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Research article

A new approach for constructing the graphs determined by their generalized Q-spectrum

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Abstract: We define the adjacency matrix $A(G)$ and the degree diagonal matrix $D(G)$ for a graph G. The Q-spectrum is the multiset of eigenvalues of the Q-matrix $Q(G) = A(G) + D(G)$, counted with their algebraic multiplicities. A class of graphs is determined by their generalized Q-spectrum (DGQS for short) if any two graphs in the class have the same Q-spectrum and so do their complements, implying that they are isomorphic. Tian et al. introduced a new method to construct DGQS graphs by considering the rooted product graphs $G \circ P_k$. They proved that when $k = 2, 3$, $G \circ P_k$ is DGQS for a special class of graphs G. In this paper, we extend this result and prove that, under the same conditions for G, the conclusion holds true for any positive integer k. While Wang et al. reported comparable results, our study arrives at these findings through an entirely different methodology. Our proof adopts the elementary properties of the resultants, providing an innovative approach to establishing the DGQS property for $G \circ P_k$.

Keywords: graph spectrum; rooted product graphs; Q-matrix; resultants; determined by their generalized Q-spectrum

1. Introduction

Throughout our paper, we consider simple and undirected graphs. Let G be a graph with the vertex set $V(G) = \{1, 2, \ldots, n\}$. We define the adjacency matrix $A = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if there is an edge between vertices i and j, and $a_{ij} = 0$ otherwise. The spectrum of a graph G refers to all its eigenvalues of A counted with their algebraic multiplicities. The question of whether a graph is uniquely determined by its spectrum (DS for short) is a fundamental one in graph theory, originating from discussions in [1]. In 2006, Wang and Xu extended this inquiry to the generalized spectrum, encompassing both the spectrum of G and its complement (denoted by $\overline{G}$) in [2,3]. A graph G is known as being determined by its generalized spectrum (DGS for short) if any graph H has the same generalized spectrum as G, which implies that H and G are isomorphic. Subsequently, in 2017, Wang proposed an effective algorithm for determining DGS in [4]: For a graph G, if its walk matrix $W(G) = [e, A(G)e, \ldots, A(G)^{n-1}e]$ satisfies the condition that $2^{-\lfloor \frac{n}{2} \rfloor} \det(W)$ is odd and square-free, then G is DGS.

Given a graph G, we have its degree diagonal matrix $D(G)$. The Q-spectrum is the set of all eigenvalues of the Q-matrix $Q(G) = A(G) + D(G)$. We denote all the Q-spectra of G as $\mathrm{Spec}_Q(G)$. A graph G is said to be determined by its generalized Q-spectrum (DGQS for short) if, for any graph H satisfying $\mathrm{Spec}_Q(G) = \mathrm{Spec}_Q(H)$ and $\mathrm{Spec}_Q(\overline{G}) = \mathrm{Spec}_Q(\overline{H})$, it implies that H is isomorphic to G. The study of the Q-spectrum originated with the work of Cvetković and Simić in [5–7]. Later, some scholars identified certain special graphs as being DGQS in [8, 9]. Finally, in [10], Qiu et al. presented a general algorithm for testing whether a graph G is determined by its generalized Q-spectrum: If its walk Q-matrix $W_Q(G) = [e, Q(G)e, \ldots, Q(G)^{n-1}e]$ (e is the all-one vector) satisfies the condition that $2^{-\lfloor \frac{n-3}{2} \rfloor} \det(W_Q)$ is odd and square-free, then G is DGQS.

