



Research article

Ricci-type symmetries and Ricci solitons on two-dimensional Riemannian manifolds with diagonal metrics in local orthogonal coordinates

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Abstract: This paper develops a unified framework for infinitesimal geometric symmetries and Ricci-type structures on two-dimensional Riemannian manifolds represented locally in orthogonal coordinates by a diagonal metric. Using a local orthonormal frame adapted to the metric coefficients, we derive explicit expressions for the Riemannian curvature tensor and the Ricci tensor in terms of the associated structure functions. Since, in dimension two, the Ricci tensor satisfies the pointwise identity $\text{Ric} = Kg$, the symmetry conditions considered can be reduced to systems of differential equations involving the Gaussian curvature, the metric coefficients, and the components of a vector field $V = V^1 E_1 + V^2 E_2$. We obtain necessary and sufficient conditions for conformal vector fields, ρ -Ricci vector fields, Ricci collineations, conformal Ricci collineations, and Ricci solitons and analyze the resulting systems under natural separability and cross-dependence assumptions. This leads to explicit solution families, rigidity results, and curvature restrictions for the corresponding symmetry classes. In the flat cases arising from the separability assumptions, the underlying metric is locally Euclidean, whereas, on connected open regions where the Gaussian curvature is nowhere zero, the corresponding symmetry equations retain an essential curvature dependence. In particular, we characterize the separable case, determine the conditions under which the orthonormal frame fields generate Ricci-type symmetries, obtain explicit formulas for conformal Ricci collineations, and derive necessary and sufficient conditions for Ricci solitons. For ρ -Ricci vector fields, the classification of the diagonal metrics for which the coordinate vector fields are ρ -Ricci is carried out under the additional assumption that ρ is a prescribed nonzero constant, yielding explicit differential constraints on the metric coefficients. Several examples are provided to illustrate the existence of non-Killing conformal fields and Ricci soliton structures. The results are local in nature and apply within the orthogonal coordinate setting and the coordinate-dependence assumptions considered in this paper.

Keywords: two-dimensional Riemannian manifold; diagonal metric; conformal vector field; ρ -Ricci vector fields; Riemannian curvature tensor; Ricci tensor

1. Introduction

The study of infinitesimal geometric symmetries plays a central role in differential geometry and geometric analysis, as it provides a systematic framework for understanding metric structures, curvature behavior, and invariant tensor fields on smooth manifolds. In particular, vector fields generating infinitesimal transformations of geometric data, such as the metric and curvature tensors, are closely related to conservation laws and geometric evolution processes arising in both Riemannian and pseudo-Riemannian geometry.

A vector field V on a Riemannian manifold (M, g) is called a conformal vector field if it satisfies the conformal Killing equation $\mathcal{L}_V g = 2\phi g$, where $\mathcal{L}_V$ denotes the Lie derivative along V and ϕ is a smooth function. Conformal vector fields generalize Killing vector fields, obtained when $\phi = 0$, and homothetic vector fields, for which ϕ is constant. In dimension two, conformal geometry exhibits a particularly flexible structure because of the close interaction between conformal transformations and scalar curvature invariants.

The systematic study of infinitesimal transformations has been developed through the works of Weyl [1], Yano [2], Obata [3], and Kühnel and Rademacher [4], among others. In particular, conformal vector fields and their rigidity properties have been extensively investigated in connection with global geometric constraints and curvature conditions.

In addition to metric symmetries, curvature-driven infinitesimal transformations have attracted considerable attention. A class of vector fields introduced by Alohali and Deshmukh [5], known as ρ -Ricci vector fields, is defined by

$$\frac{1}{2} \mathcal{L}_V g = \rho \operatorname{Ric},$$

where ρ is a smooth function and Ric denotes the Ricci tensor. This structure interpolates between Killing-type symmetries and curvature-controlled deformations of the metric.

A vector field V is called a Ricci collineation if it preserves the Ricci tensor under its local flow, that is,

$$\mathcal{L}_V \operatorname{Ric} = 0;$$

see [6, 7]. Throughout this paper, we use the term *conformal Ricci collineation* for a vector field satisfying

$$\mathcal{L}_V \operatorname{Ric} = 2\lambda g,$$

where λ is a smooth function on M .

Ricci solitons, introduced by Hamilton in the context of the Ricci flow [8], are defined by

$$\mathcal{L}_V g + 2 \operatorname{Ric} = 2\lambda g,$$

where $\lambda \in \mathbb{R}$ is the soliton constant. They represent self-similar solutions of the Ricci flow equation and play an important role in the study of singularity formation and long-time geometric evolution.

