Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(2): 890–909.
DOI: 10.3934/era.2026041
Received: 22 September 2025
Revised: 21 December 2025
Accepted: 19 January 2026
Published: 28 January 2026

Research article

Extended persistent homology of filtration of graded subgroups

Shengwen Xie, Xuezhi Zhao and Fang Sun*

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

* **Correspondence:** Email: 6716@cnu.edu.cn.

Abstract: In this work, we presented an extended persistence for filtration of graded subgroups by defining a relative homology in this setting. Our work provided a more comprehensive and flexible approach to obtaining an algebraic invariant and overcoming the limitations of the standard approach. As the application of mathematical tools in data analysis requires stability—meaning small perturbations in the input data should induce only small changes in the output—our main contribution was the development of a stability theorem for extended persistence modules of filtration of graded subgroups. This theorem was established using an extension of the definition of interleaving, along with the rectangle measure and functor extension. We demonstrated the theorem's utility by applying it to extended persistence modules obtained from the path homology of directed graphs and the homology of hypergraphs, two important examples in topological data analysis.

Keywords: persistent homology; extended persistent homology; TDA; persistence modules; path homology; hypergraph

1. Introduction

1.1. Persistent homology and extended persistent homology

Persistent homology is a mathematical tool from topological data analysis. The cornerstone of topological data analysis, persistent homology (PH), tracks the evolution of topological features—such as connected components and cycles—across a filtration of spaces.

The modern theory of persistence is built on three pillars:

- The persistence diagram, and an algorithm for computing it, were introduced by Edelsbrunne et al. [1], giving a compact representation of the size function and an effective way to compute it.
- Zomorodian and Carlsson [2] defined persistence modules in the abstract, indexed by the natural numbers and viewed as graded modules over the polynomial ring $k[t]$. This introduced tools from commutative algebra.

- Cohen-Steiner et al. [3] formulated and proved the stability theorem, which guarantees that the persistence diagram is robust to changes in the input data. Robustness is measured in terms of a "bottleneck distance" between persistence diagrams.

In recent years, the theory of persistent homology has been generalized in various directions. For example variations in parameters [7]: Multi-parameter persistence studies filtrations indexed by $\mathbb{R}^n$ rather than $\mathbb{R}$.

To capture more important topology features, Cohen-Steiner et al. [11] introduced extended persistent homology (EPH) . The idea is to grow the space from the bottom up, through sublevelsets; and then to relativise the space from the top down, with superlevelsets. Extended persistence is the persistent homology of this sequence of spaces and pairs. It is usually assumed that $(X, f : X \to \mathbb{R})$ has finitely many homological critical points (a_i). One applies a homology functor to the finite sequence

$$X^{a_0} \to X^{a_1} \to \cdots \to X^{a_{n-1}} \to X \to (X, X_{a_n}) \to \cdots \to (X, X_{a_2}) \to (X, X_{a_1})$$

where $X^{a_i} = \{x \in X \mid f(x) \leq a_i\}$ and $X_{a_i} = \{x \in X \mid f(x) \geq a_i\}$.

There are three kinds of features:

- ordinary features (which are born and die before the central X);
- relative features (which are born and die after the central X);
- extended features (which are born before the X and die after it).

The example below shows the motivation of EPH.

Example 1. *Let $T^2 \subset \mathbb{R}^3$ be a point cloud sampled from a vertical torus. Define a function $f : T^2 \to \mathbb{R}$ as the height (z-coordinate). In PH, the first connected component is born at the minimum height h_{min} and persists indefinitely, denoted by $[h_{min}, \infty)$. In EPH, the connected component born at h_{min} dies at the extended value $\bar{h}_{max}$. The interval $[h_{min}, \bar{h}_{max})$ represents the total height of the object.*

This example shows us that in many cases PH yields infinite barcodes. In data analysis pipelines (e.g., computing Wasserstein distances or persistence landscapes), infinite bars are often discarded or arbitrarily truncated, leading to information loss. So extended persistence is powerful to transform infinite barcodes into finite barcodes.

1.2. Stability

The application of a mathematical tool in data analysis typically demands its stability. That is, small perturbations at input should result in small changes at output.

The stability of persistent (simplicial) homology (c.f. [10, p182]) has been documented since the early days of the theory of PH. Stability is often formulated as an inequality stating that the bottleneck distance is not greater than the interleaving distance. The stability for persistent path homology and the stability for hypergraph homology, our principal examples, are documented in [5,14], respectively. We intend to present a stability theorem for extended persistence modules of filtration of graded groups. In [12, Section 6.2], the stability for extended persistent simplicial homology is briefly discussed. The authors of [12] said the "stability for diagram of *Ord*, *Ext*, *Rel* may be proved individually for each diagram". But they haven't provided a proof of stability theorem of extended persistence modules of filtration of complexes.

1.3. Contributions

In this article we illustrate a kind of development of a stability theorem for extended persistence modules of filtration of graded subgroups. Our specific contributions are the following:

Computing the extended persistent homology of filtration of graded groups

Proposition 7 guarantees that the supremum operation commutes with the cone construction. That is to say, computing the extended persistent homology of filtration of graded subgroups, yields the same result as computing extended persistent homology of filtration of their associated supremum chain complexes.

Stability theorem for extended persistence modules of filtration of graded groups

We show that any two ϵ-interleaved modules can be connected by a 1-parameter family of modules (Theorem 17). We achieve this by solving a functor extension problem (Theorem 16). Combining Lemma 14, we presented a stability theorem for extended persistence modules of filtration of graded groups (Theorem 18).

Stability for extended persistent path homology and extended persistent homology of hypergraphs

Applying Theorem 18, we get the stability theorem for extended persistent path homology and extended persistent homology of hypergraphs.

2. Graded subgroups of a chain complex

The study of graded subgroups and their homology is motivated by at least two subjects in TDA: path homology of directed graphs and homology of hypergraphs.

Path homology is a novel algebraic invariant for directed graphs (digraphs) developed by Grigor'yan et al. (c.f. [4]). It is capable of extracting higher dimensional features from a graph, while taking direction into account. See [5, p1154, Section 2.4] for an example of two directed graphs whose underlying undirected graphs are have identical distinct path homology groups. In addition, path homology satisfies certain functorial properties one would expect from a natural homology theory on digraphs. Namely, it is natural with respect to morphisms of digraphs and satisfies a Künneth formula for a product of digraphs. There is even a homotopy theory on digraphs that is compatible with path homology. This makes path homology an ideal feature for studying digraphs, assuming that an efficient algorithm is available.

2.1. Definition of graded subgroups of a chain complex

The concept of graded subgroups was introduced in [6]. Let

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

be a chain complex of k-vector spaces. A *graded subgroup* of $\{C_*, \partial_*\}$ is just a family of $\{D_*\}$ of k-vector spaces such that $D_p \subset C_p$ for all p. If $\partial D_p \subset D_{p-1}$ then $\{D_*\}$ would be a subcomplex, but we do not assume this, in general.

To define a proper notion of homology for graded subgroups, one has to construct a chain complex from $\{D_*\}$. It seems reasonable to either enlarge $\{D_*\}$ or excise it. Define the *supremum chain complex* of $\{D_*\}$ in $\{C_*, \partial_*\}$ to be the smallest subcomplex **containing** $\{D_*\}$, and define the *infimum chain* *complex* of $\{D_*\}$ in $\{C_*, \partial_*\}$ to be the largest subcomplex **contained in** $\{D_*\}$. As it turns out, both approaches produce the same homology groups. We now give a detailed treatment of these notions.

For each p, define $S_p = S_p(D_*; C_*) = D_p + \partial_{p+1} D_{p+1}$ and $I_p = I_p(D_*; C_*) = D_p \cap \partial_p^{-1} D_{p-1}$. Then S_* (resp. I_*) is a subcomplex of C_*, and is called the *supremum* (resp. *infimum*) *chain complex* of $\{D_*\}$ in $\{C_*, \partial_*\}$.

