



Research article

Semi-B-Browder elements and Drazin invertible elements in Banach algebras

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Abstract: This paper studies left (right) Drazin invertible elements in a semisimple Banach algebra and provides the characterizations of the left (right) Drazin invertible elements. In addition, the paper proves that the left Drazin spectrum is compact. Finally, it is shown that being Drazin invertible is equivalent to being semi-B-Browder.

Keywords: semi-B-Browder elements; semi-B-Fredholm elements; left Drazin invertible; spectrum; Drazin invertible

1. Introduction

As a core component of operator theory and functional analysis, Fredholm theory serves as a pivotal framework for investigating the solvability of linear operator equations in infinite-dimensional spaces, boasting significant cross-disciplinary application value. In the field of differential equations, it leverages the Fredholm alternative theorem to accurately determine the existence and uniqueness of solutions to linear boundary value problems [1]; in time series analysis, it enables the modeling of autoregressive processes for nonstationary data based on Hilbert and Banach spaces [2]; in signal processing, the index and perturbation properties of its operators can support the design of inverse filtering algorithms, facilitating the effective extraction of target signals from noisy datasets. Therefore, in-depth research on Fredholm theory will lay a solid theoretical foundation for theoretical breakthroughs and methodological innovations in relevant application fields. Based on this, this paper intends to conduct further research on Fredholm theory within the framework of Banach algebras. In [3], Atkinson defined Fredholm operators on a Banach space X . After that, the generalized Fredholm operators and their properties were discussed by Schmoegeer [4]. In addition, the properties of the Fredholm operators can be referred to in [5].

Now, the Fredholm theory in the context of operators is generalized to the Fredholm theory in a Banach algebra by more and more scholars. Barnes [6] studied the Fredholm theory in rings.

Fredholm theories in von Neumann algebras were studied in [7]. Next, Barnes et al. [8] studied the Riesz and Fredholm theory in Banach algebras. Subsequently, Irmatov and Mishchenko [9] developed a more general version of Fredholm operators over C^* -algebras. Such operators are defined as bounded operators that are invertible modulo all compact operators (including nonadjointable). In addition, generalized B-Fredholm Banach algebra elements in Banach algebras were characterized [10]. Thereafter, Ivković [11, 12] studied the semi-Fredholm theory in Hilbert C^* -modules and unital C^* -algebras. Then, Ivković [13] studied the nonadjointable semi- C^* -Fredholm and semi- C^* -Weyl operators on Hilbert C^* -modules and showed that the set of nonadjointable semi- C^* -Weyl operators is invariant under perturbations by general compact operators. For further research findings on Fredholm theories in Banach algebras, the reader may refer to the works cited in [14].

On the other hand, research on invertible elements in Banach algebras has attracted widespread attention. Koliha [15] studied the generalized Drazin inverse. Roch and Silbermann [16] considered the continuity of generalized inverses in Banach algebras. Then, Ren and Jiang [17] discussed the left and right-Drazin inverses in rings and operator algebras. Subsequently, Zhang et al. [18] studied the generalized Drazin inverse of the sum of two elements in a Banach algebra, and they derived explicit formulae for the generalized Drazin inverse of a matrix in a Banach algebra. On this basis, they further analyzed the Moore-Penrose (MP) inverse of a block matrix [19]. Abad and Zguitti [20] considered the generalized Drazin-Riesz invertible elements in a semisimple Banach algebra.

Since Fredholm elements in a Banach algebra are defined by virtue of the invertibility in the quotient algebra, the study of invertible elements in Banach algebras is closely related to Fredholm theory. In fact, if we weaken the invertibility of elements in the quotient algebra to Drazin invertibility, generalized Drazin invertibility, MP invertibility, and so on, we can define “weaker” Fredholm elements. Based on this motivation, the purpose of this paper is to investigate the upper semi B-Browder elements and the lower semi B-Browder elements by means of the left Drazin invertible elements and the right Drazin invertible elements in Banach algebras, thereby generalizing the semi-Fredholm operator theory and the spectral theory. In this paper, first and foremost, Section 2 introduces some symbols and concepts. Next, Section 3 defines the left (respectively, right) Drazin invertible elements and provides the characterization of the left (respectively, right) Drazin invertible elements. In addition, we prove that the left Drazin spectrum is compact. Section 4 proves that the left Drazin invertible element is equivalent to the upper semi-B-Browder element and provides the characterization of the semi-B-Browder elements.

2. Preliminaries

Throughout this paper, all algebras are defined on \mathbb{C} . We always assume that \mathcal{A} is an infinite-dimensional semisimple Banach algebra with unit 1. The socle of \mathcal{A} in this case is the algebraic sum of all the minimal left ideals of \mathcal{A} (which equals the sum of all the minimal right ideals) or $\{0\}$ if \mathcal{A} has no minimal left ideals. Also, the socle of \mathcal{A} (if it exists), denoted by $Soc(\mathcal{A})$, is a two-sided ideal in \mathcal{A} . Suppose $p \neq 0 \in \mathcal{A}$. If $p^2 = p$, and $p\mathcal{A}p$ is a division algebra, then we call p a minimal idempotent. Denote the set of all minimal idempotents of \mathcal{A} by $Min(\mathcal{A})$. Suppose $K \subseteq \mathbb{C}$, $isoK$ denotes the set of the isolated points of K , $accK$ means the set of the accumulation points of K , and \overline{K} indicates the closure of K .

It is time to provide the concept of the Fredholm elements.