Constructing rooted product graphs is a meaningful method to create a series of graphs with more vertices and edges based on an original graph. Given the graphs G and P_k, with n and k vertices, respectively, the rooted product graph $\widehat{G_k} = G \circ P_k$ is formed by associating each vertex of G with a copy of P_k. Let $V(G) = \{1, 2, \ldots, n\}$ and $V(P_k) = \{p_1, p_2, \ldots, p_k\}$ denote the vertex sets of G and P_k, respectively, and let $E(G)$ and $E(P_k)$ denote their edge sets. Thus the vertex set of $\widehat{G_k}$ is $V(\widehat{G_k}) = \{(i, p_s) | 1 \le i \le n, 1 \le s \le k\}$ and the edge set of $\widehat{G_k}$ is $\{(i, p_1) \sim (j, p_1) | ij \in E(G)\} \cup \{(i, p_s) \sim (i, p_{s+1}) | 1 \le i \le n, 1 \le s \le k-1\}$ (see Godsil and McKay [11]).

In several problems of graph theory, constructing a DGQS graph is often more challenging than determining whether a graph is DGQS. Tian et al. constructed numerous DGQS graphs using this method and proved the following theorem in [12].

Theorem 1.1. *Let G be a graph with $\det(W_Q(G)) = \pm 2^{\frac{3n-2}{2}}$ (n is even) and the constant term a_0 where the characteristic polynomial of G is equal to ± 2. Then every graph $G \circ P_k^t$ ($k = 2, 3;\ t \ge 1$) is* DGQS.

In this paper, we will prove that this conclusion about $G \circ P_k^t$ is true for any positive integer k. The paper is organized as follows. In Section 2, we will introduce some basic notations and lemmas. In Section 3, we will establish our main results and the proofs. Comparable results have previously been reported by Yan et al. [13]; however, our study provides a novel contribution by using a completely different methodology to achieve them.

2. Preliminaries

In this section, we present some notations and lemmas that will be used in the following sections. First, we give three sequences $a_k(t)$, $b_k(t)$, and $c_k(t)$, which satisfy the recursive relations of the following polynomials:

$$a_0(t) = 1, \quad a_1(t) = t - 1, \quad \text{and} \quad a_k(t) = (t-2)a_{k-1}(t) - a_{k-2}(t) \quad \text{for} \quad k \ge 2,$$

$$b_0(t) = 1, \quad b_1(t) = t - 1, \quad \text{and} \quad b_k(t) = (t-1)a_{k-1}(t) - a_{k-2}(t) \quad \text{for} \quad k \ge 2,$$

$$c_0(t) = 1, \quad c_1(t) = t - 2, \quad \text{and} \quad c_k(t) = (t-1)c_{k-1}(t) - c_{k-2}(t) \quad \text{for} \quad k \ge 2.$$

Moreover, we assume $f_k(t) = a_0(t) + a_1(t) + \ldots + a_{k-1}(t)$.

Definition 2.1. *Let $f(t) = \prod_{1\le i\le n}(t-\alpha_i)$, $g(t) = \prod_{1\le j\le m}(t-\beta_j)$. The definition of the resultant of $f(t)$ and $g(t)$, denoted $Res(f(t), g(t))$, is as follows:*

$$\prod_{1\le j\le m}\prod_{1\le i\le n}(\alpha_i-\beta_j).$$

Our paper uses the following property of the resultant frequently.

Lemma 2.2 ([13]). *Suppose $f(t) = \prod_{1\le i\le n}(t-\alpha_i)$, $g(t) = \prod_{1\le j\le m}(t-\beta_j)$, then*

$$Res(f(t), g(t)) = \prod_{1\le i\le n} g(\alpha_i) = (-1)^{mn}\prod_{1\le j\le m} f(\beta_j).$$

In particular, if $n = m + 1$, then

$$Res(f(t), g(t)) = \prod_{1\le i\le n} g(\alpha_i) = \prod_{1\le j\le m} f(\beta_j).$$

Let

$$C = \begin{bmatrix} 1 & 2-t_i^{(j)} & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2-t_i^{(j)} & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2-t_i^{(j)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2-t_i^{(j)} & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1-t_i^{(j)} \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & a_{k-1}(t_i^{(j)}) \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & a_{k-2}(t_i^{(j)}) \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{k-3}(t_i^{(j)}) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_3(t_i^{(j)}) \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_2(t_i^{(j)}) \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_1(t_i^{(j)}) \end{bmatrix},$$

be two $(s-1)\times s$ matrices with respect to $t_i^{(j)}$.

