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Research article

Least-squares solutions of generalized linear systems and the matrix equation $AXB = C$ under the general semi-tensor products

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Abstract: We investigate least-squares (LS) solutions of Sylvester-type matrix equations formulated via the general semi-tensor product (GSTP) of matrices. In particular, we consider generalized linear systems of the form $A \ltimes x = B$, where A and B are given rectangular matrices and x is an unknown column vector, with $\ltimes$ denoting the GSTP that extends both the conventional matrix product and the semi-tensor product. By analyzing the derivative of the LS error associated with the equation, we show that LS solutions can be obtained by solving an equivalent linear system under the usual matrix product. Using matrix partitioning techniques, these results are further extended to several Sylvester-type equations, including $A \ltimes X = B$, $X \ltimes \mathrm{A} = B$, and $A \ltimes X \ltimes B = C$, where X is an unknown matrix of compatible size. This framework unifies the classical and semi-tensor product cases under a generalized algebraic setting. Furthermore, we develop a gradient-descent iterative algorithm to compute approximate LS solutions efficiently. Numerical experiments confirm the convergence, capability, and effectiveness of the proposed method.

Keywords: Sylvester-type matrix equations; semi-tensor product; least-squares solution; matrix derivative; Kronecker product

1. Introduction

Linear systems and Sylvester-type matrix equations play a fundamental role in applied mathematics, engineering, control theory, and signal processing; see, e.g., [1,2]. In particular, the classical Sylvester equation $AX + XB = C$ and related matrix equations

$$AXB = C \tag{1.1}$$

play a diverse and significant role in both theoretical and practical applications, particularly in fields involving linear problems and data transformations. A classical algebraic approach to solving Sylvester-type equations such as Eq (1.1) is to apply vectorization techniques (e.g., the vector operator) to transform the matrix equation into an equivalent linear system; see, e.g., [2]. In this way, one can obtain general solutions, as well as special solutions such as the minimal-norm solution or the solution closest to a prescribed point. Least-squares solutions can also be derived by solving the normal equation associated with the linear system. In these cases, the desired solutions are expressed in a vector form involving the usual matrix product, the Kronecker product, and Moore-Penrose inverses.

A natural extension of this line of work is to replace the usual matrix product with a more general multiplication. The semi-tensor product (STP), introduced by Cheng [3], is a matrix product defined for pairs of matrices satisfying the factor-dimension condition. Consider two complex matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$. The STP of A and B can be expressed in terms of the usual matrix product and the Kronecker product $\otimes$ as follows:

$$A \ltimes B \;=\; \begin{cases} (A \otimes I_t)B, & p = nt, \\ A(B \otimes I_t), & n = pt. \end{cases} \tag{1.2}$$

For the matching-dimension case $n = p$, the STP reduces to the usual matrix product of A and B. More generally, the general semi-tensor product (GSTP) can be defined for matrices of arbitrary sizes [3], namely:

$$A \ltimes B \;=\; (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{1.3}$$

where α denotes the least common multiple (lcm) of n and p. In particular, when $n|p$ or $p|n$, the GSTP (1.3) reduces to the STP (1.2). Since the GSTP is defined in terms of the usual matrix product and the Kronecker product, it inherits several desirable algebraic properties, including associativity, distributivity over matrix addition, and compatibility with transposition, inversion, determinant, and trace; see, e.g., [4, 5]. Two remarkable properties of the (general) STP are its algebraic formulation of logical functions and its pseudo-commutativity with respect to swap matrices. Consequently, the (general) STP has served as a powerful tool in mathematics and related application areas, such as Boolean and fuzzy logic, lattice theory, finite-dimensional algebra, differential geometry, Boolean networks, finite state machines, and game theory; see [6–9] and references therein.

Over the last decade, numerous authors have developed theory and methods for obtaining exact or least-squares (LS) solutions of matrix equations involving the (general) STP. In particular, Yao et al. [10] (2015) investigated solvability criteria for the matrix equation

$$A \ltimes X \;=\; B, \tag{1.4}$$

under the GSTP. Li et al. [11] investigated exact solutions of a system of two matrix equations, $A \ltimes X = B$ and $X \ltimes C = D$, where $\ltimes$ denotes the STP. Subsequently, the work [12] employed matrix partitioning techniques to derive least-squares solutions of matrix equations of the form

$$A \ltimes X \ltimes B \;=\; C, \tag{1.5}$$

where $\ltimes$ denotes the STP. In 2022, the works [13, 14] employed matrix partitioning to solve certain Sylvester-type matrix equations under the GSTP, which also encompassed Eq (1.4). For quaternionic matrix equations, STPs served as a useful tool for solving such Sylvester-type equations, see, e.g., [15–17]. More recently, the work [18] utilized matrix derivatives to examine LS solutions of the matrix equation $A \ltimes x = B$, where $\ltimes$ is the STP and the unknown x is a column vector.

This paper continues the work of [18] by addressing least-squares solutions of the matrix equation (1.5), where A, B, C are given rectangular matrices and X is an unknown matrix of compatible size. By employing matrix partitioning techniques, the problem reduces to the study of the equation $A \ltimes x = B$, where $\ltimes$ denotes the GSTP and the unknown x is a column vector. To determine LS solutions, we analyze the derivative of the LS error associated with the matrix equation and show that the solution to the matrix equation can be obtained by solving an equivalent linear system under the usual matrix product. This framework encompasses both the classical matrix product and the STP as special cases. Our results therefore extend the vector-based analysis in [18] to a general setting of arbitrary matrix dimensions.

The remainder of this paper is organized as follows. Section 2, we recall prerequisite results on matrix algebra and derivatives that will be used in subsequent discussions. Section 3 addresses the Eq (1.4), in which the unknown is a column vector. Section 4 focuses on least-squares solutions of Eq (1.5) where the unknown is a matrix. Section 5 introduces a gradient-descent iterative (GDI) algorithm and presents numerical experiments to verify its capability and performance. Finally, Section 6 summarizes the main contributions of the paper.