Definition 2.1. [21] Suppose $a \in \mathcal{A}$. If a is invertible modulo $Soc(\mathcal{A})$, then we call it a Fredholm element. Define the Fredholm spectrum by

$$\sigma_e(a) = \{\lambda \in \mathbf{C} : a - \lambda \text{ is not a Fredholm element}\}.$$

The set of all Fredholm elements is denoted by $FE(\mathcal{A})$. Define the Fredholm spectrum by

$$\sigma_e(a) = \{\lambda \in \mathbf{C} : a - \lambda \text{ is not a Fredholm element}\}.$$

We denote the canonical homomorphism as

$$\begin{aligned} \pi : \mathcal{A} &\rightarrow \mathcal{A}/Soc(\mathcal{A}) \\ \pi : a &\mapsto a + Soc(\mathcal{A}), \end{aligned}$$

and write $\hat{a} = a + Soc(\mathcal{A})$ for the coset of a in $\mathcal{A}/Soc(\mathcal{A})$. From now on we invariably assume in the whole paper that $Soc(\mathcal{A}) \neq \{0\}$. For $a \in \mathcal{A}$, set

$$R(a) = \{x \in \mathcal{A} : ax = 0\}, \quad L(a) = \{x \in \mathcal{A} : xa = 0\}.$$

Suppose $J \subseteq \mathcal{A}$ is a right (respectively, left) ideal of \mathcal{A} . The order of J denoted by $\Theta(J)$ is defined to be the smallest number of minimal right (respectively, left) ideals which have sum J .

Definition 2.2. [21] Suppose $a \in \mathcal{A}$, let

$$\text{nul}(a) = \Theta(R(a)), \quad \text{def}(a) = \Theta(L(a)).$$

We call $\text{nul}(a)$ and $\text{def}(a)$ the nullity and defect of a .

Example. Suppose $\mathcal{A} = B(X)$, where $B(X)$ denotes the set of all bounded linear operators on a Banach space X . From [6], one can show that the Fredholm operator in $B(X)$ is a Fredholm element.

Let \mathcal{A}^{-1} and $J(\mathcal{A})$ be the sets of all invertible elements in \mathcal{A} and the Jacobson radical of \mathcal{A} , respectively. Let $p \in \mathcal{A}$ be an idempotent. For each $a \in \mathcal{A}$, one can see that

$$a = (1 - p)a(1 - p) + (1 - p)ap + pa(1 - p) + pap.$$

Then a can be represented as follows:

$$a = \begin{pmatrix} (1 - p)a(1 - p) & (1 - p)ap \\ pa(1 - p) & pap \end{pmatrix}.$$

Moreover, if p commutes with a , then

$$a = \begin{pmatrix} a(1 - p) & 0 \\ 0 & ap \end{pmatrix}_p.$$

From [20], if $pa = ap$, then $\sigma(a) = \sigma_{p\mathcal{A}p}(ap) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(a(1 - p))$. In this case, for convenience, we also write

$$a = \begin{pmatrix} a(1 - p) & 0 \\ 0 & ap \end{pmatrix}_p = a(1 - p) \oplus ap.$$

Some necessary symbols have been reviewed, and now we will study the left (respectively, right) Drazin invertible elements and the semi-B-Browder elements.

3. Left (right) Drazin invertible element

In this section, we will define the left (respectively, right) Drazin invertible elements and provide the characterization of the left (respectively, right) Drazin invertible elements. In addition, we will prove that the left Drazin spectrum is compact.

Definition 3.1. Suppose $a \in \mathcal{A}$. If there exists an idempotent p that commutes with a satisfying that $a + p$ is left (respectively, right) invertible and ap is nilpotent, then we call a a left (respectively, right) Drazin invertible element. The set of all left (respectively, right) Drazin invertible elements is denoted by $LDIE(\mathcal{A})$ (respectively, $RDIE(\mathcal{A})$).

From [15], it follows that the Drazin invertible elements must be the left Drazin invertible elements and the right Drazin invertible elements. Next, we give the definitions of the semi-B-Fredholm elements and B-Fredholm elements.

Definition 3.2. [10, Definition 2.3] Let $a \in \mathcal{A}$. If $\pi(a)$ is Drazin invertible in $\mathcal{A}/\text{Soc}(\mathcal{A})$, then a is called a B-Fredholm element. The set of all B-Fredholm elements is denoted by $BFE(\mathcal{A})$.

Define the left multiplication operator L_a on \mathcal{A} by $L_a b = ab$ for any $b \in \mathcal{A}$. For $T \in L(X)$, $p(T)$, $q(T)$ mean the ascent and descent of T , respectively. For $a \in \mathcal{A}$, the ascent of an element $a \in \mathcal{A}$ is defined as $p_l(a) = p(L_a)$, and the descent of an element $a \in \mathcal{A}$ is defined as $q_l(a) = q(L_a)$.

Definition 3.3. Suppose $a \in \mathcal{A}$. If $\pi(a) \in LDIE(\mathcal{A}/\text{Soc}(\mathcal{A}))$ (respectively, $RDIE(\mathcal{A}/\text{Soc}(\mathcal{A}))$), then we call a upper (respectively, lower) semi-B-Fredholm element. The set of all upper (respectively, lower) semi-B-Fredholm elements is denoted by $USBFE(\mathcal{A})$ (respectively, $LSBFE(\mathcal{A})$).

Definition 3.4. Suppose $a \in \mathcal{A}$.

1) If $a \in USBFE(\mathcal{A})$ and $p_l(a) < \infty$, then we call a an upper semi-B-Browder element. The set of all upper semi-B-Browder elements is denoted by $USBBE(\mathcal{A})$.

2) If $a \in LSBFE(\mathcal{A})$ and $q_l(a) < \infty$, then we call a a lower semi-B-Browder element. The set of all lower semi-B-Browder elements is denoted by $LSBBE(\mathcal{A})$.

