-like structures caused by diffusion-driven instabilities. Beyond these applications, the NWS equation offers a useful framework for explaining biological pattern development, such as pigmentation or morphogenesis, as well as assessing instability-driven structures in soft materials and thin films. The NWS equation is typically written as:

$$\mathcal{V}_t - \mathcal{V}_{xx} - m\mathcal{V} + n\mathcal{V}^3 = 0, \quad (1.1)$$

where  $\mathcal{V} = \mathcal{V}(t, x)$  is the amplitude of the pattern, and  $m$  and  $n$  are positive real numbers.

Due to the importance of the NWS equation, many authors have used a variety of techniques, such as the variational iteration method [1, 2], cubic B-spline collection method [3], differential transform method [4], adomian decomposition method [5], homotopy perturbation method [6, 7], modified homotopy perturbation transform method [8], extended direct algebraic method [9], the first integral method [10], and sine-Gordon equation expansion [11], to find the approximate and exact solutions. In general, numerous methods are employed to obtain solutions of nonlinear evolution equations, such as the enhanced algebraic method [12], the modified extended direct algebraic method [13], the dynamical system method [14], the generalizing Riccati equation mapping method [15], the Jacobi elliptic function method [16], the optimal auxiliary function method [17], and the Riccati-Bernoulli sub-ODE method [18], etc.

On the other side, external variations and inherent system uncertainties impact pattern formation near the onset of instability. Real-world physical, chemical, and biological systems are usually unpredictable; variables like the growth rates, diffusion coefficients, or reaction strengths vary due to thermal noise, external forces, or microscopic variability. Adding randomness, especially multiplicative noise, to the NWS equation lets it cope with these real-world changes and shows how noise affects nonlinear amplitude dynamics. This inclusion reveals significant phenomena, including noise-induced alterations in bifurcation thresholds, changes in pattern selection, modulation of coherent structures, and the formation of novel spatiotemporal behaviors, absent in the deterministic model. Consequently, the stochastic NWS paradigm offers a more precise, resilient, and physically significant characterization of pattern-forming systems functioning amidst real-world variability.

As a result, we consider the following stochastic NWS (SNWS) equation with multiplicative advection noise:

$$d\mathcal{V} - [\mathcal{V}_{xx} + m\mathcal{V} - n\mathcal{V}^3]dt = \rho\mathcal{V}_x dW, \quad (1.2)$$

where  $W(t)$  is a standard Wiener process and it depends only on  $t$ , and  $\rho$  is the strength of noise. Several studies have investigated the SNWS equation with different noise, including the Wiener-Hermite expansion technique [19], Nucci's reduction method [20], and Crank-Nicolson scheme [21]. All of these studies considered stochastic perturbations of the form  $\rho dW$  or  $\rho\mathcal{V}dW$ . In contrast, the present work derives exact solutions of Eq (1.2) in the presence of multiplicative advection noise of the form  $\rho\mathcal{V}_x dW$ .

The primary goal of this work is to find precise solutions to the SNWS Eq (1.2) with advection noise. This is accomplished by dividing the SNWS equation into two equations. The first equation is the deterministic NWS (DNWS) equation, and the second is the stochastic ordinary differential equation (SODE), which are obtained from the SNWS equation via Itô calculus and appropriate transformation processes. By applying the  $\exp(-\varphi(\xi))$ -expansion and  $(\frac{G'}{G})$ -expansion methods, we get the solutions of the DNWS equation. After that, by combining the results that we obtained with a solution of

the SODE, we obtain the solutions of the SNWS Eq (1.2). The purpose of studying the Newell-Whitehead-Segel equation with multiplicative noise is to determine how stochastic perturbations affect the mechanisms of amplitude modulation and mode selection in weakly nonlinear systems. In many pattern-forming situations, noise interacts nonlinearly with amplitude dynamics, which can change dominating wavelengths and promote intermittent switching between competing modes. As a result, we provide a few 3D and 2D graphics created with MATLAB tools to demonstrate the impact of the stochastic term.

The organization of this paper is as follows. The following section introduces a lemma that decomposes the SNWS Eq (1.2) into a stochastic ordinary differential equation (SODE) and a DNWS equation with an additional diffusion term. Section 3 focuses on solving the DNWS equation, whereas Section 4 offers solutions to the SNWS Eq (1.2). Section 5 discusses the physical meaning of the stochastic term and how it affects the resulting solutions. Finally, the main results of this study are summarized.

## 2. Preliminaries

The Wiener process  $\{W(t), t \geq 0\}$  is a fundamental concept in probability theory and mathematical finance [22]. It describes a continuous-time stochastic process that captures the erratic movement of fluid-suspended particles. Mathematically, a Wiener process  $W(t)$  is defined by several key properties. First, it starts at zero (i.e.,  $W(0) = 0$ ). Second, its paths are continuous. Additionally, the increments of the process are independent, meaning that the movement over one time interval does not influence the movement over another. Another crucial property is that these increments are normally distributed: for any two time points  $s$  and  $t$  where  $s < t$ , the increment  $W(t) - W(s)$  follows a normal distribution with mean zero and variance  $t - s$ .

The next lemma (see for more detail [23]) demonstrates that the solutions of the SNWS Eq (1.2) can be obtained by solving deterministic counterparts of the NWS equation and merging the result with the solution of SODEs:  $X_t(t) = x + \rho W(t)$ .