2. Prerequisite results from matrix algebra and calculus

Throughout this paper, we denote the set of all m-by-n complex matrices by $\mathbb{C}^{m\times n}$. In particular, the set of m-by-1 matrices (column vectors) is written as $\mathbb{C}^m$. The transpose, the Moor-Penrose inverse and the Frobenius norm of a complex matrix A are written by $A^T, A^\dagger$ and $\|A\|$, respectively. The identity matrix of size $n \times n$ is denoted by I_n. Let us denote the i-th column of I_n by e_i^n.

We recall two kinds of vectorizations that transform any matrix $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ into a column vector, defined as follows:

$$\begin{aligned}\operatorname{vec}(A) &= \begin{bmatrix}a_{11} & \dots & a_{m1} & a_{12} & \dots & a_{m2} & \dots & a_{1n} & \dots & a_{mn}\end{bmatrix}^T \in \mathbb{C}^{mn},\\ \operatorname{rvec}(A) &= \begin{bmatrix}a_{11} & \dots & a_{1n} & a_{21} & \dots & a_{2n} & \dots & a_{m1} & \dots & a_{mn}\end{bmatrix}^T \in \mathbb{C}^{mn}.\end{aligned}$$

The Kronecker product of $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{s\times t}$ is defined as

$$A \otimes B = [a_{ij}B]_{ij} = \begin{bmatrix}a_{11}B & a_{12}B & \dots & a_{1n}B\\ a_{21}B & a_{22}B & \dots & a_{2n}B\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}B & a_{m2}B & \dots & a_{mn}B\end{bmatrix} \in \mathbb{C}^{ms\times nt}.$$

Lemma 2.1. *(e.g., [19]) The following properties hold for any matrices A, B, C, D:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$,

2). $(A \otimes B)^T = A^T \otimes B^T$,

3). $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, *provided that all matrix products exist.*

Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ be partitioned into block submatrices A_{ij} and B_{kl} of dimension $m_i \times n_j$ and $p_k \times q_l$, respectively. Then the Tracy-Singh product [20] of A and B is defined as the block matrix:

$$A \boxtimes B \;=\; \Big[[A_{ij} \otimes B_{kl}]_{kl}\Big]_{ij} \in \mathbb{C}^{mp\times nq}. \tag{2.1}$$

We recall the following layout conventions for matrix derivatives; see, e.g., [21, Ch. 4]. Consider a column vector $y = [y_1 \; y_2 \; \cdots \; y_m]^T$ whose elements are differentiable functions of a scalar x. The derivative of y with respect to x is defined as the row vector

$$\frac{\partial y}{\partial x} \;=\; \begin{bmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} & \cdots & \frac{\partial y_m}{\partial x} \end{bmatrix}. \tag{2.2}$$

Now, let $A(x)$ be an $m \times n$ matrix-valued function whose elements are differentiable functions of a $p \times 1$ vector $x = [x_1 \; x_2 \; \cdots \; x_p]^T$. The derivative of A with respect to x is defined as a $p \times mn$ matrix given by

$$\frac{\partial}{\partial x} A(x) \;=\; \begin{bmatrix} \frac{\partial}{\partial x_1} \operatorname{vec} A(x) \\ \frac{\partial}{\partial x_2} \operatorname{vec} A(x) \\ \vdots \\ \frac{\partial}{\partial x_p} \operatorname{vec} A(x) \end{bmatrix}. \tag{2.3}$$

According to the layout convention, we have the following chain rule and certain derivative formulas.

Lemma 2.2. *(e.g., [21, Ch. 4]). Let $x = [x_1 \; x_2 \; \cdots \; x_p]^T$ be a $p \times 1$ vector variable. Let $y = [y_1(x) \; y_2(x) \; \cdots \; y_q(x)]^T$ and $z = [z_1(y) \; z_2(y) \; \cdots \; z_r(y)]^T$ be $q \times 1$ and $r \times 1$ vector functions of x and y, respectively. Then the chain rule is given by*

$$\frac{\partial z}{\partial x} \;=\; \frac{\partial y}{\partial x} \frac{\partial z}{\partial y}.$$

Lemma 2.3. *(e.g., [21, Ch. 4]). Let $x \in \mathbb{C}^{n\times 1}$ be a vector variable and $A \in \mathbb{C}^{n\times n}$ a constant matrix. Then*

$$\frac{\partial}{\partial x} x^T A x \;=\; (A + A^T)x.$$

3. Least-squares solutions of the equation $A \ltimes x = B$ where x is a vector

Throughout this paper, let m, n, p be any positive integers, and denote $\alpha = \operatorname{lcm}(n, p)$. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{\frac{\alpha m}{n} \times \frac{\alpha}{p}}$ be given. We would like to investigate the following matrix equation

$$A \ltimes x \;=\; B, \tag{3.1}$$

in an unknown $x = [x_1 \; x_2 \; \cdots \; x_p]^T \in \mathbb{C}^p$. Under this assumption, all involved matrix sizes are compatible. When $n = p$, this equation is known as the classical linear system with respect to the usual matrix product. To solve Eq (3.1), we compute the derivatives of certain matrix/scalar-valued functions as follows.

Lemma 3.1. *From the above notations, we obtain that for each* $i = 1, 2, \ldots, \alpha/p$,

$$\frac{\partial}{\partial x}(x \otimes e_i^{\alpha/p}) \;=\; I_p \boxtimes (e_i^{\alpha/p})^T.$$

Proof. For each $i = 1, 2, \ldots, \alpha/p$, we have

$$\begin{aligned} x \otimes e_i^{\alpha/p} &= [x_1 \;\; \cdots \;\; x_p]^T \otimes [0 \;\; \cdots \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0] \\ &= [0 \;\; \cdots \;\; 0 \;\; x_1 \;\; 0 \;\; \cdots \;\; 0 \;\; \cdots \;\; 0 \;\; \cdots \;\; 0 \;\; x_p \;\; 0 \;\; \cdots \;\; 0]^T. \end{aligned}$$

It follows from Definitions (2.2) and (2.3) that

$$\frac{\partial}{\partial x}(x \otimes e_i^{\alpha/p}) \;=\; \begin{bmatrix} \frac{\partial}{\partial x_1}(x \otimes e_i^{\alpha/p}) \\ \vdots \\ \frac{\partial}{\partial x_p}(x \otimes e_i^{\alpha/p}) \end{bmatrix} \;=\; \begin{bmatrix} (e_1^p)^T \otimes (e_i^{\alpha/p})^T \\ \vdots \\ (e_p^p)^T \otimes (e_i^{\alpha/p})^T \end{bmatrix} \;=\; I_p \boxtimes (e_i^{\alpha/p})^T.$$

The last equation comes from the definition (2.1) of the Tracy-Singh product.

