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Models for chain homotopy category of relative acyclic complexes

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Abstract: Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair in an abelian category $\mathcal{A}$. Denote by $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ the chain homotopy category of right $\mathcal{X}$-acyclic complexes with all items in $\mathcal{X}$, and dually by $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ the chain homotopy category of left $\mathcal{Y}$-acyclic complexes with all items in $\mathcal{Y}$. We establish realizations of $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ and $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ as homotopy categories of model categories under mild conditions. Consequently, we obtain relative versions of recollements of Krause and Neeman-Murfet. We further present applications to Gorenstein projective and Gorenstein injective modules.

Keywords: balanced pair; model category; exact category; chain homotopy category; recollement

1. Introduction

A model structure on a category C is a triple of three classes of morphisms, called cofibrations, fibrations, and weak equivalences, satisfying a few axioms; see [1,2] for details. In general, the homotopy category in the sense of Quillen (i.e., the localization of C with respect to weak equivalences) of a model structure is not a triangulated category. However, if C is weakly idempotent complete exact, then the homotopy category of an exact model structure carries a triangulated structure (see [3, Section 6]).

The Hovey correspondence [1,4] of abelian categories provides an effective tool for constructing model structures on abelian categories. It is inspired by a somewhat canonical model structure on a Frobenius category, but with two cotorsion pairs mimicking the role played by the projectives and the injectives. Furthermore, Hovey's correspondence has been extended as the one-one correspondence between exact model structures and the Hovey triples on weakly idempotent complete exact categories, by Gillespie [5] (see also Šťovíček [6]).

An important application of model category theory lies in providing systematic methods for constructing recollements of triangulated categories (see [7–11]). Recall that the notion of a recollement, introduced by Beilinson et al. in [12], can be viewed as a form of "short exact sequence" of triangulated categories, in which the functors involved admit both left and right adjoints. For example, Becker [7] recovered Krause's recollement $\mathbf{K}_{\mathrm{ac}}(\mathcal{I}) \longrightarrow \mathbf{K}(\mathcal{I}) \longrightarrow \mathbf{D}(R)$ from [13], and Gillespie [10] recovered Neeman-Murfet's recollement $\mathbf{K}_{\mathrm{ac}}(\mathcal{P}) \longrightarrow \mathbf{K}(\mathcal{P}) \longrightarrow \mathbf{D}(R)$ from [14] using the theory of abelian model categories. Here, $\mathbf{K}(\mathcal{I})$ (resp. $\mathbf{K}(\mathcal{P})$) denotes the chain homotopy category of all complexes of injective (resp. projective) modules, $\mathbf{K}_{\mathrm{ac}}(\mathcal{I})$ (resp. $\mathbf{K}_{\mathrm{ac}}(\mathcal{P})$) is the full subcategory of exact complexes of injective (resp. projective) modules, and $\mathbf{D}(R)$ is the derived category of a ring R.

Recall that a pair $(\mathcal{X}, \mathcal{Y})$ of additive subcategories in an abelian category $\mathcal{A}$ is said to be balanced if every object of $\mathcal{A}$ admits an $\mathcal{X}$-resolution that remains acyclic after applying $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ for all $Y \in \mathcal{Y}$, and also admits a $\mathcal{Y}$-coresolution that is acyclic after applying $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for all $X \in \mathcal{X}$. This condition implies a balancing phenomenon: the relative right-derived functors of $\mathrm{Hom}_{\mathcal{A}}(-,-)$ can be computed either via an $\mathcal{X}$-resolution of the first variable, or equivalently via a $\mathcal{Y}$-coresolution of the second variable. In other words, the Hom functor is right-balanced by $\mathcal{X} \times \mathcal{Y}$; see [15, §8.2]. It is straightforward to verify that $(\mathcal{P}, \mathcal{I})$ is a balanced pair when $\mathcal{A}$ has enough projective objects and injective objects. We refer to [15–17] for more examples of balanced pairs.

Let $\mathcal{A}$ be an abelian category equipped with a balanced pair $(\mathcal{X}, \mathcal{Y})$. Denote by $\mathbf{K}(\mathcal{X})$ (resp. $\mathbf{K}(\mathcal{Y})$) the chain homotopy category of complexes with all items in $\mathcal{X}$ (resp. $\mathcal{Y}$), and by $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ (resp. $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$) the full subcategory of $\mathbf{K}(\mathcal{X})$ (resp. $\mathbf{K}(\mathcal{Y})$) consisting of complexes that are acyclic with respect to the functor $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{Y})$). One then considers the following sequences of triangulated categories:

$$\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X}) \longrightarrow \mathbf{K}(\mathcal{X}) \longrightarrow \mathbf{D}_{\mathcal{X}}(\mathcal{A}) \text{ and } \mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y}) \longrightarrow \mathbf{K}(\mathcal{Y}) \longrightarrow \mathbf{D}_{\mathcal{Y}}(\mathcal{A}),$$

where $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ (resp. $\mathbf{D}_{\mathcal{Y}}(\mathcal{A})$) is the relative derived category in the sense of [16, Definition 3.1] (see also [15, 18, 19]). We have proved in [21] that the chain homotopy categories $\mathbf{K}(\mathcal{X})$ and $\mathbf{K}(\mathcal{Y})$, and the relative derived categories $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ and $\mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ can be realized as homotopy categories of model categories under certain conditions. This naturally leads us to seek realizations of $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ and $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ as homotopy categories of suitable model categories, thereby obtaining relative versions of the Krause's and Neeman-Murfet's recollements.

We now outline the results of the paper. In Section 2, we summarize some preliminaries and basic facts which will be used throughout the paper.