**Lemma 1.** *The SNWS Eq (1.2) has the solution  $\mathcal{V}(t, x) = \mathcal{P}(t, X_t)$ , where  $\mathcal{P}$  solves the DNWS equation:*

$$\mathcal{P}_t - \mathcal{P}_{xx} - m\mathcal{P} + n\mathcal{P}^3 + \frac{\rho^2}{2}\mathcal{P}_{xx} = 0, \text{ with } \mathcal{P}(0, x) = \mathcal{P}_0(x), \quad (2.1)$$

for  $(1 - \frac{\rho^2}{2}) > 0$ , and  $X_t$  solves the next SODE:

$$dX_t = \rho dW, \quad (2.2)$$

with initial values

$$X_0 = x.$$

*Proof.* We apply the Itô formula that is stated in [24] to the transformation  $\mathcal{P}(t, X_t)$  as follows:

$$d\mathcal{P}(t, X_t) = \frac{\partial \mathcal{P}}{\partial t} dt + \frac{\partial \mathcal{P}}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 \mathcal{P}}{\partial t^2} (dt)^2 + \frac{1}{2} \frac{\partial^2 \mathcal{P}}{\partial x^2} (dX_t)^2 + \frac{\partial^2 \mathcal{P}}{\partial t \partial x} dt dX_t.$$

Using (2.1) and (2.2), we obtain

$$\begin{aligned} d\mathcal{P}(t, X_t) &= [\mathcal{P}_{xx} + m\mathcal{P} - n\mathcal{P}^3 - \frac{\rho^2}{2}\mathcal{P}_{xx}]dt + \rho\mathcal{P}_x dW + \frac{1}{2}\frac{\partial^2\mathcal{P}}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2\mathcal{P}}{\partial x^2}(\rho dW)^2 + \frac{\partial^2\mathcal{P}}{\partial t\partial x}dt dX_t \\ &= [\mathcal{P}_{xx} + m\mathcal{P} - n\mathcal{P}^3]dt + \rho\mathcal{P}_x dW. \end{aligned}$$

Now, we use  $(dt)^2 = dt dW = 0$  and  $(dW)^2 = dt$  to get

$$\begin{aligned} d\mathcal{P}(t, X_t) &= [\mathcal{P}_{xx} + m\mathcal{P} - n\mathcal{P}^3 - \frac{\rho^2}{2}\mathcal{P}_{xx}]dt + \rho\mathcal{P}_x dW + \frac{\rho^2}{2}\mathcal{P}_{xx}dt \\ &= [\mathcal{P}_{xx} + m\mathcal{P} - n\mathcal{P}^3]dt + \rho\mathcal{P}_x dW. \end{aligned}$$

Since  $\mathcal{V} = \mathcal{P}$ , we obtain (1.2).

### Existence and uniqueness of the solutions:

If  $\rho^2 < 2$  and  $\mathcal{P}_0 \in \mathcal{H}^1(\mathbb{R})$  (or  $L^2(\mathbb{R})$ ), then there exists a unique local-in-time solution for the DNWS Eq (2.1):

$$\mathcal{P} \in C([0, T]; \mathcal{H}^1(\mathbb{R})) \cap C^1((0, T); L^2(\mathbb{R})),$$

for some  $T > 0$ . Moreover, the solution extends globally in time for arbitrary initial data in the dissipative case  $n > 0$ , whereas for  $n < 0$ , global existence is ensured only for sufficiently small initial data. These results follow from classical energy estimates and the locally Lipschitz nature of the nonlinear term  $m\mathcal{P} - n\mathcal{P}^3$ . For further details on the existence and uniqueness of solutions to the DNWS equation (see Eq (2.1)), the reader is referred to the references [7, 25, 26].

### 3. The wave equation of the DNWS equation

To obtain the wave equation corresponding to the DNWS Eq (2.1), we apply the following transformation:

$$\mathcal{P}(t, x) = \mathcal{R}(\xi), \quad \xi = kx + \lambda t, \quad (3.1)$$

where  $\mathcal{R}$  is a deterministic function. We see that

$$\mathcal{P}_x = k\mathcal{R}', \quad \mathcal{P}_{xx} = k^2\mathcal{R}'', \quad \text{and} \quad \mathcal{P}_t = \lambda\mathcal{R}'. \quad (3.2)$$

Plugging Eqs (3.1) and (3.2) into Eq (2.1), we have

$$\mathcal{R}'' + \hbar_0\mathcal{R}' + \hbar_1\mathcal{R} + \hbar_2\mathcal{R}^3 = 0, \quad (3.3)$$

where

$$\hbar_0 = \frac{2\lambda}{k^2(\rho^2 - 2)}, \quad \hbar_1 = \frac{-2m}{k^2(\rho^2 - 2)}, \quad \text{and} \quad \hbar_2 = \frac{2n}{k^2(\rho^2 - 2)},$$

for  $\rho \neq \sqrt{2}$ .

#### 4. Solutions of the DNWS equation

In this section, we derive exact solutions of the DNWS Eq (2.1) by employing the  $\exp(-\varphi(\xi))$ -expansion and  $(\frac{\mathcal{G}'}{\mathcal{G}})$ -expansion methods. These approaches are selected for their effectiveness in treating nonlinear dispersive wave equations. The  $\exp(-\varphi(\xi))$ -expansion method is capable of generating a wide range of solitary-wave and exponential-type solutions, whereas the  $(\frac{\mathcal{G}'}{\mathcal{G}})$ -expansion method yields closed-form solutions expressed in terms of hyperbolic, trigonometric, or exponential functions. Owing to the presence of the derivative term  $\mathcal{Z}'$  in Eq (3.3), certain techniques such as the mapping method, the Sarder method, and the  $\varphi^6$ -expansion scheme, cannot be applied in this context.

##### 4.1. The $\exp(-\varphi(\xi))$ -expansion method

Consider the  $\exp(-\varphi(\xi))$ -expansion method that is stated in [27]. Assume the solutions of Eq (3.3) are

$$\mathcal{R}(\xi) = \sum_{\kappa=0}^m \ell_{\kappa} e^{-\kappa\varphi(\xi)}, \text{ such that } \ell_m \neq 0, \quad (4.1)$$

where the constants  $\ell_0, \ell_1, \dots, \ell_m$  will be determined later, and  $\varphi = \varphi(\xi)$  satisfies:

$$\varphi' = b + a \exp(\varphi) + \exp(-\varphi), \quad (4.2)$$

where  $a$  and  $b$  are constants. Applying the homogeneous balance rule, we equate the highest powers of  $\mathcal{R}^3$  and  $\mathcal{R}''$  in (3.3) as:

$$3m = m + 2 \Rightarrow m = 1.$$

Hence, Eq (4.1) with  $m = 1$  becomes

$$\mathcal{R}(\xi) = \ell_0 + \ell_1 \exp(-\varphi(\xi)). \quad (4.3)$$

Plugging Eqs (4.3) and (4.2) into Eq (3.3), we attain

$$\begin{aligned} & (2\ell_1 + \hbar_2 \ell_1^3) e^{-3\varphi} + (3b\ell_1 - \hbar_0 \ell_1 + 3\hbar_2 \ell_0 \ell_1^2) e^{-2\varphi} + (2a\ell_1 + b^2 \ell_1 - b\hbar_0 \ell_1 \\ & + \ell_1 \hbar_1 + 3\hbar_2 \ell_0^2 \ell_1) e^{-\varphi} + (a b \ell_1 - a \hbar_0 \ell_1 + \hbar_1 \ell_0 + \hbar_2 \ell_0^3) = 0. \end{aligned}$$