Lemma 3.2. *From the above notations, we obtain*

$$\frac{\partial}{\partial x}\|A \ltimes x\|^2 = 2\sum_{i=1}^{\alpha/p}\left(I_p \boxtimes (e_i^{\alpha/p})^T\right)(A^TA \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p}). \tag{3.2}$$

Proof. Let us write the matrix $A \ltimes x$ in terms of its columns:

$$\begin{aligned} A \ltimes x &= (A \otimes I_{\alpha/n})(x \otimes I_{\alpha/p}) \;=\; (A \otimes I_{\alpha/n})\begin{bmatrix} x_1 I_{\alpha/p} \\ \vdots \\ x_p I_{\alpha/p} \end{bmatrix} \\ &= (A \otimes I_{\alpha/n})\left[x \otimes e_1^{\alpha/p} \quad \cdots \quad x \otimes e_{\alpha/p}^{\alpha/p}\right] \\ &= \left[(A \otimes I_{\alpha/n})(x \otimes e_1^{\alpha/p}) \quad \cdots \quad (A \otimes I_{\alpha/n})(x \otimes e_{\alpha/p}^{\alpha/p})\right]. \end{aligned}$$

By Lemma 2.1, we compute its squared norm as follows:

$$\begin{aligned} \|A \ltimes x\|^2 &= \sum_{i=1}^{\alpha/p} \|(A \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p})\|^2 \\ &= \sum_{i=1}^{\alpha/p}(x \otimes e_i^{\alpha/p})^T(A \otimes I_{\alpha/n})^T(A \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p}) \\ &= \sum_{i=1}^{\alpha/p}\left(x^T \otimes (e_i^{\alpha/p})^T\right)(A^TA \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p}). \end{aligned}$$

Now, its derivatives can be computed using Lemmas 2.2, 2.3 and 3.1 as follows:

$$\begin{aligned} \frac{\partial}{\partial x}\|A \ltimes x\|^2 &= \frac{\partial}{\partial x}(x \otimes e_i^{\alpha/p}) \cdot \frac{\partial}{\partial(x \otimes e_i^{\alpha/p})}\left[\left(x^T \otimes (e_i^{\alpha/p})^T\right)(A^TA \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p})\right] \\ &= \left(I_p \boxtimes (e_i^{\alpha/p})^T\right)\left[(A^TA + (A^TA)^T) \otimes I_{\alpha/n}\right](x \otimes e_i^{\alpha/p}) \\ &= 2\left(I_p \boxtimes (e_i^{\alpha/p})^T\right)(A^TA \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p}). \end{aligned}$$

From the above notations, let us partition

$$C \;:=\; B^T(A \otimes I_{\alpha/n}) \;=\; [\; C_1 \mid C_2 \mid \cdots \mid C_p \;], \tag{3.3}$$

where $C_j \in \mathbb{C}^{\frac{\alpha}{p}\times\frac{\alpha}{p}}$ for each $j = 1, 2, \ldots, p$.

Lemma 3.3. *From the above notations, we have*

$$\frac{\partial}{\partial x}\operatorname{tr}(B(x^T \ltimes A^T)) \;=\; \Big[\operatorname{tr}(C_1)\ \operatorname{tr}(C_2)\ \cdots\ \operatorname{tr}(C_p)\Big]^T.$$

Proof. We know that

$$\begin{aligned}\operatorname{tr}(B(x^T \ltimes A^T)) \;&=\; \operatorname{tr}(B^T(A \ltimes x)) \;=\; \operatorname{tr}[B^T(A \otimes I_{\alpha/n})(x \otimes I_{\frac{\alpha}{p}})] \\ &=\; \operatorname{tr}[C(x \otimes I_{\frac{\alpha}{p}})].\end{aligned}$$

For simplicity, we investigate the main diagonal of $C(x \otimes I_{\frac{\alpha}{p}})$ without the calculation of any other components else. We compute

$$\begin{aligned}\operatorname{tr}(C(x \otimes I_{\frac{\alpha}{p}})) \;&=\; (c_{11}x_1 + c_{1,\frac{\alpha}{p}+1}x_2 + \cdots + c_{1,(p-1)\frac{\alpha}{p}+1}x_p) + (c_{22}x_1 + c_{2,\frac{\alpha}{p}+2}x_2 + \cdots + c_{2,(p-1)\frac{\alpha}{p}+2}x_p) \\ &\quad + \cdots + (c_{\frac{\alpha}{p},\frac{\alpha}{p}}x_1 + c_{\frac{\alpha}{p},\frac{2\alpha}{p}}x_2 + \cdots + c_{\frac{\alpha}{p},\alpha}x_p) \\ &=\; (c_{11} + c_{22} + \cdots + c_{\frac{\alpha}{p},\frac{\alpha}{p}})x_1 + (c_{1,\frac{\alpha}{p}+1} + c_{2,\frac{\alpha}{p}+2} + \cdots + c_{\frac{\alpha}{p},\frac{2\alpha}{p}})x_2 \\ &\quad + \cdots + (c_{1,(p-1)\frac{\alpha}{p}+1} + c_{2,(p-1)\frac{\alpha}{p}+2} + \cdots + c_{\frac{\alpha}{p},\alpha})x_p \\ &=\; x_1\operatorname{tr}(C_1) + x_2\operatorname{tr}(C_2) + \cdots + x_p\operatorname{tr}(C_p).\end{aligned}$$

Therefore, according to Definition 2.3, we get

$$\begin{aligned}\frac{\partial}{\partial x}\operatorname{tr}(B(x^T \ltimes A^T)) \;&=\; \frac{\partial}{\partial x}\Big[x_1\operatorname{tr}(C_1) + x_2\operatorname{tr}(C_2) + \cdots + x_p\operatorname{tr}(C_p)\Big] \\ &=\; \Big[\operatorname{tr}(C_1)\ \operatorname{tr}(C_2)\ \cdots\ \operatorname{tr}(C_p)\Big]^T.\end{aligned}$$

For each $i = 1, 2, \ldots, \alpha/p$, denote

$$M_i \;=\; \Big(I_p \boxtimes (e_i^{\alpha/p})^T\Big)(A^TA \otimes I_{\alpha/n}) \in \mathbb{C}^{p\times\alpha}, \tag{3.4}$$

$$N_i \;=\; M_i[i, \frac{\alpha}{p} + i, \ldots, (p-1)\frac{\alpha}{p} + i] \in \mathbb{C}^{p\times p}. \tag{3.5}$$

Here, the notation N_i denotes the submatrix of M_i determined by the selected columns.

