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Research article

Existence of periodic solutions to a non-autonomous allelopathic phytoplankton model with fear effect

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Abstract: The authors considered a non-autonomous allelopathic phytoplankton competition model that incorporates the influence of fear effects that are often observed in natural biological phenomena. Based on Mawhin's coincidence degree theory, sufficient conditions for the existence of periodic solutions were obtained. The authors validated their findings with an example and simulations showing that the constant coefficient case leads to dynamics that are neither steady state nor T-periodic, while the periodic coefficient case generates sustained T-periodic oscillations.

Keywords: allelopathy; phytoplankton; competition model; fear effects; existence of periodic solutions; coincidence degree theory

1. Introduction

Phytoplankton are the autotrophic components of the plankton community and are a key part of ocean and freshwater ecosystems. Phytoplankton form the base of aquatic food webs and are crucial for ecosystem functions and services. They significantly benefit the biotechnology, carbon sequestration, pharmaceutical, and nutraceutical sectors (see [1–3]). A distinct and important phenomenon observed among phytoplankton species is the production of secondary metabolites by one species that inhibit the growth or physiological functions of another species [4]. This behavior, known as allelopathy, occurs when phytoplankton engage in competitive interactions by releasing toxic compounds.

Numerous studies have demonstrated that allelopathy plays a vital role in shaping the competitive dynamics among phytoplankton. For example, Maynard-Smith [5] incorporated an allelopathic term into the usual two-species Lotka-Volterra competition model to explain the harmful effects that one species has upon another such as in the system

$$\begin{cases} \frac{dN_1(t)}{dt} = N_1(t)\left(\alpha_1 - \beta_1 N_1(t) - v_1 N_2(t) - \gamma_1 N_1(t) N_2(t)\right), \\ \frac{dN_2(t)}{dt} = N_2(t)\left(\alpha_2 - \beta_2 N_2(t) - v_2 N_1(t) - \gamma_2 N_1(t) N_2(t)\right), \end{cases}$$

where $N_i(t)$, $i = 1, 2$, represent the density of the two competing phytoplankton species. Here, α_i is the daily cell proliferation rate, β_i denotes the intraspecific competition rate of the i-th species, v_i represents the interspecific competition rate, and γ_i is the toxicity coefficient exerted by the other species on species i. Here the initial conditions, $N_i(0)$, $i = 1, 2$, are taken to be positive.

Building upon the work of Maynard-Smith, numerous researchers have examined scenarios where only one species releases toxins. Chen et al. [6] proposed the following discrete system to model toxin release by a single species

$$\begin{cases} x_1(n+1) = x_1(n) \exp\left[r_1(n) - a_{11}(n)x_1(n) - a_{12}(n)x_2(n) - b_1(n)x_1(n)x_2(n)\right], \\ x_2(n+1) = x_2(n) \exp\left[r_2(n) - a_{21}(n)x_1(n) - a_{22}(n)x_2(n)\right]. \end{cases}$$

They established conditions for extinction and global stability of the system. It was demonstrated that at low rates of toxin release, the extinction dynamics of the system remain unaffected. This indicates that the toxic species cannot drive the non-toxic species to extinction.

A non-toxic species, however, can become extinct even when exposed to only low concentrations of toxins. This raises the question as to what factors, other than toxins, might influence the density of the competing phytoplankton species. Given that allelopathy is modeled on the classical Lotka-Volterra competition framework, we will explore the concept of competitive fear.

Wang, Zanette, and Zou [7] were the first to incorporate the fear effect into the classical two-species Lotka-Volterra predator-prey model. They defined the fear effect function as $f(k, y) = \frac{1}{1+ky}$, which represents the prey's anti-predation response induced by fear of the predator. Their study revealed that an increase in the fear level could shift the bifurcation from supercritical to subcritical, provided that the prey's birth rate also increases. Numerical simulations further indicated that anti-predator defenses of the prey intensify as the predator's attack rate increases. Additional studies on the fear effect in predator-prey models can be found in the papers [7–13].

The impact of fear on predator-prey systems has been widely explored; however, its role in competition systems has received significantly less attention. Nevertheless, compelling evidence suggests that fear can exist in purely competitive systems, even in the absence of predation effects or if, in fact, such effects are negligible (see Wiens, Anthony, and Forsman [14]).