In two-dimensional Riemannian geometry, the Ricci tensor is pointwise proportional to the metric through the Gaussian curvature:

$$\operatorname{Ric} = Kg.$$

This identity allows the curvature-based symmetry conditions considered here to be expressed in terms of the scalar function K . Nevertheless, the resulting differential systems exhibit different analytic

and geometric behavior according to the symmetry class under consideration and the coordinate-dependence assumptions imposed on the metric coefficients and the components of the vector field.

The identity $\text{Ric} = Kg$ also separates the subsequent analysis into two principal geometric regimes. On a coordinate neighborhood where $K \equiv 0$, the metric is flat and hence locally isometric to an open subset of the Euclidean plane. In this regime, $\text{Ric} = 0$, and the curvature-driven symmetry equations degenerate in different ways. In particular, the ρ -Ricci equation reduces to the Killing equation; every smooth vector field is a Ricci collineation and, with $\lambda = 0$, also a conformal Ricci collineation; and the Ricci soliton equation reduces to the homothetic vector field equation. By contrast, on a region where K is nowhere zero, the Gaussian curvature enters essentially into the corresponding differential systems, and division by K is legitimate. This distinction between flat regions and regions of nonvanishing Gaussian curvature is maintained throughout the paper. In particular, the separability assumption $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$ necessarily yields a flat and hence locally Euclidean metric, which explains the simpler form of the corresponding symmetry equations.

This distinction is especially relevant for ρ -Ricci vector fields. In dimension two, every ρ -Ricci vector field is automatically conformal, with conformal factor $\phi = \rho K$. If ρ is allowed to vary freely as a smooth function, this relation is purely algebraic on a region where K is nowhere zero, since, for a given conformal vector field with conformal factor ϕ , one may set $\rho = \frac{\phi}{K}$. In that case, the ρ -Ricci condition imposes no additional restriction beyond conformality. Accordingly, to obtain a genuinely curvature-dependent result, in Theorem 2 we classify the diagonal metrics for which the coordinate vector field $\partial/\partial x^j$ is a ρ -Ricci vector field under the assumption that $\rho = \rho_0 \neq 0$ is a prescribed constant. This restriction is specific to that classification and is not required in the preceding flat separable cases, where $K \equiv 0$.

It is well known that every two-dimensional Riemannian metric admits local isothermal coordinates, in which the metric can be written in the conformally flat form $g = e^{2u}((dx^1)^2 + (dx^2)^2)$. Nevertheless, in the present study, we deliberately retain the more general orthogonal representation involving two independent positive functions f_1 and f_2 . In the notation used here, adopting isothermal coordinates would impose $f_1 = f_2$, so that the structure functions I_{12} and I_{21} would no longer arise from independent metric coefficients but from different directional derivatives of the same function. Moreover, under some of the cross-dependence assumptions considered in this paper, such as $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$, the additional condition $f_1 = f_2$ would substantially restrict the corresponding class and, on a connected coordinate neighborhood, force the common metric coefficient to be locally constant. Retaining two independent functions therefore makes it possible to track the separate coordinate dependence of the metric coefficients and to analyze the separability and cross-dependence assumptions underlying the classification results. Thus, the use of local orthogonal coordinates is a structural choice rather than a restriction on the local existence of such coordinates. The resulting classifications should consequently be understood as local, coordinate-dependent results rather than as global or topological classifications of Riemannian surfaces; see, for comparison, [9] for nonexistence phenomena concerning orthogonal coordinates in higher dimensions and [10] for the use of isothermic coordinate systems in a Lorentzian setting.

Related investigations of conformal geometric solitons, Ricci-type symmetries, Ricci solitons associated with diagonal metrics, and soliton structures on tangent and cotangent bundles can be found in [11–14]. Among the studies most closely related to the present work, Blaga [15] investigates Killing vector fields for a family of diagonal Riemannian metrics on $\mathbb{R}^2$ under specific restrictions on the

Lamé coefficients. In particular, Theorem 2.10 of [15] gives an explicit classification under the cross-dependence assumptions $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$ and $V^1 = V^1(x^2)$, $V^2 = V^2(x^1)$. The formulas obtained there are recovered from Theorem 1 below by setting $\phi = 0$, whereas the present result determines the additional compatibility conditions required in the general conformal case. The analysis is then extended to ρ -Ricci vector fields, Ricci collineations, conformal Ricci collineations, and Ricci solitons. Our purpose is not to provide a coordinate-independent classification of all two-dimensional Riemannian metrics but to derive and analyze the governing systems under explicitly stated coordinate-dependence assumptions.

