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Research article

A relationship between the algebraic multiplicity of eigenvalues of fractional integral operators and the simplicity of zeros of Mittag–Leffler functions

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Abstract: In this short note, we establish a connection between the algebraic multiplicity of eigenvalues of compact integral operators and the simplicity of zeros of a class of the Mittag–Leffler functions $E_{\alpha,\alpha}(z)$ and $E_{\alpha,2}(z)$ for $1 < \alpha < 2$. Although these are two seemingly unrelated concepts in mathematics, our approach provides a functional-analytic criterion for determining the simplicity of zeros in this class of Mittag–Leffler functions.

Keywords: algebraic simplicity; fractional Sturm–Liouville; Mittag–Leffler function; simplicity of zeros

1. Preliminary

The fractional eigenvalue problems

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) + \lambda u(t) = 0, t \in (0,1) \\ u(0) = u(1) = 0, 1 < \alpha < 2 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}u(t) + \lambda u(t) = 0, t \in (0,1) \\ u(0) = u(1) = 0, 1 < \alpha < 2 \end{cases} \tag{1.2}$$

are two fundamental problems in fractional spectral analysis and have been investigated from both perspectives theoretically and numerically in the past decades. We use ${}^{C}\mathcal{D}^{\alpha}_{0+}$ and $\mathcal{D}^{\alpha}_{0+}$ to denote the standard Caputo and Riemann–Liouville fractional derivative of order α, respectively. See Definition 2.4.15, p. 92, [1] and Definition 2.2, p. 35, [2]:

Definition 1. *For $1 < \alpha < 2$, the Riemann–Liouville fractional integral, the Caputo fractional derivative and the Riemann–Liouville fractional derivative are defined as*

$$\mathcal{I}^{\alpha}_{0+}u = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)dt, \quad x \in (a,b),$$
$${}^{C}\mathcal{D}^{\alpha}_{0+}u = \frac{1}{\Gamma(2-\alpha)} \int_a^x (x-t)^{1-\alpha} u''(t)dt, \quad x \in (a,b),$$
$$\mathcal{D}^{\alpha}_{0+}u = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_a^x (x-t)^{1-\alpha} u(t)dt, \quad x \in (a,b),$$

respectively.

Both (1.1) and (1.2) generalize the classical Sturm–Liouville problem with homogeneous boundary conditions. Moreover, when $\alpha = 2$, both ${}^{C}\mathcal{D}^{\alpha}_{0+}$ and $\mathcal{D}^{\alpha}_{0+}$ reduce to the conventional second order derivative.

On one hand, it is well-known that $\lambda \in \mathbb{C}$ is an eigenvalue of (1.1) if and only if λ is a zero of $E_{\alpha,2}(-z)$, i.e., $E_{\alpha,2}(-\lambda) = 0$ (see, e.g., [3]); and $\lambda \in \mathbb{C}$ is an eigenvalue of (1.2) if and only if λ is a zero of $E_{\alpha,\alpha}(-z)$, i.e., $E_{\alpha,\alpha}(-\lambda) = 0$ (see, e.g., [4]). Since $E_{\alpha,2}(z)$ and $E_{\alpha,\alpha}(z)$ are entire functions with countably many zeros, there are always countably infinite eigenvalues $\lambda \in \mathbb{C}$ in the complex plane for any $1 < \alpha < 2$ in both cases of (1.1) and (1.2). Here, the two-parametric Mittag–Leffler functions are defined as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \mathrm{Re}\alpha > 0, \quad \beta \in \mathbb{C}, \quad z \in \mathbb{C}, \tag{1.3}$$

which is an entire function of order $\frac{1}{\mathrm{Re}\alpha}$.