Example. Suppose $\mathcal{A} = B(X)$, where $B(X)$ denotes the set of all bounded linear operators on a Banach space X . From [6, 21], it follows that the B-Fredholm operator (respectively, semi-B-Fredholm operator and semi-B-Browder operator) in $B(X)$ is a B-Fredholm element (respectively, semi-B-Fredholm element and semi-B-Browder element).

The circle operation \circ is defined by $x \circ y = x + y - xy$ for all $x, y \in \mathcal{A}$. If there exist $u, v \in \mathcal{A}$ such that $u \circ x = x \circ v = 0$, then we call $x \in \mathcal{A}$ a quasi-invertible element. Suppose $a \in \mathcal{A}$. If there exist $u, v \in \mathcal{A}$ such that $u \circ x \in \text{Soc}(\mathcal{A})$, $x \circ v \in \text{Soc}(\mathcal{A})$, then we call a a quasi-Fredholm element.

Suppose $\mathcal{B} \subseteq \mathcal{A}$, then we let $L[\mathcal{B}] = \{a \in \mathcal{A} : a\mathcal{B} = 0\}$ and $R[\mathcal{B}] = \{a \in \mathcal{A} : \mathcal{B}a = 0\}$.

Definition 3.5. Suppose $a \in \mathcal{A}$, and $a \in \mathcal{A}$ is called left (respectively, right) quasi-Fredholm in \mathcal{A} if there exists $v \in \mathcal{A}$ such that $v \circ a \in \text{Soc}(\mathcal{A})$ (respectively, $a \circ v \in \text{Soc}(\mathcal{A})$).

Evidently, if $u, v \in \mathcal{A}$ are quasi-Fredholm, then $v \circ u$ is also a quasi-Fredholm element.

Next, we give the characterization of the left Drazin invertible elements and the right Drazin invertible elements, which generalizes the results in [22, Theorems 3.2 and 3.4]. In fact, Theorems 3.2 and 3.4 in reference [22] provide equivalent characterizations of left Drazin invertible operators

defined on a Banach space X . Subsequently, our results, Theorems 3.6 and 3.7, generalize the conclusions from the operator case $B(X)$ to general semisimple Banach algebras and give characterizations of left Drazin invertible elements and right Drazin invertible elements.

Theorem 3.6. *Suppose $a \in \mathcal{A}$, then the following are equivalent.*

1) $a \in \text{LDIE}(\mathcal{A})$, i.e., there exists an idempotent p that commutes with a , satisfying that $a + p$ is left invertible and ap is nilpotent.

2) There exists an idempotent p that commutes with a , satisfying that $a_1 = a(1 - p)$ is left invertible in $(1 - p)\mathcal{A}(1 - p)$ and $a_2 = ap$ is nilpotent such that $a = a_1 + a_2$.

Proof. 1) \Rightarrow 2). Suppose there exists $b \in \mathcal{A}$ satisfying that $b(a + p) = 1$ and $a_2 = ap$ is nilpotent, then one can get

$$b(a + p)(1 - p) = (ba + bp)(1 - p) = ba(1 - p) = 1 - p.$$

Hence, one can obtain

$$(1 - p)b(1 - p)a(1 - p) = (1 - p)(ba(1 - p)) = 1 - p.$$

Thus, $(1 - p)b(1 - p) \in (1 - p)\mathcal{A}(1 - p)$ is the left inverse of $a_1 = a(1 - p)$. It is evident that $a_2 = ap$ is nilpotent.

2) \Rightarrow 1). Since $a_1 = a(1 - p)$ is left invertible in $(1 - p)\mathcal{A}(1 - p)$, we can arrive at the conclusion that there exists $c \in (1 - p)\mathcal{A}(1 - p)$ such that $ca_1 = 1 - p$. Since $a_2 = ap$ is a nilpotent element, we obtain that a_2 is nilpotent in $p\mathcal{A}p$. Since the unit $p \in p\mathcal{A}p$ is invertible, it can be deduced that $p + a_2$ is invertible. Suppose the inverse of $p + a_2$ is $d \in p\mathcal{A}p$, then $d(p + a_2) = p$. Let $u = (1 - p)c(1 - p) + d$. Notice that $dp = pd = d$ and $(1 - p)c(1 - p) \in (1 - p)\mathcal{A}(1 - p)$. Through calculations, one can get

$$u(a + p) = (1 - p)c(1 - p)a(1 - p) + d(a_2 + p) = (1 - p)(1 - p) + p = 1,$$

which implies that $a + p$ is a left invertible element, and a_2 is a nilpotent element. \square

Following a proof procedure analogous to that of the above theorem, the conclusion below can be readily derived.

Theorem 3.7. *Suppose $a \in \mathcal{A}$, then the following are equivalent:*

1) $a \in \text{RDIE}(\mathcal{A})$, i.e., there exists an idempotent p that commutes with a , satisfying that $a + p$ is right invertible and ap is nilpotent.

2) There exists an idempotent p that commutes with a , satisfying that $a_1 = a(1 - p)$ is right invertible in $(1 - p)\mathcal{A}(1 - p)$, and $a_2 = ap$ is nilpotent such that $a = a_1 + a_2$.

Now, we define the left (respectively, right) Drazin spectrum of $a \in \mathcal{A}$ to be the following set:

$$\sigma_{ld}(a) = \{\lambda \in \mathbf{C} : a - \lambda \text{ is not left Drazin invertible}\}.$$

It is evident that $\sigma_{ld}(a) \subseteq \sigma_l(a) \subseteq \sigma(a)$, where $\sigma_l(a)$ denotes the left spectrum, and $\sigma(a)$ is the spectrum of a .

At the end of this section, we study the left Drazin spectrum.