One well-known example of "fear" as a factor in a predator-prey relationship is between wolves (*Canis lupus*) and elk (*Cervus canadensis*) in Yellowstone National Park in the USA [12]. This relationship illustrates how the fear of predation can significantly impact prey behavior and even the ecosystem structure beyond just the direct impact of being hunted. After wolves were reintroduced to Yellowstone in 1995, elk populations had to adapt not only to the risk of predation, but also to the heightened stress and vigilance required to avoid wolves. Studies have shown that elk alter their grazing patterns by avoiding open areas where wolves are more likely to spot them, and so tend to spend more time in dense forest cover. This behavioral shift reduced the browsing pressure on young aspen and willow trees, which allowed these plants to grow taller. It even led to a resurgence of certain riparian ecosystems that depend on these trees for structural stability and habitat. "Fear" effects, where prey animals modify their behavior due to the perceived risk of predation, demonstrates how the presence of predators can influence ecosystems by inducing fear in the prey species even if the actual predation rates remain low.

The work of Srivastava et al. [15] appears to be the first one to consider fear in competitive systems. Inspired by the works [6,10,15–17], Chen et al. [9] investigated how the fear parameter affects competitive allelopathic planktonic systems by introducing a fear effect term in the case where the non-toxic species is "fearful" of the toxic species. In [9], they analyzed the model

$$\frac{dx_1}{dt} = r_1 x_1\left(1-\frac{x_1}{k_1}\right) - \beta_1 x_1 x_2, \tag{1.1}$$

$$\frac{dx_2}{dt} = r_2 x_2\left(\frac{1}{1+w_1x_1}-\frac{x_2}{k_2}\right) - \beta_2 x_1 x_2 - w_2 x_1 x_2^2, \tag{1.2}$$

where r_i, k_i, β_i, $i = 1, 2$, w_1, and w_2, respectively, denote the intrinsic growth rates, the carrying capacitiers, the interspecific competition rates, the fear effect parameter, and the toxicity coefficient.

To more accurately reflect the real-world influences of climate change and seasonal variations, many researchers have studied non-autonomous models, examining their behavior, including permanence and the existence and stability of positive periodic solutions (see [18–23]). In [24–26], the authors obtained sufficient conditions for permanence along with the existence of an almost periodic solution of non-autonomous ecological models. In [27], a non-autonomous allelopathic phytoplankton model with nonlinear inter-inhibition terms and feedback controls was studied. However, no study has yet explored the non-autonomous allelopathic phytoplankton model with fear effects. A non-autonomous model can effectively incorporate these time-dependent effects.

Motivated by the above-mentioned works, in this paper, we formulate a non-autonomous allelopathic phytoplankton model with fear effect and investigate the existence of positive periodic solutions. The model, as constructed from (1.1) and (1.2) using non-autonomous parameters, is

$$\frac{dx_1}{dt} = r_1(t)x_1(t)\left(1-\frac{x_1(t)}{k_1}\right) - \beta_1(t)x_1(t)x_2(t), \tag{1.3}$$

$$\frac{dx_2}{dt} = r_2(t)x_2(t)\left(\frac{1}{1+w_1x_1(t)}-\frac{x_2(t)}{k_2}\right) - \beta_2(t)x_1(t)x_2(t) - w_2 x_1(t)x_2^2(t), \tag{1.4}$$

where $r_1(t)$, $r_2(t)$, $\beta_1(t)$, and $\beta_2(t)$ are positive T-periodic functions. The ecological interpretation of the parameters in the model (1.3) and (1.4) remains the same as in (1.1) and (1.2), that is, $x(t)$ and $y(t)$ represent the toxic and non-toxic species, respectively, r_i, k_i, β_i, $i = 1, 2$, w_1, and w_2, respectively, denote the intrinsic growth rates, the carrying capacities, the interspecific competition rates, the fear effect parameter, and the toxicity coefficient. That is to say, the parameters retain their ecological meaning as previously described, but are now time dependent.

Many mathematical models have been formulated using differential equations or systems of differential equations, and this has attracted numerous researchers to explore the dynamical behavior of solutions with a particular focus on the existence and uniqueness of solutions. Several studies in the literature employ Schauder's fixed point theorem, Krasnosel'skii's fixed point theorem, or Mawhin's coincidence degree theory to investigate the existence of positive periodic solutions in a variety of biological models. In recent years, the coincidence degree method has become a powerful and effective tool for addressing the existence of periodic solutions in both differential and difference equation models. Recent studies employing coincidence degree theory for various biological models include those in [18–23,28,29]. The present paper aims to find suitable conditions for the existence of positive periodic solutions for system (1.3) and (1.4).

The remainder of this paper is organized as follows. Section 2 introduces the fundamental concepts of coincidence degree theory and some useful notations. In Section 3, we establish sufficient conditions for the existence of a positive periodic solution using the continuation theorem of coincidence degree theory (Theorem 2.1 below). In Section 4, we give an example with numerical simulations to support our results.

2. Preliminaries

Before developing conditions for the existence of periodic solutions of system (1.3) and (1.4), we briefly introduce coincidence degree theory and some related notation.