Theorem 3.4. *From the above notations, the LS solutions of $A \ltimes x = B$ can be obtained by solving the linear system*

$$Kx \;=\; f, \tag{3.6}$$

where $K = \sum_{i=1}^{\alpha/p} N_i \in \mathbb{C}^{p\times p}$ *and* $f = \begin{bmatrix}\operatorname{tr}(C_1) & \operatorname{tr}(C_2) & \cdots & \operatorname{tr}(C_p)\end{bmatrix}^T \in \mathbb{C}^p$*. With any LS solution* x^**, the LS error is given by*

$$||A \ltimes x^* - B||^2 \;=\; ||A \ltimes K^{-1}f||^2 - 2f^TK^{-1}f + ||B||^2. \tag{3.7}$$

Moreover, the equation $A \ltimes x = B$ *has a unique LS solution if and only if* $\operatorname{rank} K = p$*.*

Proof. Consider the LS error $||(A\ltimes x)-B||^2$ associated with Eq (3.1). From the norm formula, we obtain

$$\begin{aligned} ||(A \ltimes x) - B||^2 &= \operatorname{tr}[(A \ltimes x) - B)((A \ltimes x) - B)^T] \\ &= \operatorname{tr}[(A \ltimes x)(A \ltimes x)^T - B(x^T \ltimes A^T) - (A \ltimes x)B^T + BB^T] \\ &= ||A \ltimes x||^2 - 2\operatorname{tr}(B(x^T \ltimes A^T)) + ||B||^2. \end{aligned} \tag{3.8}$$

To find such vector x that minimizes the above error, we evaluate the derivative with respect to x and set it equal to zero, i.e.,

$$\frac{\partial}{\partial x}||(A \ltimes x) - B||^2 \;=\; 0.$$

From Eq (3.8) and Lemma 3.3, it follows that

$$\frac{\partial}{\partial x}||A \ltimes x||^2 \;=\; 2\frac{\partial}{\partial x}\operatorname{tr}(B(x^T \ltimes A^T)) \;=\; 2f.$$

From Lemma 3.2 and the Eqs (3.4) and (3.5), we have

$$\begin{aligned} f &= \sum_{i=1}^{\alpha/p}\Big(I_p \boxtimes (e_i^{\alpha/p})^T\Big)(A^TA \otimes I_{\alpha/n})(x \otimes e_i^{\alpha/p}) \;=\; \sum_{i=1}^{\alpha/p} M_i(x \otimes e_i^{\alpha/p}) \\ &= \sum_{i=1}^{\alpha/p}\Big(M_i[i, \tfrac{\alpha}{p} + i, ..., (p-1)\tfrac{\alpha}{p} + i]\Big)x \;=\; \sum_{i=1}^{\alpha/p} N_i x \;=\; Kx. \end{aligned}$$

Thus, the desire LS solutions can be obtained from the system $Kx = f$. Hence, the equation $A \ltimes x = B$ has a unique LS solution if and only if the matrix K is invertible or $\operatorname{rank} K = p$. From the proof of Lemma 3.3, note that $\operatorname{tr}(B(x^T \ltimes A^T)) = f^Tx$. The LS is therefore given by

$$\begin{aligned} ||A \ltimes x^* - B||^2 &= ||A \ltimes x^*||^2 - 2\operatorname{tr}(B(x^{*T} \ltimes A^T)) + ||B||^2 \\ &= ||A \ltimes K^{-1}f||^2 - 2f^TK^{-1}f + ||B||^2, \end{aligned}$$

where x^* is a LS solution.

Remark 3.5. From Theorem 3.4, we can find the general/minimal-norm LS solutions of the equation $A \ltimes x = B$ as follows. The equivalent linear system (3.6) has the general solutions given by

$$x^* \;=\; K^\dagger f + (I_p - K^\dagger K)w,$$

where $w \in \mathbb{C}^p$ is arbitrary. Among the solutions, the minimal-norm one is given by $x^* = K^\dagger f$.

Theorem 3.4 includes the study of the equation $A \ltimes x = b$ where $\ltimes$ is the STP. In the next result, we observe the factor-dimension case when $p = nt$ for some integer t.

Corollary 3.6. *Let $A \in \mathbb{C}^{m\times n}$ and $b = [b_1 \ b_2 \ \cdots \ b_{mt}]^T \in \mathbb{C}^{mt}$. Then the LS solutions $x \in \mathbb{C}^{nt}$ of the equation $A \ltimes x = b$ can be obtained by solving the linear system*

$$A^T A \ltimes x = A^T \ltimes b. \tag{3.9}$$

Moreover, if $\operatorname{rank} A = n$*, then Eq* (3.9) *has a unique solution given by*

$$x^* = (A^T A)^{-1} A^T \ltimes b. \tag{3.10}$$

The LS error is given by

$$\|A \ltimes x^* - b\|^2 = \|b\|^2 - (b^T \ltimes A)x^* = \|b\|^2 - (b^T \ltimes A)[(A^T A)^{-1} A^T \ltimes b]. \tag{3.11}$$

Proof. Since $p = nt$, we get $\operatorname{lcm}(n, p) = p$. From Theorem 3.4, we obtain

$$\begin{aligned} K &= N_1 = M_1[1, 2, \ldots, p] = M_1 \\ &= \left(I_{nt} \boxtimes (e_1^1)^T\right)(A^T A \otimes I_t) = A^T A \otimes I_t. \end{aligned}$$