Theorem 3.8. *The left Drazin spectrum $\sigma_{ld}(a)$ is a compact set.*

Proof. Since $\sigma_{ld}(a) \subseteq \sigma(a)$ and $\sigma(a)$ is bounded, it suffices to show that $\sigma_{ld}(a)$ is closed. Then, we only need to show that if a is left Drazin invertible, then for all $\lambda \in \mathbf{C}$, there exists $\varepsilon > 0$ such that $a - \lambda$ is left Drazin invertible when $0 < |\lambda| < \varepsilon$. Suppose $a \in LDIE(\mathcal{A})$, then there exists $p^2 = p \in \mathcal{A}$ with $ap = pa$ satisfying that $a_1 = a(1 - p)$ is left invertible in $(1 - p)\mathcal{A}(1 - p)$, and $a_2 = ap$ is nilpotent such that $a = a_1 + a_2$. Since $\sigma_l(x)$ is closed for any $x \in \mathcal{A}$, for all $\lambda \in \mathbf{C}$, there exists $\varepsilon > 0$ satisfying that $a_1 - \lambda(1 - p)$ is left invertible when $0 < |\lambda| < \varepsilon$. Thus,

$$a - \lambda = a_1 + a_2 - \lambda = (a - \lambda)(1 - p) + a_2.$$

It is evident $(a - \lambda)(1 - p)$ is a left invertible element in $(1 - p)\mathcal{A}(1 - p)$, and a_2 is a nilpotent element in $p\mathcal{A}p$. Therefore, $a - \lambda \in LDIE(\mathcal{A})$. Then $\sigma_{ld}(a)$ is closed, which implies that $\sigma_{ld}(a)$ is a compact set. \square

Similarly, we define the right Drazin spectrum of $a \in \mathcal{A}$ by

$$\sigma_{rd}(a) = \{\lambda \in \mathbf{C} : a - \lambda \text{ is not right Drazin invertible}\}.$$

Obviously, the following relation holds: $\sigma_{rd}(a) \subseteq \sigma_r(a) \subseteq \sigma(a)$, where $\sigma_r(a)$ means the right spectrum of a . Similar to the proof of the above theorem, one can get the following conclusion.

Proposition 3.9. *The right Drazin spectrum $\sigma_{rd}(a)$ is a compact set in \mathbf{C} .*

4. Upper semi-B-Browder elements

We will prove that being left Drazin invertible is equivalent to being upper semi-B-Browder. Suppose V is a linear space. We denote the set of all linear maps from V on V by $End_{\mathbf{C}}(V)$. We first provide some Lemmas.

Lemma 4.1. *Suppose $a \in \mathcal{A}$ and $p^2 = p \in \mathcal{A}$ with $pa = ap$, then a can be written as*

$$a = \begin{pmatrix} a(1 - p) & 0 \\ 0 & ap \end{pmatrix}_p.$$

Proof. It is evident that

$$a = (1 - p)a(1 - p) + (1 - p)ap + pa(1 - p) + pap.$$

From [20], it follows that a can be represented as

$$a = \begin{pmatrix} (1 - p)a(1 - p) & (1 - p)ap \\ pa(1 - p) & pap \end{pmatrix}.$$

If p commutes with a , then

$$a = \begin{pmatrix} a(1 - p) & 0 \\ 0 & ap \end{pmatrix}_p.$$

\square

Lemma 4.2. *Suppose $a \in \mathcal{A}$, $Soc(\mathcal{A})$ can be written as the sum of the finite minimal two-sided ideal, and suppose $Soc(\mathcal{A}) = \bigoplus S_i$, where S_i is a minimal two-sided ideal. Let $p \in \mathcal{A}$ be an idempotent commuting with $a \in \mathcal{A}$, such that $\hat{a} + \hat{p}$ is left invertible in $\mathcal{A}/Soc(\mathcal{A})$. Define $\Gamma : S_i \rightarrow S_i$ by $\Gamma(t) = t(a + p)$. Then Γ is surjective.*

Proof. It is evident that each minimal two-sided ideal S_i is finite-dimensional and simple; by the Wedderburn-Artin structure theorem of semisimple Banach algebra, one can get that $S_i \cong M_n(\mathbf{C})$ for some n . In other words, there exists a finite-dimensional vector space V and an irreducible representation $\rho : \mathcal{A} \rightarrow End_{\mathbf{C}}(V)$ such that $S_i \cong End_{\mathbf{C}}(V)$, where i is fixed, and S_i acts on V , identified as a linear operator on V . For any $x \in \mathcal{A}$, define the right multiplication operator R_x on $End_{\mathbf{C}}(V)$ by

$$R_x(\varphi) = \varphi \circ \rho(x), \varphi \in End_{\mathbf{C}}(V).$$

Since $S_i \cong End_{\mathbf{C}}(V)$, we define $\tilde{\Gamma} : End_{\mathbf{C}}(V) \rightarrow End_{\mathbf{C}}(V)$ by $\tilde{\Gamma}(\varphi) = \varphi \circ \rho(a + p)$.

Hence, it suffices to prove that $\tilde{\Gamma}$ is surjective. We claim that the map $\varphi \rightarrow \varphi \circ u$ on $End_{\mathbf{C}}(V)$ is surjective if and only if $u \in End_{\mathbf{C}}(V)$ is invertible. Indeed, if u is invertible for any $\psi \in End_{\mathbf{C}}(V)$, we take $\varphi = \psi \circ u^{-1}$, which implies that $\varphi \circ u = \psi$. Conversely, if $\varphi \rightarrow \varphi \circ u$ is surjective, then there exists φ such that $\varphi \circ u = Id$, where Id denotes the identity, so u is left invertible. Since V is finite-dimensional, one can get that u is invertible.