In Section 3, we realize the chain homotopy categories of complexes $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ and $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ as homotopy categories of certain model categories. For the given balanced pair $(\mathcal{X}, \mathcal{Y})$, we denote by $\mathcal{E}$ the class of short exact sequences in $\mathcal{A}$ which remain exact by applying $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for any $X \in \mathcal{X}$. It follows that $(\mathcal{A}, \mathcal{E})$ is an exact category. Therefore, the category $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ of complexes over $(\mathcal{A}, \mathcal{E})$ with respect to the class $\mathrm{Ch}(\mathcal{E})$ of short exact sequences of complexes which belong to $\mathcal{E}$ in each degree, is also an exact category (see [20, Lemma 9.1]). By the Hovey correspondence between exact model structures and the Hovey triples on weakly idempotent complete exact categories (see [5, 6]), we will denote the model structure $\mathcal{M}$ by the corresponding Hovey triples, and denote the homotopy category of model categories by $\mathrm{Ho}(\mathcal{M})$. Under the assumption that $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ is closed under direct sums, we establish a hereditary model structure $\mathcal{M}_{ac\mathcal{X}} = (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}, \mathrm{Ch}(\mathcal{A}))$ on the exact category $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ with a triangle equivalence $\mathrm{Ho}(\mathcal{M}_{ac\mathcal{X}}) \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ (see Theorem 3.9).

Dually, if ${}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ is closed under direct products, then we have a hereditary model structure $\mathcal{M}_{ac\mathcal{Y}} = (\mathrm{Ch}(\mathcal{A}), {}^{\perp}(\mathcal{E}\text{-ac}\widetilde{\mathcal{Y}}), \mathcal{E}\text{-ac}\widetilde{\mathcal{Y}})$ on $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ with a triangle equivalence $\mathrm{Ho}(\mathcal{M}_{ac\mathcal{Y}}) \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ (see Remark 3.10). In the specific case of $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}, \mathcal{I})$, $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ (resp. $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$) is exactly the injective (resp. projective) stable derived category which have been studied by Gillespie in [10] and Krause in [13].

In Section 4, we obtain relative versions of Krause's and Neeman-Murfet's recollements. This is based on the models for $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ and $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ in Section 3 and the models for $\mathbf{K}(\mathcal{X})$, $\mathbf{K}(\mathcal{Y})$, $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ and $\mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ in [21]. It is proved in Corollary 4.3 that if $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ is closed under direct sums, then there is a recollement:

$$\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{X}) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{K}(\mathcal{X}) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{D}_{\mathcal{X}}(\mathcal{A}).$$

Dually, if ${}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ is closed under direct products, then it is shown in Corollary 4.4 that there is a recollement:

$$\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{Y}) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{K}(\mathcal{Y}) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{D}_{\mathcal{Y}}(\mathcal{A}).$$

These recollements generalize the Krause's recollement in [22, Theorem 7.7] and the Neeman-Murfet's recollement in [14,23]. Denote by $\mathcal{GP}$ (resp. $\mathcal{GI}$) the subcategory consisting of all Gorenstein projective (resp. injective) modules over a ring R. Let R be a ring with finite Gorenstein weak dimension. It follows from [24, Theorem 4.2] and [21, Lemma 5.7] that $(\mathcal{GP}, \mathcal{GI})$ is a balanced pair such that $(\mathcal{E}\text{-dw}\widetilde{\mathcal{GP}})^{\perp} = {}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{GI}})$ is closed under direct sums and direct products. In combination with this, we obtain recollements $\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{GP}) \longrightarrow \mathbf{K}(\mathcal{GP}) \longrightarrow \mathbf{D}_{\mathcal{GP}}(R)$ and $\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{GI}) \longrightarrow \mathbf{K}(\mathcal{GI}) \longrightarrow \mathbf{D}_{\mathcal{GI}}(R)$ (see Corollary 4.5). Here $\mathbf{D}_{\mathcal{GP}}(R) = \mathbf{D}_{\mathcal{GI}}(R)$ are called Gorenstein derived categories by Gao and Zhang in [25] (see Remark 2.1). The principal technique we employ comes from the work of Becker [7] and Gillespie [9, 10]. They provided a method to construct recollement from three interrelated hereditary Hovey triples.

2. Preliminaries

Let $\mathcal{A}$ be a complete and cocomplete abelian category. A class of objects in $\mathcal{A}$ will be always assumed to be closed under isomorphisms and under finite direct sums. An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{E}$ is a class of "short exact sequences" in $\mathcal{A}$, that is, kernel-cokernel pairs (i, p) depicted by $A' \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} A''$, satisfying some axioms; see Quillen's original definition in [26]. A map, such as i, is called an *admissible monomorphism* while p is called an *admissible epimorphism*. Recall that an exact category $(\mathcal{A}, \mathcal{E})$ is *weakly idempotent complete* if every split monomorphism has a cokernel and every split epimorphism has a kernel; see [5, Definition 2.2] or [20, Definition 7.2]. We refer to a readable exposition [20] for details on exact categories.

2.1. Cotorsion pairs

Analogously to abelian categories, the axioms of exact categories allow for the usual construction of the Yoneda Ext bifunctor $\mathrm{Ext}^1_{\mathcal{E}}(M, N)$. It is the abelian group of equivalence classes of short exact sequences $N \rightarrowtail L \twoheadrightarrow M$. In particular, we get that $\mathrm{Ext}^1_{\mathcal{E}}(M, N) = 0$ if and only if every short exact sequence $N \rightarrowtail L \twoheadrightarrow M$ is isomorphic to the split exact sequence $N \rightarrowtail N \oplus M \twoheadrightarrow M$.

The definition of a cotorsion pair readily generalizes to exact categories; see [5, Definition 2.1]. Specifically, a pair of classes $(\mathcal{F}, \mathcal{C})$ in $(\mathcal{A}, \mathcal{E})$ is a *cotorsion pair* provided that $\mathcal{F} = {}^{\perp}\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^{\perp}$, where the left orthogonal class ${}^{\perp}\mathcal{C}$ consists of F such that $\mathrm{Ext}^1_{\mathcal{E}}(F, X) = 0$ for all $X \in \mathcal{C}$, and the right orthogonal class $\mathcal{F}^{\perp}$ is defined similarly. We say the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is *hereditary* if $\mathcal{F}$ is closed under taking kernels of admissible epimorphisms between objects of $\mathcal{F}$, and if C is closed under taking cokernels of admissible monomorphisms between objects of C.