Substituting the coefficients of  $\exp(-\kappa\varphi)$  equal to zero, for  $\kappa = 3, 2, 1$ , and  $0$ , we obtain

$$2\ell_1 + \hbar_2 \ell_1^3 = 0,$$

$$3b\ell_1 - \hbar_0 \ell_1 + 3\hbar_2 \ell_0 \ell_1^2 = 0,$$

$$2a\ell_1 + b^2 \ell_1 - b\hbar_0 \ell_1 + \ell_1 \hbar_1 + 3\hbar_2 \ell_0^2 \ell_1 = 0,$$

and

$$a b \ell_1 - a \hbar_0 \ell_1 + \hbar_1 \ell_0 + \hbar_2 \ell_0^3 = 0. \quad (4.4)$$

We obtain by solving these equations:

$$\ell_0 = \mp \frac{\hbar_0}{6} \sqrt{\frac{-2}{\hbar_2}}, \ell_1 = \pm \sqrt{\frac{-2}{\hbar_2}}, a = -\frac{\hbar_1}{8} \text{ and } , b = 0, \quad (4.5)$$

for  $\hbar_2 < 0$ . Substituting the values of  $\ell_0$  and  $\ell_1$  into Eq (4.3), we have

$$\mathcal{R}(\xi) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \exp(-\varphi(\xi)) \right). \quad (4.6)$$

Equation (4.2) possesses various solutions determined by the parameters  $a$  and  $b$ , which are given as follows:

**Case I:** If  $\Omega = b^2 - 4a = -4a > 0$  (i.e.,  $a < 0$ ), then Eq (4.2) has the solutions:

$$\varphi(\xi) = \ln \left( \frac{1}{\sqrt{-a}} \tanh \left( \sqrt{-a}(\xi + C) \right) \right), \quad (4.7)$$

and

$$\varphi(\xi) = \ln \left( \frac{1}{\sqrt{-a}} \coth \left( \sqrt{-a}(\xi + C) \right) \right), \quad (4.8)$$

where  $C$  is the constant of integration.

Now, the solutions of the DNWS Eq (2.1) are acquired by substituting Eqs (4.7) and (4.8) into (4.6) and applying Eq (3.1), yielding for  $\hbar_2 < 0$ :

$$\mathcal{P}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \frac{\sqrt{-a}}{\tanh \left( \sqrt{-a}(kx + \lambda t + C) \right)} \right), \quad (4.9)$$

and

$$\mathcal{P}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \frac{\sqrt{-a}}{\coth \left( \sqrt{-a}(kx + \lambda t + C) \right)} \right). \quad (4.10)$$

**Case II:** If  $\Omega = -4a < 0$  (i.e.,  $a > 0$ ), then Eq (4.2) has the solutions:

$$\varphi(\xi) = \ln \left( \frac{1}{\sqrt{a}} \tan \left( \sqrt{a}(\xi + C) \right) \right), \quad (4.11)$$

and

$$\varphi(\xi) = \ln \left( \frac{-1}{\sqrt{a}} \cot \left( \sqrt{a}(\xi + C) \right) \right). \quad (4.12)$$

Thus, the solutions of the DNWS Eq (2.1) are obtained by substituting Eqs (4.11) and (4.12) into (4.6) and applying Eq (3.1), yielding for  $\hbar_2 < 0$ :

$$\mathcal{P}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \frac{\sqrt{a}}{\tan \left( \sqrt{a}(kx + \lambda t + C) \right)} \right), \quad (4.13)$$

and

$$\mathcal{P}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} + \frac{\sqrt{a}}{\cot \left( \sqrt{a}(kx + \lambda t + C) \right)} \right). \quad (4.14)$$

#### 4.2. The $(\frac{G'}{G})$ -expansion method

Now, we employ the  $(\frac{G'}{G})$ -expansion method [28, 29] to derive the solutions of DNWS Eq (2.1). Assume that, for  $m = 1$ , the solutions of Eq (3.3) can be expressed as follows:

$$\mathcal{R}(\xi) = \alpha_0 + \alpha_1 \left[ \frac{G'}{G} \right], \quad (4.15)$$

where  $G$  solves

$$G'' + \lambda G' + \nu G = 0, \quad (4.16)$$

where  $\lambda$  and  $\nu$  are unknown constants. Substituting Eq (4.15) into Eq (3.3), we obtain

$$\begin{aligned} & (2\alpha_1 + \hbar_2 \alpha_1^3) \left[ \frac{G'}{G} \right]^3 + (3\lambda \alpha_1 - \alpha_1 \hbar_0 + 3\hbar_2 \alpha_1^2 \alpha_0) \left[ \frac{G'}{G} \right]^2 \\ & + (\lambda^2 \alpha_1 + 2\alpha_1 \nu - \lambda \hbar_0 \alpha_1 + 3\hbar_2 \alpha_1 \alpha_0^2 + \alpha_1 \hbar_1) \left[ \frac{G'}{G} \right] \\ & + (\nu \lambda \alpha_1 - \nu \alpha_1 \hbar_0 + \alpha_0 \hbar_1 + \hbar_2 \alpha_0^3) = 0. \end{aligned}$$

Equating the coefficients of  $[\frac{G'}{G}]^J$  to zero for  $J = 3, 2, 1$ , and 0 yields

$$2\alpha_1 + \hbar_2 \alpha_1^3 = 0,$$

$$3\lambda \alpha_1 - \alpha_1 \hbar_0 + 3\hbar_2 \alpha_1^2 \alpha_0 = 0,$$

$$\lambda^2 \alpha_1 + 2\alpha_1 \nu - \lambda \hbar_0 \alpha_1 + 3\hbar_2 \alpha_1 \alpha_0^2 + \alpha_1 \hbar_1 = 0,$$

and

$$\nu \lambda \alpha_1 - \nu \alpha_1 \hbar_0 + \alpha_0 \hbar_1 + \hbar_2 \alpha_0^3 = 0.$$