Lemma 2.3 ([14]). *If $\boldsymbol{x}$ is a column vector in $\mathbb{R}^s$, then $\boldsymbol{C}\boldsymbol{x} = 0$ if and only if $\boldsymbol{D}\boldsymbol{x} = 0$.*

According to the natural partition of the vertex set of the rooted product graph $\widehat{G_k}$ into k layers, each of size n, the adjacency matrix $A(\widehat{G_k})$ can be written as a block matrix of order $kn\times kn$, where each block is an $n\times n$ matrix. With respect to this partition, the adjacency matrix of $\widehat{G_k}$ has the form

$$A(\widehat{G_k}) = \begin{bmatrix} A(G) & I_n & 0 & 0 & \dots & 0 & 0 & 0 \\ I_n & 0 & I_n & 0 & \dots & 0 & 0 & 0 \\ 0 & I_n & 0 & I_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & \dots & I_n & 0 & I_n \\ 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 \end{bmatrix},$$

and the diagonal degree matrix is $D(\widehat{G_k}) = \text{diag}\,(D(G) + I_n, 2I_n, \ldots, I_n)$, where the diagonal consists of k blocks of size $n \times n$. Therefore, the Q-matrix of $\widehat{G_k}$, defined by $Q(\widehat{G_k}) = A(\widehat{G_k}) + D(\widehat{G_k})$, is the following block matrix:

$$Q(\widehat{G_k}) = \begin{bmatrix} Q(G)+I_n & I_n & 0 & 0 & \ldots & 0 & 0 & 0 \\ I_n & 2I_n & I_n & 0 & \ldots & 0 & 0 & 0 \\ 0 & I_n & 2I_n & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2I_n & I_n & 0 \\ 0 & 0 & 0 & 0 & \ldots & I_n & 2I_n & I_n \\ 0 & 0 & 0 & 0 & \ldots & 0 & I_n & I_n \end{bmatrix}.$$

Lemma 2.4. *Let λ_i be the Q-spectrum of the graph G with respect to the normalized eigenvector x_i $(i = 1, 2, \ldots, n)$ and $t_i^{(j)}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ be a group of real numbers. Then $t_i^{(j)}$ is the Q-spectrum of the rooted product graph $\widehat{G_k}$ if and only if $t_i^{(j)}$ satisfies the following equation: $\varphi(t_i^{(j)}) = b_k(t_i^{(j)}) - \lambda_i a_{k-1}(t_i^{(j)})$.*

Proof. The sufficient condition has been already proved in the proof of [14, Lemma 3.1]. It suffices to show that the necessary condition holds. Suppose that $t_i^{(j)}$ satisfies $b_k(t_i^{(j)}) - \lambda_i a_{k-1}(t_i^{(j)}) = 0$ for the eigenvalue λ_i of $Q(G)$.

Claim 1: $a_{k-1}(t_i^{(j)}) \neq 0$.

The proof of the claim is by contradiction. Otherwise, we have $\lambda_i a_{k-1}(t_i^{(j)}) = 0$ and, by $\varphi(t_i^{(j)}) = b_k(t_i^{(j)}) - \lambda_i a_{k-1}(t_i^{(j)}) = 0$, we have $b_k(t_i^{(j)}) = 0$. It follows that $a_{k-2}(t_i^{(j)}) = (t_i^{(j)} - 2)a_{k-1}(t_i^{(j)}) - a_k(t_i^{(j)}) = 0$. Repeating the process, we finally get $a_2(t_i^{(j)}) = a_1(t_i^{(j)}) = 0$, and so $a_0(t_i^{(j)}) = 0$. However, as $a_0(t) \equiv 1$, this leads to a contradiction. This proves Claim 1.