Remark 2. *It was shown ([6, Proposition 2.1, 2.2]) that the S_* is the minimal subcomplex containing $\{D_*\}$, while I_* is the maximal subcomplex being contained in $\{D_*\}$. In particular, replacing C_* by a larger chain complex (containing it) does not affect S_* or I_*. Similar ideas in considering path homology can be found in [4, Section 3].*

Define the homology of S_* (resp. I_*) by $H_*^{\sup}(D_*; C_*)$ (resp. $H_*^{\inf}(D_*; C_*)$). An important result is:

Proposition 3. *([6, Proposition 2.4]) The inclusion of $I_* \hookrightarrow S_*$ is a chain map that induces isomorphisms $H_*^{\inf}(D_*; C_*) \to H_*^{\sup}(D_*; C_*)$.*

Proof. With the help of the isomorphism theorem of groups, we have

$$\begin{aligned} & H_*^{\sup}(D_*; C_*) \\ = \; & \operatorname{Ker}(\partial_n|_{D_n + \partial_{n+1} D_{n+1}}) / \operatorname{Im}(\partial_{n+1}|_{D_{n+1} + \partial_{n+2} D_{n+2}}) \\ = \; & (\partial_{n+1} D_{n+1} + \operatorname{Ker}(\partial_n|_{D_n})) / \partial_{n+1} D_{n+1} \\ \cong \; & \operatorname{Ker}(\partial_n|_{D_n}) / \operatorname{Ker}(\partial_n|_{D_n}) \cap \partial_{n+1} D_{n+1} \\ = \; & \operatorname{Ker}(\partial_n|_{D_n}) / (D_n \cap \partial_{n+1} D_{n+1}) \\ = \; & H_*^{\inf}(D_*; C_*). \end{aligned}$$

The assertion follows. □

The above isomorphism is natural in the following sense: Suppose that we are given two graded subgroups $\{D_*\}$ and $\{D'_*\}$ of $\{C_*, \partial_*\}$, and $D_p \subset D'_p$ for all p. Let S_*, I_* (resp. S'_*, I'_*) be the supremum and infimum subcomplex of D_* (resp. D'_*), then S_*, I_* are embedded in S'_*, I'_* respectively, so that the following diagram

$$\begin{array}{ccc} H_*^{\inf}(D_*; C_*) & \longrightarrow & H_*^{\sup}(D_*; C_*) \\ \downarrow & & \downarrow \\ H_*^{\inf}(D'_*; C_*) & \longrightarrow & H_*^{\sup}(D'_*; C_*) \end{array}$$

commutes, where all homomorphisms are induced by inclusions.

There are at least two settings where graded subgroups arise naturally. We present both of them in the following.

2.2. Path homology

The concept of path homology was introduced in [5]. The first case concerns the notion of path homology of directed graphs (digraphs). Formally, a (finite) digraph is a pair $G = (X, E)$ where X is a finite set and $E \subset X \times X$. We deal exclusively with digraphs without self-loops.

Given a non-negative integer p and a field k, an *elementary p-path* on the set X is a sequence $x_0 \cdots x_p$ of $p+1$ elements of X. Denote by $\Lambda_p = \Lambda_p(X)$ the k-vector space of all formal linear combinations of elementary p-paths on X. An elementary path $x_0 \cdots x_p$ as an element of Λ_p is denoted $e_{x_0 \cdots x_p}$. Define the boundary operator $\overline{\partial}\colon \Lambda_p \to \Lambda_{p-1}$ by

$$\overline{\partial}(e_{x_0 \cdots x_p}) = \sum_{i=0}^{p} (-1)^i e_{x_0 \cdots \hat{x}_i \cdots x_p},$$

where $\hat{x}_i$ denotes omission as usual. It can be shown ([4, Lemma 2.1]) that $\overline{\partial}^2 = 0$, thus $\{\Lambda_*\}$ is a chain complex.

An elementary path $x_0 \cdots x_p$ is called *regular* if $x_{k-1} \neq x_k, 1 \leq k \leq p$. Define $\mathcal{R}_p = \mathcal{R}_p(X)$ (resp. $\mathcal{I}_p = \mathcal{I}_p(X)$) as the subspace of Λ_p consisting of the formal linear combinations of regular (resp. irregular) elementary p-paths. It is not difficult to check that $\mathcal{I}_*$ is a subcomplex of Λ_*, so the quotient chain complex $(\Lambda/\mathcal{I})_*$ is well-defined. Since $(\Lambda/\mathcal{I})_p$ and $\mathcal{R}_p$ are canonically isomorphic, we obtain a chain complex $\{\mathcal{R}_*, \partial_*\}$ via this identification.

The digraph structure enters the scene in the following way: an elementary regular path $x_0 \cdots x_p$ is called *allowed* if each $x_{k-1}x_k$ belongs to the edge set E. In other words, it is a head-to-tail concatenation of arrows. The set of formal linear combinations of allowed elementary p-paths of G is denoted $\mathcal{A}_p = \mathcal{A}_p(G)$. Noting that $\mathcal{A}_*$ is a graded subgroup of $\mathcal{R}_*, \partial_*\}$, we define the p-dimensional path homology of G as $H_p(G) := H_p^{\sup}(\mathcal{A}_*; \mathcal{R}_*) \cong H_*^{\inf}(\mathcal{A}_*; \mathcal{R}_*)$.

Given $G = (X, E)$ and $G' = (X, E')$ with $E \subset E'$, we have an embedding $\mathcal{A}_p(G) \subset \mathcal{A}_p(G')$ for each p, and thus an induced homomorphism $H_*(G) \to H_*(G')$.

2.3. Hypergraph homology

The second case concerns the homology of hypergraphs which have been studied by [6]. Let V be a finite set. The power set $P(V)$ of V is the collection of all non-empty subsets of V. A *hypergraph* $\mathcal{H}$ on V is just a subset of $P(V)$. The elements of $\mathcal{H}$ are called to be *hyperedges*. If $\tau \subset \sigma \in \mathcal{H}$ implies $\tau \in \mathcal{H}$, then $\mathcal{H}$ would be an abstract simplicial complex, but we do not assume this in general.

Any hypergraph $\mathcal{H}$ can be embedded in a simplicial complex K. The most economical choice being $K_{\mathcal{H}} = \{\tau | \tau \subset \sigma \text{ for some } \sigma \in \mathcal{H}\}$. Actually, $K_{\mathcal{H}}$ is the minimal one among all such kinds of K's.

For a simplicial complex L, let $\Delta_*(L)$ denote the (oriented) simplicial chain complex of L. Given a hypergraph $\mathcal{H}$, define $\Delta_*(\mathcal{H})$ as the graded subgroup of $\Delta_*(K_{\mathcal{H}})$ spanned by the hyperedges of $\mathcal{H}$, and define the *embedded homology* of $\mathcal{H}$ as $H_*(\mathcal{H}) = H_*^{\sup}(\Delta_*(\mathcal{H}), \Delta_*(K_{\mathcal{H}})) \cong H_*^{\inf}(\Delta_*(\mathcal{H}), \Delta_*(K_{\mathcal{H}}))$. When $\mathcal{H}$ is embedded in a simplicial complex K, there are canonical embeddings of $\Delta_*(K_{\mathcal{H}})$ and $\Delta_*(\mathcal{H})$ into $\Delta_*(K)$. By Remark 2, we could replace $K_{\mathcal{H}}$ with any such K in the definition of $H_*(\mathcal{H})$. This enables us to formulate naturality for hypergraph homology. Given hypergraphs $\mathcal{H}' \subset \mathcal{H}$, choose an embedding of $\mathcal{H}$ in a simplicial complex K. There is an induced homomorphism

$$H_*(\mathcal{H}') = H_*(\Delta_*(\mathcal{H}'), \Delta_*(K)) \to H_*(\Delta_*(\mathcal{H}), \Delta_*(K)) = H_*(\mathcal{H}).$$

Note that this homomorphism is independent of the choice of K.

2.4. Persistent homology

Here is a brief review of the standard persistent homology from an algebraic perspective.

2.4.1. Persistence module and interval decomposition

For our purpose, a *persistence module* $V = (\{V^i\}, \{\phi^i\})_{i\in\mathbb{Z}}$ is a sequence of k-vector spaces and k-linear maps

$$\cdots \xrightarrow{\phi^{i-1}} V^i \xrightarrow{\phi^i} V^{i+1} \to \cdots .$$

A *morphism* of persistence module from $(\{V^i\}, \{\phi^i\})_{i\in\mathbb{Z}}$ to $(\{W^i\}, \{\psi^i\})_{i\in\mathbb{Z}}$ is a sequence of linear maps $\eta^i : V^i \to W^i$ such that $\psi^i \circ \eta^i = \eta^{i+1} \circ \phi^i$. If each η^i is an isomorphism, we call it an *isomorphism of persistence modules*. Given a family $\{V_\lambda\}$ of persistence modules, define their *direct sum* $\oplus_\lambda V_\lambda$ by $(\oplus_\lambda V^i_\lambda, \oplus_\lambda \phi^i_\lambda)$.

We are mainly interested in decomposing a persistence module into simple building blocks up to isomorphism. The simplest non-trivial modules are the interval modules.

For an interval I, the *interval module* k_I is defined by

$$k_I^m = \begin{cases} k, & m \in I, \\ 0, & \text{otherwise,} \end{cases}$$

with the identity map joining nonzero vector spaces (the other maps are automatically zero). Most persistence modules we encounter in practice are decomposable into interval modules:

Proposition 4. *([9, Theorem 1.2]) If a persistence module V is of pointwise finite-dimensional, i.e.,* $\dim V^i < \infty$ *for all i, then V is isomorphic to a direct sum of interval modules. Moreover, such a decomposition is unique.*

The collection of intervals in the decomposition of V is called the *persistent barcode* of V. The definition of persistent diagrams involves multisets. For our purposes, a *multiset* is a pair $A = (S, m)$ where S is a set and

$$m : S \to \{1, 2, ...\} \cup \{\infty\},$$

is a function called *the multiplicity function*, which counts the occurrence of an element in A. The *cardinality* card(A) is defined to be the sum $\sum_{s\in S} m(s)$ if this sum is well-defined and finite, and ∞ otherwise.