The cotorsion pair $(\mathcal{F}, C)$ is said to be *complete* if for any object $M \in \mathcal{A}$, there exist short exact sequences $C \rightarrowtail F \twoheadrightarrow M$ and $M \rightarrowtail C' \twoheadrightarrow F'$ with $F, F' \in \mathcal{F}$ and $C, C' \in C$. In this case, $F \twoheadrightarrow M$ is called a *special right $\mathcal{F}$-approximation* (or, *special $\mathcal{F}$-precover*) of M, and $M \rightarrowtail C'$ is called a *special left C-approximation* (or, *special C-preenvelope*) of M.

2.2. Approximation and balanced pairs

Let X be a subcategory of the abelian category $\mathcal{A}$ which is closed under taking direct summands and M an object in $\mathcal{A}$. A morphism $f : X \to M$ (resp. $f : M \to X$) with $X \in \mathcal{X}$ is called a *right $\mathcal{X}$-approximation* (resp. *left $\mathcal{X}$-approximation*) of M, if any morphism from an object in $\mathcal{X}$ to M (resp. M to $\mathcal{X}$) factors through f. The subcategory $\mathcal{X}$ is called *contravariantly finite* (resp. *covariantly finite*) if each object in $\mathcal{A}$ has a right $\mathcal{X}$-approximation (resp. *left $\mathcal{X}$-approximation*).

Recall that a complex is *right $\mathcal{X}$-acyclic* (resp. *left $\mathcal{Y}$-acyclic*) if it remains acyclic after applying $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for all $X \in \mathcal{X}$ (resp. $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ for all $Y \in \mathcal{Y}$).

A pair $(\mathcal{X}, \mathcal{Y})$ of subcategory in an abelian category $\mathcal{A}$ is called a *balanced pair* if the following conditions are satisfied (see [16, Definition 1.1]):

1) the subcategory $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite;
2) for each object $M \in \mathcal{A}$, there is a complex $\cdots \to X_1 \to X_0 \to M \to 0$ with each $X_i \in \mathcal{X}$ which is both right $\mathcal{X}$-acyclic and left $\mathcal{Y}$-acyclic;
3) for each object $N \in \mathcal{A}$, there is a complex $0 \to M \to Y_0 \to Y_1 \to \cdots$ with each $Y_i \in \mathcal{Y}$ which is both right $\mathcal{X}$-acyclic and left $\mathcal{Y}$-acyclic;

The balanced pair is called *admissible* if each right $\mathcal{X}$-approximation is an epimorphism and each left $\mathcal{Y}$-approximation is a monomorphism. It follows from [16, Proposition 2.6] that if there exist two complete and hereditary cotorsion pairs $(\mathcal{X}, \mathcal{Z})$ and $(\mathcal{Z}, \mathcal{Y})$ in $\mathcal{A}$, then the pair $(\mathcal{X}, \mathcal{Y})$ is an admissible balanced pair. In this case, $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ is called a *cotorsion triple*. It follows from [17, Theorem 4.4] that the existence of complete and hereditary cotorsion triple in $\mathcal{A}$ is equivalent to that $\mathcal{A}$ has enough projective objects and injective objects.

2.3. Relative derived categories

Let $\mathcal{X}$ be a contravariantly finite subcategory of an abelian category $\mathcal{A}$. Denote by $\mathbf{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$ and $\widetilde{\mathcal{E}}$ the subcategory of right $\mathcal{X}$-acyclic complexes, we recall the *relative derived category* $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ of $\mathcal{A}$ with respect to $\mathcal{X}$ (see [16, Definition 3.1]) is defined to be the Verdier quotient of $\mathbf{K}(\mathcal{A})$ modulo the subcategory consisting of objects in $\widetilde{\mathcal{E}}$, that is,

$$\mathbf{D}_{\mathcal{X}}(\mathcal{A}) := \mathbf{K}(\mathcal{A})/\widetilde{\mathcal{E}}.$$

Remark 2.1. Note that the derived category of exact category in the sense of [27, Construction 1.5] is an example of relative derived category. In particular, if $\mathcal{X}$ is the full subcategory of Gorenstein projective objects in the sense of Enochs and Jenda in [15], $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ is the Gorenstein derived category in the sense of Gao and Zhang in [25].

Dually, for a covariantly finite subcategory $\mathcal{Y}$, one can define the relative derived category $\mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ of $\mathcal{A}$ with respect to $\mathcal{Y}$. Under the assumption that $(\mathcal{X},\mathcal{Y})$ is a balanced pair, it follows from [16, Proposition 2.2] that $\widetilde{\mathcal{E}}$ is exactly the complexes which is left $\mathcal{Y}$-acyclic, thus $\mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ coincides with $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$. Moreover, we have realized it as a homotopy category of a model structure, see [21, Theorem 3.10] for details.

2.4. *Hovey triples and model structures*

The notion of model structure is introduced by Quillen [2], which refers to three specified classes of morphisms, called fibrations, cofibrations and weak equivalences, satisfing a few axioms; see [1, 2] for details. A *model category* is a complete and cocomplete category equipped with a model structure.

Now suppose the exact category $(\mathcal{A},\mathcal{E})$ has a model structure. An object $M\in\mathcal{A}$ is called *trivial* (resp. *cofibrant*, *fibrant*) if $0\rightarrowtail M$ (resp. $0\rightarrowtail M$, $M\twoheadrightarrow 0$) is a weak equivalence(resp. cofibration, fibration). We say M is trivially cofibrant (resp. trivially fibrant) if it is both trivial and cofibrant (resp. fibrant). The subcategories of trivial, cofibrant, and fibrant objects will be denoted by $\mathcal{A}_{tri}$, $\mathcal{A}_c$, and $\mathcal{A}_f$, respectively.