Hence, if we want to prove Γ is surjective, we only need to show that $\rho(a + p)$ is invertible in $End_{\mathbf{C}}(V)$. Since $\hat{a} + \hat{p} \in \mathcal{A}/Soc(\mathcal{A})$ is left invertible, there exists \hat{b} such that $\hat{b}(\hat{a} + \hat{p}) = \hat{1}$. Let $P = \ker \rho$, then P is a primitive ideal and $\mathcal{A}/P \cong End_{\mathbf{C}}(V)$, and one can get that $\hat{a} + \hat{p}$ still has a left inverse in \mathcal{A}/P . Then $\rho(a + p) \in End_{\mathbf{C}}(V)$ is left invertible, which implies that $\rho(a + p)$ is invertible since $End_{\mathbf{C}}(V)$ is a finite-dimensional algebra. Therefore, because $\rho(a + p)$ is invertible, we can obtain that $\tilde{\Gamma}$ is surjective, which implies that Γ is surjective. \square

Lemma 4.3. *Suppose $a \in \mathcal{A}$, $\hat{a}\hat{p} = \hat{p}\hat{a}$, and $\hat{p}^2 = \hat{p} \in \mathcal{A}/Soc(\mathcal{A})$, then there exists $p^2 = p \in \mathcal{A}$ with $\pi(p) = \hat{p}$ and $ap = pa$.*

Proof. It is evident that π has the Riesz property and it is surjective, according to Line 17 on Page 3735 of reference [10], thus π has the lifting property. In other words, there exists $p_1^2 = p_1 \in \mathcal{A}$ satisfying that $\pi(p_1) = \hat{p}$. Due to $\hat{a}\hat{p} = \hat{p}\hat{a}$, we can deduce that $d = ap_1 - p_1a \in Soc(\mathcal{A})$. Next, we will find an element $z \in Soc(\mathcal{A})$ such that $a(p_1 + z) - (p_1 + z)a = 0$.

$Soc(\mathcal{A}) = \bigoplus S_j$, where S_j is the minimal two-sided ideal. In order to find an element z such that $az - za = -d$, suppose $d = \sum d_j$, $z = \sum z_j$, where $d_j, z_j \in S_j$. On each block, it suffices to consider $az_j - z_ja = -d_j$, where $z_j \in S_j$. Define the linear map $\varphi : S_j \rightarrow S_j$ by $z_j \mapsto az_j - z_ja$. It suffices to prove $-d_j \in Range(\varphi)$, where $Range(T)$ means the range of the operator T . Indeed, if $-d_j \in Range(\varphi)$, then this implies that there exists z_j such that $\varphi(z_j) = az_j - z_ja = -d_j$. Hence, one can get that $z = \sum z_j$, satisfying $az - za = -d$.

Define $L_a : S_j \rightarrow S_j$ by $s \mapsto as$ and $R_a : S_j \rightarrow S_j$ by $s \mapsto sa$. Then, one can see that $\varphi = L_a - R_a$ on S_j . Next we show $-d_j \in Range(L_a - R_a)$. Since

$$d_j = (ap_1 - p_1a)|_{S_j} = L_a(p_1|_{S_j}) - R_a(p_1|_{S_j}) = (L_a - R_a)(p_1|_{S_j}),$$

where $(ap_1 - p_1a)|_{S_j}$ denotes the corresponding decomposition element of $ap_1 - p_1a \in Soc(\mathcal{A})$ on block S_j . Define $z_j = -p_1|_{S_j} \in S_j$, and it satisfies $(L_a - R_a)(z_j) = -d_j$. Let $z = \sum z_j$. It is evident that

$z \in Soc(\mathcal{A})$ and

$$a(p_1 + z) - (p_1 + z)a = 0.$$

Define $p = p_1z$, then $ap = pa$, $\pi(p) = \pi(p_1 + z) = \hat{p}_1$, and $p^2 = p$ (the idempotent persists because $z \in Soc(\mathcal{A})$ and $S_j \cong M_n(\mathbf{C})$). □

Lemma 4.4. *Suppose $a \in \mathcal{A}$, $\pi : \mathcal{A} \rightarrow \mathcal{A}/Soc(\mathcal{A})$ is the quotient map, $p^2 = p \in \mathcal{A}$, and $pa = ap$, satisfying that $\pi(a)\pi(p)$ is nilpotent, so ap is nilpotent in \mathcal{A} .*

Proof. Since $Soc(\mathcal{A})$ is the sum of all minimal two-sided ideals of \mathcal{A} , we suppose $Soc(\mathcal{A}) = \bigoplus S_j$, where S_j is finite-dimensional and simple, and $S_j \cong M_n(\mathbf{C})$. Given that $\pi(a)\pi(p)$ is nilpotent, there exists k such that $(\pi(a)\pi(p))^k = 0$ in $\mathcal{A}/Soc(\mathcal{A})$, which implies that $(ap)^k \in Soc(\mathcal{A})$. Suppose $(ap)^k = \sum_{j \in J} s_j$, where only finitely many s_j are nonzero. Here, the sum is the algebraic direct sum of the minimal two-sided ideals, i.e.,

$$\bigoplus_{j \in J} s_j = \left\{ \sum_{j \in F} x_j : F \subseteq J \text{ is finite, } x_j \in S_j \right\}.$$

Let $\rho : \mathcal{A} \rightarrow End_{\mathbf{C}}(V)$ be the irreducible representation, and R_x is the right multiplication operator defined by $R_x : S_j \rightarrow S_j$ by $R_x(\varphi) = \varphi \circ \rho(x)$ and $\varphi \in End_{\mathbf{C}}(V)$. Define $T = \rho(ap) \in End_{\mathbf{C}}(V)$. It is clear that $\rho((ap)^m) = T^m$. Owing to $(ap)^k \in Soc(\mathcal{A})$, we can establish that $T^k \in End_{\mathbf{C}}(V)$. Next we prove T is nilpotent.