By solving these equations, we obtain

$$\alpha_0 = \frac{\hbar_0 - 3\lambda}{3\hbar_2 \alpha_1}, \quad \alpha_1 = \pm \sqrt{\frac{-2}{\hbar_2}}, \quad \lambda = \lambda, \quad \text{and} \quad \nu = \frac{1}{12} \hbar_0^2 + \frac{1}{4} \lambda^2 - \frac{1}{2} \hbar_1. \quad (4.17)$$

The roots of auxiliary Eq (4.16) are:

$$\frac{-\lambda}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\nu}. \quad (4.18)$$

Based on the value of  $\Delta = \lambda^2 - 4\nu = 2\hbar_1 - \frac{1}{3}\hbar_0^2$ , the solutions of Eq (4.16) can be grouped into three separate cases as follows:

**Case I:** For  $\Delta = 0$ , the solution of Eq (4.16) is given by:

$$G(\xi) = (c_1 + c_2 \xi) \exp\left(\frac{-\lambda}{2} \xi\right), \quad (4.19)$$

where  $c_1$  and  $c_2$  are constants. Accordingly, utilizing Eqs (4.15) and (4.19), the solution of Eq (3.3) can be written as:

$$\mathcal{R}(\xi) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{c_2}{c_1 + c_2 \xi} \right), \quad \text{for } \hbar_2 < 0. \quad (4.20)$$

Hence, by applying Eq (3.1), the solution of the DNWS Eq (2.1) becomes:

$$\mathcal{P}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{c_2}{c_1 + c_2(kx + \lambda t)} \right). \quad (4.21)$$

**Case II:** For  $\Delta < 0$  and  $\hbar_2 < 0$ , the solution of Eq (4.16) is given by:

$$G(\xi) = \left( c_1 \sin \left( \frac{1}{2} \sqrt{-\Delta} \xi \right) + c_2 \cos \left( \frac{1}{2} \sqrt{-\Delta} \xi \right) \right) \exp \left( \frac{-\lambda}{2} \xi \right), \quad (4.22)$$

where  $c_1$  and  $c_2$  are constants. Thus, utilizing Eqs (4.15) and (4.19), the solution of Eq (3.3) can be written as:

$$\mathcal{R}(\xi) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{-\Delta} c_1 \cos \left( \frac{1}{2} \sqrt{-\Delta} \xi \right) - c_2 \sin \left( \frac{1}{2} \sqrt{-\Delta} \xi \right)}{2 c_1 \sin \left( \frac{1}{2} \sqrt{-\Delta} \xi \right) + c_2 \cos \left( \frac{1}{2} \sqrt{-\Delta} \xi \right)} \right). \quad (4.23)$$

Hence, by applying Eq (3.1), the solution of the DNWS Eq (2.1) becomes:

$$\mathcal{P}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{-\Delta} c_1 \cos \left( \frac{1}{2} \sqrt{-\Delta} (kx + \lambda t) \right) - c_2 \sin \left( \frac{1}{2} \sqrt{-\Delta} (kx + \lambda t) \right)}{2 c_1 \sin \left( \frac{1}{2} \sqrt{-\Delta} (kx + \lambda t) \right) + c_2 \cos \left( \frac{1}{2} \sqrt{-\Delta} (kx + \lambda t) \right)} \right). \quad (4.24)$$

**Case III:** For  $\Delta > 0$  and  $\hbar_2 < 0$ , the solution of Eq (4.16) is given by:

$$G(\xi) = \left( c_1 \sinh \left( \frac{1}{2} \sqrt{\Delta} \xi \right) + c_2 \cosh \left( \frac{1}{2} \sqrt{\Delta} \xi \right) \right) \exp \left( \frac{-\lambda}{2} \xi \right), \quad (4.25)$$

where  $c_1$  and  $c_2$  are constants. Consequently, utilizing Eqs (4.15) and (4.19), the solution of Eq (3.3) can be written as:

$$\mathcal{R}(\xi) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{\Delta} c_1 \cosh \left( \frac{1}{2} \sqrt{\Delta} \xi \right) + c_2 \sinh \left( \frac{1}{2} \sqrt{\Delta} \xi \right)}{2 c_1 \sinh \left( \frac{1}{2} \sqrt{\Delta} \xi \right) + c_2 \cosh \left( \frac{1}{2} \sqrt{\Delta} \xi \right)} \right). \quad (4.26)$$

Hence, by applying Eq (3.1), the solution of the DNWS Eq (2.1) becomes:

$$\mathcal{P}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{\Delta} c_1 \cosh \left( \frac{1}{2} \sqrt{\Delta} (kx + \lambda t) \right) + c_2 \sinh \left( \frac{1}{2} \sqrt{\Delta} (kx + \lambda t) \right)}{2 c_1 \sinh \left( \frac{1}{2} \sqrt{\Delta} (kx + \lambda t) \right) + c_2 \cosh \left( \frac{1}{2} \sqrt{\Delta} (kx + \lambda t) \right)} \right). \quad (4.27)$$

## 5. Exact solutions of the SNWS equation

To obtain the exact solutions of the SNWS Eq (1.2), we integrate Eq (2.2) from 0 to  $t$ :

$$X_t = x + \rho W(t). \quad (5.1)$$

Now, by applying the transformation  $\mathcal{V}(t, x) = \mathcal{P}(t, X_t)$  that is stated in Lemma 1 and the results from the previous section, the solutions of the SNWS Eq (1.2) are obtained as follows.