We then set x_1 the corresponding eigenvector of λ_i. Let $x_i = \dfrac{a_{s-i}(t_i^{(j)})}{a_{s-1}(t_i^{(j)})} x_1$, $2 \leq i \leq s$ and $\tilde{x} = (x_1^T, x_2^T, \ldots, x_s^T)^T$.

Claim 2: $\tilde{x}$ is an eigenvector corresponding to $t_i^{(j)}$ of the rooted product graph $Q(\widehat{G_k})$.

We easily find that $Q(\widehat{G_k})\tilde{x} = t_i^{(j)}\tilde{x}$ is equivalent to

$$Q(G)x_1 + x_2 = (t_i^{(j)} - 1)x_1,$$

and

$$\begin{cases} x_1 + x_3 = (t_i^{(j)} - 2)x_2 \\ x_2 + x_4 = (t_i^{(j)} - 2)x_3 \\ \vdots \\ x_{s-3} + x_{s-1} = (t_i^{(j)} - 2)x_{s-2} \\ x_{s-2} + x_s = (t_i^{(j)} - 2)x_{s-1} \\ x_{s-1} = (t_i^{(j)} - 1)x_s \end{cases}. \tag{2.1}$$

The equalities (2.1) are also equivalent to $C\tilde{x} = 0$. By Lemma 2.3, they are equivalent to $D\tilde{x} = 0$. This is obviously deduced by the formula of x_i and

$$Q(G)x_1 + x_2 = \lambda_i x_1 + x_2 = \left(\frac{b_s(t_i^{(j)}) + a_{s-2}(t_i^{(j)})}{a_{s-1}(t_i^{(j)})}\right)x_1 = (t_i^{(j)} - 1)x_1.$$

This proves Claim 2, and we have thus proved the lemma.

To prove the main result, we need to study some properties of $a_k(t)$, $b_k(t)$, and $f_k(t)$.

Lemma 2.5. *Let λ_i, $t_i^{(j)}$ be as in Lemma 2.4. Then for any $1 \le i \le n$, we have*

$$\prod_{1\le k_2\le k} a_{k-1}(t_i^{(k_2)}) = (-1)^{\frac{k(k-1)}{2}}. \tag{2.2}$$

Proof. According to Lemma 2.4, $t_i^{(j)}$ indicates all roots of the equation $\varphi(t) = b_k(t) - \lambda_i a_{k-1}(t)$. Let $\varphi(t) = \prod_{1\le k_2\le k}(t - t_i^{(k_2)})$ and $\alpha_1, \ldots, \alpha_{k-1}$ be all roots of $a_{k-1}(t)$; by Lemma 2.2, we then have

$$\prod_{1\le k_2\le k} a_{k-1}(t_i^{(k_2)}) = \prod_{1\le j\le k-1} \varphi(\alpha_j).$$

We then have $\varphi(\alpha_j) = b_k(\alpha_j) - \lambda_i a_{k-1}(\alpha_j) = a_k(\alpha_j) = -a_{k-2}(\alpha_j)$. It suffices to show that $\prod_{1\le j\le k-1} a_{k-2}(\alpha_j) = (-1)^{\frac{(i-2)(i-1)}{2}}$.

Claim: For any i, if $\alpha_1^{(i)}, \ldots, \alpha_i^{(i)}$ are the roots of $a_i(t)$, then

$$\prod_{1\le j\le i} a_{i-1}(\alpha_j^{(i)}) = (-1)^{\frac{i(i-1)}{2}}.$$

The claim is proved by induction on i. It is clear if $i = 1, 2$, and we suppose that when $i = n-1$, the claim holds, i.e.,

$$\prod_{1\le j\le i-1} a_{i-2}(\alpha_j^{(i-1)}) = (-1)^{\frac{(i-2)(i-1)}{2}}.$$

If $i = n$, we have

$$\prod_{1\le j\le i} a_{(i-1)}(\alpha_j^{(i)}) = \prod_{1\le j\le i-1} a_i(\alpha_j^{(i-1)}) = (-1)^{i-1} \prod_{1\le j\le i-1} a_{i-2}(\alpha_j^{(i-1)}) = (-1)^{\frac{i(i-1)}{2}}.$$

This proves the claim and we can get the desired result.