Because $(\pi(a)\pi(p))^k = 0$ in $\mathcal{A}/Soc(\mathcal{A})$, for any representation $\sigma : \mathcal{A} \rightarrow End_{\mathbf{C}}(V)$, there exists a homomorphism $\widehat{\sigma} : \mathcal{A}/Soc(\mathcal{A}) \rightarrow End_{\mathbf{C}}(V)$ such that $\sigma = \widehat{\sigma} \circ \pi$. Since $\widehat{\sigma}$ is a homomorphism, a direct calculation shows that

$$\sigma((ap)^k) = \widehat{\sigma}(\pi((ap)^k)) = \widehat{\sigma}(0) = 0.$$

Hence, for representation ρ , T^k is nilpotent in $End_{\mathbf{C}}(V)$, which implies that T is nilpotent in $End_{\mathbf{C}}(V)$. Equivalently, there exists r_j such that $(\rho(ap))^{r_j} = 0$. It follows that $((ap)_j)^{r_j} = 0$, where $(ap)_j$ denotes the decomposition of ap on S_j . Define $N = \max_j(k + r_j)$, then $((ap)_j)^N = 0$. Therefore, $(ap)^N = 0$; in other words, ap is nilpotent. □

Lemma 4.5. *Suppose $\hat{a} \in \mathcal{A}/Soc(\mathcal{A}) = \mathcal{B}$, then \hat{a} is quasi-invertible if and only if $1 - \hat{a}$ is invertible, where 1 is the unit of \mathcal{B} .*

Proof. Suppose there exist $\hat{u}, \hat{v} \in \mathcal{B}$ such that $\hat{u} \circ \hat{a} = 0$, $\hat{a} \circ \hat{v} = 0$, which implies that $\hat{u} + \hat{a} - \hat{u}\hat{a} = 0$, $\hat{a} + \hat{v} - \hat{a}\hat{v} = 0$, where $\hat{u} \circ \hat{a} = \hat{u} + \hat{a} - \hat{u}\hat{a}$. Hence, it follows that

$$(1 - \hat{u})(1 - \hat{a}) = 1 - \hat{u} - \hat{a} + \hat{u}\hat{a} = 1,$$

$$(1 - \hat{a})(1 - \hat{v}) = 1 - \hat{a} - \hat{v} + \hat{a}\hat{v} = 1,$$

hence, $1 - \hat{a}$ is invertible in \mathcal{B} .

Conversely, suppose $1 - \hat{a}$ is invertible, then define $\hat{b} = (1 - \hat{a})^{-1} \in \mathcal{B}$. Let $\hat{u} = \hat{v} = 1 - \hat{b}$. A calculation shows that

$$\hat{u} \circ \hat{a} = (1 - \hat{b}) + \hat{a} - (1 - \hat{b})\hat{a} = 1 - \hat{b} + \hat{a} - \hat{a} + \hat{b}\hat{a} = 1 - \hat{b} + \hat{b}\hat{a} = 1 - \hat{b}(1 - \hat{a}) = 0.$$

Similarly, we have $\hat{a} \circ \hat{v} = 0$. Hence, \hat{a} is quasi-invertible. □

Lemma 4.6. *Suppose $p_l(a) < \infty$, if \hat{a} is quasi-invertible in $\mathcal{B} = \mathcal{A}/\text{Soc}(\mathcal{A})$ and $0 \in \text{iso}\sigma_e(a)$, then \hat{a} is left Drazin invertible in \mathcal{B} .*

Proof. Suppose $p_l(a) = p < \infty$. In view of $0 \in \text{iso}\sigma(\hat{a})$, one can reach the conclusion that there exists a spectral idempotent \hat{p} in \mathcal{B} commuting with \hat{a} satisfying that $\mathcal{B} = (1 - \hat{p})\mathcal{B} \oplus \hat{p}\mathcal{B}$ and $\hat{a} = \hat{a}_1 + \hat{a}_2$, where \hat{a}_1 is invertible and \hat{a}_2 is quasinilpotent.

For any $y \in \hat{p}\mathcal{B}$, there exists $x \in \mathcal{A}$ satisfying that $\pi(x) = y$. Define $x_n = a^n x$; in other words, the sequence $x, ax, a^2x, \dots, a^n x, \dots$, denotes $\overline{x_n} = \pi(x_n) \in \mathcal{B}$. It is evident $\overline{x_n} = L_{\hat{a}}^n(y)$. Since $\ker(L_a^p) = \ker(L_a^{p+1})$, it follows that $a^{p+1}z = 0$ when $a^p z = 0$ for $z \in \mathcal{A}$. Consider the finite-dimensional vector space:

$$V = \text{span}\{x_0, x_1, \dots, x_p\}, W = V + \text{Soc}(\mathcal{A}).$$

Then, there exists $\alpha_0, \alpha_1, \dots, \alpha_p$ such that $\overline{x_{p+1}} = \sum_{j=0}^p \alpha_j \overline{x_j}$ in $W/\text{Soc}(\mathcal{A})$, which implies that

$$a^{p+1}x - \sum_{j=0}^p \alpha_j a^j x \in \text{Soc}(\mathcal{A}).$$

It follows that $a^m(a^{p+1}x - \sum_{j=0}^p \alpha_j a^j x) \in \text{Soc}(\mathcal{A})$ for all m . Hence, $\hat{a}^{p+1}y - \sum_{j=0}^p \alpha_j \hat{a}^j y = 0$ holds in \mathcal{B} . Since y is arbitrary, we take y to be the identity in \mathcal{B} . One can get that $\hat{a}^{p+1} - \sum_{j=0}^p \alpha_j \hat{a}^j = 0$. Define $p(x) = 1 - \sum_{j=0}^p \alpha_j x$. One can check that $p(L_{\hat{a}}) = 0$, which implies that $L_{\hat{a}}$ is algebraic of degree $\leq p + 1$.