Against this background, we study the differential systems governing infinitesimal symmetries on a two-dimensional Riemannian manifold (M, g) represented locally in an orthogonal coordinate neighborhood with coordinates (x^1, x^2) by the diagonal metric

$$g = \frac{1}{f_1^2(x^1, x^2)} dx^1 \otimes dx^1 + \frac{1}{f_2^2(x^1, x^2)} dx^2 \otimes dx^2,$$

where f_1 and f_2 are smooth positive functions. For notational convenience, we identify this coordinate neighborhood with an open subset of $\mathbb{R}^2$ and, with a slight abuse of notation, denote the resulting local model by $(\mathbb{R}^2, g)$. Accordingly, all classification results obtained below are local and are to be understood within such a coordinate neighborhood of (M, g) . We employ the adapted local orthonormal frame $\{E_1, E_2\}$ and express a vector field as $V = V^1 E_1 + V^2 E_2$.

The main objective of this paper is to derive necessary and sufficient conditions for conformal vector fields, ρ -Ricci vector fields, Ricci collineations, conformal Ricci collineations, and Ricci solitons in this setting. We obtain explicit systems of partial differential equations governing these symmetry classes and analyze their dependence on the coordinate structure of the metric coefficients. Under the stated separability and cross-dependence assumptions on the metric functions and, where required, on the vector-field components, these systems yield explicit solution families, rigidity statements, and curvature restrictions.

The paper is organized as follows. Section 2 introduces the geometric preliminaries and derives the structure equations associated with the adapted orthonormal frame. Section 3 contains the main results and is divided into subsections devoted to conformal vector fields, ρ -Ricci vector fields, Ricci collineations, conformal Ricci collineations, and Ricci solitons. Finally, Section 4 summarizes the principal conclusions and discusses the scope of the results.

2. Preliminaries

In this section, we present the geometric framework and the structural equations used throughout the paper. As explained in the Introduction, let (M, g) be a two-dimensional Riemannian manifold and let $U \subset M$ be an orthogonal coordinate neighborhood with local coordinates (x^1, x^2) , in which the metric takes the diagonal form

$$g = \frac{1}{f_1^2(x^1, x^2)} dx^1 \otimes dx^1 + \frac{1}{f_2^2(x^1, x^2)} dx^2 \otimes dx^2,$$

where $f_1, f_2 \in C^\infty(U)$ are positive. As noted above, we identify U with an open subset of $\mathbb{R}^2$ and denote the resulting local model by $(\mathbb{R}^2, g)$.

The associated local orthonormal frame $\{E_1, E_2\}$ is defined by

$$E_1 = f_1 \frac{\partial}{\partial x^1}, \quad E_2 = f_2 \frac{\partial}{\partial x^2}. \quad (2.1)$$

It satisfies $g(E_1, E_1) = 1$, $g(E_2, E_2) = 1$, $g(E_1, E_2) = 0$.

Remark 1. Throughout this paper, (x^1, x^2) denotes a local orthogonal coordinate system, and all subsequent computations are carried out within the corresponding coordinate neighborhood. Accordingly, the results are local and apply to any two-dimensional Riemannian manifold on each orthogonal coordinate neighborhood where the stated coordinate-dependence assumptions are satisfied. Compactness and other global topological properties of the underlying manifold do not affect the validity of these local formulas. If M has a nonempty boundary, the results apply directly on interior coordinate neighborhoods and, at a boundary point, whenever a compatible local orthogonal boundary chart is available. No global orthogonal coordinate system on the entire manifold is required, and no global topological classification is intended.

We introduce the functions

$$I_{12} := \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} = E_2(\ln f_1), \quad I_{21} := \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} = E_1(\ln f_2),$$

which determine the structure coefficients of the adapted frame. The Lie bracket of the frame fields is given by

$$[E_1, E_2] = -I_{12}E_1 + I_{21}E_2. \quad (2.2)$$

The relation $[E_2, E_1] = -[E_1, E_2]$ follows immediately.