Recall that a *thick subcategory* means a class $\mathcal{W}$ of objects which is closed under direct summands, and such that if two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third; see [5, Definition 3.2]. Recall that a triple $(\mathcal{C},\mathcal{W},\mathcal{F})$ of subcategories in $(\mathcal{A},\mathcal{E})$ is called a *(hereditary) Hovey triple*, if $\mathcal{W}$ is thick and both $(\mathcal{C},\mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ are complete (hereditary) cotorsion pairs. It is well known that there is a correspondence between Hovey triples and model structures stated as follows:

Lemma 2.2. *[5, Theorem 3.3] If the exact category $(\mathcal{A},\mathcal{E})$ admits a model structure, then the triple $(\mathcal{A}_c,\mathcal{A}_{tri},\mathcal{A}_f)$ of subcategories becomes a Hovey triple. If $(\mathcal{A},\mathcal{E})$ is weakly idempotent complete, then the converse holds. In this case, a map is a (trivial) cofibration if and only if it is an admissible monomorphism with a (trivially) cofibrant cokernel, and a map is a (trivial) fibration if and only if it is an admissible epimorphism with a (trivially) fibrant kernel. A map is weak equivalence if and only if it factors as a trivial cofibration followed by a trivial fibration.*

Throughout this paper, we always denote a model structure by its corresponding Hovey triple $(\mathcal{A}_c,\mathcal{A}_{tri},\mathcal{A}_f)$. Recall that the class $\mathcal{A}_c\cap\mathcal{A}_{tri}\cap\mathcal{A}_f$ is called the *core* of the model structure $(\mathcal{A}_c,\mathcal{A}_{tri},\mathcal{A}_f)$.

Let $\mathcal{A}$ be a model category with a hereditary (i.e., its corresponding Hovey triple is hereditary) model structure $\mathcal{M}=(\mathcal{A}_c,\mathcal{A}_{tri},\mathcal{A}_f)$. Its *homotopy category*, denoted by $\mathrm{Ho}(\mathcal{M})$, is the localization of $\mathcal{A}$ with respect to the collection of weak equivalences. It is well known that $\mathcal{A}_{cf}=\mathcal{A}_c\cap\mathcal{A}_f$ is a Frobenius category, with core $\omega=\mathcal{A}_c\cap\mathcal{A}_{tri}\cap\mathcal{A}_f$ being the class of projective-injective objects. Then, the stable category $\underline{\mathcal{A}_{cf}}=\mathcal{A}_{cf}/\omega$ is a triangulated category. In this case, one has a triangle equivalence $\mathrm{Ho}(\mathcal{A})\simeq\underline{\mathcal{A}_{cf}}$; see [28, Theorem 1.3], [1, Theorem 1.2.10], [5, Proposition 4.4], or [7, Proposition 1.1.13].

2.5. *Exact category of complexes*

For a complex $\cdots\to C_{n+1}\overset{d_{n+1}}{\to}C_n\overset{d_n}{\to}C_{n-1}\to\cdots$ we denote $\mathrm{Ker}d_n$ by Z_nC, $\mathrm{Im}d_{n+1}$ by B_nC, and the nth homology $\mathrm{Z}_nC/\mathrm{B}_nC$ by H_nC. For an object $A\in\mathcal{A}$, denote by S^nA the complex with A in degree n and all other entries 0, and D^nA the complex with A in degree n and $n-1$ and all other entries 0, with all maps 0 except $d_n=1_A$. We refer to [29, Lemma 3.1] and [5, Lemma 4.2] for some useful isomorphisms with respect to complexes of the form S^nA and D^nA. The suspension functor over complexes is denoted by Σ.

Given two complexes C and D and a chain map $f : C \to D$, denote by $\mathrm{Con}(f)$ the *mapping cone* of f. Recall that f is *null homotopic*, denoted by $f \sim 0$, if there are maps $s_n : C_n \to D_{n+1}$ such that $f_n = d^D_{n+1}s_n + s_{n-1}d^C_n$. Chain maps $f, g : C \to D$ are called *chain homotopic*, denoted by $f \sim g$ if $f - g \sim 0$. In this sense $\{s_n\}$ are called a *chain homotopy*.

The *Hom-complex* $\mathrm{Hom}_{\mathcal{A}}(C, D)$ is defined with nth component $\mathrm{Hom}_{\mathcal{A}}(C, D)_n = \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X_k, Y_{k+n})$ and differential $(\delta_n f)_k = d^D_{k+n} f_k - (-1)^n f_{k-1} d^C_k$ for morphisms $f_k : C_k \to D_{k+n}$.

Let $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ be the exact category of chain complexes with respect to the class $\mathrm{Ch}(\mathcal{E})$ of short exact sequences of complexes which are in $\mathcal{E}$ degreewise. Denote by $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{A})}(C, D)$ the group of equivalence classes of short exact sequences $0 \to D \to E \to C \to 0$ of complexes. Let $\mathrm{Ext}^1_{dw}(C, D)$ and $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{E})}(C, D)$ be the subgroups of $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{A})}(C, D)$ consisting of those short exact sequences which are in each degree split, and in $\mathcal{E}$ respectively. The following is well known; see [29, Lemma 2.1].

Lemma 2.3. *For chain complexes C and D, one has*

$$\mathrm{Ext}^1_{dw}(C, \Sigma^{-n-1}D) \cong \mathrm{H}_n\mathrm{Hom}_{\mathcal{A}}(C, D) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C, \Sigma^{-n}D)/\sim .$$

3. Models for relative acyclic complexes

Throughout the paper, let $\mathcal{A}$ be a Grothendieck category, and let $(\mathcal{X}, \mathcal{Y})$ be an admissible balanced pair in $\mathcal{A}$.

Recall that a complex C is *right $\mathcal{X}$-acyclic* if it remains acyclic by applying $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for any $X \in \mathcal{X}$, and dually, one has the notion of *left $\mathcal{Y}$-acyclic*; see [16, pp. 2721]. We begin with the following observation, which will lead to [16, Proposition 2.2] by a different and more straightforward proof.

Lemma 3.1. *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence. Then, it is right $\mathcal{X}$-acyclic if and only if it is left $\mathcal{Y}$-acyclic.*

In the following, $\mathcal{E}$ will denote the class of short exact sequences in $\mathcal{A}$ which are right $\mathcal{X}$-acyclic (equivalently, left $\mathcal{Y}$-acyclic). Then, $(\mathcal{A}, \mathcal{E})$ is an exact category.