Lemma 2.6 ([14]). *Let s be an integer ≥ 1. Then*

$$f_s(t) = \begin{cases} tc_{k-1}(t) & if\ \ s = 2k,\ \ k \ge 1 \\ a_k^2(t) & if\ \ s = 2k+1,\ \ k \ge 0. \end{cases}$$

Lemma 2.7 ([14]). *For $s \ge 2$, we have*

$$tc_{s-1}(t) = (t-1)a_{s-1}(t) - a_{s-2}(t) = b_s(t).$$

Lemma 2.8. *If all roots of f_i are $\beta_1^{(i)}, \ldots, \beta_i^{(i)}$, then for any $1 \leq j \leq i$, $b_i(\beta_j^{(i)}) = 0$.*

Proof. First, we prove the following claim.
Claim: $a_{k+i}(t) + a_{k-i}(t) = (a_i(t) - a_{i-1}(t))a_k(t), (0 < i < k+1)$
By induction, when $i = 1$, we have

$$a_{k+1}(t) + a_{k-1}(t) = (t-2)a_k(t) = (a_1(t) - a_0(t))a_k(t).$$

Assuming that it is true for $i = n-1$ and considering the case $i = n$, we obtain

$$\begin{aligned} a_{k+n}(t) + a_{k-n}(t) &= (t-2)(a_{k+n-1}(t) + a_{k-n+1}(t)) - (a_{k+n-2}(t) + a_{k-n+2}(t)) \\ &= (t-2)(a_{n-1}(t) - a_{n-2}(t))a_k(t) - (a_{n-2}(t) - a_{n-3}(t))a_k(t) \\ &= (a_n(t) - a_{n-1}(t))a_k(t). \end{aligned}$$

Thus we prove the claim.

We now turn to the proof of Lemma 2.8. We define $a_{-1} = -1$ and easily get $a_1(t) = (t-2)a_0(t) - a_{-1}(t)$. The sequence $\{a_{-1}(t), a_0(t), \ldots, a_k(t), \ldots\}$ also satisfies the iterative relationship $a_k(t) = (t-2)a_{k-1}(t) - a_{k-2}(t)$.

When $i = 2k+1$, it follows that

$$\begin{aligned} b_{2k+1}(t) &= a_{2k+1}(t) + a_{2k}(t) \\ &= (a_{2k+1}(t) + a_{-1}(t)) + (a_{2k}(t) + a_0(t)) \\ &= (a_{k+1}(t) - a_{k-1}(t))a_k(t). \end{aligned}$$

Thus we can get $a_k(t) | b_{2k+1}(t)$. Combining with Lemma 2.6, when $i = 2k+1$, the conclusion is established.

When $i = 2k$, we easily observe that f_k and $tc_{k-1}(t)$ have the same roots. Then, by Lemma 2.7, $b_k(t) = tc_k(t)$. Therefore, we get the desired result by combining the discussion above.

Lemma 2.9 ([12])**.** *For any $k \geq 1$, let $\beta_j^{(k)}, (1 \leq j \leq k)$ be all the roots of $f_k(t)$. Then $\prod_{1\leq j\leq k} a_{k-1}(\beta_j^{(k)}) = (-1)^k$.*

Proof. We shall prove the result by induction on i. It is trivial if $i = 1, 2$, and we may assume that it holds for $i = k-1$, i.e.,

$$\prod_{1\leq j\leq k-1} a_{k-2}(\beta_j^{(k-1)}) = (-1)^{k-1}.$$

When $i = k$, it follows that

$$\prod_{1\leq j\leq k} a_{k-1}(\beta_j^{(k)}) = (-1)^k \prod_{1\leq j\leq k} f_{k-1}(\beta_j^{(k)}) = (-1)^k \prod_{1\leq j\leq k-1} f_k(\beta_j^{k-1}) = (-1)^k \prod_{1\leq j\leq k-1} a_{k-1}(\beta_j^{(k-1)}).$$