### 5.1. The $\exp(-\varphi(\xi))$ -expansion method

**Case I:** If  $\Omega = -4a > 0$  (i.e.,  $a < 0$ ) and  $\hbar_2 < 0$ , hence the solutions of the SNWS Eq (1.2) are

$$\mathcal{V}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \frac{\sqrt{-a}}{\tanh(\sqrt{-a}(kx + \rho kW(t) + \lambda t + C))} \right), \quad (5.2)$$

and

$$\mathcal{V}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \frac{\sqrt{-a}}{\coth(\sqrt{-a}(kx + \rho kW(t) + \lambda t + C))} \right). \quad (5.3)$$

**Case II:** If  $\Omega = -4a < 0$ ,  $a > 0$ , and  $\hbar_2 < 0$ , then the solutions of the SNWS Eq (1.2) are

$$\mathcal{V}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} - \frac{\sqrt{a}}{\tan(\sqrt{a}(kx + \rho kW(t) + \lambda t + C))} \right), \quad (5.4)$$

and

$$\mathcal{V}(t, x) = \mp \sqrt{\frac{-2}{\hbar_2}} \left( \frac{\hbar_0}{6} + \frac{\sqrt{a}}{\cot(\sqrt{a}(kx + \rho kW(t) + \lambda t + C))} \right). \quad (5.5)$$

### 5.2. The $(\frac{G'}{G})$ -expansion method

**Case I:** If  $\Delta = 0$  and  $\hbar_2 < 0$ , then the solution of the SNWS Eq (1.2) is

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{c_2}{c_1 + c_2(kx + \rho kW(t) + \lambda t)} \right). \quad (5.6)$$

**Case II:** If  $\Delta < 0$  and  $\hbar_2 < 0$ , then SNWS Eq (1.2) has the solution

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{-\Delta}}{2} \frac{c_1 \cos\left(\frac{1}{2} \sqrt{-\Delta} \Theta\right) - c_2 \sin\left(\frac{1}{2} \sqrt{-\Delta} \Theta\right)}{c_1 \sin\left(\frac{1}{2} \sqrt{-\Delta} \Theta\right) + c_2 \cos\left(\frac{1}{2} \sqrt{-\Delta} \Theta\right)} \right), \quad (5.7)$$

where  $\Theta = kx + \rho kW(t) + \lambda t$ .

**Case III:** If  $\Delta > 0$  and  $\hbar_2 < 0$ , then SNWS Eq (1.2) has the solution

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{\Delta}}{2} \frac{c_1 \cosh\left(\frac{1}{2} \sqrt{\Delta} \Theta\right) + c_2 \sinh\left(\frac{1}{2} \sqrt{\Delta} \Theta\right)}{c_1 \sinh\left(\frac{1}{2} \sqrt{\Delta} \Theta\right) + c_2 \cosh\left(\frac{1}{2} \sqrt{\Delta} \Theta\right)} \right), \quad (5.8)$$

where  $\Theta = kx + \rho kW(t) + \lambda t$ .

**Special cases:**

**Case I:** Putting  $c_1 = 0$  in Eq (5.7), we get

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} - \frac{\sqrt{-\Delta}}{2} \tan\left(\frac{1}{2} \sqrt{-\Delta}(kx + \rho kW(t) + \lambda t)\right) \right). \quad (5.9)$$

**Case II:** Putting  $c_2 = 0$  in Eq (5.7), we get

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{-\Delta}}{2} \cot \left( \frac{1}{2} \sqrt{-\Delta} (kx + \rho kW(t) + \lambda t) \right) \right). \quad (5.10)$$

**Case III:** Putting  $c_1 = 0$  in Eq (5.8), we get

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{\Delta}}{2} \tanh \left( \frac{1}{2} \sqrt{\Delta} (kx + \rho kW(t) + \lambda t) \right) \right). \quad (5.11)$$

**Case IV:** Putting  $c_2 = 0$  in Eq (5.8), we get

$$\mathcal{V}(t, x) = \pm \sqrt{\frac{-2}{\hbar_2}} \left( \frac{-\hbar_0}{6} + \frac{\sqrt{\Delta}}{2} \coth \left( \frac{1}{2} \sqrt{\Delta} (kx + \rho kW(t) + \lambda t) \right) \right). \quad (5.12)$$

## 6. Physical meaning and noise impacts

**Physical meaning:** In this work, we derived exact stochastic solutions of the SNWS equation. By employing two different analytical methods, we obtained a variety of solution types, including hyperbolic, trigonometric, and rational forms. These solutions correspond to distinct physical wave modes commonly observed in pattern-forming systems.