On the other hand, by incorporating the boundary conditions, the eigenvalue problems (1.1) and (1.2) can be reformulated as equivalent fractional integral equations, respectively:

$$Au = \frac{1}{\lambda}u, \quad \text{where} \quad Au := -\mathcal{I}^{\alpha}_{0+}u + cx, \quad c := \mathcal{I}^{\alpha}_{0+}u(1) \tag{1.4}$$

and

$$Au = \frac{1}{\lambda}u, \quad \text{where} \quad Au := -\mathcal{I}^{\alpha}_{0+}u + cx^{\alpha-1}, \quad c := \mathcal{I}^{\alpha}_{0+}u(1). \tag{1.5}$$

Notation $\mathcal{I}^{\alpha}_{0+}u(1)$ represents the evaluation of $\mathcal{I}^{\alpha}_{0+}u$ at $x = 1$, that is, $(\mathcal{I}^{\alpha}_{0+}u)|_{x=1}$, and $\mathcal{I}^{\alpha}_{0+}$ represents the Riemann–Liouville fractional integral of order α (see Definition 2.1, p. 33, [1]). Clearly, λ is an eigenvalue of (1.1) if and only if the reciprocal, $\frac{1}{\lambda}$, is an eigenvalue of (1.4); similarly, λ is an eigenvalue of (1.2) if and only if the reciprocal, $\frac{1}{\lambda}$, is an eigenvalue of (1.5). Furthermore, it is easy to see that the operator A is compact from the Hilbert space $L^2(\Omega), \Omega = (0,1)$, to itself in both cases by rewriting A in terms of the Green's function. That is,

$$(Au)(x) = \int_0^1 G(\alpha, x, t)u(t)dt,$$

where $G(\alpha, x, t)$ reads

$$G(\alpha, x, t) = \begin{cases} \frac{1}{\Gamma(\alpha)}[(1-t)^{\alpha-1}x - (x-t)^{\alpha-1}], & 1 > x > t > 0; \\ \frac{1}{\Gamma(\alpha)}(1-t)^{\alpha-1}x, & 1 > t \geq x > 0, \end{cases}$$

in (1.4) and $G(\alpha, x, t)$ reads

$$G(\alpha, x, t) = \begin{cases} \frac{1}{\Gamma(\alpha)}[(1-t)^{\alpha-1}x^{\alpha-1} - (x-t)^{\alpha-1}], & 1 > x > t > 0; \\ \frac{1}{\Gamma(\alpha)}(1-t)^{\alpha-1}x^{\alpha-1}, & 1 > t \geq x > 0, \end{cases}$$

in (1.5).

In this short work, we establish a relationship between the algebraic multiplicity of the eigenvalues of the integral operator A and the simplicity of the zeros of the Mittag–Leffler functions $E_{\alpha,2}(z)$ and $E_{\alpha,\alpha}(z)$, as presented in Theorems 1 and 2. This connection is somewhat surprising, since the algebraic multiplicity is a fundamental concept in operator theory, whereas, the simplicity of zeros arises primarily in the theory of entire functions–two areas that are seemingly unrelated. Roughly speaking, we show that the zeros of these Mittag–Leffler functions are simple if and only if the algebraic multiplicity of the corresponding eigenvalues of the integral operator A is equal to 1. To the best of the authors' knowledge, this is the first work to relate the multiplicity of zeros of Mittag–Leffler functions with the algebraic multiplicity of compact integral operators.

The study of the simplicity of zeros of Mittag–Leffler functions is an important yet challenging topic in the theory of special functions and has a long history. In 1905, using the integral representation first introduced by Mittag Leffler [5], A. Wiman proved that $\alpha \geq 2, \beta = 1$, all zeros of $E_{\alpha,\beta}(z)$ are real, negative, and simple. Since then, many mathematicians have contributed to this direction over the past century. Among various achievements, M. M. Dzhrbashyan's pioneering work [6] was particularly remarkable, presenting a relatively mature theory on the asymptotic analysis of $E_{\alpha,\beta}$. Further historical notes and survey results on this topic can be found in Chapter 4 of [7] and [3].

The novel idea of this work provides a new possibility to study the distribution and simplicity of zeros of Mittag–Leffler functions from a functional perspective, rather than through the traditional complex–analytic approach based on their Mittag–Leffler functions' integral representations in the complex plane. This represents a fundamental different angle and opens the door to subsequent extensions and generalizations.