On the other hand, by Lemma 2.8, we have

$$0 = b_{k-1}(\beta_j^{(k-1)}) = a_{k-1}(\beta_j^{k-1}) + a_{k-2}(\beta_j^{k-1}).$$

So

$$-a_{k-1}(\beta_j^{(k-1)}) = a_{k-2}(\beta_j^{(k-1)}).$$

Then

$$\prod_{1\leq j\leq k} a_{k-1}(\beta_j^{(k)}) = (-1)^{2k-1} \prod_{1\leq j\leq k-1} a_{k-2}(\beta_j^{(k-1)}) = (-1)^k.$$

3. Main result

In this section, let G be a graph and a_0 be the constant term with respect to the characteristic polynomial of the matrix $Q(G)$.

3.1. *The determinant of* $W_Q(\widehat{G_k})$

Lemma 3.1 ([12]). *Let λ_i be the Q-eigenvalues of the graph G with respect to the normalized eigenvectors x_i $(i = 1, 2, \ldots, n)$. Let $t_i^{(j)}$ be the Q-eigenvalues of the rooted product graph $G \circ P_k$ with respect to the eigenvectors ζ_i^j $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, k)$. In this case,*

$$\det(W_Q(G \circ P_k)) = \pm(\det(W_Q(G))^k \prod_{1\le i\le n} \prod_{1\le k_2\le k} a_{k-1}(t_i^{(k_2)}) f_k(t_i^{(k_2)}). \tag{3.1}$$

Theorem 3.2. *For any $k \geq 1$,* $\det(W_Q(G \circ P_k)) = (-1)^{\frac{nk(k+1)}{2}} a_0^{k-1} (\det(W_Q(G)))^k$.

Remark 3.3. *During the writing of this article, the author found that Theorem 3.2 was proved by Yan et al. in [13] using a different method.*

Proof of Theorem 3.2. By Lemma 3.1

$$\det(W_Q(G \circ P_k)) = \pm(\det(W_Q(G))^k \prod_{1\le i\le n} \prod_{1\le k_2\le k} a_{k-1}(t_i^{(k_2)}) f_k(t_i^{(k_2)}).$$

Let $\beta_j^{(k)} (1 \leq j \leq k)$ be all roots of $f_k(t)$. Then, by Lemmas 2.2, 2.5, and 2.9, we have

$$\begin{aligned}
&\prod_{1\le i\le n} \prod_{1\le k_2\le k} a_{k-1}(t_i^{(k_2)}) f_k(t_i^{(k_2)}) \\
&= (-1)^{\frac{nk(k-1)}{2}} \prod_{1\le i\le n} \prod_{1\le k_2\le k-1} \varphi_k(\beta_j^{(k)}) \\
&= (-1)^{\frac{nk(k-1)}{2}} \prod_{1\le i\le n} (-\lambda_i)^{k-1} (-1)^k \\
&= (-1)^{\frac{nk(k+1)}{2}} a_0^{k-1},
\end{aligned}$$

which completes the proof of Theorem 3.2.

3.2. *Constructing an infinite family of* DGQS *graphs*

Lemma 3.4 ([10]). *If* $2^{-\lfloor \frac{3n-2}{2} \rfloor} \det(W_Q(G))$ *is odd and square-free, then G is* DGQS.

Lemma 3.5 ([12]). *Let $a_0^{(t)}$ be the constant term with respect to the characteristic polynomial of the matrix $Q(G \circ P_k^t)$. If $a_0 = \pm 2$, then $a_0^{(t)} = \pm 2$.*

Theorem 3.6. *Let n be an even positive integer and G be a graph with n vertices. If* $\det(W_Q(G)) = \pm 2^{\frac{3n-2}{2}}$ *and $a_0 = \pm 2$, then all graphs in the infinite sequence $G \circ P_k^t$ $(k \geq 2;\ t \geq 1)$ are* DGQS.