The Levi-Civita connection ∇ is uniquely determined by the Koszul formula (see [16]):

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned}$$

Using (2.1) and (2.2), we obtain

$$\begin{aligned} \nabla_{E_1} E_1 &= I_{12}E_2, & \nabla_{E_1} E_2 &= -I_{12}E_1, \\ \nabla_{E_2} E_1 &= -I_{21}E_2, & \nabla_{E_2} E_2 &= I_{21}E_1. \end{aligned}$$

We adopt the following convention for the Riemannian curvature tensor:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

A direct computation gives

$$R(E_1, E_2)E_1 = (-E_1(I_{21}) - E_2(I_{12}) + I_{12}^2 + I_{21}^2)E_2, \quad (2.3)$$

$$R(E_1, E_2)E_2 = (E_1(I_{21}) + E_2(I_{12}) - I_{12}^2 - I_{21}^2)E_1. \quad (2.4)$$

The Gaussian curvature K is therefore given by

$$\begin{aligned} K &= g(R(E_1, E_2)E_2, E_1) \\ &= E_1(I_{21}) + E_2(I_{12}) - I_{12}^2 - I_{21}^2. \end{aligned} \quad (2.5)$$

Equivalently, $R(E_1, E_2)E_1 = -KE_2$, $R(E_1, E_2)E_2 = KE_1$.

The Ricci tensor is obtained by tracing the Riemannian curvature tensor:

$$\text{Ric}(X, Y) = \sum_{i=1}^2 g(R(E_i, X)Y, E_i).$$

In dimension two, the Ricci tensor is pointwise proportional to the metric. More precisely,

$$\text{Ric} = Kg. \quad (2.6)$$

Consequently, with respect to the orthonormal frame $\{E_1, E_2\}$,

$$\text{Ric}(E_1, E_1) = K, \quad \text{Ric}(E_2, E_2) = K, \quad \text{Ric}(E_1, E_2) = 0. \quad (2.7)$$

Remark 2. Equations (2.3) and (2.4) show that, in dimension two, the Riemannian curvature tensor is completely determined by the Gaussian curvature K and the metric g through

$$R(X, Y)Z = K(g(Y, Z)X - g(X, Z)Y).$$

Consequently, on a coordinate neighborhood where $K \equiv 0$, the curvature tensor vanishes identically. Thus, $(U, g|_U)$ is flat and is locally isometric to an open subset of the Euclidean plane; see, for example, [16]. In this case, $\text{Ric} = Kg = 0$. By contrast, whenever division by K is used in the subsequent sections, the corresponding argument is understood to take place on a connected open region where K is nowhere zero. In particular, the separability condition $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$ implies $I_{12} = I_{21} = 0$ and hence, by (2.5), $K \equiv 0$. Moreover, the local change of variables

$$u = \int \frac{dx^1}{f_1(x^1)}, \quad v = \int \frac{dx^2}{f_2(x^2)}$$

transforms the metric into $g = du^2 + dv^2$. Therefore, the separable case considered in this paper constitutes a distinguished flat and locally Euclidean subclass. The converse is not asserted: A flat diagonal metric need not be separable with respect to the original orthogonal coordinates.

Finally, when a metric coefficient f_i or a vector-field component V^i depends on a single coordinate variable, we use the notation $f_i(x^j)$ or $V^i(x^j)$ to indicate this dependence. Equations (2.5)–(2.7) provide the basic curvature identities used throughout the subsequent sections.

3. Main results

In this section, we provide a systematic characterization of several classes of infinitesimal geometric symmetries on $(\mathbb{R}^2, g)$, the local coordinate model introduced in Section 2, endowed with a diagonal metric. Working with the orthonormal frame $\{E_1, E_2\}$ associated with the metric, we derive necessary and sufficient differential conditions for vector fields $V = V^1E_1 + V^2E_2$ to generate conformal, curvature-related, and soliton-type symmetries.

In dimension two, the Ricci tensor is pointwise proportional to the metric via the Gaussian curvature, that is, $\text{Ric} = Kg$. This structural property allows the symmetry equations considered here,

including those for conformal Killing fields, ρ -Ricci vector fields, Ricci collineations, conformal Ricci collineations, and Ricci solitons, to be expressed within a unified system of partial differential equations involving the metric coefficients and the components of V .

We first derive the full system of equations characterizing conformal vector fields in the orthonormal frame formulation. This system serves as the foundational structure for all subsequent symmetry analyses. In particular, we determine how the behavior of the metric coefficients f_1 and f_2 influences the existence of non-trivial conformal vector fields and identify precise separability conditions under which the orthonormal frame fields become Killing vector fields.