Let Z and W be real Banach spaces and let $L : dom(L) \subset Z \to W$ be a Fredholm operator of index zero. If $P : Z \to Z$ and $Q : W \to W$ are two continuous projectors such that

$$Im(P) = Ker(L),\ Ker(Q) = Im(L),\ Z = Ker(L) \oplus Ker(P),\ W = Im(L) \oplus Im(Q),$$

then we know that the inverse operator of $L|_{dom(L)\cap Ker(P)} : dom(L) \cap Ker(P) \to Im(L)$ exists, and we denote it by K_p (the generalized inverse operator of L). If Ω is a bounded open subset of Z such that $dom(L) \cap \Omega \neq 0$, then the mapping $N : Z \to W$ is said to be L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to Z$ is compact. The abstract equation $Lx = Nx$ is then solvable in view of the following result of Mawhin.

Theorem 2.1. ([30]) *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(dom(L)\backslash Ker(L)) \cap \partial\Omega] \times (0, 1)$*;*
2) $Nx \notin Im(L)$ *for every* $x \in Ker(L) \cap \partial\Omega$*;*
3) $deg(QN|_{Ker(L)}, Ker(L) \cap \Omega, 0) \neq 0$, *where* $Q : W \to W$ *is a projector with* $Im(L) = Ker(Q)$*.*

Then, the equation $Lx = Nx$ *has at least one solution in* $dom(L) \cap \overline{\Omega}$*.*

For convenience, we introduce the notation

$$\overline{f} = \frac{1}{T}\int_0^T f(t)dt,\ \ f^L = \min_{t\in[0,T]} f(t),\ \text{ and }\ f^M = \max_{t\in[0,T]} f(t),$$

where f is any continuous T-periodic function. We set

$$m_\epsilon = \frac{k_2}{1 + w_1 k_1} + \epsilon \quad \text{and} \quad g_\epsilon = k_1\left(1 - \frac{\beta_1^M m_0}{r_1^L}\right) + \epsilon$$

for any $\epsilon \geq 0$.

3. Existence of periodic solutions

In order to apply Mawhin's coincidence degree theory, we need to make a change of variables. Consider

$$z_1(t) = \ln x_1(t) \text{ so that } x_1(t) = e^{z_1(t)},$$
$$z_2(t) = \ln x_2(t) \text{ so that } x_2(t) = e^{z_2(t)}.$$

Then, the system (1.3) and (1.4) becomes

$$\frac{dz_1}{dt} = r_1(t)\left(1 - \frac{e^{z_1(t)}}{k_1}\right) - \beta_1(t)e^{z_2(t)}, \tag{3.1}$$
$$\frac{dz_2}{dt} = r_2(t)\left(\frac{1}{1 + w_1 e^{z_1(t)}} - \frac{e^{z_2(t)}}{k_2}\right) - \beta_2(t)e^{z_1(t)} - w_2 e^{z_1(t)} e^{z_2(t)}. \tag{3.2}$$

The spaces Z and W are defined as follows:

$$Z = W = \{z = (z_1, z_2)^\tau \in (\mathbb{R}, \mathbb{R}^2) \mid z(t+T) = z(t)\},$$

where each element $z = (z_1, z_2)^\tau$ (τ means transpose) satisfies the periodicity condition $z_i(t+T) = z_i(t)$, $i = 1, 2$. These spaces are equipped with the norm

$$\|z\| = \max_{t\in[0,T]} \sum_{i=1}^{2} |z_i|, \quad z = (z_1, z_2)^\tau \in Z \text{ or } W.$$

Since uniform limits of continuous T-periodic functions are also continuous and T-periodic, a sequence of periodic, continuous functions that gets uniformly closer and closer will converge to a continuous function, that is also periodic, since periodicity passes to the limit. Because the limit remains inside the same space, the space is complete. This ensures that both Z and W are complete, and hence are Banach spaces.

Define the operators L, P, and Q by

$$L : \operatorname{dom}(L) \cap Z \to W, \quad Lz = \left(\frac{dz_1}{dt}, \frac{dz_2}{dt}\right)^\tau,$$

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Q\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\int_0^T z_1(t)dt \\ \frac{1}{T}\int_0^T z_2(t)dt \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z = W,$$

where

$$\operatorname{dom}(L) = \{z \in Z : z(t) \in C^1(\mathbb{R}, \mathbb{R}^2)\}.$$

We now define $N : Z \times [0, 1] \to W$ by

$$N\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1(z, t) \\ \Gamma_2(z, t) \end{pmatrix},$$

where

$$\Gamma_1(z, t) = r_1(t)\left(1 - \frac{e^{z_1(t)}}{k_1}\right) - \beta_1(t)e^{z_2(t)}$$

and

$$\Gamma_2(z,t) = r_2(t)\left(\frac{1}{1+w_1 e^{z_1(t)}} - \frac{e^{z_2(t)}}{k_2}\right) - \beta_2(t)e^{z_1(t)} - w_2 e^{z_1(t)}e^{z_2(t)}.$$

These functions are T-periodic. In fact,

$$\begin{aligned}\Gamma_1(z(t+T),t+T) &= r_1(t+T)\left(1-\frac{e^{z_1(t+T)}}{k_1}\right) - \beta_1(t+T)e^{z_2(t+T)}\\ &= r_1(t)\left(1-\frac{e^{z_1(t)}}{k_1}\right) - \beta_1(t)e^{z_2(t)}.\end{aligned}$$

Similarly, $\Gamma_2(z,t)$ is also T-periodic.