2. Main results

2.1. Algebraic multiplicity

Let A be a compact operator over $L^2(\Omega)$ and λ be an eigenvalue, i.e., $A \in \mathcal{K}(L^2(\Omega))$, $\Omega = (0, 1)$, and $\lambda \in \sigma(A)\backslash\{0\}$. From the operator theory, it is known that the sequence of the null sets, $\mathcal{N}((A - \lambda I)^k), k = 1, 2, \cdots$, is strictly increasing up to some positive integer N, and then it stays constant; that is

$$\mathcal{N}((A - \lambda I)) \subset \mathcal{N}((A - \lambda I)^2) \subset \cdots \subset \mathcal{N}((A - \lambda I)^N) = \mathcal{N}((A - \lambda I)^{N+1}) = \cdots . \tag{2.1}$$

The dimension of $\mathcal{N}((A-\lambda I))$ is called the geometric multiplicity of λ, and the dimension of $\mathcal{N}((A-\lambda I)^N)$ is called the algebraic multiplicity of λ. Obviously, both the geometric and algebraic multiplicity are finite and the geometric multiplicity is less than or equal to the algebraic multiplicity in general. In particular, they coincide if A is a self-adjoint operator, which is not the case in our problems.

2.2. Simplicity of zeros

Suppose $\lambda \in \mathbb{C}$ is a zero of $E_{\alpha,\beta}(-z)$, i.e., $E_{\alpha,\beta}(-\lambda) = 0$, we say the zero λ is simple (or prime), if $\frac{d}{dz}E_{\alpha,\beta}(-z)\big|_{z=\lambda} \neq 0$ (or, equivalently, $\frac{d}{dz}E_{\alpha,\beta}(z)\big|_{z=-\lambda} \neq 0$).

2.3. Theorems

Let $\lambda \in \mathbb{C}$ be an eigenvalue of problem (1.1). We have the following.

Theorem 1. *λ, as the root of the Mittag–Leffler function $E_{\alpha,2}(-\lambda) = 0$, is prime (simple) if and only if the algebraic multiplicity of $1/\lambda$ in (1.4) with respect to A is equal to 1.*

Proof. 1) Since λ is a (nonzero) eigenvalue of (1.1), by directly solving the equivalent integral equation (1.4), we obtain the unique corresponding eigenfunction $u_\lambda = xE_{\alpha,2}(-\lambda x^\alpha)$ (up to a multiplicative constant). To prove Theorem 1, it is therefore sufficient to show:

$$(A - \mu I)u = u_\lambda, \quad \mu := 1/\lambda \tag{2.2}$$

has a solution in $L^2(\Omega)$ if and only if

$$\frac{d}{dz}E_{\alpha,2}(z)\Big|_{z=-\lambda} = 0. \tag{2.3}$$

2) $\Rightarrow$. Rewrite (2.2) as

$$u = \lambda Au - \lambda u_\lambda = -\lambda \mathcal{I}_{0+}^{\alpha} u + \lambda \mathcal{I}_{0+}^{\alpha} u(1) \cdot x - \lambda u_\lambda. \tag{2.4}$$

It is clear that $u(1) = 0$ due to $u_\lambda(1) = 0$. Furthermore, the Abel equation (2.4) is solvable in $L^2(\Omega)$ (Theorem 4.2, [7]) and the solution u is uniquely determined by the Neumann series

$$\begin{aligned} u &= \lambda \mathcal{I}_{0+}^{\alpha} u(1) \cdot \sum_{n=0}^{\infty} (-\lambda \mathcal{I}_{0+}^{\alpha})^n x - \lambda \sum_{n=0}^{\infty} (-\lambda \mathcal{I}_{0+}^{\alpha})^n u_\lambda \\ &= \lambda \mathcal{I}_{0+}^{\alpha} u(1) \cdot \sum_{n=0}^{\infty} (-\lambda)^n \mathcal{I}_{0+}^{n\alpha} x - \lambda u_\lambda - \lambda \sum_{n=1}^{\infty} (-\lambda)^n \mathcal{I}_{0+}^{n\alpha} u_\lambda \\ &= \lambda \mathcal{I}_{0+}^{\alpha} u(1) \cdot u_\lambda - \lambda u_\lambda + \lambda^2 \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(x-t)^\alpha) t E_{\alpha,2}(-\lambda t^\alpha) dt. \end{aligned}$$

Upon carrying out the last term (see Eq (63) and bottom of p. 25, [8])), we arrive at

$$u = \lambda(\mathcal{I}_{0+}^{\alpha} u(1) - 1)u_\lambda + \frac{\lambda^2}{\alpha} x^{1+\alpha} \left(E_{\alpha,1+\alpha}(-\lambda x^\alpha) - E_{\alpha,2+\alpha}(-\lambda x^\alpha)\right).$$