Next, we analyze ρ -Ricci vector fields defined by [5],

$$\frac{1}{2} \mathcal{L}_V g = \rho \text{ Ric},$$

and show that in dimension two, this condition reduces to a coupling between the conformal factor and the Gaussian curvature. This yields explicit integrability conditions for the components of V , leading to solvable differential systems under suitable assumptions on the metric coefficients.

We then consider Ricci collineations and conformal Ricci collineations. By expressing $\mathcal{L}_V \text{ Ric}$ in terms of the Gaussian curvature and its derivatives, we obtain necessary and sufficient conditions for the existence of such vector fields. In particular, we describe the degeneration of the symmetry equations in the flat regime and identify non-trivial solution classes under additional structural constraints on the metric (see, e.g., [7, 17]).

Finally, we study Ricci solitons defined by

$$\mathcal{L}_V g + 2 \text{ Ric} = 2\lambda g$$

and derive the corresponding system of differential equations in the orthonormal frame. We show that the soliton condition imposes strong restrictions on the metric coefficients, leading to explicit families for the components of the potential vector field together with differential restrictions on the metric coefficients.

Together, these results provide a unified framework for the classification of several curvature-related symmetry structures in two-dimensional diagonal Riemannian geometry.

3.1. Conformal vector fields for diagonal metrics in local orthogonal coordinates

Let (M, g) be a two-dimensional Riemannian manifold and let (x^1, x^2) be local orthogonal coordinates on a coordinate neighborhood $U \subset M$ in which the metric takes the diagonal form

$$g = \frac{1}{f_1^2(x^1, x^2)} dx^1 \otimes dx^1 + \frac{1}{f_2^2(x^1, x^2)} dx^2 \otimes dx^2, \quad (3.1)$$

where f_1 and f_2 are positive smooth functions on U . Let $\{E_1, E_2\}$ be the local orthonormal frame associated with the metric (3.1), given by

$$E_1 = f_1 \frac{\partial}{\partial x^1}, \quad E_2 = f_2 \frac{\partial}{\partial x^2}.$$

By definition, since $\{E_1, E_2\}$ constitutes an orthonormal basis, we have $g(E_1, E_1) = 1$, $g(E_2, E_2) = 1$, and $g(E_1, E_2) = 0$.

A vector field $V = V^1 E_1 + V^2 E_2$ on U is said to be a conformal vector field if it satisfies the conformal Killing equation

$$(\mathcal{L}_V g)(X, Y) = 2\phi g(X, Y) \quad (3.2)$$

for any vector fields X, Y on U , where $\mathcal{L}_V$ denotes the Lie derivative along V and ϕ is a smooth function on U , called the conformal factor. If $\phi = 0$, then V is a Killing vector field.

As established in Section 2, the Ricci tensor satisfies the pointwise identity $\text{Ric} = Kg$ in dimension two. Consequently, the conformal Killing system derived below provides a useful framework for the subsequent analysis of curvature-related infinitesimal symmetries and Ricci solitons.

By evaluating the conformal Killing equation (3.2) on the pairs (E_1, E_1) , (E_2, E_2) , and (E_1, E_2) , we obtain the following system of differential equations characterizing conformal vector fields on $(U, g|_U)$.

Lemma 1. *Let $(\mathbb{R}^2, g)$ be as in (3.1). A vector field $V = V^1 E_1 + V^2 E_2$ is conformal with conformal factor ϕ if and only if the component functions V^1, V^2 satisfy the system:*

$$(i) \quad f_1 \frac{\partial V^1}{\partial x^1} - \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^2 = \phi, \quad (3.3)$$

$$(ii) \quad f_2 \frac{\partial V^2}{\partial x^2} - \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^1 = \phi, \quad (3.4)$$

$$(iii) \quad f_1 \frac{\partial V^2}{\partial x^1} + f_2 \frac{\partial V^1}{\partial x^2} + \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^1 + \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^2 = 0. \quad (3.5)$$

Proof. The Lie derivative of the metric tensor is given by

$$(\mathcal{L}_V g)(E_i, E_j) = V(g(E_i, E_j)) - g([V, E_i], E_j) - g(E_i, [V, E_j]).$$