Proof of Theorem 3.6. By Theorem 3.2, Lemma 3.5, we have

$$\det(W_Q(G \circ P_k^t)) = \pm 2^{k-1} (\det(W_Q(G \circ P_k^{t-1}))^k) = \ldots = \pm 2^{\frac{3k^t n}{2} - 1}.$$

Meanwhile, the number of the vertices in $G \circ P_k^t$ is $k^t n$. Thus we have

$$\frac{\det(W_Q(G \circ P_k^t))}{2^{\frac{3k^t n}{2}-1}} = \pm 1.$$

Therefore, $G \circ P_k^t$ are DGQS from Lemma 3.4.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there are no conflicts of interest.

References

1. H. H. Günthard, H. Primas, Zusammenhang von Graphenteorie und MO-Theorie von Molekeln mit Systemen konjugierter, *Helv. Chim. Acta*, **39** (1956), 1645–1653. https://doi.org/10.1002/hlca.19560390623
2. W. Wang, C. X. Xu, A sufficient condition for a family of graphs being determined by their generalized spectra, *Eur. J. Comb.*, **27** (2006), 826–840. https://doi.org/10.1016/j.ejc.2005.05.004
3. W. Wang, C. X. Xu, An excluding algorithm for testing whether a family of graphs are determined by their generalized spectra, *Linear Algebra Appl.*, **418** (2006), 62–74. https://doi.org/10.1016/j.laa.2006.01.016
4. W. Wang, A simple arithmetic criterion for graphs being determined by by their generalized spectra, *J. Comb. Theory Ser. B*, **122** (2017), 438–451. https://doi.org/10.1016/j.jctb.2016.07.004
5. D. M. Cvetkovć, S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, *Publ. Inst. Math.*, **85** (2009), 19–33. https://doi.org/10.2298/PIM0999019C
6. D. M. Cvetković, S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian II, *Linear Algebra Appl.*, **432** (2010), 2257–2272. https://doi.org/10.1016/j.laa.2009.05.020
7. D. M. Cvetković, S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian III, *Appl. Anal. Discrete Math.*, **4** (2010), 156–166. https://doi.org/10.2298/AADM1000001C
8. G. R. Omidi, On a signless Laplacian spectral characterization of T-shape trees, *Linear Algebra Appl.*, **431** (2009), 1607–1615. https://doi.org/10.1016/j.laa.2009.05.035
9. A. Schrijver, *The Theory of Linear and Integer Programming*, John Wiley & Sons, 1998.
10. L. Qiu, Y. Ji, W. Wang, A new arithmetic criterion for graphs being determined by their generalized Q-spectrum, *Discrete Math.*, **342** (2019), 2770–2782. https://doi.org/10.1016/j.disc.2018.08.008
11. C. D. Godsil, B. D. McKay, A new graph product and its spectrum, *Bull. Austral. Math. Soc.*, **18** (1978), 21–28. https://doi.org/10.1017/S0004972700007760

12. G. X. Tian, J. X. Wu, S. Y. Cui, H. L. Sun, Construction of graphs being determined by their generalized Q-spectra, preprint, arxiv: 2307.14832.

13. Z. D. Yan, L. H. Mao, W. Wang, On the determinant of the Q-walk matrix of rooted product with a path, *Bull. Malays. Math. Sci. Soc.*, **47** (2024), 174. https://doi.org/10.1007/s40840-024-01774-5

14. Z. Z. Lou, Q. X. Huang, X. Y. Huang, On the construction of Q-controllable graphs, *Electron. J. Linear Algebra*, **32** (2017), 365–379. https://doi.org/10.13001/1081-3810.3298