Now, from Eq (3.3), we have

$$C := b^T (A \otimes I_t) = \begin{bmatrix} b_1 a_{11} + b_{1+t} a_{21} + \cdots + b_{1+(m-1)t} a_{m1} \\ b_2 a_{11} + b_{2+t} a_{21} + \cdots + b_{2+(m-1)t} a_{m1} \\ \vdots \\ b_t a_{11} + b_{2t} a_{21} + \cdots + b_{mt} a_{m1} \\ \vdots \\ b_1 a_{1n} + b_{1+t} a_{2n} + \cdots + b_{1+(m-1)t} a_{mn} \\ b_2 a_{1n} + b_{2+t} a_{2n} + \cdots + b_{2+(m-1)t} a_{mn} \\ \vdots \\ b_t a_{1n} + b_{2t} a_{2n} + \cdots + b_{mt} a_{mn} \end{bmatrix}^T.$$

We partition $C = [C_1 \ \cdots \ C_p]$ so that each block is of size 1×1. It follows that

$$f = \begin{bmatrix} \operatorname{tr}(C_1) \\ \operatorname{tr}(C_2) \\ \vdots \\ \operatorname{tr}(C_{nt}) \end{bmatrix} = C^T = (A^T \otimes I_t) b.$$

Hence, we arrive at the equation

$$(A^T A \otimes I_t) x = (A^T \otimes I_t) b. \tag{3.12}$$

If $\operatorname{rank} A = n$, then $A^T A$ is invertible, and thus Eq (3.12) has a unique solution given by

$$x^* = [(A^T A)^{-1} A^T \otimes I_t] b.$$

To compute the LS error, note that according to Eq (3.9), we get

$$\begin{aligned}||A \ltimes x^*||^2 &= (x^{*T} \ltimes A^T)(A \ltimes x^*) = x^{*T} \ltimes (A^T A \ltimes x^*) \\ &= x^{*T} \ltimes (A^T \ltimes b) = (A \ltimes x^*)^T b = b^T (A \ltimes x^*).\end{aligned}$$

Thus, the LS error can be obtained from Eq (3.8) as follows:

$$\begin{aligned}||A \ltimes x^* - b||^2 &= ||A \ltimes x^*||^2 - 2\,\mathrm{tr}(b(x^{*T} \ltimes A^T)) + ||b||^2 \\ &= b^T(A \ltimes x^*) - 2b(x^{*T} \ltimes A^T) + ||b||^2 \\ &= ||b||^2 - b^T(A \ltimes x^*) \\ &= ||b||^2 - (b^T \ltimes A)x^*.\end{aligned}$$

Hence, we arrive at Eq (3.11).

The second factor-dimension case of the STP is that when p divides n as in the next corollary.

Corollary 3.7. *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times t}$, where $n = pt$ for some integer t. Then the LS solutions $x \in \mathbb{C}^p$ of the equation $A \ltimes x = B$ can be obtained by solving the linear system*

$$\left(\sum_{i=1}^{t} N_i\right) x = \begin{bmatrix} a_{1,1} & \cdots & a_{1,t} & \cdots & a_{m,1} & \cdots & a_{m,t} \\ a_{1,t+1} & \cdots & a_{1,2t} & \cdots & a_{m,t+1} & \cdots & a_{m,2t} \\ & \vdots & & \vdots & & \vdots & \\ a_{1,(p-1)t+1} & \cdots & a_{1,pt} & \cdots & a_{m,(p-1)t+1} & \cdots & a_{m,pt} \end{bmatrix} \mathrm{rvec}\,(B), \tag{3.13}$$

where $N_i = M_i[i, t+i, ..., (p-1)t+i]$ and $M_i = \left(I_p \boxtimes (e_i^t)^T\right)(A^T A)$ for each $i = 1, \ldots, t$.

Proof. From $n = pt$, we have $\alpha = \mathrm{lcm}(n, p) = n$. From Theorem 3.4, we obtain

$$K = \sum_{i=1}^{t} N_i = \sum_{i=1}^{t} M_i[i, t+i, ..., (p-1)t+i].$$

According to Theorem 3.4, we compute

$$C := B^T A = \begin{bmatrix} b_{11}a_{11} + \cdots + b_{m1}a_{m1} & \ldots & b_{11}a_{1n} + \cdots + b_{m1}a_{mn} \\ b_{12}a_{11} + \cdots + b_{m2}a_{m1} & \ldots & b_{12}a_{1n} + \cdots + b_{m2}a_{mn} \\ \vdots & \ddots & \vdots \\ b_{1t}a_{11} + \cdots + b_{mt}a_{m1} & \ldots & b_{1t}a_{1n} + \cdots + b_{mt}a_{mn} \end{bmatrix}.$$

Then, we partition $C = [C_1\ C_2\ \cdots\ C_p]$ where

$$C_j = \begin{bmatrix} a_{1,(j-1)t+1}b_{11} + \cdots + a_{m,(j-1)t+1}b_{m1} & \cdots & a_{1,jt}b_{11} + \cdots + a_{m,jt}b_{m1} \\ \vdots & \ddots & \vdots \\ a_{1,(j-1)t+1}b_{1t} + \cdots + a_{m,(j-1)t+1}b_{mt} & \cdots & a_{1,jt}b_{1t} + \cdots + a_{m,jt}b_{mt} \end{bmatrix},$$

for each $i = 1, \ldots, p$. Thus, we get

$$\begin{aligned}\operatorname{tr}(C_j) &= (a_{1,(j-1)t+1}b_{11} + \cdots + a_{m,(j-1)t+1}b_{m1}) + \cdots + (a_{1,jt}b_{1t} + \cdots + a_{m,jt}b_{mt}) \\ &= a_{1,(j-1)t+1}b_{11} + a_{1,(j-1)t+2}b_{12} + \cdots + a_{1,jt}b_{1t} \\ &\quad + \cdots + a_{m,(j-1)t+1}b_{m1} + a_{m,(j-1)t+2}b_{m2} + \cdots + a_{m,jt}b_{mt}.\end{aligned}$$

Hence,

$$f = \begin{bmatrix}\operatorname{tr}(C_1) \\ \operatorname{tr}(C_2) \\ \vdots \\ \operatorname{tr}(C_p)\end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,t} & \cdots & a_{m,1} & \cdots & a_{m,t} \\ a_{1,t+1} & \cdots & a_{1,2t} & \cdots & a_{m,t+1} & \cdots & a_{m,2t} \\ & \vdots & & \vdots & & \vdots & \\ a_{1,(p-1)t+1} & \cdots & a_{1,pt} & \cdots & a_{m,(p-1)t+1} & \cdots & a_{m,pt}\end{bmatrix} \begin{bmatrix} b_{11} \\ \vdots \\ b_{1t} \\ \vdots \\ b_{m1} \\ \vdots \\ b_{mt}\end{bmatrix}$$

Therefore, we arrive at Eq (3.13).