It is also useful to recall the equivalent expression

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X),$$

which provides an alternative way of computing the components of $\mathcal{L}_V g$. Let $V = V^1 E_1 + V^2 E_2$ and recall that $\{E_1, E_2\}$ is an orthonormal frame. The conformal Killing equation

$$\mathcal{L}_V g = 2\phi g$$

is now evaluated componentwise. For (E_1, E_1) , since $g(E_1, E_1) = 1$, we obtain

$$(\mathcal{L}_V g)(E_1, E_1) = 2\phi,$$

which yields Eq (3.3). Similarly, for (E_2, E_2) with $g(E_2, E_2) = 1$, we obtain

$$(\mathcal{L}_V g)(E_2, E_2) = 2\phi,$$

which gives Eq (3.4). Finally, for the mixed component (E_1, E_2) , since $g(E_1, E_2) = 0$, we have

$$(\mathcal{L}_V g)(E_1, E_2) = 0.$$

A direct computation of the Lie derivative in this case yields Eq (3.5). Hence, the system (3.3)–(3.5) follows. $\square$

Remark 3. *The class of conformal vector fields is, in general, considerably larger than the class of Killing vector fields, even on a flat two-dimensional plane ($f_1 = f_2 = 1$). This does not, by itself, determine when the adapted orthonormal frame fields E_1 and E_2 are themselves conformal for a general choice of f_1, f_2 ; the following proposition settles this question precisely and shows that the resulting condition forces E_1, E_2 to be Killing vector fields.*

To establish a mathematically rigorous link between the basis fields and the conformal framework, we introduce the following proposition, which characterizes the corresponding geometric constraints on the metric functions.

Before Proposition 1, we emphasize that the results presented above provide only partial information concerning conformal vector fields, since the system given in Lemma 1 has not been solved in full generality. In particular, suppose that $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$. By introducing the change of variables

$$\bar{x} = \int \frac{dx}{f_1(x)}, \quad \bar{y} = \int \frac{dy}{f_2(y)},$$

the metric takes the form

$$g = d\bar{x}^2 + d\bar{y}^2,$$

which is the Euclidean metric. Moreover, under this coordinate transformation, the orthonormal frame fields become

$$E_1 = f_1 \frac{\partial}{\partial x} = f_1 \frac{\partial \bar{x}}{\partial x} \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial \bar{x}}, \quad E_2 = f_2 \frac{\partial}{\partial y} = f_2 \frac{\partial \bar{y}}{\partial y} \frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}}.$$

Thus, in this case, E_1 and E_2 are clearly Killing vector fields on the resulting Euclidean coordinate domain. Consequently, the conformality of E_1 and E_2 on $(\mathbb{R}^2, g)$ necessarily implies that they are Killing vector fields, and the corresponding rigidity result is established in the following proposition.

Proposition 1. *The orthonormal basis vector fields E_1 and E_2 are conformal vector fields on $(\mathbb{R}^2, g)$ if and only if the metric coefficients are separable, meaning $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$. In this specific configuration, the conformal factor identically vanishes ($\phi = 0$), implying that E_1 and E_2 are inherently Killing vector fields.*

Proof. Suppose that the basis vector fields E_1 and E_2 are conformal. First, choosing E_1 as V , we readily find its components as $V^1 = 1$ and $V^2 = 0$. Substituting these components into Lemma 1, condition (3.3) directly yields $\phi = 0$. Since $\phi = 0$, E_1 reduces to a Killing vector field. Next, substituting $V^1 = 1$, $V^2 = 0$, and $\phi = 0$ into condition (3.4) of Lemma 1, we obtain

$$-\frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1}(1) = 0 \implies \frac{\partial f_2}{\partial x^1} = 0,$$

which proves that f_2 depends solely on x^2 , i.e., $f_2 = f_2(x^2)$. Lastly, verifying condition (3.5) under the same substitutions leads to $\frac{\partial f_1}{\partial x^2} = 0$, meaning $f_1 = f_1(x^1)$. An analogous application for E_2 ($V^1 = 0$, $V^2 = 1$) via Lemma 1 symmetrically yields $\phi = 0$, $\frac{\partial f_1}{\partial x^2} = 0$, and $\frac{\partial f_2}{\partial x^1} = 0$. Conversely, assuming $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$, direct back-substitution into Lemma 1 reveals that the system is identically satisfied for both E_1 and E_2 with $\phi = 0$. This completes the proof. $\square$

Proposition 2. Consider the vector field $V = V^1 E_1 + V^2 E_2$ on $\mathbb{R}^2$ and the diagonal metric (3.1). For a fixed $j \in \{1, 2\}$, assume that $f_i = f_i(x^j)$ for $i = 1, 2$. Then the coordinate vector field $\frac{\partial}{\partial x^j}$ is a conformal vector field on $(\mathbb{R}^2, g)$ if and only if there exists a non-zero scaling constant $c_j \in \mathbb{R}$ such that

$$f_1(x^j) = c_j f_2(x^j).$$

Moreover, since f_1 and f_2 are positive, the constant c_j must be positive.