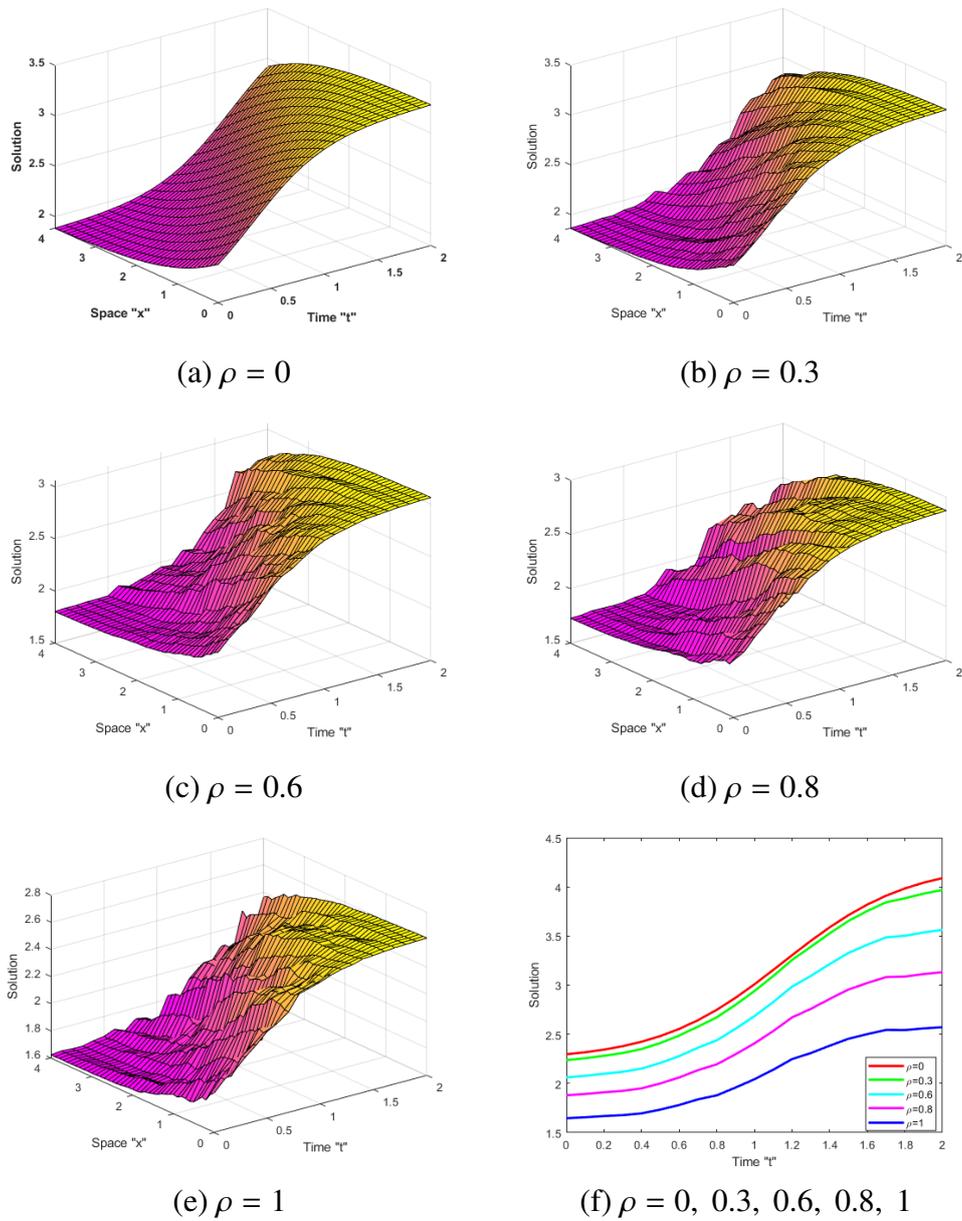
The hyperbolic-function solutions describe localized wave structures such as solitary waves and propagating fronts. In particular, the tanh-type solutions represent smooth transition fronts connecting two different steady states, which frequently arise in convection phenomena and bistable optical systems. The coth-type solutions correspond to strongly localized or singular modes and may be associated with sharp interfaces or defect-like structures. The presence of multiplicative Wiener noise induces stochastic modulation of the wave position, leading to random drift of these localized structures while preserving their overall shape.

The trigonometric-function solutions represent periodic wave patterns. Such solutions are characteristic of roll-type or stripe patterns in Rayleigh–Bénard convection and periodic intensity modulations in nonlinear optical media. Under stochastic forcing, these periodic modes experience random phase shifts and amplitude fluctuations, which are consistent with experimentally observed noise-induced perturbations of regular patterns.

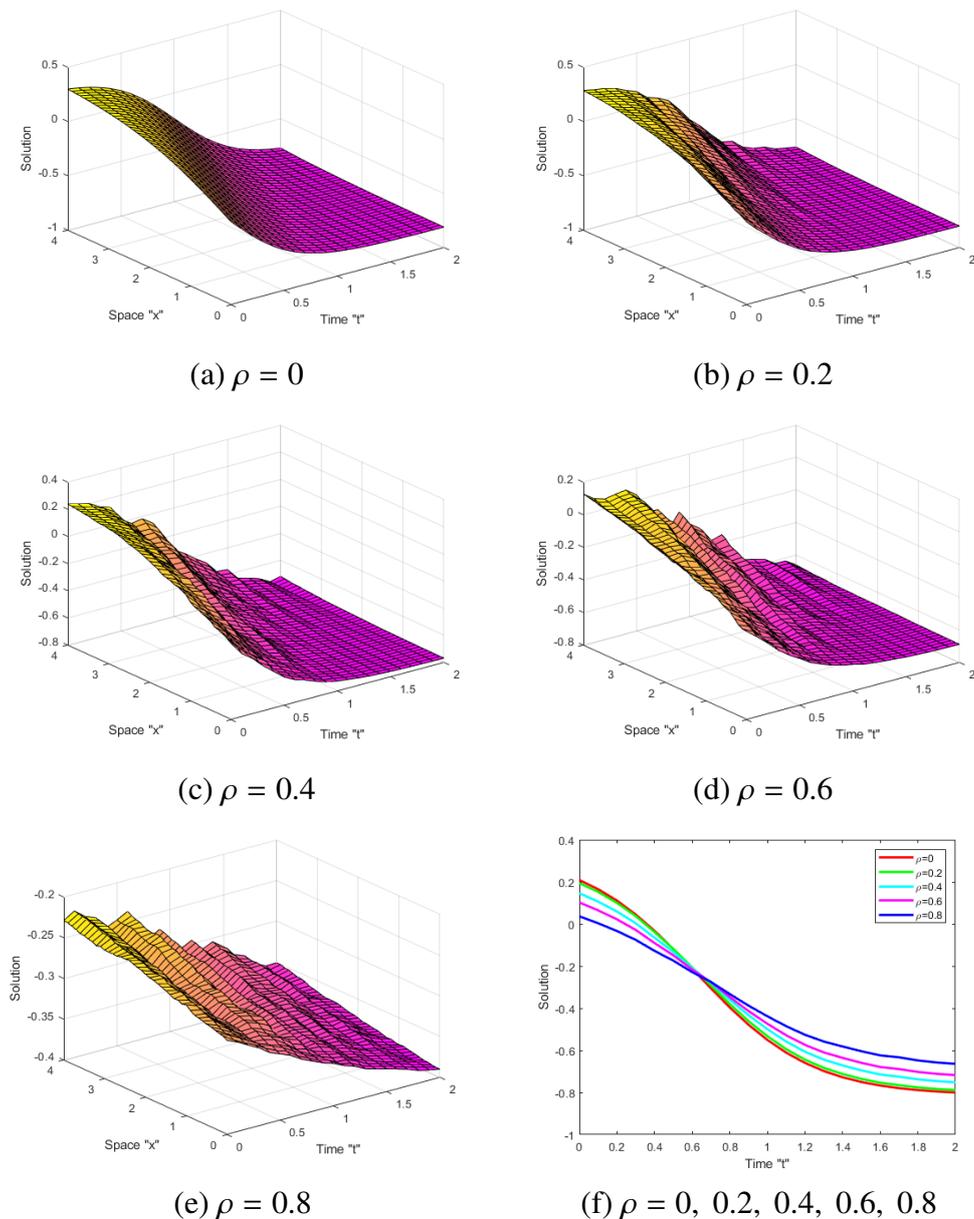
The rational solutions correspond to algebraically localized modes and may be viewed as limiting cases between periodic and solitary wave solutions. These solutions are particularly relevant for modeling transient localized structures, intermittent events, or defect-like patterns in noisy environments. In both optical and fluid systems, such modes are often associated with localized bursts or extreme events driven by stochastic effects.