Remark 3.8. When $t = 1$, Corollaries 3.6 and 3.7 are both reduced to the following classical result. Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then the LS solutions $x \in \mathbb{C}^n$ of the linear system $Ax = b$ can be obtained by solving the associated normal equation

$$A^T Ax = A^T b.$$

4. Least-squares solutions of the equation $A \ltimes X \ltimes B = C$ where X is a matrix

First, we consider the matrix equation

$$A \ltimes X = B, \tag{4.1}$$

where $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{\frac{\alpha m}{n}\times\frac{\alpha q}{p}}$ are given matrices, and $x \in \mathbb{C}^{p\times q}$ is an unknown matrix. Here, recall that $\alpha = \operatorname{lcm}(n, p)$. Note that all involved matrix sizes are compatible and full of generality. Let us partition

$$X = [X_1\ X_2\ \cdots\ X_q] \quad \text{and} \quad \mathrm{B} = [B_1\ B_2\ \cdots\ B_q],$$

where $X_j \in \mathbb{C}^p$ and $B_j \in \mathbb{C}^{\frac{\alpha m}{n}\times\frac{\alpha}{p}}$, respectively. Observe that

$$A \ltimes X = \begin{bmatrix} A \ltimes X_1 & A \ltimes X_2 & \cdots & A \ltimes X_q\end{bmatrix} = \begin{bmatrix} B_1 & B_2 & \cdots & B_q\end{bmatrix}.$$

The equality between the corresponding block matrices gives rise to

$$A \ltimes X_j = B_j$$

for each $j = 1, 2, \ldots, q$. Hence, we can apply Theorem 3.4 to get the LS solution X_j for each j. Therefore, we obtain the desire solution X.

Now, consider the matrix equation

$$X \ltimes A \ = \ B,$$

where A and B are given rectangular matrices, and X is an unknown matrix of compatible size. Note that this equation is equivalent to the equation

$$A^T \ltimes X^T \ = \ B^T. \tag{4.2}$$

We can solve Eq (4.2) for X^T using Theorem 3.4, and then we get the desire solution X.

In general, let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{a\times b}$ and $C \in \mathbb{C}^{r\times s}$ be given matrices, and let $X \in \mathbb{C}^{p\times q}$ be an unknown matrix of compatible size. We consider the matrix equation

$$A \ltimes X \ltimes B \ = \ C. \tag{4.3}$$

From the previous discussion, we can find the LS solution for Eq (4.3) as follows. The first step is to find the LS solution Y of the equation $A \ltimes Y = C$. The second step is to find the LS solution X of the equation $X \ltimes B \ = \ Y$. Thus we get the desired solution X.

5. GDI algorithm and numerical experiments

In this section, we provide an effective computational method associated with the main result, Theorem 3.4. Numerical experiments are demonstrated to verify the effectiveness of the proposed algorithm.

As discussed in Section 3, solving the matrix equation (3.1) is equivalent to solving the linear system (3.6). To solve this linear system, a gradient-descent iterative solver takes the form

$$x^{(i+1)} \ = \ x^{(i)} - \gamma_{i+1}\nabla f(x), \tag{5.1}$$

where $f(x) := \|Kx - f\|^2$ is the associated LS error. More precisely, the next approximate solution $x^{(i+1)}$ is equal to the sum between the current approximation $x^{(i)}$ and the direction vector $-\nabla f(x)$ with step size γ_{i+1}. Indeed, the gradient of f can be computed using a process as that for Eq (3.8):

$$\begin{aligned} \frac{\partial}{\partial x}\|Kx - f\|^2 &= \frac{\partial}{\partial x}\left(\|Kx\|^2 - 2\,\mathrm{tr}(fx^TK^T) + \|f\|^2\right) \\ &= \frac{\partial}{\partial x}(x^TK^TKx) - 2\frac{\partial}{\partial x}(x^TK^Tf) \\ &= 2K^TKx - 2K^Tf \\ &= 2K^T(Kx - f). \end{aligned}$$

The suitable step size γ_{i+1} can be obtained from an optimization technique as described in [22]. So, we arrive at a gradient-descent iterative (GDI) algorithm for solving Eq (3.1) as Algorithm 1.

We implement all simulations using MATLAB R2017a on the same PC environment: AMD A9-9425 RADEON R5@3.10 GHz with RAM 4 GB. The performance of the algorithm is analyzed through the residual error

$$R^{(i)} \ = \ \|r^{(i)}\| \ = \ \|f - Kx^{(i)}\|,$$

together with the CPU time measured by the *tic-toc* function on MATLAB.

Algorithm 1: GDI Algorithm for the matrix equation (3.1)

```
$A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{\frac{\alpha m}{n}\times\frac{\alpha}{p}}$; Choose $x^{(0)} \in \mathbb{C}^{p}$ and $\epsilon > 0$.
for $i = 1,2,3,\dots,p$ do
    compute $C_i$ according to Eq (3.3).
end
for $i = 1,2,3,\dots,\alpha/p$ do
    compute $N_i$ according to Eq (3.5).
end
Compute $f = \begin{bmatrix}\mathrm{tr}(C_1) & \mathrm{tr}(C_2) & \cdots & \mathrm{tr}(C_p)\end{bmatrix}^T$, $K = \sum_{i=1}^{\alpha/p} N_i$, and $M = KK^T$.
for $i = 0,1,2,3,\dots$ do
    $r^{(i)} = f - Kx^{(i)}$;
    if $\|r^{(i)}\| \leqslant \epsilon$ then
        $x^{(i)}$ is a solution; break;
    else
        $m_i = Mr^{(i)}$; $\gamma_{i+1} = m_i^T r^{(i)}/(2m_i^T m_i)$ ;
        $x^{(i+1)} = x^{(i)} + \gamma_{i+1}K^T r^{(i)}$ ;
    end
    update $i$;
end
```