Proof. We can write $\frac{\partial}{\partial x^1} = V^1 E_1 + V^2 E_2 = V^1 f_1 \frac{\partial}{\partial x^1} + V^2 f_2 \frac{\partial}{\partial x^2}$. Since $\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\}$ are linearly independent, it follows that $V^1 = \frac{1}{f_1}$ and $V^2 = 0$. Substituting these values into Lemma 1, the following equations are obtained:

$$\begin{aligned} f_1 \frac{\partial}{\partial x^1} \left(\frac{1}{f_1} \right) &= \phi \implies -\frac{f_1'(x^1)}{f_1(x^1)} = \phi, \\ -\frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} \frac{1}{f_1} &= \phi \implies -\frac{f_2'(x^1)}{f_2(x^1)} = \phi, \\ f_2 \frac{\partial}{\partial x^2} \left(\frac{1}{f_1} \right) + \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} \frac{1}{f_1} &= 0. \end{aligned}$$

Since the functions f_1 and f_2 depend only on the variable x^1 by hypothesis, the third equation is automatically satisfied. Combining the remaining two relations yields

$$\frac{f_1'(x^1)}{f_1(x^1)} = \frac{f_2'(x^1)}{f_2(x^1)} \implies \partial_1(\ln f_1) = \partial_1(\ln f_2).$$

Integrating both sides of the equation leads to $\ln f_1 = \ln f_2 + \ln c_1$, which yields $f_1(x^1) = c_1 f_2(x^1)$ for a non-zero constant c_1 . From the above equalities, it follows that there exists a function ϕ satisfying

$$\mathcal{L}_{\frac{\partial}{\partial x^1}} g = 2\phi g.$$

Hence, the vector field $\frac{\partial}{\partial x^1}$ is a conformal vector field. Similarly using an argument analogous to the one above, one can show that $\frac{\partial}{\partial x^2}$ is a conformal vector field with $f_1(x^2) = c_2 f_2(x^2)$. This completes the proof. $\square$

Proposition 3. Consider the vector field $V = V^1 E_1 + V^2 E_2$ and the diagonal metric (3.1) on $\mathbb{R}^2$. If $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$, then V is a conformal vector field on $(\mathbb{R}^2, g)$ if and only if

$$\begin{aligned} f_1 \frac{\partial V^1}{\partial x^1} &= f_2 \frac{\partial V^2}{\partial x^2} = \phi, \\ f_1 \frac{\partial V^2}{\partial x^1} + f_2 \frac{\partial V^1}{\partial x^2} &= 0. \end{aligned}$$

Proof. Substituting the identities

$$\frac{\partial f_1}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial f_2}{\partial x^1} = 0$$

into Lemma 1, the statement of the proposition follows immediately. $\square$

Under the assumptions $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$, the metric can be transformed locally into the Euclidean metric by a suitable change of variables. Consequently, Proposition 3 may be viewed as a characterization of conformal vector fields on a locally Euclidean coordinate domain expressed in the adapted orthonormal frame.

Proposition 4. Consider the vector field $V = V^1 E_1 + V^2 E_2$ and the diagonal metric (3.1) on $\mathbb{R}^2$. Assume that $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$ and $V^1 = V^1(x^1)$, $V^2 = V^2(x^2)$. Then V is a conformal vector field on $(\mathbb{R}^2, g)$ if and only if

$$f_1(x^1) \frac{dV^1}{dx^1} = f_2(x^2) \frac{dV^2}{dx^2} = c,$$

where $c \in \mathbb{R}$ is a constant. In particular, the conformal factor satisfies $\phi = c$.