Because $\hat{a}|_{\hat{p}\mathcal{B}} = \hat{a}_2$ is quasinilpotent, we can obtain that $L_{\hat{a}}|_{\hat{p}\mathcal{B}}$ is a quasinilpotent operator. From [10, Proposition 4.1], $L_{\hat{a}}|_{\hat{p}\mathcal{B}}$ is nilpotent operator, so there exists m such that $(L_{\hat{a}}|_{\hat{p}\mathcal{B}})^m = 0$, which implies that $\hat{a}\hat{p}$ is nilpotent element in $\hat{p}\mathcal{B}\hat{p}$. Also, the other component $\hat{a}_1 = \hat{a}(1 - \hat{p})$ is invertible in $(1 - \hat{p})\mathcal{B}(1 - \hat{p})$. Thus, \hat{a} is left Drazin invertible in \mathcal{B} . □

Next, we provide the characterization of the upper semi-B-Browder elements.

Theorem 4.7. *Suppose $a \in \mathcal{A}$. Then the following are equivalent:*

- 1) $a \in \text{LDIE}(\mathcal{A})$.
 - 2) $a \in \text{USBBE}(\mathcal{A})$.
- If $0 \notin \text{iso}\sigma_e(a)$ and $1 \notin \sigma_e(a)$, then the above statements are equivalent to the following.*
- 3) a is quasi-Fredholm and $p_l(a) < \infty$.

Proof. (1) \Rightarrow (2). Owing to $a \in \text{LDIE}(\mathcal{A})$, one can derive that there exists $p^2 = p \in \mathcal{A}$, and $ap = pa$ satisfying that $a + p$ is left invertible and ap is nilpotent. Since

$$a = \begin{pmatrix} a(1 - p) & 0 \\ 0 & ap \end{pmatrix}_p,$$

and

$$a + p = \begin{pmatrix} a(1 - p) & 0 \\ 0 & ap + p \end{pmatrix}_p.$$

From the fact that $a + p$ is left invertible in \mathcal{A} , it follows that there exists $b \in \mathcal{A}$ satisfying that $b(a + p) = 1$, hence $\pi(b)\pi(a + p) = 1$, which implies that $\pi(a + p)$ is left invertible in $\mathcal{A}/\text{Soc}(\mathcal{A})$. Since

ap is nilpotent and π is a homomorphism, it follows that $\pi(ap)$ is nilpotent, $\pi(p)^2 = \pi(p)$, and $\pi(p)$ commutes with $\pi(a)$. Hence, $\pi(a) \in LDIE(\mathcal{A}/Soc(\mathcal{A}))$. Thus, $a \in USBFE(\mathcal{A})$.

Notice that the block decomposition relative to p gives

$$a = \begin{pmatrix} a(1-p) & 0 \\ 0 & ap \end{pmatrix}_p.$$

Because $a_2 = ap$ is nilpotent, we can derive that there exists k such that $a_2^k = 0$; then $\ker(L_{a_2^k}) = \ker(L_{a_2^{k+1}})$, which implies that

$$\ker(L_{a_2^k}) = \ker(L_{a_2^{k+1}}).$$

Since $a_1 = a(1-p)$ is left invertible, let $p = k$. Then, we can arrive at $\ker(L_a^p) = \ker(L_a^{p+1})$, which implies that $p_l(a) < \infty$.

2) \Rightarrow 1). On account of $a \in USBFE(\mathcal{A})$, we can obtain $\pi(a) \in LDIE(\mathcal{A}/Soc(\mathcal{A}))$; thus, there exists $\hat{p}^2 = \hat{p}$ with $\hat{p}\hat{a} = \hat{a}\hat{p}$ satisfying that $\hat{a} + \hat{p}$ is left invertible and $\hat{a}\hat{p}$ is nilpotent. From the above Lemmas 4.3 and 4.4, there exists $p^2 = p \in \mathcal{A}$ with $\pi(p) = \hat{p}$, and it commutes with a , and thus ap is nilpotent. Next, it suffices to prove $a + p$ is left invertible. By applying the representation theory and Wedderburn Artin structure theorem and considering that $\hat{a} + \hat{p}$ is left invertible, we can show that there exists b_0 satisfying

$$s = b_0(a + p) - 1 \in Soc(\mathcal{A}).$$

Next we construct $t \in Soc(\mathcal{A})$ such that $(b_0 + t)(a + p) = 1$. Here $b_0 + t$ is the left inverse of $a + p$. Hence, it follows that

$$(b_0 + t)(a + p) = 1 \Leftrightarrow t(a + p) = -s.$$

Suppose $s = \sum s_j$ and $t = \sum t_j$, then $t(a + p) = -s$ is equivalent to $(t_j(a + p))_j = -s_j$, where $(t_j(a + p))_j$ denotes the component of $t(a + p)$ on S_j . Next, we claim that the above linear equation has a solution $t_j \in S_j$ on each S_j . Fixing j , we define the mapping $T_j : S_j \rightarrow S_j$ by $t \mapsto (t(a + p))_j$. From the above Lemma 4.2, it follows that T_j is surjective onto S_j ; hence, there exists t_j such that $T_j(t_j) = -s_j$. Define $t = \sum t_j$. Then $t \in Soc(\mathcal{A})$ and $t(a + p) = -s$. Let $b = b_0 + t$. Then, $b(a + p) = 1$. Hence, $a + p$ is left invertible. Therefore, a is left Drazin invertible.