To satisfy the conditions of Theorem 2.1, we first present two lemmas.

Lemma 3.1. *L is a Fredholm operator of index zero, and N is L-compact.*

Proof. It is easy to see that

$$Ker(L) = \{z \in Z | z = c_0,\ c_0 \in \mathbb{R}^2\}$$

and that

$$Im(L) = \left\{ z \in W \,\middle|\, \int_0^T z(t)dt = 0 \right\}$$

is closed in W. Furthermore, both P and Q are continuous projections satisfying

$$Im(P) = Ker(L) \text{ and } Im(L) = Ker(Q) = Im(I-Q).$$

To see that $Im(P) = Ker(L)$, first note that $ImP \subseteq KerL$. Then, for any $z \in Z$, the vector Pz is constant, so

$$\frac{d}{dt}(Pz) = 0,$$

which implies $Pz \in KerL \subseteq ImP$. If $z \in KerL$, then $z'(t) = 0$ for all t, so z is a constant, say, $z(t) \equiv c = (c_1, c_2)^\tau$. Since P fixes constant functions, i.e.,

$$Pz = P(c) = c = z,$$

we have $z \in Im\,P$. Therefore, $Im\,P = KerL$. For any $z \in W$, let $\hat{z} = z - Qz$; we can obtain that

$$\int_0^T \hat{z}dp = \int_0^T z(p)dp - \int_0^T \frac{1}{T}\int_0^T z(t)dtdp = 0,$$

so $\hat{z} \in Im(L)$. It follows that $W = Im(L) + Im(Q) = Im(L) + \mathbb{R}^2$. Since $Im(L) \cap \mathbb{R}^2 = \emptyset$, we conclude that $W = Im(L) \oplus \mathbb{R}^2$, which means $dim\ Ker(L) = codim\ Im(L) = dim\ (\mathbb{R}^2) = 2$. Thus, L is a Fredholm operator of index zero, which implies that L has a unique generalized inverse operator (see [30, §3.2]).

Next, we show that N is L-compact. The inverse of L is $K_P : Im(L) \to Ker(P) \cap dom(L)$ and is given by

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{T}\int_0^T \int_0^t z(s)dsdt.$$

Therefore, for any $z(t) \in Z$, we have

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\int_0^T \Gamma_1(z,t)dt \\ \frac{1}{T}\int_0^T \Gamma_2(z,t)dt \end{pmatrix}$$

and

$$\begin{aligned} K_P(I-Q)Nz &= \int_0^t Nz(s)ds - \frac{1}{T}\int_0^T\int_0^t Nz(s)dsdt - \frac{1}{T}\int_0^t\int_0^T QNz(s)dtds \\ &\quad + \frac{1}{T^2}\int_0^T\int_0^t\int_0^T QNz(s)dtdsdt \\ &= \int_0^t Nz(s)ds - \frac{1}{T}\int_0^T\int_0^t Nz(s)dsdt - \left(\frac{t}{T}-\frac{1}{2}\right)\int_0^T QNz(s)ds. \end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Since Z is a Banach space of continuous T-periodic functions, and N maps bounded sets into equicontinuous and uniformly bounded sets in W, from the Arzelà-Ascoli theorem, it can be shown that N is L-compact on $\overline{U}$ for any open bounded set $U \subset Z$ (see [30, p. 87]).

In our next lemma, we give upper and lower bounds for solutions.

Lemma 3.2. *For $\lambda \in (0,1)$, consider the family of systems*

$$\begin{cases} \dfrac{dz_1}{dt} = \lambda\left[r_1(t)\left(1-\frac{e^{z_1}}{k_1}\right)-\beta_1(t)e^{z_2}\right], \\ \dfrac{dz_2}{dt} = \lambda\left[r_2(t)\left(\frac{1}{1+w_1e^{z_1}}-\frac{e^{z_2}}{k_2}\right)-\beta_2(t)e^{z_1}-w_2e^{z_1}e^{z_2}\right]. \end{cases} \tag{3.3}$$

If the following conditions hold

(A1) $k_1 > 1$,
(A2) $k_2 > 1 + w_1k_1$,
(A3) $r_1^L > \beta_1^M m_0$,
(A4) $r_2^L > (1+w_1k_1)\left(\frac{r_2^M}{k_2} + w_2k_1 + \beta_2^M k_1\right)$,

then for any periodic solution $(z_1,z_2)^\tau$ of (3.3)*, there exist positive constants L_i, $i = 1,2,\ldots,4$, such that*

$$L_2 \le z_1(t) \le L_1 \quad \textit{and} \quad L_4 \le z_2(t) \le L_3. \tag{3.4}$$

The values of L_i in the above lemma will be computed in the proof.