If $\mathbf{L}$ is a multiset and $V = \oplus_{I\in\mathbf{L}} k_I$, we define $\mathrm{Dgm}(V) := \mathbf{L}$ and define $\mathrm{dgm}(V) = \{int(I), I \in \mathbf{L} - \Delta\}$ where $int(I)$ is the interior of I.

2.4.2. Persistent homology

In topological data analysis, persistence modules usually arise as homology groups of filtrations of subcomplexes. Given a chain complex $\{C_*, \partial_*\}$ and a filtration

$$\cdots \hookrightarrow C^i_* \hookrightarrow C^{i+1}_* \hookrightarrow \cdots \tag{2.1}$$

of subcomplexes. The inclusions induce a persistence module on homology

$$\cdots \to H_p(C^i_*) \to H_p(C^{i+1}_*) \to \cdots$$

for each dimension p, which is said to be the *persistent homology* (PH) of (2.1).

In practice, filtration (2.1) usually comes from a filtration

$$\cdots \hookrightarrow K^i \hookrightarrow K^{i+1} \hookrightarrow \cdots$$

of subcomplexes of an ambient simplicial complex K, as the (oriented) simplicial chain complexes. The most common scenario is: we are given a real valued function $f : K \to \mathbb{R}$ defined on the vertex set of a simplicial complex K. Let $-\infty = a_0 < a_1 < a_2 < \cdots < a_n$ be the values of f. The sublevel subcomplexes $K^i = \{\sigma \in K \mid f(v) \leq a_i \text{ for all } v \in \sigma\}$ constitutes a filtration. The persistent homology of their simplicial chain complexes is called the persistent homology of K with respect to f.

3. PH for filtration of graded subgroups

3.1. Background

Suppose that we are given a filtration

$$\cdots \hookrightarrow D^i_* \hookrightarrow D^{i+1}_* \hookrightarrow \cdots$$

of graded subgroups of a chain complex $\{C_*, \partial_*\}$. Define the persistent (p-dimensional) homology of this filtration to be the persistence module:

$$\cdots \to H^{\sup}_p(D^i_*) \to H^{\sup}_p(D^{i+1}_*) \to \cdots$$

By the discussion in Proposition 3 at Section 2, this is equivalent (up to isomorphism) to

$$\cdots \to H^{\inf}_p(D^i_*) \to H^{\inf}_p(D^{i+1}_*) \to \cdots .$$

3.2. An algebraic reformulation

In this subsection, we give an algebraic reformulation of extended persistence for filtration graded subgroups.

Let $\{C_*, \partial_*\}$ be a chain complex. Let

$$S^1_* \hookrightarrow \cdots \hookrightarrow S^i_* \hookrightarrow S^{i+1}_* \hookrightarrow \cdots \hookrightarrow S^M_*,$$
$$T^1_* \hookrightarrow \cdots \hookrightarrow T^j_* \hookrightarrow T^{j+1}_* \hookrightarrow \cdots \hookrightarrow T^N_*,$$

be two filtrations of subcomplexes in C_* such that $S^M_* = T^N_*$ (in general, S^i_* and T^i_* are not required to contain one another). Define the *extended persistent homology* of $(\{S^i_*\}_{1\leq i\leq M}, \{T^j_*\}_{1\leq j\leq N})$ as the persistence modules

$$H_p(S^1_*) \to \cdots \to H_p(S^M_*) \to H_p(S^M_*/T^1_*) \to \cdots \to H_p(S^M_*/T^N_*) = 0. \tag{3.1}$$

where S^M_*/T^j_*, $j = 1, 2, \ldots N$ is the quotient of two Abelian groups. That is, the extended persistent homology of $(\{S^i_*\}_{1\leq i\leq M}, \{T^j_*\}_{1\leq j\leq N})$ is the persistent homology for

$$S^1_* \hookrightarrow \cdots \hookrightarrow S^i_* \hookrightarrow S^{i+1}_* \hookrightarrow \cdots \hookrightarrow S^M_* = T^N_* \to T^N_*/T^1_* \cdots \hookrightarrow T^N_*/T^N_* = 0$$

The definition of the extended PH can be found in [12, p112], In order to compute the decomposition of (3.1), we need the notion of mapping cones of chain complexes. Let $\{C_*, \partial_*\}$, $\{C'_*, \partial'_*\}$ be chain complexes and $\varphi : C'_* \to C_*$ be a chain map. The *mapping cone* of φ is the chain complex $\mathrm{Cone}(\varphi) = \{\overline{C}_*, \overline{\partial}_*\}$ where $\overline{C}_p = C'_{p-1} \oplus C_p$ and

$$\overline{\partial}_p(c', c) = (-\partial'_{p-1}(c'), \varphi(c') + \partial_p(c)) \in C'_{p-2} \oplus C_{p-1} \text{ for } c' \in C'_{p-1}, c \in C_p$$

As is the case with topological cones, there is an embedding $C_* \hookrightarrow \overline{C}_*$ taking $c \in C_p$ to $(0, c) \in \overline{C}_p$. We are interested in the case where φ is the inclusion of a subcomplex. When there is no ambiguity on C_*, we will denote $\mathrm{Cone}(\varphi)$ by $\mathrm{Cone}(C'_*)$. The following is an algebraic analogy of the contractibility of cones, which can be proved by a standard argument in homology algebra.

Lemma 5. *Let C'_* be a subcomplex of C_*. Denote $\mathrm{Cone}(C'_*)$ by $\{\overline{C}_*, \overline{\partial}_*\}$. Then the map*

$$\begin{array}{rcl} h : H_p(C_*/C'_*) & \Longrightarrow & H_p(\overline{C}_*), \\ [\bar{x}] & \Longrightarrow & [(-\partial x, x)], \end{array}$$

is an isomorphism for all p, where $\bar{x} \in C_p/C'_p$ is represented by $x \in C_p$ and $[\cdot]$ means taking the homology class.

From the construction in the above proof, we can see that the isomorphism h is natural with respect to C'_*: if $C^1_* \subset C^2_*$ are two subcomplexes of C_*, then $\overline{C}^1_* = \mathrm{Cone}(C^1_*)$ is a subcomplex of $\overline{C}^2_* = \mathrm{Cone}(C^2_*)$ and the diagram

$$\begin{array}{ccc} H_*(C_*/C^1_*) & \xrightarrow{h} & H_*(\overline{C}^1_*) \\ \downarrow & & \downarrow \\ H_*(C_*/C^2_*) & \xrightarrow{h} & H_*(\overline{C}^2_*), \end{array}$$

commutes, where the left vertical map is induced by natural projection while the right vertical map is induced by inclusion. This implies:

Corollary 6. *Suppose $\{S^i_*\}_{1\le i\le M}, \{T^j_*\}_{1\le j\le N}$ are two filtrations of subcomplexes of C_* such that $S^M_* = T^N_*$. Denote by $\overline{T}^j_*$ the mapping cone of $T^j_* \hookrightarrow S^M_*$. Then $S^1_* \subset \cdots \subset S^M_* \subset \overline{T}^1_* \subset \cdots \subset \overline{T^N_*}$ is a filtration on $\overline{T}^N_*$, and the persistence module (3.1) is isomorphic to*

$$H_p(S^1_*) \to \cdots \to H_p(S^M_*) \to H_p(\overline{T}^1_*) \to \cdots \to H_p(\overline{T}^N_*) = 0. \tag{3.2}$$

□

Again, we postpone the detail here to the next subsection, where we study extended PH of filtrations of graded groups.

3.3. Extended PH for graded subgroups

We are interested in extended PH of filtrations of graded subgroups. The first step is to define relative homology in this setting.

Let $E_* \subset D_*$ be graded subgroups of a chain complex C_*. Define the *relative supremum homology group* as

$$H^{\sup}_p(D_*, E_*) = H^{\sup}_p(D_*, E_*; C_*) \overset{\Delta}{=} H_p(S_*, T_*) = H_p(S_*/T_*),$$

where S_*, T_* denote the supremum complexes of D_*, E_* in C_* respectively. The relative infimum homology group $H_p^{\inf}(D_*, E_*; C_*)$ is defined analogously. An easy argument by the five lemma shows that $H_p^{\sup}(D_*, E_*)$ is naturally isomorphic to $H_p^{\inf}(D_*, E_*)$. By Remark 2, the relative homology groups are not affected by replacing C_* with a larger chain complex. Naturality with respect to inclusion is also available: if $D_*^i, E_*^i, i = 1, 2$ are graded subgroups of C_* and $D_*^1 \subset D_*^2, E_*^1 \subset E_*^2, E_*^i \subset D_*^i, i = 1, 2$, then there is a canonical homomorphism $H_p^{\sup}(D^1, E^1) \to H_p^{\sup}(D^2, E^2)$ induced by inclusion.