Example 5.1. Consider the generalized linear system $A \ltimes x = B$, where

$$A = \begin{bmatrix}-1 & 1 & 4\\ 2 & 0 & -2\end{bmatrix} \in \mathbb{R}^{2\times 3}, \quad B = \begin{bmatrix}-2 & -4 & 2\\ -1 & -2 & -4\\ 4 & 2 & 0\\ 0 & 4 & 2\end{bmatrix} \in \mathbb{R}^{4\times 3}.$$

Our task is to find an LS solution $X = \begin{bmatrix}x_1 & x_2\end{bmatrix}^T \in \mathbb{R}^2$ satisfying the above equation. Due to Theorem 3.4, it suffices to find a solution of the associated linear system $Kx = f$. Indeed, we compute

$$N_1 = \begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix}, \quad N_2 = \begin{bmatrix}5 & 0\\ 0 & 20\end{bmatrix}, \quad N_3 = \begin{bmatrix}1 & 0\\ 0 & 20\end{bmatrix}, C_1 = \begin{bmatrix}10 & 1 & -2\\ 8 & 10 & -4\\ -2 & 8 & 2\end{bmatrix}, \quad C_2 = \begin{bmatrix}-1 & -16 & -4\\ -2 & -20 & -16\\ -4 & 8 & -20\end{bmatrix},$$

and thus obtain

$$K = \sum_{i=1}^{3} N_i = \begin{bmatrix}11 & 0\\ 0 & 41\end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix}\mathrm{tr}(C_1)\\ \mathrm{tr}(C_2)\end{bmatrix} = \begin{bmatrix}22\\ -41\end{bmatrix}.$$

We execute Algorithm 1 with an initial vector $x = \begin{bmatrix}0.5 & 0.5\end{bmatrix}^T \in \mathbb{R}^2$, and set a tolerance error to be $\epsilon = 0.03$. The simulation results are illustrated numerically via Table 1 and graphically via Figure 1.

Table 1. The updated solutions for each iteration.

i	x_1	x_2	$r^{(i)}$
1	0.5	0.5	63.6750
2	0.5542	−0.2536	34.4878
3	0.6072	−0.6335	21.4590
4	0.6608	−0.8294	16.3070
5	0.7230	−0.9394	14.2653
6	0.8532	−1.0252	12.6575
7	1.1261	−0.9418	9.9047
8	1.1890	−1.0000	8.9210
9	1.5945	−0.9998	4.4605
10	1.7971	−1.0011	2.2318
⋮	⋮	⋮	⋮
25	1.9976	−0.9998	0.0278

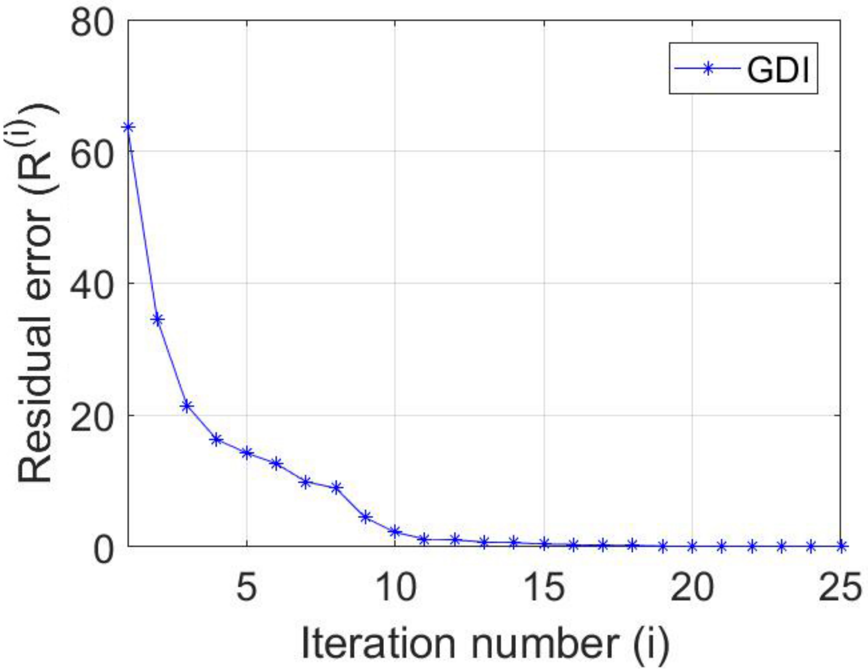

Figure 1. The relative error at each iteration for Example 5.1.

It turns out that this procedure consumes 0.039715 seconds to reach an approximate LS solution

$$x^{(25)} = \begin{bmatrix} 1.9976 \\ -0.9998 \end{bmatrix}$$

within 25 iterations. The associated LS error is given by

$$||A \ltimes x^{(25)} - B||^2 \;=\; 0.0278.$$

Thus, $x^{(25)}$ is a desired LS solution. Hence, the proposed algorithm is capable and effective.

Example 5.2. Consider the generalized linear system $A \ltimes x \;=\; B$, where

$$A \;=\; \text{triu}(\text{rand}(50, 20)), \quad B \;=\; 0.5 \times \text{ones}(150, 5).$$

Here, A taken as a unit tridiagonal random matrix of dimension 50×20. The symbol ones(150, 50) denotes the 150×50 matrix whose entries are all equal to 1. We seek a least-squares solution $x \in \mathbb{R}^{12}$ that satisfies the above equation. In view of Theorem 3.4, the problem is equivalent to solving the linear system $Kx \;=\; f$. We run the Algorithm 1 with the initial vector $x^{(0)} \;=\; 0.5 \times \text{ones} \in \mathbb{R}^{12}$, and set the tolerance error to $\epsilon \;=\; 0.5$. The simulation results are presented graphically in Figure 2. The algorithm converges in 100 iterations, requires 0.134697 seconds, and yields an LS error of