Inspired by [29, Definition 3.3], we have the following:

Definition 3.2. 1) $\widetilde{\mathcal{E}}$: the class of right $\mathcal{X}$-acyclic (left $\mathcal{Y}$-acyclic) complexes.

2) $\widetilde{\mathcal{X}}_{\mathcal{E}}$: the class of complexes $X \in \widetilde{\mathcal{E}}$ with all $\mathrm{Z}_nX \in \mathcal{X}$.

3) $\mathcal{E}$-dw$\widetilde{\mathcal{X}}$: the class of complexes X for which each item $X_n \in \mathcal{X}$.

4) $\mathcal{E}$-dg$\widetilde{\mathcal{X}}$: the class of complexes $X \in \mathcal{E}$-dw$\widetilde{\mathcal{X}}$ and for which every map $X \to E$ is null homotopic whenever $E \in \widetilde{\mathcal{E}}$.

5) $\mathcal{E}$-ac$\widetilde{\mathcal{X}}$: $= \mathcal{E}$-dw$\widetilde{\mathcal{X}} \cap \widetilde{\mathcal{E}}$ the class of complexes X which are right $\mathcal{X}$-acyclic with each item $X_n \in \mathcal{X}$.
Dually, $\widetilde{\mathcal{Y}}_{\mathcal{E}}$, $\mathcal{E}$-dw$\widetilde{\mathcal{Y}}$, $\mathcal{E}$-dg$\widetilde{\mathcal{Y}}$, and $\mathcal{E}$-ac$\widetilde{\mathcal{Y}}$ are defined.

The prefix "$\mathcal{E}$" in "$\mathcal{E}$-dw$\widetilde{\mathcal{X}}$" is used to indicate that we consider the right orthogonal $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ with respect to $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{E})}(-, -)$.

We need the following facts.

Lemma 3.3. *(i) For any $X \in \mathcal{X}$, one has $S^nX \in \mathcal{E}\text{-dg}\widetilde{\mathcal{X}}$, and $D^nX \in \widetilde{\mathcal{X}}_{\mathcal{E}}$.*

(ii) For any $Y \in \mathcal{Y}$, one has $S^nY \in \mathcal{E}\text{-dg}\widetilde{\mathcal{Y}}$, and $D^nY \in \widetilde{\mathcal{Y}}_{\mathcal{E}}$.

(iii) Let $0 \to X' \to X'' \to X \to 0$ be a short exact sequence in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ with $X \in \mathcal{E}\text{-dg}\widetilde{\mathcal{X}}$. Then, $X' \in \mathcal{E}\text{-dg}\widetilde{\mathcal{X}}$ if and only if $X'' \in \mathcal{E}\text{-dg}\widetilde{\mathcal{X}}$

Proof. We only proof (ii), that is, for any complex $X \in \widetilde{\mathcal{E}}$ and any object $Y \in \mathcal{Y}$, the chain map $X \to S^nY$ is null homotopic. One can check (i) and (iii) directly.

It follows from $f : X \to S^nY$ is chain map that $f_n d^X_{n+1} = 0$. Then, $f_n : X_n \to Y$ induces a map $g : X_n/B_nX \cong Z_{n-1}X \to Y$. Since $X \in \widetilde{\mathcal{E}}$, we infer from Lemma 3.1 that the short exact sequence $0 \to Z_{n-1}X \to X_{n-1} \to Z_{n-2}X \to 0$ is left $\mathcal{Y}$-acyclic. Hence, $\mathrm{Hom}_{\mathcal{A}}(X_{n-1}, Y) \to \mathrm{Hom}_{\mathcal{A}}(Z_{n-1}X, Y)$ is epic, and then, there is a preimage of $g \in \mathrm{Hom}_{\mathcal{A}}(Z_{n-1}X, Y)$, that is, a map $s : X_{n-1} \to Y$, such that $f_n = sd^X_n$. Note that all f_i other than f_n are 0. Then, it follows that the chain map $f : X \to S^nY$ is null homotopic. □

Let $\mathbf{K}(\mathcal{A})$ be the homotopy category of $\mathcal{A}$ and $\mathbf{K}(\mathcal{X})$ the subcategory of $\mathbf{K}(\mathcal{A})$ whose objects are complexes in $\mathcal{E}\text{-dw}\widetilde{\mathcal{X}}$; see [16, Proposition 3.5].

Denote by $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ and $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ the subcategory of complexes in $\mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$ and $\mathcal{E}\text{-ac}\widetilde{\mathcal{Y}}$, respectively. In this section, we intend to find model structures to realize $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$ and $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$; see Theorem 3.9 and Remark 3.10. For this order we need the following results, which imply model structures for the chain homotopy categories and relative derived category, see [21].

Lemma 3.4. *[21, Proposition 3.9] There are complete cotorsion pairs $(\widetilde{\mathcal{X}}_{\mathcal{E}}, \mathrm{Ch}(\mathcal{A}))$ and $(\mathrm{Ch}(\mathcal{A}), \widetilde{\mathcal{Y}}_{\mathcal{E}})$.*

This result implies the following model structures:

Lemma 3.5. *[21, Theorem 4.10] For the exact category $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$;*

If $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ is closed under direct sums, then $\mathcal{M}_{dw\mathcal{X}} = (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}, \mathrm{Ch}(\mathcal{A}))$ is a hereditary model structure with $\mathrm{Ho}(\mathcal{M}_{dw\mathcal{X}}) \simeq \mathbf{K}(\mathcal{X})$;

If ${}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ is closed under direct products, then $\mathcal{M}_{dw\mathcal{Y}} = (\mathrm{Ch}(\mathcal{A}), {}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}}), \mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ is a hereditary model structure with $\mathrm{Ho}(\mathcal{M}_{dw\mathcal{Y}}) \simeq \mathbf{K}(\mathcal{Y})$.