To compute relative homology, we need the notion of cones for graded subgroups. Denote the mapping cone Cone($id : C_* \to C_*$) by $\overline{C}_*$. Define the *mapping cone* of $E_* \hookrightarrow D_*$ as the graded subgroup Cone($E_* \hookrightarrow D_*$) $= \overline{E}_*$ of $\overline{C}_*$, where $\overline{E}_p = E_{p-1} \oplus D_p \subseteq \overline{C}_p$ for each p. The following proposition shows that the cone construction commutes with taking supremum subcomplex.

Proposition 7. *In the above setting, let S_* (resp. T_*) be the supremum subcomplex of D_* (resp. E_*) and set $\overline{T}_* =$ Cone$(T_* \hookrightarrow S_*)$. Then with respect to the natural embedding $\overline{T}_* \hookrightarrow \overline{C}_*$, $\overline{T}_*$ is the supremum subcomplex of $\overline{E}_* =$ Cone$(E_* \hookrightarrow D_*)$ in $\overline{C}_*$.*

Proof. For $\overline{E}_* = (E_{*-1} \oplus D_*)$ and $\overline{T}_* = (T_{*-1} \oplus S_*)$ there exists an embedding from $\overline{E}_*$ to $\overline{T}_*$ induced by the embedding from E_{*-1} to T_{*-1} and the embedding from D_* to S_*.We need to check that $\overline{T}_p$ coincide with $\overline{E}_p + \partial\overline{E}_{p+1}$. By definition,

$$\overline{T}_p = T_{p-1} \oplus S_p = (E_{p-1} + \partial E_p) \oplus (D_p + \partial D_{p+1}),$$

while

$$\overline{E}_p + \partial\overline{E}_{p+1} = E_{p-1} \oplus D_p + \partial(E_p \oplus D_{p+1}),$$

It is obvious that $\overline{E}_p + \partial\overline{E}_{p+1} \subset \overline{T}_p$. Conversely, take $(x + \partial y, z + \partial w) \in \overline{T}_p$, where $x \in E_{p-1}, y \in E_p, z \in D_p, w \in D_{p+1}$. We compute

$$\begin{aligned}(x, y + z) + \partial(-y, w) &= (x, y + z) + (\partial y, -y + w)\\ &= (x + \partial y, z + \partial w),\end{aligned}$$

Since $E_p \subset D_p$, we have $(x, y + z) + \partial(-y, w) \in \overline{E}_p + \partial\overline{E}_{p+1}$. □

Proposition 7 guarantees that the supremum operation commutes with the cone construction. That is to say, computing the extended persistent homology of filtration of graded subgroups—yields the same result as computing extended persistent homology of filtration of their associated supremum chain complexes. Suppose that we are given a chain complex C_* and two filtrations of graded subgroups

$$\begin{gathered}D_*^1 \hookrightarrow \cdots \hookrightarrow D_*^i \hookrightarrow D_*^{i+1} \hookrightarrow \cdots \hookrightarrow D_*^M,\\ E_*^1 \hookrightarrow \cdots \hookrightarrow E_*^j \hookrightarrow E_*^{j+1} \hookrightarrow \cdots \hookrightarrow E_*^N,\end{gathered}$$

with $D_*^M = E_*^N$. As is expected, p-dimensional extended persistent homology of $(\{D_*^i\}_{1\le i\le M}, \{E_*^j\}_{1\le j\le N})$ is defined as the persistence module

$$H_p^{\sup}(D_*^1) \to \cdots \to H_p^{\sup}(D_*^M) \to H_p^{\sup}(D_*^M, E_*^1) \to \cdots \to H_p^{\sup}(D_*^M, E_*^N) = 0.s \tag{3.3}$$

where all maps are induced by inclusion. The extended PH of filtrations of graded groups $\{D_*^i\}_{1\le i\le M}, \{E_*^j\}_{1\le j\le N}$ is the extended PH of filtrations of supremum complexes of the graded groups.

In other words, if S_*^i and T_*^j $(1 \le i \le M, 1 \le j \le N)$ denote the supremum complexes of D_*^i and E_*^j respectively, then (3.3) is just the extended PH of $(\{S_*^i\}_{1\le i\le M}, \{T_*^j\}_{1\le j\le N})$. If the filtrations of graded groups $\{D_*^i\}_{1\le i\le M}, \{E_*^j\}_{1\le j\le N}$ are filtrations of complexes, then the extended PH as filtrations of graded groups is isomorphic extended PH of filtrations of complex. Note that if $H^{\inf}$ is used in (3.3) instead of $H^{\sup}$, we would get a persistence module isomorphic to (3.3).

Our goal is to compute the interval decomposition of (3.3). Define $\overline{E}_*^j = \mathrm{Cone}(E_*^j \hookrightarrow D_*^M) \subset \overline{C}_* = \mathrm{Cone}(id : C_* \to C_*)$ and identify D_*^i with $0 \oplus D_*^i \subset \overline{C}_*$. Then

$$D_*^1 \subset \cdots \subset D_*^M \subset \overline{E}_*^1 \subset \cdots \subset \overline{E}_*^N, \tag{3.4}$$

is a filtration of graded subgroups of $\overline{C}_*$. By Proposition 7, taking supremum complexes in (3.4) gives

$$S_*^1 \subset \cdots \subset S_*^M \subset \overline{T}_*^1 \subset \cdots \subset \overline{T}_*^N,$$

By definition, the persistence module (3.3) is the same as (3.1), which is in turn isomorphic (Corollary 6) to (3.2). The above discussion shows that (3.2) is the persistent homology of the filtration (3.4).

4. Stability theorem of extended persistent modules of filtration of graded groups

We will present a stability theorem for extended persistence modules (defined later) of filtration of graded groups in this section and apply this theorem to the extended persistence of path homology and hypergraph homology.

In Section 4.1, we present the basic notions necessary for formulating the stability results. In Section 4.2, we discuss the bottleneck distance and interleaving distance in the extended setting, which are measurements for small changes and prove the stability theorem (Theorem 18). In Section 4.2 we show that any two ϵ-interleaved modules can be connected by a 1-parameter family of modules (Theorem 17). We achieve this by solving a functor extension problem (Theorem 16). Combining Lemma 14, we presented a stability theorem for extended persistence modules of filtration of graded groups (Theorem 18). In the remaining two subsections, we state and prove stability theorems for path homology and hypergraph homology, respectively.

4.1. Continuous extended persistence modules

We define two posets $\mathbb{R}^o$ and E as follows.

Definition 8. *The poset $\mathbb{R}^o$ is defined to be the set $\mathbb{R}$ with reversed order. For $b \in \mathbb{R}$, denote the corresponding element of $\mathbb{R}^o$ as $\bar{b}$. Hence, $\bar{a} \le \bar{b} \in R^o$ if and only if $b \le a \in R$. The order of the poset $E = \mathbb{R} \cup \{\infty\} \cup \mathbb{R}^o$ is given by $s < \infty < \bar{t}$ for all $s, t \in \mathbb{R}$.*

A *(continuous) extended persistence module V* is a functor from poset E (regarded as a category) to category Vect_k of finite-dimensional vector spaces over k. In other words, each $x \in E$ is assigned a vector space V_x, and each pair $x \le y$ is assigned a linear map $V_{x,y} : V_x \to V_y$ such that

(i) $V_{x,x} = \mathrm{id}$,

(ii) $V_{x,z} = V_{x,y} \circ V_{y,z}$,

The definitions of morphism, direct sum, and interval module are analogous to the discrete case. Note that intervals here are those with respect to the poset E (e.g., $(3, \bar{2}], (\infty, \bar{1})$).

Here is how extended persistence modules arise from discrete ones defined earlier in this paper.

For our purpose, it suffices to deal with a smaller class of persistence modules. An extended persistence module V is called *decomposable* if it is isomorphic to a direct sum of finitely many interval modules over E, none of which has ∞ as an endpoint. Such a decomposition is always unique (see [12, Theorem 2.7], note that E is isomorphic to $\mathbb{R}$ as a poset). The decomposability can be reformulated in the following way:

Proposition 9. *An extended persistence module V is decomposable if and only if the following statement hold:*

(i) V_t is finite-dimensional for all $t \in E$,

(ii) there exist $a, b \in \mathbb{R}$, such that $V_{s,t}$ is an isomorphism for all $a < s < t < \overline{b}$ (i.e., V is locally constant *near ∞).*

Proof. For the "if" part, condition (i) implies the existence of interval decomposition (c.f. [12, Theorem 2.8]). Locally constant near ∞ guarantees that none of the intervals has ∞ as endpoint. The "only if" part is trivial. □

We now move to define the *extended persistence diagrams* of a decomposable module V. For simplicity, we denote an interval in E of the form $(a, b), (a, b], [a, b), [a, b]$ as $(a^+, b^-), (a^+, b^+), (a^-, b^-), (a^-, b^+)$ respectively. By Section 6.2 of [12] there are then three types of intervals in the decomposition :

(i) ordinary: $(a^\pm, a'^\pm)$,

(ii) relative: $(\bar{b}^\pm, \bar{b}'^\pm)$,

(iii) extended: $(a^\pm, \bar{b}^\pm)$.