$$||A \ltimes x^{(100)} - B||^2 \;=\; 0.5376.$$

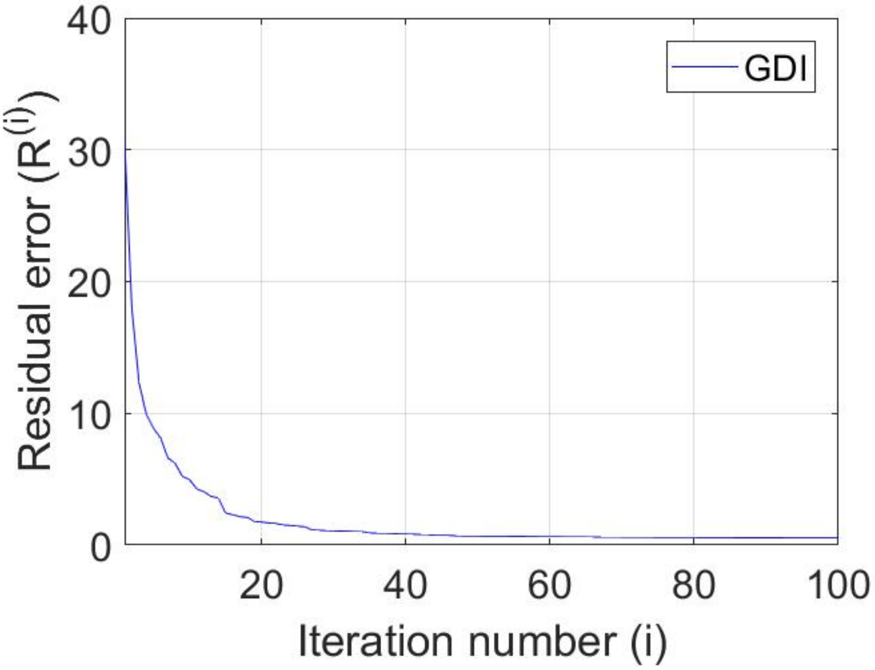

Figure 2. The relative error at each iteration for Example 5.2.

Example 5.3. Consider the matrix equation $A \ltimes X \ltimes B = C$, where

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 3}, \quad B = \begin{bmatrix} -1 & 0 \\ 2 & -1 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \in \mathbb{R}^{4\times 2}, \quad C = \begin{bmatrix} -3 & 0 & 2 & 0 & 0 & 1 \\ -5 & -3 & 0 & 5 & 0 & 0 \\ 0 & -5 & -3 & 0 & 5 & 0 \\ 6 & 0 & -5 & -3 & 0 & 5 \\ -2 & 2 & -2 & 0 & -1 & -1 \\ 5 & -2 & 2 & -5 & 0 & -1 \\ -2 & 5 & -2 & -1 & -5 & 0 \\ 4 & -2 & 5 & -2 & -1 & -5 \end{bmatrix} \in \mathbb{R}^{8\times 6}.$$

Our task is to find an LS solution $X \in \mathbb{R}^{2\times 2}$ satisfying the above equation.

Step 1: Find the LS solution $Y = [y_1 \;\; y_2] \in \mathbb{R}^{4\times 2}$ of the equation $A \ltimes Y = C$. We obtain

$$A \ltimes Y = \begin{bmatrix} A \ltimes y_1 & A \ltimes y_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

where C_1 and C_2 denote the 1st–3rd columns and the 4th–6th columns of C, respectively. We apply Theorem 3.4 to compute the LS solutions y_1 and y_2 of the equations $A \ltimes y_1 = C_1$ and $A \ltimes y_2 = C_2$, respectively. In Algorithm 1, the initial vector is chosen as $[1 \;\; 1 \;\; 1 \;\; 1]^T$, and the tolerance error is set to $\epsilon = 0.05$. The resulting matrix Y is obtained as

$$Y = \begin{bmatrix} -0.9998 & 0.0001 \\ 2.000 & -1.0000 \\ -1.9911 & -0.9876 \\ 5.0000 & -5.0000 \end{bmatrix}.$$

Step 2: Find the LS solution X of the equation $X \ltimes B = Y$, or equivalently $B^T \ltimes X^T = Y^T$. By writing $X^T = [x_1 \;\; x_2]$, we thus get

$$\begin{bmatrix} B^T \ltimes x_1 & B^T \ltimes x_2 \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix}, \quad \text{where}$$
$$Z_1 = \begin{bmatrix} -0.9998 & 2.0000 \\ 0.0001 & -1.0000 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} -1.9911 & 5.0000 \\ -0.9876 & -5.0000 \end{bmatrix}.$$

We apply Theorem 3.4 to solve the equations $B^T \ltimes x_1 = Z_1$ and $B^T \ltimes x_2 = Z_2$ for the LS solutions x_1 and x_2, respectively. The computations are carried out via Algorithm 1 using the initial vector $[1 \;\; 1]^T$ and a tolerance error of $\epsilon = 0.05$. This yields the matrix X as

$$X = \begin{bmatrix} 1.0127 & 0.0070 \\ 2.0101 & -0.9934 \end{bmatrix}.$$

6. Conclusions

This paper has presented a unified framework for deriving least-squares (LS) solutions of Sylvester-type matrix equations formulated under the general semi-tensor product (GSTP). By employing matrix partitioning techniques, the problem was reduced to the fundamental equation $A \ltimes x = B$, where $\ltimes$ denotes the GSTP and x is a column vector. Through the analysis of matrix derivatives associated with the LS error function, we established that the desired LS solutions can be obtained by solving an equivalent linear system defined by the conventional matrix product. The proposed formulation naturally encompasses both the classical matrix product and the semi-tensor product as particular cases.

Furthermore, a gradient-descent iterative (GDI) algorithm was developed to compute approximate LS solutions. Numerical experiments demonstrated the convergence, stability, and effectiveness of the proposed algorithm, even for generalized linear systems of large or mismatched dimensions.

The theoretical framework admits potential applications across several domains. Owing to the close connection between the (general) semi-tensor product and logical representations, the results are applicable to Boolean networks and finite-state systems, where GSTP serves as an algebraic tool for system modeling and optimization. In control theory, GSTP-based Sylvester-type equations arise naturally in the analysis and design of logical control systems, switched systems, and networked dynamical systems, particularly when matrix dimensions are incompatible under conventional matrix multiplication. In signal processing, the ability to accommodate generalized linear models with dimension mismatch renders the proposed LS framework relevant to problems of system identification, filtering, and data fusion.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Pattrawut Chansangiam and Yang Zhang are guest editors for [Electronic Research Archive (ERA)] and were not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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