Next, we shall prove the equivalence of 2) and 3), where this conclusion holds under the premise of condition “ $0 \notin iso\sigma_e(a)$ and $1 \notin \sigma_e(a)$ ”.

2) \Rightarrow 3). Since $1 \notin \sigma_e(a) = \sigma(\hat{a})$, which implies that $1 - \hat{a}$ is invertible, then from the above Lemma 4.5, it is evident that \hat{a} is quasi-invertible. Hence, a is quasi-Fredholm. It is clear that $p_l(a) < \infty$ holds.

3) \Rightarrow 2). Given that a is quasi-Fredholm, we can reach the conclusion that \hat{a} is quasi-invertible. From $0 \in iso\sigma_e(a)$, it follows that $0 \in iso\sigma(\hat{a})$. According to Lemma 4.6, it follows that $\hat{a} \in LDIE(\mathcal{B})$. Equivalently, $a \in USBFE(\mathcal{A})$. Hence, the relation $a \in USBBE(\mathcal{A})$ holds from the fact that $p_l(a) < \infty$. \square

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

References

1. G. S. Karkarashvili, Fredholm Integro-differential equation, *J. Sov. Math.*, **66** (1993), 2236–2242. <https://doi.org/10.1007/BF01229590>
2. K. Tanaka, *Brownian Motion, the Fredholm Determinant, and Time Series Analysis*, Cambridge University Press, 2024. <https://doi.org/10.1017/9781009567008>
3. F. V. Atkinson, The normal solubility of linear equations in normed spaces, *Mat. Sb.*, **28** (1951), 3–14.
4. C. Schmoeger, On a class of generalized Fredholm operators, VII, *Demonstr. Math.*, **33** (2000), 123–130. <https://doi.org/10.1515/dema-2000-0115>
5. P. Aiena, *Fredholm and Local Spectral Theory II—With Application to Weyl-type Theorems*, Springer, 2018. <https://doi.org/10.1007/978-3-030-02266-2>
6. B. A. Barnes, The Fredholm elements of a ring, *Can. J. Math.*, **21** (1969), 84–95. <https://doi.org/10.4153/CJM-1969-009-1>
7. M. Breuer, Fredholm theories in von Neumann algebras. II, *Math. Ann.*, **180** (1969), 313–325. <https://doi.org/10.1007/BF01351884>
8. B. A. Barnes, G. J. Murphy, M. R. F. Smyth, T. T. West, Riesz and Fredholm theory in Banach algebras, *Res. Notes Math.*, (1982), 123.
9. A. A. Irmatov, A. S. Mishchenko, On compact and Fredholm operators over C^* -algebras and a new topology in the space of compact operators, *J. K-Theory*, **2** (2008), 329–351. <https://doi.org/10.1017/is008004001jkt034>
10. M. D. Cvetković, E. Boasso, S. Č. Živković-Zlatanović, Generalized B-Fredholm Banach algebra elements, *Mediterr. J. Math.*, **13** (2016), 3729–3746. <https://doi.org/10.1007/s00009-016-0711-y>
11. S. Ivković, Semi-Fredholm theory on Hilbert C^* -modules, *Banach J. Math. Anal.*, **13** (2019), 989–1016. <https://doi.org/10.1215/17358787-2019-0022>
12. S. Ivković, Semi-Fredholm theory in C^* -algebras, *Banach J. Math. Anal.*, **17** (2023), 51. <https://doi.org/10.1007/s43037-023-00277-y>
13. S. Ivković, On non-adjointable semi- C^* -Fredholm and semi- C^* -Weyl operators, *Filomat*, **37** (2023), 5523–5539. <https://doi.org/10.2298/FIL2317523I>

14. Z. Wu, Q. Zeng, Left and right Browder elements in Banach algebras, *J. Math. Anal. Appl.*, **541** (2025), 128716. <https://doi.org/10.1016/j.jmaa.2024.128716>
15. J. J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.*, **38** (1996), 367–381. <https://doi.org/10.1017/S0017089500031803>
16. S. Roch, B. Silbermann, Continuity of generalized inverses in Banach algebras, *Stud. Math.*, **136** (1999), 197–227.
17. Y. Ren, L. Jiang, Left and right-Drazin inverses in rings and operator algebras, *J. Algebra Appl.*, **23** (2024), 2450064. <https://doi.org/10.1142/S0219498824500646>
18. D. Zhang, Y. Zhao, D. Mosić, The generalized Drazin inverse of the sum of two elements in a Banach algebra, *J. Comput. Appl. Math.*, **470** (2025), 116701. <https://doi.org/10.1016/j.cam.2025.116701>
19. D. Zhang, L. Ma, D. Mosić, Representations of the Moore-Penrose inverse of a 2×2 block matrix based on Schur complements, *Appl. Math. Comput.*, **512** (2026), 129776. <https://doi.org/10.1016/j.amc.2025.129776>
20. O. Abad, H. Zguitti, Generalized Drazin-Riesz invertible elements in a semisimple Banach algebra, *Afr. Mat.*, **36** (2025), 8. <https://doi.org/10.1007/s13370-024-01227-z>
21. Y. Y. Kong, L. N. Jiang, Y. X. Ren, The Weyl's theorem and its perturbations in semisimple Banach algebra, *Acta Math. Sin. Engl. Ser.*, **37** (2021), 675–688. <https://doi.org/10.1007/s10114-021-0434-2>
22. A. Ghorbel, M. Mnif, On a generalization of left and right invertible operators on Banach spaces, *Oper. Matrices*, **16** (2022), 401–413. <https://doi.org/10.7153/oam-2022-16-31>



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