Therefore, we see

$$0 = u(1) = E_{\alpha,1+\alpha}(-\lambda) - E_{\alpha,2+\alpha}(-\lambda). \tag{2.5}$$

Apply the well-known identity

$$E_{t_1,t_1+t_2}(z) = \frac{E_{t_1,t_2}(z)}{z} - \frac{1}{z\Gamma(t_2)}, \quad z \neq 0 \tag{2.6}$$

to each of the terms on the right-hand side of (2.5), (2.5) gives

$$0 = E_{\alpha,1}(-\lambda) - E_{\alpha,2}(-\lambda). \tag{2.7}$$

Then apply the differentiation formula

$$\frac{d}{dz}E_{t_1,t_2}(z) = \frac{E_{t_1,t_2-1}(z) - (t_2-1)E_{t_1,t_2}(z)}{t_1 z}, \quad z \neq 0, \tag{2.8}$$

(2.7) yields

$$\frac{d}{dz}E_{\alpha,2}(z)|_{z=-\lambda} = \frac{E_{\alpha,1}(-\lambda) - E_{\alpha,2}(-\lambda)}{-\alpha\lambda} = 0.$$

3) $\Leftarrow$. Suppose $\frac{d}{dz}E_{\alpha,2}(z)\Big|_{z=-\lambda} = 0$. With the aid of (2.6) and (2.8), we deduce

$$E_{\alpha,1+\alpha}(-\lambda) - E_{\alpha,2+\alpha}(-\lambda) = 0.$$

Define $u(x) = \frac{\lambda^2}{\alpha}x^{1+\alpha}\left(E_{\alpha,1+\alpha}(-\lambda x^\alpha) - E_{\alpha,2+\alpha}(-\lambda x^\alpha)\right)$. It is directly verified that u is a solution of

$$\lambda \mathcal{I}^{\alpha}_{0+}u = \lambda x - u - \lambda u_\lambda. \tag{2.9}$$

It follows that

$$\mathcal{I}^{\alpha}_{0+}u(1) = 1 \tag{2.10}$$

by evaluating both sides of (2.9) at the boundary point $x = 1$ and dividing by λ. Substitute (2.10) back into (2.9), it reads

$$\lambda \mathcal{I}^{\alpha}_{0+}u = \lambda \mathcal{I}^{\alpha}_{0+}u(1)\cdot x - u - \lambda u_\lambda,$$

which is the same as the desired equation (2.2). The proof is complete.

Likewise, let $\lambda \in \mathbb{C}$ be an eigenvalue of problem (1.2). We have

Theorem 2. *λ, as the root of the Mittag–Leffler function $E_{\alpha,\alpha}(-\lambda) = 0$, is prime (simple) if and only if the algebraic multiplicity of* $1/\lambda$ *in* (1.5) *with respect to A is equal to* 1.

Proof. The procedure of the proof is exactly the same as in the proof of Theorem 1 with slight modification in calculations, by replacing