The *ordinary persistence diagram* of V is obtained by placing a point (a, a') on the plane $\mathbb{R} \times \mathbb{R}$ for each ordinary interval $(a^\pm, a'^\pm)$, counting multiplicity. The *relative* (resp. *extended*) *persistence diagram* is defined similarly, except that it lies in the plane $\mathbb{R} \times \mathbb{R}^o$ (resp. $\mathbb{R}^o \times \mathbb{R}^o$). The three types of persistence diagrams are denoted by $\mathrm{Ord}(V)$, $\mathrm{Rel}(V)$ and $\mathrm{Ext}(V)$ respectively. Note that they are multisets on the respective planes.

4.2. Bottleneck distance and interleaving

There are two notions of distance between persistence modules: bottleneck distance and interleaving distance [12]. Roughly speaking, the bottleneck distance measures how close the persistence modules are by comparing their persistence diagrams, while the interleaving distance measures how far away they are from being isomorphic and typically can be related to the input data. In the unextended case, the two distances are equated by the Isometry Theorem (see [12, Theorem 5.14]) if certain finiteness assumptions are satisfied.

We now formulate these notions in the extended setting, starting with Bottleneck distance. Throughout this discussion, we will be using the l^∞ distance on the plane:

$$d((x, y), (z, w)) = \max(|x - z|, |y - w|).$$

Given two multisets A, B, a *partial matching* is a bijection $\phi : A' \longleftrightarrow B'$ between subsets $A' \subset A, B' \subset B$. If $A' = A, B' = B$, we call ϕ a *perfect matching*. We call a and $\phi(a)$ matched pairs where $a \in A'$ and $a \notin A'$ or $b \notin B'$ are unmatched points. Let V, W be decomposable persistence modules and $\delta > 0$. A δ-matching between V and W is a triple $\Phi = \{\phi_O, \phi_R, \phi_E\}$ where ϕ_O (resp. ϕ_R) is a partial matching between $\mathrm{Ord}(V)$ and $\mathrm{Ord}(W)$ (resp. $\mathrm{Rel}(V)$ and $\mathrm{Rel}(W)$) while ϕ_E is a perfect matching between $\mathrm{Ext}(V)$ and $\mathrm{Ext}(W)$ such that all matched pairs are δ-close and unmatched points are δ-close to the diagonal means the distance in the plane. The *bottleneck distance* between V, W is defined by

$$d_B(V, W) = \inf\{\, \delta \mid \text{there is a } \delta\text{-matching between } V \text{ and } W\,\}.$$

In particular, the requirement of perfect matching for the extended diagram means that $d_B(V, W) = +\infty$ unless $\dim V_\infty = \dim W_\infty$.

We now turn to interleaving.

Definition 10. *Let V, W be extended persistence modules and ε be a positive real number. A ε-interleaving between V and W is a quadruple of families of maps $(\{\varphi_a\}, \{\varphi_{\bar b}\}, \{\psi_a\}, \{\psi_{\bar b}\})$ where*

$$\begin{array}{ll} \varphi_a : V_a \longrightarrow W_{a+\varepsilon} & \varphi_{\bar b} : V_{\overline{b}} \longrightarrow W_{\overline{b-\varepsilon}}, \\ \psi_a : W_a \longrightarrow V_{a+\varepsilon} & \psi_{\bar b} : W_{\overline{b}} \longrightarrow V_{\overline{b-\varepsilon}}, \end{array}$$

for $a \in \mathbb{R}, \overline{b} \in \mathbb{R}^o$ satisfying the following conditions:

(i) For $a < a' \in \mathbb{R}, \overline{b} < \overline{b'} \in \mathbb{R}^o$, we have the following naturality conditions:

$$\begin{array}{c} W_{a+\varepsilon,a'+\varepsilon} \circ \varphi_a = \varphi_{a'} \circ V_{a,a'}, \\ W_{a+\varepsilon,\overline{b-\varepsilon}} \circ \varphi_a = \varphi_{\overline{b}} \circ V_{a,\overline{b}}, \\ W_{\overline{b-\varepsilon},\overline{b'-\varepsilon}} \circ \varphi_{\overline{b}} = \varphi_{\overline{b'}} \circ V_{\overline{b},\overline{b'}}. \end{array}$$

In other words, the following diagrams commute:

$$\begin{array}{cccc} & W_{a+\varepsilon} & \xrightarrow{W_{a+\varepsilon,a'+\varepsilon}} & W_{a'+\varepsilon} \\ \nearrow^{\varphi_a} & & \nearrow^{\varphi_{a'}} & \\ V_a & \xrightarrow{V_{a,a'}} & V_{a'} & \end{array}$$

$$\begin{array}{cccc} & W_{a+\varepsilon} & \xrightarrow{W_{a+\varepsilon,\overline{b-\varepsilon}}} & W_{\overline{b-\varepsilon}} \\ \nearrow^{\varphi_a} & & \nearrow^{\varphi_{\overline{b}}} & \\ V_a & \xrightarrow{V_{a,\overline{b}}} & V_{\overline{b}} & \end{array}$$

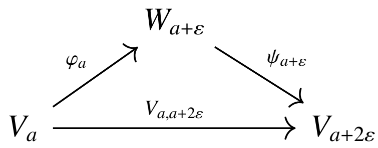
$$\begin{array}{cccc} & W_{\overline{b-\varepsilon}} & \xrightarrow{W_{\overline{b-\varepsilon},\overline{b'-\varepsilon}}} & W_{\overline{b'-\varepsilon}} \\ \nearrow^{\varphi_{\overline{b}}} & & \nearrow^{\varphi_{\overline{b'}}} & \\ V_{\overline{b}} & \xrightarrow{V_{\overline{b},\overline{b'}}} & V_{\overline{b'}} & \end{array}$$

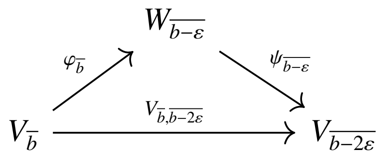
(i') The three equations in (i) remains true with V, W exchanged and φ replaced by ψ.

(ii) For $a \in \mathbb{R}, \overline{b} \in \mathbb{R}^o$, the following equations hold:

$$\psi_{a+\varepsilon} \circ \varphi_a = V_{a,a+\varepsilon}$$
$$\psi_{\overline{b-\varepsilon}} \circ \varphi_{\overline{b}} = V_{\overline{b},\overline{b-2\varepsilon}}$$

These amounts to commutativity of diagrams:





(ii') The two equations in (ii) remains true with the roles of (V, φ) and (W, ψ) reversed.

If there is a ε-interleaving between V and W, we call that V and W are ε- interleaved.

The *interleaving distance* between V, W is defined by:

$$d_I(V, W) = \inf\{\varepsilon \mid \text{there is a } \varepsilon\text{-interleaving between } V \text{and } W.\}$$

We shall prove that the assumptions of decomposability are sufficient for stability, i.e., the bottleneck distance of two extended persistence modules is bounded above by their interleaving distance.

The proof of our stability theorem makes use of rectangle measures [12]. For our purpose, we define an *admissible rectangle* as a planar rectangle of one of the following forms:

$$[a, b] \times [c, d] \subseteq \mathbb{R} \times \mathbb{R}, a < b < c < d \in \mathbb{R},$$
$$[\overline{a}, \overline{b}] \times [\overline{c}, \overline{d}] \subseteq \mathbb{R}^o \times \mathbb{R}^o, a > b > c > d \in \mathbb{R},$$
$$[a, b] \times [\overline{c}, \overline{d}] \subseteq \mathbb{R} \times \mathbb{R}^o, a < b, c > d \in \mathbb{R}.$$

We define three rectangle measures, one for each plane, that assign a nonnegative integer or $+\infty$ to an admissible rectangle T:

$$u_V^O(T) := \mathrm{card}(\mathrm{Ord}(V) \mid_T),$$
$$u_V^R(T) := \mathrm{card}(\mathrm{Rel}(V) \mid_T),$$
$$u_V^E(T) := \mathrm{card}(\mathrm{Ext}(V) \mid_T).$$

Given $\delta > 0$, the δ-thickening of a rectangle is defined by

$$([a, b] \times [c, d])^\delta = [a - \delta, b + \delta] \times [c - \delta, d + \delta],$$
$$([\overline{a}, \overline{b}] \times [\overline{c}, \overline{d}])^\delta = [\overline{a + \delta}, \overline{b - \delta}] \times [\overline{c + \delta}, \overline{d - \delta}],$$
$$([a, b] \times [\overline{c}, \overline{d}])^\delta = [a - \delta, b + \delta] \times [\overline{c + \delta}, \overline{d - \delta}].$$

The following lemmas are analogies of Lemma 5.26 and Theorem 5.29 in [12] in the extended setting.