In summary, the obtained exact solutions describe solitary, periodic, and rational wave modes of the SNWS equation under stochastic perturbations. This classification provides a clear physical interpretation of the analytical results and demonstrates their relevance to noise-driven pattern formation in convection, optics, and related physical systems.

**Impact of noise:** Let us simulate several solutions, such as the solutions described in Eqs (5.2) and (5.11), to see how multiplicative diffusion noise impacts them.



**Figure 1.** (a–e) provide the 3D shape for the solution  $\mathcal{V}(t, x)$  stated in Eq (5.2) with  $n = m = k = 1$ ,  $b = \sqrt{5}$ ,  $a = 1$ ,  $\lambda = -2$ ,  $C = 0$ ,  $x \in [0, 3]$ ,  $t \in [0, 2]$ , and with various  $\rho$ . (f) provides the 2D shape for this solution with a distinct value of  $\rho$ .



**Figure 2.** (a–e) provide the 3D shape for the solution  $\mathcal{V}(t, x)$  stated in Eq (5.11) with  $n = m = k = 1$ ,  $b = 2$ ,  $a = 1$ ,  $\lambda = -4$ ,  $x \in [0, 4]$ ,  $t \in [0, 2]$ , and with different  $\rho$ . (f) provides the 2D shape for this solution with a distinct value of  $\rho$ .

## 7. Discussion

The numerical simulations demonstrate that the inclusion of multiplicative noise has a clear and systematic impact on the evolution of the wave profile. In the deterministic case ( $\rho = 0$ ) as in Figure 1(a) and Figure 2(a), the solution exhibits a smooth and coherent front with a well-defined amplitude and monotonic growth. As the noise intensity appears as in Figure 1(b) and Figure 2(b),

small oscillations emerge along the spatial and temporal directions, indicating mild perturbations to the wave's regularity. Under stronger noise levels as in Figure 1(c–e) and Figure 2(c–e), these fluctuations become more pronounced, leading to noticeable roughness, local amplitude suppression, and irregular deformation of the wavefront.

Overall, the stochastic Newell–Whitehead–Segel equation models pattern formation near instability thresholds in dissipative systems such as Rayleigh–Bénard convection, nonlinear optical cavities, and reaction–diffusion media. In this work, multiplicative advection noise represents random transport effects arising from environmental fluctuations. The exact solutions obtained in this work describe coherent wave structures whose positions and phases undergo stochastic modulation while their overall profiles remain intact. In particular, localized solutions exhibit noise-induced wandering of fronts and solitary waves, periodic solutions display phase fluctuations and spatial irregularity, and rational solutions correspond to transient or defect-like modes. These results highlight how stochastic transport modifies pattern stability and coherence in physically relevant systems governed by the Newell–Whitehead–Segel equation.

## 8. Conclusions

This work examined the SNWS Eq (1.2) affected by multiplicative advection noise. Using transformation methods and Itô calculus, we broke down the SNWS equation into the deterministic DNWS equation Eq (2.1) and the SODE (2.2). The  $(\frac{G'}{G})$ -expansion approach and the  $\exp(-\varphi(\xi))$ -expansion method are two different techniques that we used to determine the solutions of the DNWS equation. Next, we combined our results with the SODE solution to find the solutions to the SNWS Eq (1.2). Additionally, we examined the effect of the multiplicative advection noise on the obtained solutions by simulating certain solutions. Finally we deduced that when the noise intensity increases, the originally smooth wave profile becomes progressively more irregular and fluctuating, indicating reduced stability and stronger stochastic distortion of the solution.

To clearly define the scope of the present study, we note that the analytical approach developed herein is restricted to the stochastic Newell-Whitehead-Segel equation driven by multiplicative advection noise within the Itô framework. The exact solutions obtained rely critically on this specific form of stochastic perturbation and on the structure of the associated wave transformation. As a result, the findings are not directly applicable to models involving additive noise, spatially dependent noise intensities, or stochastic forcing acting on the reaction terms. Extending the present analysis to such more general noise configurations, as well as to higher-dimensional settings, remains an open challenge and represents an important direction for future research.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This research has been funded by the Scientific Research Deanship at the University of Ha'il-Saudi Arabia through project number RG-24065.

## Conflict of interest

The authors declare there are no conflicts of interest.