$$u_\lambda = xE_{\alpha,2}(-\lambda x^\alpha)$$

with

$$u_\lambda = x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^\alpha),$$

and replacing

$$\frac{d}{dz}E_{\alpha,2}(z)\Big|_{z=-\lambda} = 0$$

with

$$\frac{d}{dz}E_{\alpha,\alpha}(z)\Big|_{z=-\lambda} = 0$$

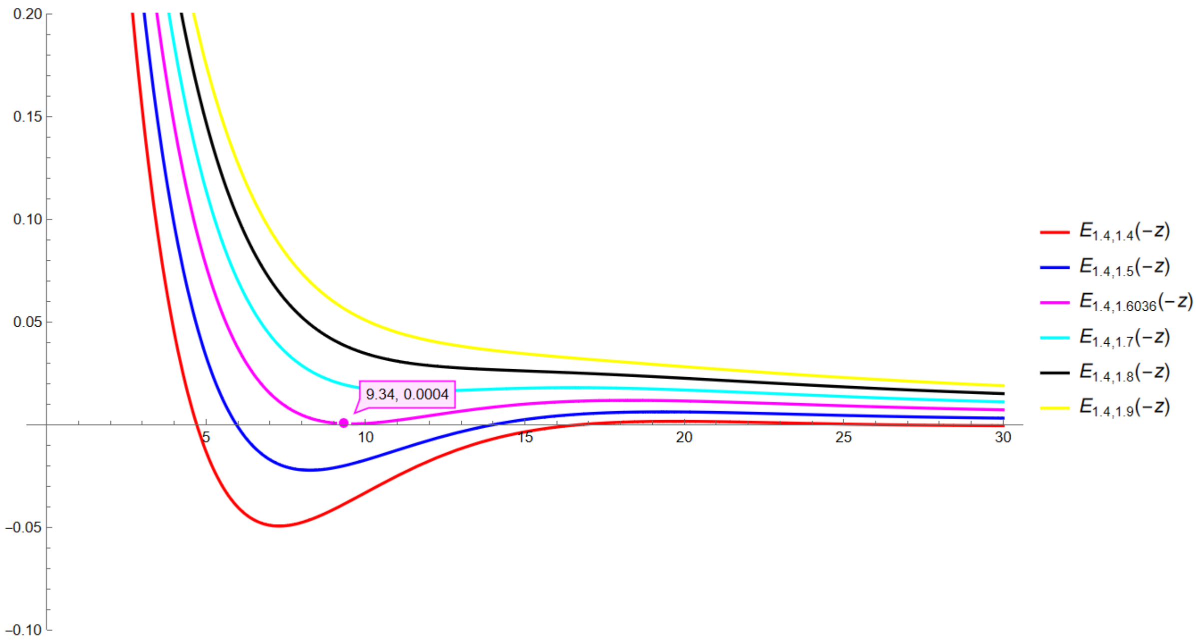
in the first step of the proof of Theorem 1.

Steps 2 and 3 can be justified in an analogous manner. For brevity, the verification is left to the interested readers and is therefore not repeated here.

3. Discussion and conclusions

As noted at the beginning, the study of the simplicity of zeros of Mittag–Leffler functions in the complex plane is a challenging task, and there is no uniform formula to prove or disprove such properties. Most of the existing results rely either on asymptotic analysis (see, for example, [3, 7, 9], etc.) and non-asymptotic analysis (see, for example, [4, 10, 11], etc.) of Mittag–Leffler functions. Indeed, depending on the parameters α, β, the functions $E_{\alpha,\beta}(z)$ may exhibit quite different profiles. For example, the first real root of $E_{\alpha,\alpha}(-z)$, $1 < \alpha < 2$ always exists, is smallest in modulus among all the complex roots and must be simple [12]; however, $E_{\alpha,2}(-z)$, $1 < \alpha < 2$ may or may not have a real root (Theorem 6.1.5, [3], see also [11] for a different proof) and the first real root may not be simple, if it exists. (See Figure 1(a),(b) for numerical illustration and comparison. The graphs are plotted using Wolfram Mathematica with the built-in function "MittagLefflerE"). Regarding the simplicity of complex roots of Mittag–Leffler functions, the results are considerably more complicated and remain incomplete, and thus will not be discussed here. For these reasons, the connection established in Theorems 1 and 2 becomes particularly attractive, since it enables the study of the simplicity of zeros of Mittag–Leffler functions in a unified manner from the perspective of operator theory. In other words, in order to prove the simplicity of roots of Mittag–Leffler functions, it suffices to show that the algebraic multiplicity of the integral operator A with the associated eigenvalues is equal to 1. This provides a functional-analytic criterion for the proving simplicity of zeros.

Our discussion has been restricted to $E_{\alpha,\alpha}(z)$ and $E_{\alpha,2}(z)$ with $1 < \alpha < 2$, which are two of the most important classes of Mittag–Leffler functions. These functions behave in many ways like the exponential function and are closely related to superdiffusive transport and anomalous diffusion processes in modeling. Nevertheless, it is possible to extend this work to $E_{\alpha,\beta}(z)$ for a wider range of parameters α and β. In such cases, one would need to re-construct the fractional integral equations with general parameters α, β which may or may not correspond to fractional differential equations with clear physical meaning and applications as in (1.1) and (1.2). Another possible direction of extension is to establish that the algebraic multiplicity of the integral operators coincides exactly with the multiplicity of zeros of Mittag–Leffler functions. The purpose of this short note is to provide a starting point; a careful examination of these generalizations will be undertaken in future work.