Lemma 11. *([12, Lemma 5.26]) Let U, V be a δ-interleaved pair of extended persistence modules. Let T be an admissible rectangle in $\mathbb{R}\times\mathbb{R}$ whose δ-thickening is above the digonal. Then $u^*_U(T) \leq u^*_V(T^\delta)$, $u^*_V(T) \leq u^*_U(T^\delta)$*

Lemma 12. *Let U, V be a δ-interleaved pair of extended persistence modules. Let T be an admissible rectangle in $\mathbb{R}\times\mathbb{R}, \mathbb{R}\times\mathbb{R}^o$ or $\mathbb{R}^o\times\mathbb{R}^o$ whose δ-thickening T^δ is also admissible. Then $u^*_U(T) \leq u^*_V(T^\delta)$, $u^*_V(T) \leq u^*_U(T^\delta)$.*

The proof of the equality $u^*_U(T) \leq u^*_V(T^\delta)$, $u^*_V(T) \leq u^*_U(T^\delta)$ for $T \subset \mathbb{R}\times\mathbb{R}$ has been written in the proof of Lemma 5.26 of [12]. And the proof of the equality for $T \subset \mathbb{R}\times\mathbb{R}^o$ or for $T \subset \mathbb{R}^o\times\mathbb{R}^o$ is similar to the proof of Lemma 5.26 in [12]. Therefore we omit the proof here and refer the interest reader to [12].

Theorem 13. *([12, Theorem 5.29]) Suppose $(u_x|x \in [0,\varepsilon])$ is a 1-parameter family of finite r-measures on an open set D in $\mathbb{R}\times\mathbb{R}$. Suppose for all $x, y \in [0,\varepsilon]$ the equality holds*

$$u_x(T) \leq u_y(T^{|y-x|}),$$

holds for all rectangles T whose $|y-x|$-thickening $T^{|y-x|}$ belongs to $Rect(D)$. Then there exists a ε-matching between the diagrams $\mathrm{dgm}(u_0)$ *and* $\mathrm{dgm}(u_\varepsilon)$.

Lemma 14. *For $\varepsilon > 0$, if there exists a family $\{V^t|t \in [0,\varepsilon]\}$ of decomposable extended persistence modules such that V^s, V^t are $|s-t|$-interleaved for all $s, t \in [0,\varepsilon]$, then $d_B(V^0, V^\varepsilon) \leq \varepsilon$.*

Proof. For any $s, t \in [0,\varepsilon]$ and any rectangle T such that $T^{|s-t|}$ is admissible, we have

$$u^*_{V^s}(T) \leq u^*_{V^t}(T^{|s-t|}), \quad u^*_{V^t}(T) \leq u^*_{V^s}(T^{|s-t|}),$$

by Lemma 12. It suffices to find an ε-matching $\Phi = \{\phi_O, \phi_R, \phi_E\}$ between V^0 and V^ε. The partial matching ϕ_O can be obtained by applying [12, Theorem 5.29] to $(u^O_{V^s}|s \in [0,\varepsilon])$ with $\mathcal{D}$ being the open half plane above the diagonal (not including the infinity). The matching ϕ_R is obtained analogously, using the natural symmetry between $\mathbb{R}\times\mathbb{R}$ and $\mathbb{R}^o\times\mathbb{R}^o$ for $\mathbb{R}^o$ as the poset $\mathbb{R}$ with reversed order. To get ϕ_E, we utilize the symmetry between $\mathbb{R}\times\mathbb{R}$ and $\mathbb{R}\times\mathbb{R}^o$ and apply the above-mentioned theorem to $(u^E_{V^s}|s \in [0,\varepsilon])$ with $\mathcal{D}$ being the entire plane (not including infinity). The resulted ϕ_E has to be perfect by our choice of $\mathcal{D}$. □

By Lemma 14, given two ε-interleaved extended persistence modules, it suffices to join them by a family of mutually interleaved modules (as in Lemma 14). Next we will construct a family of mutually interleaved modules. This family can be constructed using functorextension.

Definition 15. *Given $\varepsilon \geq 0$, we define a translation Ω_ε on the subset $\mathbb{R}\cup\mathbb{R}^o$ of E as follows*

$$\Omega_\varepsilon(x) = \begin{cases} x+\varepsilon, & x \in \mathbb{R}, \\ x-\varepsilon, & x \in \mathbb{R}^o. \end{cases}$$

The order of the poset $E\times\{0,\varepsilon\}$ is given by setting $(x, a) \leq (y, b)$ if one of the following holds:

(1) $x \leq y$, $a = b$,

(2) $\Omega_\varepsilon(x) \leq y$, $a \neq b$, $x \neq \infty$,

(3) $x = \infty$, $y \in \mathbb{R}^o$, $a \neq b$.

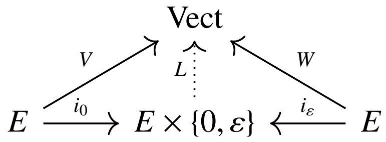
The order of the poset $E \times [0, \varepsilon]$ *is given by setting* $(x, a) \leq (y, b)$ *if one of the following holds:*

(1) $x \leq y$, $a = b$,

(2) $\Omega_{|a-b|}(x) \leq y$, $a \neq b$, $x \neq \infty$,

(3) $x = \infty$, $y \in \mathbb{R}^o$, $a \neq b$.

Theorem 16. *Two extended persistence module V, W are ε-interleaved if and only if there exists a functor L such that the following diagram commutes:*



see Definition 8 and Definition 15 for the definition of two posets E and $E \times \{0, \varepsilon\}$ *.*

Proof. The functor i_0 is an embedding from E to $E \times \{0\}$ such that $i_0(x) = (x, 0), x \in E$ and functor i_ε is an embedding from E to $E \times \{\varepsilon\}$ such that $i_\varepsilon(x) = (x, \varepsilon), x \in E$. We denote the extension functor L and denote the morphism from $L((x, a))$ to $L((y, b))$ as $L_{(x,a),(y,b)}$ for any (x, a), $(y, b) \in E \times \{0, \varepsilon\}$. If V, W is decomposable, then L is finite dimensional, and has locally constant near ∞ which means there exist $a \in \mathbb{R}$ $b \in \mathbb{R}$, such that $V_{s,t}$ $W_{s,t}$ is an isomorphism for all $a < s < t < \overline{b}$.

For the "if" part, the existence of the solution can make us take a quadruple of families of maps

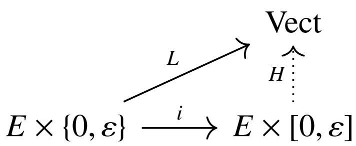
$$\left(\{L_{(a,0),(a+\varepsilon,\varepsilon)}\}, \{L_{(\bar{b},0),(\overline{b-\varepsilon},\varepsilon)}\}, \{L_{(a,\varepsilon),(a+\varepsilon,0)}\}, \{L_{(\bar{b},\varepsilon),(\overline{b-\varepsilon},0)}\}\right),$$

as a quadruple of families of maps $(\{\varphi_a\}, \{\varphi_{\bar{b}}\}, \{\psi_a\}, \{\psi_{\bar{b}}\})$ in the Definition 10.The functoriality of L implies the commutativity of diagrams in Definition 10.

For the "only if" part, extended persistence module V, W are ε-interleaved then there exist a quadruple of families of maps $(\{\varphi_a\}$, $\{\varphi_{\bar{b}}\}, \{\psi_a\}$, $\{\psi_{\bar{b}}\})$ in the Definition 10 that can be taken as ($\{L_{(a,0),(a+\varepsilon,\varepsilon)}\}$, $\{L_{(\bar{b},0),(\overline{b-\varepsilon},\varepsilon)}\}$, $\{L_{(a,\varepsilon),(a+\varepsilon,0)}\}$, $\{L_{(\bar{b},\varepsilon),(\overline{b-\varepsilon},0)}\}$). The commutativity of diagrams in the Definition 10 implies the functoriality of L and the commutativity of diagram in the Theorem 16. □

Theorem 17. *If two decomposable extended persistence modules V, W are ε- interleaved, then there exists a family* $\{V^t | t \in [0, \varepsilon]\}$ *of decomposable extended persistence modules such that* V^s, V^t *are* $|s - t|$*-interleaved for all* $s, t \in [0, \varepsilon]$, $V^0 = V$, $V^\varepsilon = W$.