Proof. Assume that $z = (z_1,z_2)^\tau \in Z$ is a T-periodic solution of the system (3.3) for any fixed $\lambda \in (0,1)$. Since $(z_1,z_2)^\tau \in Z$, there exist η_i, $\xi_i \in [0,T]$ such that

$$z_i(\eta_i) = \max_{t\in[0,T]} z_i(t) \quad \text{and} \quad z_i(\xi_i) = \min_{t\in[0,T]} z_i(t), \quad i = 1,2.$$

Clearly,

$$\left.\frac{dz_1}{dt}\right|_{t=\eta_1} = \left.\frac{dz_1}{dt}\right|_{t=\xi_1} = 0 \quad \text{and} \quad \left.\frac{dz_2}{dt}\right|_{t=\eta_2} = \left.\frac{dz_2}{dt}\right|_{t=\xi_2} = 0.$$

Applying these to (3.3), we obtain

$$r_1(\eta_1)\left(1-\frac{e^{z_1(\eta_1)}}{k_1}\right)-\beta_1(\eta_1)e^{z_2(\eta_1)}=0, \tag{3.5}$$

$$r_2(\eta_2)\left(\frac{1}{1+w_1e^{z_1(\eta_2)}}-\frac{e^{z_2(\eta_2)}}{k_2}\right)-\beta_2(\eta_2)e^{z_1(\eta_2)}-w_2e^{z_1(\eta_2)}e^{z_2(\eta_2)}=0, \tag{3.6}$$

and

$$r_1(\xi_1)\left(1-\frac{e^{z_1(\xi_1)}}{k_1}\right)-\beta_1(\xi_1)e^{z_2(\xi_1)}=0, \tag{3.7}$$

$$r_2(\xi_2)\left(\frac{1}{1+w_1e^{z_1(\xi_2)}}-\frac{e^{z_2(\xi_2)}}{k_2}\right)-\beta_2(\xi_2)e^{z_1(\xi_2)}-w_2e^{z_1(\xi_2)}e^{z_2(\xi_2)}=0. \tag{3.8}$$

From (3.5), we obtain

$$r(\eta_1)-\frac{r(\eta_1)e^{z_1(\eta_1)}}{k_1}>0,$$

which implies that

$$z_1(t)\leq z_1(\eta_1)<\ln(k_1)=L_1. \tag{3.9}$$

In view of (3.6), we see that

$$\frac{1}{1+w_1e^{z_1(\eta_2)}}>\frac{e^{z_2(\eta_2)}}{k_2}. \tag{3.10}$$

Since $z_1(\eta_2)\leq z_1(\eta_1)<\ln k_1$, we have

$$\frac{k_2}{1+w_1e^{z_1(\eta_2)}}>\frac{k_2}{1+w_1k_1}.$$

Now, the left-hand side of (3.10) is independent of z_2, the right-hand side is independent of z_1, and both sides are positive, so there exists $\lambda_0\in(0,1)$ with $\lambda_0<\frac{1}{1+w_1k_1}$ such that

$$\frac{e^{z_2(\eta_2)}}{k_2}<\lambda_0<\frac{1}{1+w_1e^{z_1(\eta_2)}}.$$

Hence,

$$z_2(t)\leq z_2(\eta_2)<\ln\left(\frac{k_2}{1+w_1k_1}\right)=\ln m_0=L_3. \tag{3.11}$$

From (3.7),

$$r_1(\xi_1)\left(1-\frac{e^{z_1(\xi_1)}}{k_1}\right)=\beta_1(\xi_1)e^{z_2(\xi_1)},$$

and applying (3.11), we obtain

$$\frac{e^{z_1(\xi_1)}}{k_1}>1-\frac{\beta_1^M m_0}{r_1^L},$$

which implies that

$$z_1(t)\geq z_1(\xi_1)>\ln\left(k_1\left(1-\frac{\beta_1^M m_0}{r_1^L}\right)\right)=\ln(g_0)=L_2. \tag{3.12}$$