Furthermore, we obtain the following realization for the relative derived category $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$,

Lemma 3.6. *[21, Theorem 3.10] There are hereditary model structures $(\mathcal{E}\text{-dg}\widetilde{\mathcal{X}}, \widetilde{\mathcal{E}}, \mathrm{Ch}(\mathcal{A}))$ and $(\mathrm{Ch}(\mathcal{A}), \widetilde{\mathcal{E}}, \mathcal{E}\text{-dg}\widetilde{\mathcal{Y}})$ on the exact category $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$, with homotopy categories $\mathbf{D}_{\mathcal{X}}(\mathcal{A}) \simeq \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$.*

Note that $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ coincides with Neeman's derived category of the exact category $(\mathcal{A}, \mathcal{E})$ in [27, Construction 1.5].

In order to establish the model structure for $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X})$, we need the following:

Proposition 3.7. *If $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ is closed under direct sums, then $(\mathcal{E}\text{-ac}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp})$ is a complete cotorsion pair in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$.*

Proof. Let $\widehat{C}$ be the collection of all complexes C satisfying any chain map $X \to C$ from complexes $X \in \mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$ is null homotopic. Analogous to

$$\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{E})}(X, C) = \mathrm{Ext}^1_{dw}(X, C) \cong \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, \Sigma C)/\sim = 0$$

we can prove that $(\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp} = \widehat{C}$ and $\mathcal{E}\text{-ac}\widetilde{\mathcal{X}} \subseteq {}^{\perp}(\widehat{C}) \subseteq \mathcal{E}\text{-dw}\widetilde{\mathcal{X}}$.

Let $X \in {}^{\perp}(\widehat{C})$. Let Y be any object in $\mathcal{Y}$, and consider the short exact sequence $0 \to \mathrm{S}^{n+1}Y \to \mathrm{D}^{n+1}Y \to \mathrm{S}^{n}Y \to 0$. It follows from of Lemma 3.3(ii) that $\mathrm{S}^{n+1}Y \in \widehat{C}$, and then any chain map $X \to \mathrm{S}^{n}Y$ can be lifted to $X \to \mathrm{D}^{n+1}Y$. By the natural isomorphisms in [29, Lemma 3.1], we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, \mathrm{S}^{n}Y) & \to & \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, \mathrm{D}^{n+1}Y) & \to & \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, \mathrm{S}^{n+1}Y) & \to 0 \\
 & \cong\downarrow & & \downarrow\cong & & \downarrow\cong & \\
0 \to & \mathrm{Hom}_{\mathcal{A}}(X_n/\mathrm{B}_nX, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X_n, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X_{n+1}/\mathrm{B}_{n+1}X, Y) & \to 0
\end{array}$$

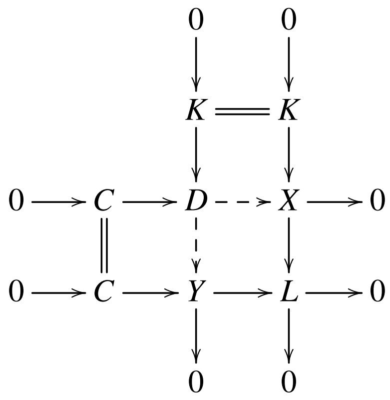
Then, every sequence $0 \to X_{n+1}/\mathrm{B}_{n+1}X \to X_n \to X_n/\mathrm{B}_nX \to 0$ is left $\mathcal{Y}$-acyclic. This yields that the complex X is left $\mathcal{Y}$-acyclic, that is, $X \in \widetilde{\mathcal{E}}$. Hence, we have $X \in {}^{\perp}(\widehat{C}) \subseteq \mathcal{E}\text{-dw}\widetilde{\mathcal{X}} \cap \widetilde{\mathcal{E}} = \mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$. This implies that $(\mathcal{E}\text{-ac}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp})$ is a cotorsion pair in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$.

Also from Lemma 3.6, it follows that for any complex C, there is a short exact sequence $0 \to Y \to E \to C \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$, for which $E \in \widetilde{\mathcal{E}}$ and $Y \in \mathcal{E}\text{-dg}\widetilde{\mathcal{Y}}$. For E, by Lemma 3.5, we have a short exact sequence $0 \to Z \to X \to E \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$, where $X \in \mathcal{E}\text{-dw}\widetilde{\mathcal{X}}$ and $Z \in (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$. Consider the following pullback of $Y \to E$ and $X \to E$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Z & = & Z & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \dashrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $Z \in (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp} \subseteq \widetilde{\mathcal{E}}$, we infer from the middle column that $X \in \widetilde{\mathcal{E}} \cap \mathcal{E}\text{-dw}\widetilde{\mathcal{X}} = \mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$. Since $\mathcal{E}\text{-dg}\widetilde{\mathcal{Y}} = (\widetilde{\mathcal{E}})^{\perp}$ and $\mathcal{E}\text{-ac}\widetilde{\mathcal{X}} \subseteq \widetilde{\mathcal{E}}$, it follows that $Y \in \mathcal{E}\text{-dg}\widetilde{\mathcal{Y}} \subseteq (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}$. We infer from $\mathcal{E}\text{-ac}\widetilde{\mathcal{X}} \subseteq \mathcal{E}\text{-dw}\widetilde{\mathcal{X}}$ that $Z \in (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp} \subseteq (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}$. The left column then implies that $K \in (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}$. Hence, for any complex C, we have constructed a short exact sequence $0 \to K \to X \to C \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$, where $X \in \mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$ and $K \in (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}$.

Furthermore, we will apply a standard argument (known as Salce's trick) to prove the other part. For any complex C, by Lemma 3.4, there is a short exact sequence $0 \to C \to Y \to L \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$, where $Y \in \widetilde{\mathcal{Y}}_{\mathcal{E}}$. For L we have a short exact sequence $0 \to K \to X \to L \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$, where $X \in \mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$ and $K \in (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}$. Consider the following pullback of $X \to L$ and $Y \to L$:



Note that $Y \in \widetilde{\mathcal{Y}}_{\mathcal{E}} \subseteq \widehat{C} = (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$ since the complexes in $\widetilde{\mathcal{Y}}_{\mathcal{E}}$ are contractible. Thus, $0 \to C \to D \to X \to 0$ is in $\mathcal{E}$ in each degree with $X \in \mathcal{E}\text{-ac}\widetilde{X}$ and $D \in (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$. This completes the proof. □

It is direct to check the following fact:

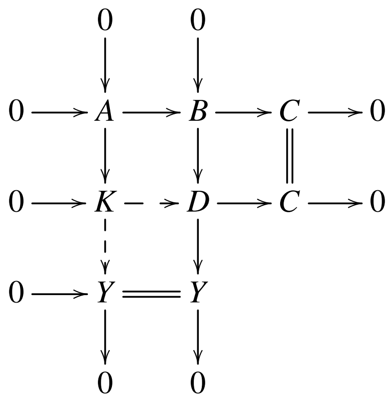
Lemma 3.8. $\mathcal{E}\text{-ac}\widetilde{X} \cap (\mathcal{E}\text{-ac}\widetilde{X})^{\perp} = \widetilde{X}_{\mathcal{E}}$.