Proof. By Theorem 16 there exists a family $\{V^t | t \in [0, \varepsilon]\}$ of decomposable extended persistence modules such that V^s, V^t are $|s - t|$-interleaved for all $s, t \in [0, \varepsilon]$ if and only if there exists a functor H such that the following diagram commutes



For any $(e, b) \in E \times [0, \varepsilon]$, define H by:

$$H(e,b) := \operatorname*{colim}_{i(x,t)\to(e,b)} L((x,t)) = \begin{cases} L((e,b)), & (e,b) \in E \times \{0,\varepsilon\}, \\ L((\Omega_b^{-1}(e),0) \bigoplus L(\Omega_{\varepsilon-b}^{-1}(e),\varepsilon))/G, & \text{otherwise,} \end{cases}$$

where G is a subgroup generated by the following element :

$$\{(L_{(\Omega_\varepsilon^{-1}(\Omega_b^{-1}(e)),\varepsilon),(\Omega_b^{-1}(e),0)}(x), -L_{(\Omega_\varepsilon^{-1}(\Omega_b^{-1}(e)),\varepsilon),(\Omega_{\varepsilon-b}^{-1}(e),\varepsilon)}(x))|x \in L_{(\Omega_\varepsilon^{-1}(\Omega_b^{-1}(e)),\varepsilon)}\},$$

$$\{(L_{(\Omega_\varepsilon^{-1}(\Omega_{\varepsilon-b}^{-1}(e)),0),(\Omega_b^{-1}(e),0)}(y), -L_{(\Omega_\varepsilon^{-1}(\Omega_{\varepsilon-b}^{-1}(e)),0),(\Omega_{\varepsilon-b}^{-1}(e),\varepsilon)}(y))|y \in L_{(\Omega_\varepsilon^{-1}(\Omega_{\varepsilon-b}^{-1}(e)),0)}\}.$$

We denote the morphism from $H((x, a))$ to $H((y, b))$ as $H_{(x,a),(y,b)}$ for any (x, a), $(y, b) \in E \times [0, \varepsilon]$. So $V^t(e) := H((e, t))$ has finite dimension, and has locally constant near ∞ which means there exist $a \in \mathbb{R}$ $b \in \mathbb{R}$, such that $V^t_{l,h} := H_{(l,t),(h,t)}$ is an isomorphism for all $a < l < h < \overline{b}$ for the image of L has finite dimension, and is locally constant near ∞. Then by Proposition 9 $\{V^t | t \in [0, \varepsilon]\}$ is a family of decomposable extended persistence modules. □

Theorem 18. *If two decomposable extended persistence modules V and W are ε-interleaved then $d_B(V, W) \leq \varepsilon$.*

Proof. Theorem 18 is a corollary of Lemma 14 and Theorem 17. □

4.3. Stability for extended persistent path homology

We shall need the following definition of relative path homology of digraphs. Let $G = (X, E), G' = (X, E')$ be two graphs on the same vertex set X such that $E' \subset E$, then we have $\mathcal{A}_*(G') \subset \mathcal{A}_*(G) \subset \mathcal{R}_*(X)$. Define the *relative path homology group* of (G, G') as $H_p(G, G') = H_p^{\sup}(\mathcal{A}_*(G), \mathcal{A}_*(G'); \mathcal{R}_*(X))$.

For digraphs, a (continuous) persistent module usually arises from a weight function. We define a *weighted digraph* as a triple $D = (X, E, A)$ where X is a finite set with a weight function $A : E \to \mathbb{R}, E \subseteq X \times X - \{(x, x)|x \in X\}$. Let A, A' be two weight functions on E, we define the *distance* between them as

$$d_E(A, A') = \max_{(x,y)\in E} |A(x, y) - A'(x, y)|,$$

Given a weighted digraph $D = (X, E, A)$ and $a \in \mathbb{R}$, define digraphs $G_a = (X, E_a)$, $G_{\overline{a}} = (X, E_{\overline{a}})$ by

$$E_a := \{(x, y) \in E : A(x, y) \leq a\},$$

$$E_{\overline{a}} := \{(x, y) \in E : A(x, y) \geq a\}.$$

For any $a' \geq a \in \mathbb{R}$, we have natural inclusions $E_a \hookrightarrow E_{a'}$, $E_{\overline{a'}} \hookrightarrow E_{\overline{a}}$. The nested families of subdigraphs $\{G_a\}$, $\{G_{\overline{a}}\}$ of $G = (X, E)$ induces nested families of subgroups $\mathcal{A}_*(G_a)$, $\mathcal{A}_*(G_{\overline{a}})$ of $\mathcal{R}_*(X)$ (see Section 2.2.2). Denote the supremum complexes of $\mathcal{A}_*(G_*)$ as $\mathcal{S}_*(G_*)$. Then we have nested families of subcomplexs $\mathcal{S}_*(G_a)$, $\mathcal{S}_*(G_{\overline{a}})$ of $\mathcal{R}_*(X)$ .Thus for each p, we get an extended persistence module V_p^D by defining

$$V_p^D(a) := H_p(\mathcal{S}_*(G_a)) = H_p^{\sup}(\mathcal{A}_*(G_a); \mathcal{R}_*(X)) = H_p(G_a),$$

$$V_p^D(\infty) := H_p(\mathcal{S}_*(G)) = H_p^{\sup}(\mathcal{A}_*(G); \mathcal{R}_*(X)) = H_p(G),$$

$$V_p^D(\overline{a}) := H_p(\mathcal{S}_*(G)/\mathcal{S}_*(G_{\overline{a}})) = H_p^{\sup}(\mathcal{A}_*(G), \mathcal{A}_*(G_{\overline{a}}); \mathcal{R}_*(X)) = H_p(G, G_{\overline{a}}),$$

for $a \in \mathbb{R}$ and $\overline{a} \in \mathbb{R}^o$, and joining them by homomorphisms induced by inclusions.

Finiteness of X implies that $\mathcal{R}_p(X)$ and $\mathcal{S}_p(G)$ are finite-dimensional for all $p \in \mathbb{N}$. Thus, V_p^D is finite dimensional everywhere. For sufficiently large $a \in \mathbb{R}$, $b \in \mathbb{R}$, we have $G_a = G$, $G_{\overline{b}} = (X, \emptyset)$. Thus, $(V_p^D)_{s,t}$ is an isomorphism for $a < s < t < \overline{b}$. By Proposition 9, V_p^D is decomposable for any weighted graph D and any $p \in \mathbb{N}$.

We are now able to formulate and prove the stability of extended persistent path homology.

Theorem 19. *Let $D = (X, E, A^0)$, $D' = (X, E, A^\delta)$ be two weighted digraphs with $d_E(A^0, A^\delta) = \delta$.*

We have

$$d_B(\mathrm{dgm}(V_p^D), \mathrm{dgm}(V_p^{D'})) \leq d_E(A^0, A^\delta),$$

for any $p \in \mathbb{Z}_+$.

Proof. For any $h \in \{0, \delta\}$, $a \in \mathbb{R}$, define E_a^h, $E_{\overline{a}}^h$ by

$$E_a^h := \{(x, y) \in E : A^h(x, y) \leq a\},$$

$$E_{\overline{a}}^h := \{(x, y) \in E : A^h(x, y) \geq a\},$$

and define $G_a^h = (X, E_a^h)$, $G_{\overline{a}}^h = (X, E_{\overline{a}}^h)$.

Since $d_E(A^0, A^\delta) = \delta$, we have the following natural inclusions of digraphs.

$$G_a^0 \hookrightarrow G_{a+\delta}^\delta, G_{\overline{a}}^0 \hookrightarrow G_{\overline{a-\delta}}^\delta,$$

$$G_a^\delta \hookrightarrow G_{a+\delta}^0, G_{\overline{a}}^\delta \hookrightarrow G_{\overline{a-\delta}}^0. \qquad (*)$$

Consider now the following commutative diagrams of chain complexes, where all arrows are induced by inclusion or natural projection (the slant arrows are justified by $(*)$):

(i) For any $a < a', b < b' \in \mathbb{R}$, we have diagrams:

$$\begin{array}{ccccc} \mathcal{S}_*(G)/\mathcal{S}_*(G^s_{\overline{b'}}) & \longrightarrow & & \mathcal{S}_*(G)/\mathcal{S}_*(G^s_{\overline{b}}) & \\ & \searrow & & & \searrow \\ & \mathcal{S}_*(G)/\mathcal{S}_*(G^t_{\overline{b'-\delta}}) & \longrightarrow & & \mathcal{S}_*(G)/\mathcal{S}_*(G^t_{\overline{b-\delta}}) \end{array}$$

$$\begin{array}{ccccc} \mathcal{S}_*(G)/\mathcal{S}_*(G^0_{\overline{b'}}) & \longrightarrow & & \mathcal{S}_*(G)/\mathcal{S}_*(G^0_{\overline{b}}) & \\ & \searrow & & & \searrow \\ & \mathcal{S}_*(G)/\mathcal{S}_*(G^\delta_{\overline{b'-\delta}}) & \longrightarrow & & \mathcal{S}_*(G)/\mathcal{S}_*(G^\delta_{\overline{b'-\delta}}) \end{array}$$

$$\begin{array}{ccccc} \mathcal{S}_*(G)/\mathcal{S}_*(G^0_{\overline{b'}}) & \longrightarrow & & \mathcal{S}_*(G)/\mathcal{S}_*(G^0_{\overline{b}}) & \\ & \searrow & & & \searrow \\ & \mathcal{S}_*(G)/\mathcal{S}_*(G^\delta_{\overline{b'-\delta}}) & \longrightarrow & & \mathcal{S}_*(G)/\mathcal{S}_*(G^\delta_{\overline{b-\delta}}) \end{array}$$

(i') Same diagrams as (i) but with s, t reversed.