(a) The first real root of $E_{1.4,1.4}(-z)$ is simple (for the theoretical proof, see [12]). As the value of β increases, the simplicity of the first real root of $E_{1.4,\beta}(-z)$, becomes subtle and is not guaranteed; in particular, when β is around 1.6036, the tangent line at the first root is horizontal, and thus, the first root of $E_{1.4,1.6036}(-z)$ is not simple.

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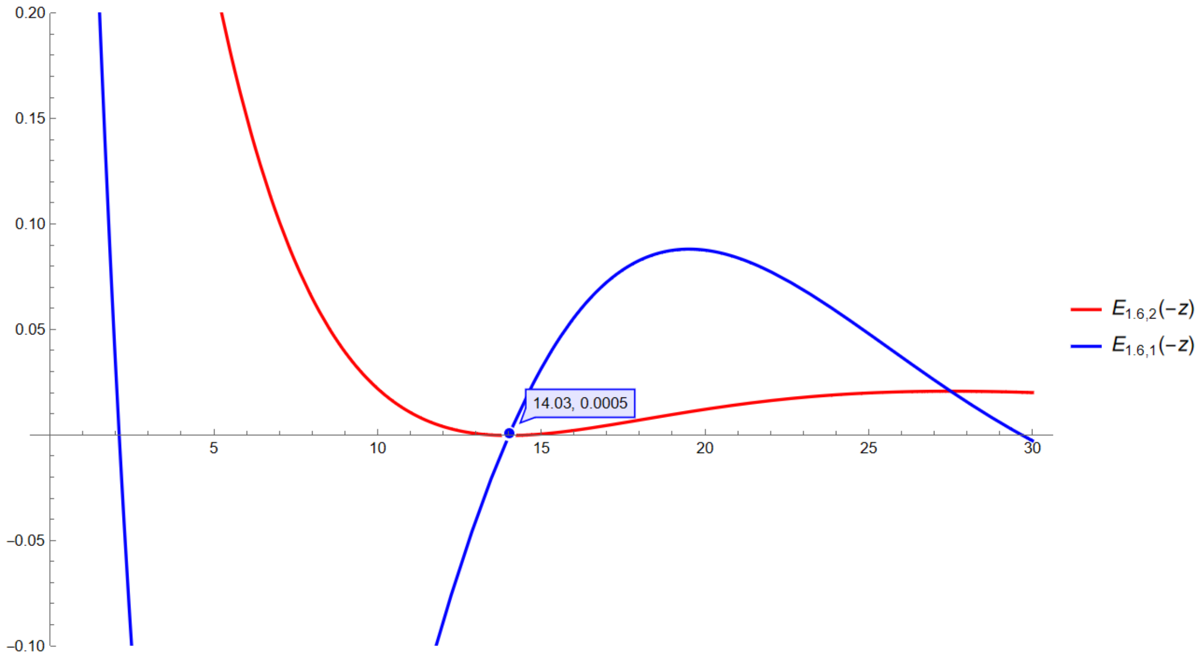

(b) The simplicity of the first zero of $E_{\alpha,2}(z)$, $1 < \alpha < 2$ is theoretically unclear. The numerical experiment shows that non-simple zeros may (or may not) exist for certain values of α around 1.6, and other researchers conjectured that $E_{a^*,2}(-z)$ has the double zero around $a^* \approx 1.599115$; see [13]. From the graph, it seems that $E_{1.6,2}(-14.03) = 0$ and $E_{1.6,1}(-14.03) = 0$, which by (2.8) implies $E'_{1.6,2}(-14.03) = 0$, and meanwhile, $E''_{1.6,2}(-14.03) \neq 0$ since the tangent line of $E'_{1.6,2}$ at $z = 14.03$ is obviously not horizontal; that is, $E_{1.6,2}(-z)$ appears to have a double zero. However, this is just a numerical observation. Note that minor numerical inaccuracies may arise due to the implementation of the built-in function "MittagLefflerE" with working precision 20 in Wolfram Mathematica. A theoretical proof or disproof is needed in the future work for a precise value of a^*.

Figure 1. Numerical illustration and comparison of the simplicity of some real roots of $E_{\alpha,\beta}(-z)$ for some values of α and β.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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