Theorem 3.9. *If* $(\mathcal{E}\text{-dw}\widetilde{X})^{\perp}$ *is closed under direct sums, then there exists a hereditary model structure* $\mathcal{M}_{acX} = (\mathcal{E}\text{-ac}\widetilde{X}, (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}, \mathrm{Ch}(\mathcal{A}))$ *on the exact category* $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ *with a triangle equivalence*

$$\mathrm{Ho}(\mathcal{M}_{acX}) \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(X).$$

Proof. We claim that $(\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$ is a thick subcategory. First, we note that the cotorsion pair $(\mathcal{E}\text{-ac}\widetilde{X}, (\mathcal{E}\text{-ac}\widetilde{X})^{\perp})$ is hereditary. It suffices to prove that $(\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$ is closed under taking kernels of admissible epimorphisms. That is, for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ for which $B, C \in (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$, we need to show that $A \in (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$.

It follows from Lemma 3.7 that there is a short exact sequence $0 \to A \to K \to Y \to 0$ in $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ with $K \in (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$ and $Y \in \mathcal{E}\text{-ac}\widetilde{X}$. Consider the pushout of $A \to B$ and $A \to K$ we have the following commutative diagram



with $B, D \in (\mathcal{E}\text{-ac}\widetilde{X})^{\perp}$, thus $Y \in (\mathcal{E}\text{-ac}\widetilde{X})^{\perp} \cap \mathcal{E}\text{-ac}\widetilde{X} = \widetilde{X}_{\mathcal{E}}$, and then $0 \to A \to K \to Y \to 0$ is split degreewise. Since Y is contractible, $A \to K$ is a homotopy equivalence, this proves the above claim.

Then, by Lemmas 3.4, 3.8 and Proposition 3.7, together with the correspondence stated in Lemma 2.2, the model structure $\mathcal{M}_{acX}$ follows. The class of cofibrant-fibrant objects of the model structure is precisely $\mathcal{E}\text{-ac}\widetilde{X}$. Then we get the equivalence $\mathrm{Ho}(\mathcal{M}_{acX}) \simeq \mathcal{E}\text{-ac}\widetilde{X}/\widetilde{X}_{\mathcal{E}} \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(X)$. □

Dually, we obtain the model structure $\mathcal{M}_{ac\mathcal{Y}}$ as follows:

Remark 3.10. If ${}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ is closed under direct products, then there is a model structure $\mathcal{M}_{ac\mathcal{Y}} = (\mathrm{Ch}(\mathcal{A}), {}^{\perp}(\mathcal{E}\text{-ac}\widetilde{\mathcal{Y}}), \mathcal{E}\text{-ac}\widetilde{\mathcal{Y}})$ on the exact category $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ with a triangle equivalence

$$\mathrm{Ho}(\mathcal{M}_{ac\mathcal{Y}}) \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y}).$$

Corollary 3.11. *Assume that* $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ *and* ${}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ *are closed under direct sums and direct products, respectively. If* $(\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp} = {}^{\perp}(\mathcal{E}\text{-ac}\widetilde{\mathcal{Y}})$*, then there is a triangle-equivalence*

$$\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X}) \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y}).$$

Proof. Under the assumption, together with Theorem 3.9 and Remark 3.10, we obtain model structures $\mathcal{M}_{ac\mathcal{X}}$ and $\mathcal{M}_{ac\mathcal{Y}}$ with common trivial objects. It follows from [28, Corollary 1.4] that there is a Quillen equivalence between the model categories $\mathcal{M}_{ac\mathcal{X}}$ and $\mathcal{M}_{ac\mathcal{Y}}$, which yields an equivalence of the corresponding homotopy categories $\mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{X}) \simeq \mathrm{Ho}(\mathcal{M}_{ac\mathcal{X}}) \simeq \mathrm{Ho}(\mathcal{M}_{ac\mathcal{Y}}) \simeq \mathbf{K}_{\mathcal{E}\text{-ac}}(\mathcal{Y})$ and completes the proof. □

The referee pointed us to the following consequence of Theorem 3.9 and [30, Theorem 1.2].

Corollary 3.12. *If* $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ *is closed under direct sums, then there is a unique thick class* $\mathcal{W}$ *for which* $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}}, \mathcal{W}, (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp})$ *is a hereditary model structure on the exact category* $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$.

Proof. By [21, Lemma 4.9] and Lemma 3.8, we obtain inclusions

$$\mathcal{E}\text{-ac}\widetilde{\mathcal{X}} \subseteq \mathcal{E}\text{-dw}\widetilde{\mathcal{X}}, \quad (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp} \subseteq (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp},$$

as well as the equality

$$\mathcal{E}\text{-dw}\widetilde{\mathcal{X}} \cap (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp} = \mathcal{E}\text{-ac}\widetilde{\mathcal{X}} \cap (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}.$$

Applying Lemma 3.5 and Theorem 3.9, we conclude that the two complete hereditary cotorsion pairs $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp})$ and $(\mathcal{E}\text{-ac}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp})$ satisfy the hypotheses of [30, Theorem 1.2] or [31, Corollary 3.11]. This completes the verification. □

4. Applications

Throughout this section, let $\mathcal{A}$ be a Grothendieck category with an admissible balanced pair $(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{E}$ be the class of short exact sequences that are right $\mathcal{X}$-acyclic, as mentioned. We assume that $(\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}$ is closed under direct sums and ${}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{Y}})$ is closed under direct products on the exact category $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$.