(ii) For any $a, b \in \mathbb{R}$, we have diagrams:

$$\begin{array}{ccc} \mathcal{S}_*(G^0_a) & \longrightarrow & \mathcal{S}_*(G^0_{a+2\delta}) \\ & \searrow \quad \nearrow & \\ & \mathcal{S}_*(G^\delta_{a+\delta}) & \end{array}$$

$$\begin{array}{ccc} \mathcal{S}_*(G)/\mathcal{S}_*(G^0_{\overline{b}}) & \longrightarrow & \mathcal{S}_*(G)/\mathcal{S}_*(G^0_{\overline{b-2\delta}}) \\ & \searrow \quad \nearrow & \\ & \mathcal{S}_*(G)/\mathcal{S}_*(G^\delta_{\overline{b-\delta}}) & \end{array}$$

(ii') Same diagrams as (ii) but with $0, \delta$ reversed.

Passing to homology, the above diagrams constitute a δ-interleaving between V^D_p and $V^{D'}_p$. Since all of these modules are decomposable, we have $d_B(\mathrm{dgm}_p(V^D_p), \mathrm{dgm}_p(V^{D'}_p) \leq \delta = d_E(A, A')$ by Theorem 18. □

4.4. Stability for extended persistent homology of hypergraphs

We shall need the following definition of relative embedded homology for hypergraph pairs. Let V be a finite set and $\mathcal{H}' \subseteq \mathcal{H}$ be hypergraphs on V. Then we have $\Delta_*(\mathcal{H}') \subseteq \Delta_*(\mathcal{H}) \subseteq \Delta_*(K_{\mathcal{H}})$. Define the relative embedded homology group of $(\mathcal{H}, \mathcal{H}')$ as $H_p(\mathcal{H}, \mathcal{H}') = H^{\sup}_p(\Delta_*(\mathcal{H}), \Delta_*(\mathcal{H}'); \Delta_*(K_{\mathcal{H}}))$. Note that we could replace $K_{\mathcal{H}}$ by any larger simplicial complex. In particular, an inclusion of hypergraph pairs induces a well-defined homomorphism between their homology groups.

Now suppose $\mathcal{H}$ is a hypergraph defined on a finite set V, let f be a real valued function on $\mathcal{H}$. For each $a \in \mathbb{R}$, let

$$\mathcal{H}^f_a := f^{-1}((-\infty, a]),$$
$$\mathcal{H}^f_{\bar{a}} := f^{-1}([a, +\infty)).$$

For each $p \in \mathbb{N}$, we define an extended persistence module V^f_p by defining

$$V^f_p(a) := H_p(\mathcal{H}^f_a) = H^{\sup}_p(\Delta_*(\mathcal{H}^f_a); \Delta_*(K_{\mathcal{H}})),$$
$$V^f_p(\infty) := H_p(\mathcal{H}) = H^{\sup}_p(\Delta_*(\mathcal{H}); \Delta_*(K_{\mathcal{H}})),$$
$$V^f_p(\bar{a}) := H_p(\mathcal{H}, \mathcal{H}^f_{\bar{a}}) = H^{\sup}_p(\Delta_*(\mathcal{H}), \Delta_*(\mathcal{H}^f_{\bar{a}}); \Delta_*(K_{\mathcal{H}})).$$

for any $a \in \mathbb{R}$ and $\overline{a} \in \mathbb{R}^o$, and joining them by homomorphisms induced by inclusions. Suppose f, g are two real valued functions on $\mathcal{H}$. Define the L^∞ distance between f and g by

$$\|f - g\|_\infty = \sup_{\sigma \in \mathcal{H}} |f(\sigma) - g(\sigma)|$$

We are now able to formulate and prove the stability of extended persistent embedded homology of the hypergraph.

Theorem 20. *Let f, g be two real valued functions on $\mathcal{H}$. We have*

$$d_B(\mathrm{dgm}(V_p^f), \mathrm{dgm}(V_p^g)) \leq \|f-g\|_\infty.$$

Proof. The proof of this theorem is similar to the proof of Theorem 19. Let $\delta = \|f-g\|_\infty$. Then we have

$$\mathcal{H}_a^f \to \mathcal{H}_{a+\delta}^g, \mathcal{H}_a^g \to \mathcal{H}_{a+\delta}^f,$$

$$\mathcal{H}_{\bar{a}}^f \to \mathcal{H}_{\bar{a}-\delta}^g, \mathcal{H}_a^g \to \mathcal{H}_{\bar{a}-\delta}^f. \qquad (**)$$

A δ-interleaving between V_p^f, V_p^g is obtained by (**) in the same way that the interleaving is obtained by (*) in Theorem 19. Since these modules are decomposable by Proposition 9, we have

$$d_B(\mathrm{dgm}(V^f), \mathrm{dgm}(V^g) \leq \delta = \|f-g\|_\infty.$$

by Theorem 18. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We gratefully acknowledge the support of the NSF of China, funding reference number 12331003. We are grateful for the suggestions and comments of the reviewers, which help us to improve the presentation of this paper significantly.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. H. Edelsbrunner, D. Letscher, A. Zomorodian, Topological persistence and simplification, *Discrete Comput. Geom.*, **28** (2002), 511–533. https://doi.org/10.1007/s00454-002-2885-2
2. A. Zomorodian, G. Carlsson, Computing persistent homology, in *Proceedings of the Twentieth Annual Symposium on Computational Geometry*, (2004), 347–356. https://doi.org/10.1145/997817.997870
3. D. Cohen-Steiner, H. Edelsbrunner, J. Harer, Stability of persistence diagrams, in *Proceedings of the Twenty-First Annual Symposium on Computational Geometry*, (2005), 263–271. https://doi.org/10.1145/1064092.1064133
4. A. Grigor'yan, Y. Lin, Y. Muranov, S. Yau, Path complexes and their homologies, *J. Math. Sci.*, **248** (2020), 564–599. http://dx.doi.org/10.1007/s10958-020-04897-9

5. S. Chowdhury, F. Mémoli, Persistent path homology of directed networks, in *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, (2018), 1152–1169. http://dx.doi.org/10.1137/1.9781611975031.75
6. S. Bressan, J. Li, S. Ren, J. Wu, The embedded homology of hypergraphs and applications, *Asian J. Math.*, **23** (2019), 479–500. http://dx.doi.org/10.4310/AJM.2019.v23.n3.a6
7. G. Carlsson, A. Zomorodian, The theory of multidimensional persistence, *Discrete Comput. Geom.*, **42** (2009), 71–93. https://doi.org/10.1145/1247069.1247105
8. C. Chen, M. Kerber, Persistent homology computation with a twist, in *Proceedings 27th European Workshop on Computational Geometry*, **11** (2011), 197–200.
9. M. Botnan, W. Crawley-Boevey, Decomposition of persistence modules, in *Proceedings of the American Mathematical Society*,**148** (2020), 4581–4596. http://dx.doi.org/10.1090/proc/14790
10. G. Rote, G. Vegter, Computational topology: An introduction, in *Effective Computational Geometry for Curves and Surfaces*, (2006), 277–312. http://dx.doi.org/10.1007/978-3-540-33259-6_7
11. D. Cohen-Steiner, H. Edelsbrunner, J. Harer, Extending persistence using Poincaré and Lefschetz duality, *Found. Comput. Math.*, **9** (2009), 133–134. http://dx.doi.org/10.1007/s10208-008-9038-9
12. F. Chazal, V. Silva, M. Glisse, S. Oudot, *The Structure and Stability of Persistence Modules*, Berlin, Springer, 2016. http://dx.doi.org/10.1007/978-3-319-42545-0
13. P. Bubenik, J. Scott, Categorification of persistent homology, *Discrete Comput. Geom.*, **51** (2014), 600–627. http://dx.doi.org/10.1007/s00454-014-9573-x
14. S. Ren, J. Wu, Stability of persistent homology for hypergraphs, perprint, arXiv:2002.02237. https://doi.org/10.48550/arXiv.2002.02237
15. A. Patel, Generalized persistence diagrams, *J. Appl. Comput. Topol.*, **1** (2018), 397–419. http://dx.doi.org/10.1007/s41468-018-0012-6