Finally, from (3.8) and applying (3.9),

$$\begin{aligned} e^{z_2(\xi_2)}\left[\frac{r_2^M}{k_2}+w_2e^{L_1}\right] &\ge e^{z_2(\xi_2)}\left[\frac{r_2(\xi_2)}{k_2}+w_2e^{z_1(\xi_2)}\right] \\ &= \frac{r_2(\xi_2)}{1+w_1e^{z_1(\xi_2)}} - \beta_2(\xi_2)e^{z_1(\xi_2)} \\ &\ge \frac{r_2^L}{1+w_1e^{L_1}} - \beta_2^M e^{L_1}. \end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned} z_2(t) \ge z_2(\xi_2) &> \ln\left(\frac{\frac{r_2^L}{1+w_1e^{L_1}} - \beta_2^M e^{L_1}}{\frac{r_2^M}{k_2}+w_2e^{L_1}}\right) \\ &= \ln\left(\frac{\frac{r_2^L}{1+w_1k_1} - \beta_2^M k_1}{\frac{r_2^M}{k_2}+w_2k_1}\right) = L_4. \end{aligned} \tag{3.14}$$

This completes the proof of the lemma.

Now observe the following. From (3.9), (3.11), (3.12), and (3.14) (or see (3.4)), we see that

$$|z_1(t)| < max\{|L_1|, |L_2|\} = \Lambda_1$$

and

$$|z_2(t)| < max\{|L_3|, |L_4|\} = \Lambda_2,$$

where Λ_1 and Λ_2 are independent of λ in system (3.3).

Let $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$, where Λ_3 is chosen sufficiently large that each solution (z_1^*, z_2^*) of the system

$$\overline{r_1} - \frac{\overline{r_1}e^{z_1(t)}}{k1} - \overline{\beta_1}e^{z_2(t)} = 0, \tag{3.15}$$

$$\frac{\overline{r_2}}{1+w_1e^{z_1(t)}} - \frac{\overline{r_2}e^{z_2(t)}}{k_2} - \overline{\beta_2}e^{z_1(t)} - w_2e^{z_1(t)}e^{z_2(t)} = 0 \tag{3.16}$$

satisfies $|z_1^*| + |z_2^*| < \Lambda$. Now consider $\Omega = \{(z_1, z_2)^T \in Z : \|(z_1, z_2)\| < \Lambda\}$; it is clear that Ω satisfies the first condition of Theorem 2.1.

We wish to show that $QN(z_1, z_2)^\tau \neq (0, 0)^\tau$ for each $(z_1, z_2)^\tau \in \partial\Omega \cap Ker(L)$. If $(z_1, z_2)^\tau \in \partial\Omega \cap Ker(L) = \partial\Omega \cap \mathbb{R}^2$, then $(z_1, z_2)^\tau$ is a constant vector in $\mathbb{R}^2$, and $|z_1| + |z_2| = \Lambda$. If the system (3.15) and (3.16) has a solution, then

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \overline{r_1} - \frac{\overline{r_1}e^{z_1(t)}}{k1} - \overline{\beta_1}e^{z_2(t)} \\ \frac{\overline{r_2}}{1+w_1e^{z_1(t)}} - \frac{\overline{r_2}e^{z_2(t)}}{k_2} - \overline{\beta_2}e^{z_1(t)} - w_2e^{z_1(t)}e^{z_2(t)} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since (3.15) and (3.16) does not have solution, it is clear that $QN(z_1, z_2)^\tau \neq 0$ for all $z = (z_1, z_2)^\tau \in \partial\Omega \cap Ker(L)$ and so the second condition of Theorem 2.1 is satisfied.

Having provided the necessary lemmas, we now present a theorem guaranteeing the existence of a positive periodic solution of the system (1.3) and (1.4).

Theorem 3.1. *If the conditions (A1)–(A4) hold, then system* (1.3) *and* (1.4) *has at least one positive T-periodic solution.*

Proof. It follows from Lemma 3.1 that L is a Fredholm operator of index zero and N is L-compact. From the discussion following the proof of Lemma 3.2, we see that the first two conditions of Theorem 2.1 are satisfied. We only need to prove that the last condition of Theorem 2.1 holds. To do so, we define the mapping $\Psi_\mu : dom(L) \times [0,1] \to Z$ by

$$\Psi(z_1, z_2, \mu) = \begin{pmatrix} \overline{r_1} - \frac{\overline{r_1} e^{z_1(t)}}{k1} \\ \frac{\overline{r_2}}{1+w_1 e^{z_1(t)}} - \frac{\overline{r_2} e^{z_2(t)}}{k_2} \end{pmatrix} + \mu \begin{pmatrix} -\overline{\beta_1} e^{z_2(t)} \\ -\overline{\beta_2} e^{z_1(t)} - w_2 e^{z_1(t)} e^{z_2(t)} \end{pmatrix},$$