## References

1. A. Prakash, M. Kumar, He's variational iteration method for the solution of non-linear Newell–Whitehead–Segel equation, *J. Appl. Anal. Comput.*, **6** (2016), 738–748. <https://doi.org/10.11948/2016048>
2. M. Nadeem, S. W. Yao, N. Parveen, Solution of Newell-Whitehead-Segel equation by variational iteration method with He's polynomials, *J. Math. Comput. Sci.*, **20** (2020), 21–29. <https://doi.org/10.22436/jmcs.020.01.03>
3. W. K. Zahra, W. A. Ouf, M. S. El-Azab, Cubic B-spline collocation algorithm for the numerical solution of Newell Whitehead Segel type equations, *Electron. J. Math. Anal. Appl.*, **2** (2014), 81–100. <https://doi.org/10.21608/ejmaa.2014.310179>
4. A. Aasaraai, Analytic solution for Newell–Whitehead–Segel equation by differential transform method, *Middle-East J. Sci. Res.*, **10** (2011), 270–273.
5. S. A. Manaa, An approximate solution to the Newell–Whitehead–Segel equation by the Adomian decomposition method, *AL-Rafidain J. Comput. Sci. Math.*, **8** (2011), 171–180. <https://doi.org/10.33899/csmj.2011.163617>
6. S. S. Nourazar, M. Soori, A. Nazari-Golshan, On the exact solution of Newell–Whitehead–Segel equation using the homotopy perturbation method, preprint, arXiv:1502.08016.
7. J. Patade, S. Bhalekar, Approximate analytical solutions of Newell–Whitehead–Segel equation using a new iterative method, *World J. Modell. Simul.*, **11** (2015), 94–103.
8. N. Iqbal, A. M. Albalahi, M. S. Abdo, W. W. Mohammed, Analytical analysis of Fractional-Order Newell-Whitehead-Segel equation: a modified homotopy perturbation transform method, *J. Funct. Spaces*, **2022** (2022), 1–10. <https://doi.org/10.1155/2022/3298472>
9. H. Ur Rehman, M. A. Imran, N. Ullah, A. Akgul, On solutions of the Newell–Whitehead–Segel equation and Zeldovich equation, *Math. Methods Appl. Sci.*, **44** (2021), 7134–7149. <https://doi.org/10.1002/mma.7249>
10. Y. M. Chu, S. Javeed, D. Baleanu, S. Riaz, H. Rezazadeh, New exact solutions of Kolmogorov Petrovskii Piskunov equation, Fitzhugh Nagumo equation, and Newell-Whitehead equation, *Adv. Math. Phys.*, **2020** (2020), 5098329. <https://doi.org/10.1155/2020/5098329>
11. A. Korkmaz, Complex wave solutions to mathematical biology models I: Newell-Whitehead-Segel and Zeldovich equations, *J. Comput. Nonlinear Dyn.*, **13** (2018), 081004. <https://doi.org/10.1115/1.4040411>
12. T. Han, Y. Liang, W. Fan, Dynamics and soliton solutions of the perturbed Schrödinger-Hirota equation with cubic-quintic-septic nonlinearity in dispersive media, *AIMS Math.*, **10** (2025), 754–776. <https://doi.org/10.3934/math.2025035>

13. T. Han, Y. Jiang, H. Fan, Exploring shallow water wave phenomena: A fractional approach to the Whitham-Broer-Kaup-Boussinesq-Kupershmidt system, *Ain Shams Eng. J.*, **16** (2025), 103700. <https://doi.org/10.1016/j.asej.2025.103700>
14. T. Han, K. Zhang, Y. Jiang, H. Rezazadeh, Chaotic pattern and solitary solutions for the (21)-dimensional beta-fractional double-chain DNA system, *Fractal Fract.*, **8** (2024), 415. <https://doi.org/10.3390/fractalfract8070415>
15. W. W. Mohammed, F. M. Al-Askar, New stochastic solitary solutions for the modified Korteweg-de Vries equation with stochastic term/random variable coefficients, *AIMS Math.*, **9** (2024), 20467–20481. <https://doi.org/10.3934/math.2024995>
16. W. W. Mohammed, N. Iqbal, R. Sidaoui, E. E. Ali, Dynamical behavior of the fractional nonlinear Kadoma equation in plasma physics and optics, *Mod. Phys. Lett. B*, **39** (2025), 2450434. <https://doi.org/10.1142/S0217984924504347>
17. N. Iqbal, S. Hussain, M. N. Tufail, W. W. Mohammed, Solving the fractional Fornberg-Whitham equation within Caputo framework using the optimal auxiliary function method, *Phys. Scr.*, **99** (2024), 055234. <https://doi.org/10.1088/1402-4896/ad3488>
18. N. Iqbal, S. Mukhtar, A. M. Saeed, R. Shah, S. Hussain, New solitary and soliton wave solutions of the fractional Higgs system using a Riccati-Bernoulli and Bäcklund framework, *Nonlinear Dyn.*, **113** (2025), 26505–26519. <https://doi.org/10.1007/s11071-025-11430-7>
19. J. Hussain, On the series solution of the stochastic Newell Whitehead Segel equation, *AIMS Math.*, **8** (2023), 21591–21605. <https://doi.org/10.3934/math.20231100>
20. M. S. Hashemi, M. Mirzazadeh, Exact solutions of nonlinear stochastic Newell-Whitehead-Segel equation by a reduction technique, *Eur. Phys. J. Plus*, **138** (2023), 1016. <https://doi.org/10.1140/epjp/s13360-023-04648-0>
21. M. S. Iqbal, M. W. Yasin, N. Ahmed, A. Akgül, M. Rafiq, A. Raza, Numerical simulations of nonlinear stochastic Newell-Whitehead-Segel equation and its measurable properties, *J. Comput. Appl. Math.*, **418** (2023), 114618. <https://doi.org/10.1016/j.cam.2022.114618>
22. O. Calin, *An Informal Introduction to Stochastic Calculus with Applications*, World Scientific, Singapore, 2015. <https://doi.org/10.1142/9620>
23. K. Adjibi, A. Martinez, M. Mascorro, C. Montes, T. Oraby, R. Sandoval, et al., Exact solutions of stochastic Burgers–Korteweg de Vries type equation with variable coefficients, *Partial Differ. Equations Appl. Math.*, **11** (2024), 100753. <https://doi.org/10.1016/j.padiff.2024.100753>
24. B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Universitext, Springer-Verlag, Berlin, 1998.
25. J. Hussain, M. Ali, Global well-posedness and exponential decay estimates for semilinear Newell-Whitehead-Segel equation, *Nonlinear Eng.*, **13** (2024), 20240041. <https://doi.org/10.1515/nleng-2024-0041>
26. P. W. M. Chin, The analysis of an efficient numerical scheme for the Allen–Cahn equations using the Galerkin method, *Commun. Nonlinear Sci. Numer. Simul.*, **105** (2022), 106061. <https://doi.org/10.1016/j.cnsns.2021.106061>

27. K. Khan, M. A. Akbar, Application of  $\exp(-\varphi(\xi))$ -expansion method to find the exact solutions of modified Benjamin-Bona-Mahony equation, *World Appl. Sci. J.*, **24** (2013), 1373–1377. <https://doi.org/10.5829/idosi.wasj.2013.24.10.1130>
28. M. L. Wang, X. Z. Li, J. L. Zhang, The  $(\frac{G'}{G})$ -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A*, **372** (2008), 417–423. <https://doi.org/10.1016/j.physleta.2007.07.051>
29. H. Zhang, New application of the  $(\frac{G'}{G})$ -expansion method, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 3220–3225. <https://doi.org/10.1016/j.cnsns.2009.01.006>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)