We recall the definition of recollement of triangulated categories, see [12].

Definition 4.1. Let $\mathcal{T}_1$, $\mathcal{T}$, and $\mathcal{T}_2$ be triangulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by

$$\mathcal{T}_1 \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*=i_!\;} \\ \xleftarrow{\;i^!\;} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^!=j^*\;} \\ \xleftarrow{\;j_*\;} \end{array} \mathcal{T}_2$$

such that

(R1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs of triangle functors;
(R2) i_*, $j_!$ and j_* are full embeddings;
(R3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);
(R4) for each $X \in \mathcal{T}$, there are triangles

$$j_! j^! X \to X \to i_* i^* X \to$$
$$i_! i^! X \to X \to j_* j^* X \to$$

where the arrows to and from X are the counits and the units of the adjoint pairs, respectively.

Gillespie has obtained the following method to construct recollements.

Lemma 4.2. *[10, Theorem 8.3] Let $\mathcal{A}$ be an abelian category with three hereditary model structures*

$$\mathcal{M}_1 = (\mathcal{Q}_1, \mathcal{W}_1, \mathcal{R}), \;\; \mathcal{M}_2 = (\mathcal{Q}_2, \mathcal{W}_2, \mathcal{R}), \;\; \mathcal{M}_3 = (\mathcal{Q}, \mathcal{W}_3, \mathcal{R})$$

with cores all coincide and $\mathcal{W}_3 \cap \mathcal{Q}_1 = \mathcal{Q}_2$ and $\mathcal{Q}_3 \subseteq \mathcal{Q}_1$, then the sequence

$$\mathrm{Ho}(\mathcal{M}_2) \to \mathrm{Ho}(\mathcal{M}_1) \to \mathrm{Ho}(\mathcal{M}_3)$$

induces a recollement:

$$\mathrm{Ho}(\mathcal{M}_2) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathrm{Ho}(\mathcal{M}_1) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathrm{Ho}(\mathcal{M}_3)$$

Combining Lemma 3.5, Theorem 3.9, and Lemma 3.6, we get three hereditary model structures on $\mathrm{Ch}(\mathcal{A}, \mathcal{E})$ as follows:

$$\mathcal{M}_1 = \mathcal{M}_{dw\mathcal{X}} = (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-dw}\widetilde{\mathcal{X}})^{\perp}, \mathrm{Ch}(\mathcal{A})),$$
$$\mathcal{M}_2 = \mathcal{M}_{ac\mathcal{X}} = (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}}, (\mathcal{E}\text{-ac}\widetilde{\mathcal{X}})^{\perp}, \mathrm{Ch}(\mathcal{A})),$$
$$\mathcal{M}_3 = \mathcal{M}_{dg\mathcal{X}} = (\mathcal{E}\text{-dg}\widetilde{\mathcal{X}}, \widetilde{\mathcal{E}}, \mathrm{Ch}(\mathcal{A})),$$

whose cores are both $\widetilde{\mathcal{X}}_{\mathcal{E}}$. Since $\widetilde{\mathcal{E}} \cap \mathcal{E}\text{-dw}\widetilde{\mathcal{X}} = \mathcal{E}\text{-ac}\widetilde{\mathcal{X}}$ and $\mathcal{E}\text{-dg}\widetilde{\mathcal{X}} \subseteq \mathcal{E}\text{-dw}\widetilde{\mathcal{X}}$, we have the following relative type of the Krause's recollement, compare to [22, Theorem 7.7].

Corollary 4.3. *For $\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{X})$, $\mathbf{K}(\mathcal{X})$, and $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ as mentioned above, we have an induced recollement:*

$$\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{X}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{K}(\mathcal{X}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{D}_{\mathcal{X}}(\mathcal{A})$$

Dually, together with [10, Theorem 8.2] we obtain the following relative type of Neeman-Murfet's recollement, compare to [14, 23]:

Corollary 4.4. *For $\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{Y})$, $\mathbf{K}(\mathcal{Y})$, and $\mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ as mentioned above, we have an induced recollement:*

$$\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{Y}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{K}(\mathcal{Y}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$$

Let R be an associative ring with identity. Recall a left R-module M is *Gorenstein projective* if $M \cong \mathrm{Z}_0 C$ for some totally $\mathcal{P}$-acyclic complex C, that is, C is both right and left $\mathcal{P}$-acyclic with each item belonging to $\mathcal{P}$. Similarly, *Gorenstein injective* modules are defined. Denote by $\mathcal{GP}$ (resp. $\mathcal{GI}$) the subcategory consisting of all Gorenstein projective (resp. injective) modules over R, see [32] for details.

Recall that the *Gorenstein weak dimension* of R is defined to be the supremum of Gorenstein flat dimension of all left R-modules. Throughout this section, let R be with finite Gorenstein weak dimension, and $\mathcal{A}$ be the category of left R-modules, it follows from [21, Lemmas 5.6 and 5.7] that $(\mathcal{GP}, \mathcal{GI})$ is an admissible balanced pair and $(\mathcal{E}\text{-dw}\widetilde{\mathcal{GP}})^{\perp} = {}^{\perp}(\mathcal{E}\text{-dw}\widetilde{\mathcal{GI}})$. Together with Corollaries 4.3 and 4.4, we get the following result.

Corollary 4.5. *Let R be a ring with finite Gorenstein weak dimension. Then, we have recollements:*

$$\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{GP}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbf{K}(\mathcal{GP}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbf{D}_{\mathcal{GP}}(\mathcal{A})$$

and

$$\mathbf{K}_{\mathcal{E}\text{-}ac}(\mathcal{GI}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbf{K}(\mathcal{GI}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbf{D}_{\mathcal{GI}}(\mathcal{A})$$

It is worth noting that $\mathbf{D}_{\mathcal{GP}}(\mathcal{A})$ coincides with $\mathbf{D}_{\mathcal{GI}}(\mathcal{A})$, which is exactly the Gorenstein derived category, see [25].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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