where $\mu \in [0,1]$ is a parameter. If $(z_1, z_2)^T \in \partial\Omega \cap Ker(L) = \partial\Omega \cap R^2$, then $(z_1, z_2)^T$ is a constant vector in R^2 with $\|(z_1, z_2)^T\| = W$. We will show that if $(z_1, z_2)^T \in \partial\Omega \cap Ker(L)$, then $\Psi((z_1, z_2)^T, \mu) \neq 0$. The algebraic equation

$$\Psi(z_1, z_2, 0) = 0$$

has a unique solution $z_1^* = \ln(k_1)$ and $z_2^* = \ln\left(\frac{k_2}{1+w_1 k_1}\right)$. Define the homomorphism $J : Im(Q) \to Ker(L)$ by $Jz \equiv z$. A direct calculation shows that

$$\begin{aligned} \deg(JQN(z_1, z_2)^T, \Omega \cap Ker(L), (0,0)^T) &= deg(QN(z_1, z_2)^T, \Omega \cap Ker(L), (0,0)^T) \\ &= deg(\Psi(z_1, z_2, 1)^T, \Omega \cap Ker(L), (0,0)^T) \\ &= deg(\Psi(z_1, z_2, 0)^T, \Omega \cap Ker(L), (0,0)^T) \neq 0. \end{aligned}$$

Therefore, part three of the coincidence degree theorem (Theorem 2.1) holds, and so the system (1.3) and (1.4) has at least one positive T-periodic solution. This proves the theorem.

4. An example and numerical simulations

Example 1. *We now present an example to illustrate the applicability of our results. Consider the system* (1.3) *and* (1.4) *with the following selected parameters:*

$$r_1(t) = 3.5 + 0.5(1+\cos t), \qquad r_2(t) = 7.5 + 0.5(1+\sin t),$$

$$\beta_1(t) = 0.8 + 0.02\sin t, \qquad \beta_2(t) = 0.005 + 0.05\sin t,$$

$$k_1 = 4, \qquad k_2 = 4.85, \qquad w_1 = 0.4, \qquad w_2 = 0.12.$$

By substituting these values, system (1.3) *and* (1.4) *becomes*

$$\begin{cases} \frac{dx_1}{dt} = (3.5 + 0.5(1+\cos t))x_1\left(1 - \frac{x_1}{4}\right) - (0.8 + 0.02\sin t)x_1x_2, \\ \frac{dx_2}{dt} = (7.5 + 0.5(1+\sin t))x_2\left(\frac{1}{1+0.4x_1} - \frac{x_2}{4.85}\right) \\ \qquad\qquad - (0.005 + 0.05\sin t)x_1x_2 - 0.12x_1x_2^2. \end{cases} \tag{4.1}$$

For the chosen time-periodic functions, we have the obvious bounds

$$r_1^L = 3.5, \qquad r_2^L = 7.5,$$

$$r_1^M = 4.5, \qquad r_2^M = 8.5,$$

$$\beta_1^M = 0.82, \qquad \beta_2^M = 0.055.$$

we get

$$m_0 = \frac{k_2}{1 + w_1 k_1} = \frac{4.85}{1 + 0.4 \cdot 4} = \frac{4.85}{2.6} \approx 1.86538,$$

and

$$g_0 = k_1 \left(1 - \frac{\beta_1^M m_0}{r_1^L}\right) = 4\left(1 - \frac{0.82 \times 1.86538}{3.5}\right) \approx 2.25187.$$

It is easy to see that conditions (A1)–(A4) are satisfied for the parameter choice above. Hence, by Theorem 3.1, system (4.1) *admits at least one positive* 2π*-periodic solution. Figure 1 illustrates these findings for our results.*

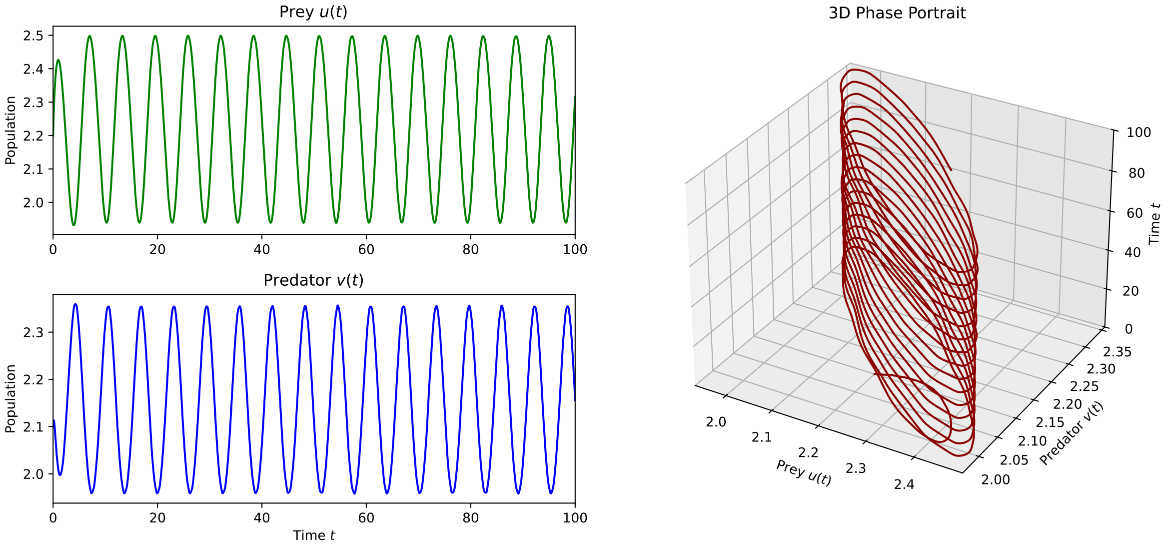

Figure 1. Periodic dynamics for system (4.1) with initial condition $x_1(0) = 2.2$ and $x_2(0) = 2.1$.

Remark 4.1. Notice that for Example 1, if we suppress the time-periodic forcing by setting the sin and cos terms to zero, the coefficient functions reduce to the constants $r_1 \equiv 4$, $r_2 \equiv 8$, $\beta_1 \equiv 0.8$, and $\beta_2 \equiv 0.005$. The system is then autonomous. Numerical simulations reveal that the solutions exhibit neither T-periodic oscillations nor convergence to a positive steady state (see Figure 2). This highlights the fact that the periodicity of $r_1(t)$, $r_2(t)$, $\beta_1(t)$, and $\beta_2(t)$ in (4.1) is essential for sustaining T-periodic dynamics.

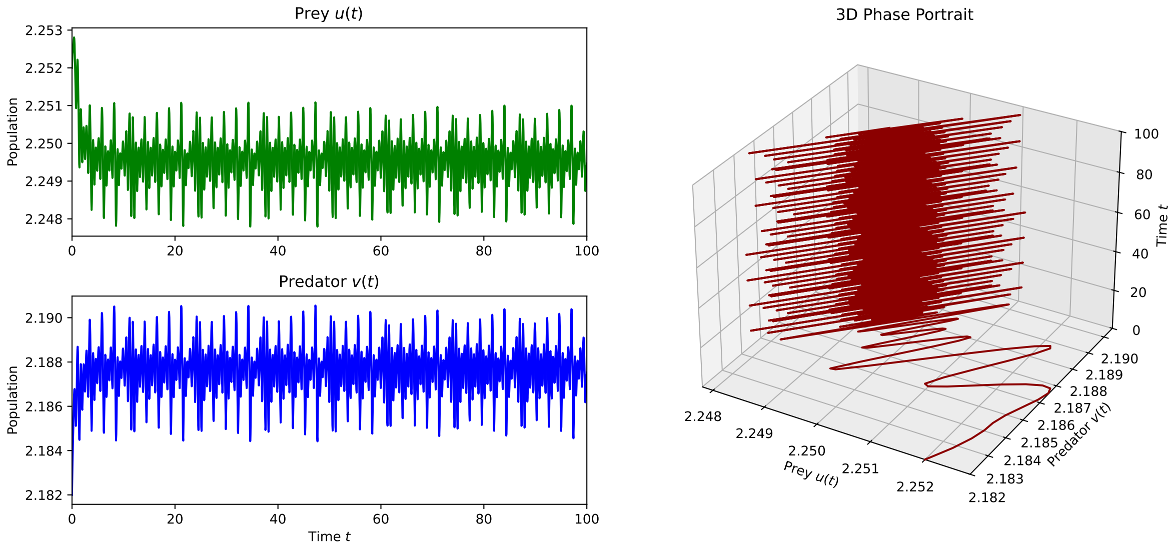

Figure 2. Simulation of the autonomous system with constant coefficients and initial conditions $x_1(0) = 2.252$ and $x_2(0) = 2.182$. The solutions neither show T-periodic oscillations nor converge to a steady state, highlighting the role of time-periodicity.

5. Suggestions for future research

Modifying our model to include the situation where "fear" affects both species would be especially of interest. Expanding the model to include three interacting species, although more complicated, would also be interesting. The inclusion of time delays to account for maturation or for acquiring the "fear" is also of possible interest.

6. Conclusions

In this paper, we considered a non-autonomous allelopathic phytoplankton predator-prey population model that incorporates fear effects. We established sufficient conditions for the existence of a positive periodic solution to the system using the coincidence degree theory due to Mawhin. An example and numerical simulations were included to support the findings by showing that the time-periodic coefficients lead to sustained oscillations, while the constant coefficient case previously studied did not do this.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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