Proof. Since $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$, Proposition 3 implies that V is a conformal vector field if and only if

$$\begin{aligned} f_1(x^1) \frac{dV^1}{dx^1} &= f_2(x^2) \frac{dV^2}{dx^2} = \phi, \\ f_1(x^1) \frac{\partial V^2}{\partial x^1} + f_2(x^2) \frac{\partial V^1}{\partial x^2} &= 0. \end{aligned} \quad (3.6)$$

Because $V^1 = V^1(x^1)$, $V^2 = V^2(x^2)$, Eq (3.6) is satisfied identically. Moreover, $f_1(x^1) \frac{dV^1}{dx^1}$ depends only on x^1 , whereas $f_2(x^2) \frac{dV^2}{dx^2}$ depends only on x^2 . Since these expressions are equal, both must be equal to the same constant $c \in \mathbb{R}$. Consequently,

$$f_1(x^1) \frac{dV^1}{dx^1} = f_2(x^2) \frac{dV^2}{dx^2} = c,$$

and therefore the conformal factor is constant, $\phi = c$.

Conversely, if the above equalities hold, then the equations of Proposition 3 are satisfied, and hence V is a conformal vector field. $\square$

Example 1. Consider the diagonal metric

$$g = \frac{1}{(x^1)^2} dx^1 \otimes dx^1 + \frac{1}{(x^2)^2} dx^2 \otimes dx^2$$

on the domain $D = \{(x^1, x^2) \in \mathbb{R}^2 : x^1 > 0, x^2 > 0\}$. Thus, $f_1(x^1) = x^1$, $f_2(x^2) = x^2$, and the associated orthonormal frame is $E_1 = x^1 \frac{\partial}{\partial x^1}$, $E_2 = x^2 \frac{\partial}{\partial x^2}$. Let $V = V^1 E_1 + V^2 E_2$, where $V^1(x^1) = \lambda \ln x^1 + c_1$, $V^2(x^2) = \lambda \ln x^2 + c_2$, with $\lambda, c_1, c_2 \in \mathbb{R}$.

Since $E_1(V^1) = x^1 \frac{\partial}{\partial x^1} (\lambda \ln x^1 + c_1) = \lambda$ and $E_2(V^2) = x^2 \frac{\partial}{\partial x^2} (\lambda \ln x^2 + c_2) = \lambda$, while $E_1(V^2) = 0$, $E_2(V^1) = 0$, the equations of Proposition 4 are satisfied with conformal factor $\phi = \lambda$. Consequently, $\mathcal{L}_V g = 2\lambda g$.

Hence V is a homothetic vector field. In particular, if $\lambda \neq 0$, then V provides a non-Killing conformal vector field on (D, g) .

Proposition 5. Let $V = V^1 E_1 + V^2 E_2$ be a vector field on $\mathbb{R}^2$ endowed with the diagonal metric (3.1). Assume that $f_i = f_i(x^i)$, $V^i = V^i(x^i)$, $i \in \{1, 2\}$. Then V is a conformal vector field if and only if there exist constants $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$V^1(x^1) = \frac{c_1}{f_2(x^1)},$$

$$V^2(x^1) = \frac{c_2}{f_2(x^1)},$$

and

$$V^1(x^1) = \int \frac{\phi(x^1)}{f_1(x^1)} dx^1 + c_3.$$

Moreover, $V^1(x^1)V^2(x^1) = \frac{c_1c_2}{f_2(x^1)^2}$, and the conformal factor is given by $\phi(x^1) = -\frac{c_1f_1(x^1)f_2'(x^1)}{f_2(x^1)^2}$.

Proof. Substituting $\frac{\partial f_1}{\partial x^2} = \frac{\partial f_2}{\partial x^2} = \frac{\partial V^1}{\partial x^2} = \frac{\partial V^2}{\partial x^2} = 0$ into Lemma 1, we obtain

$$f_1(x^1)(V^1(x^1))' = \phi, \quad (3.7)$$

$$-\frac{f_1(x^1)}{f_2(x^1)} f_2'(x^1)V^1(x^1) = \phi, \quad (3.8)$$

$$(V^2(x^1))' + \frac{f_2'(x^1)}{f_2(x^1)} V^2(x^1) = 0. \quad (3.9)$$

Comparing (3.7) and (3.8), we obtain

$$f_2(x^1)(V^1(x^1))' + f_2'(x^1)V^1(x^1) = 0,$$

or equivalently, $(f_2(x^1)V^1(x^1))' = 0$. Hence $f_2(x^1)V^1(x^1) = c_1$, which yields $V^1(x^1) = \frac{c_1}{f_2(x^1)}$. Similarly, Eq (3.9) gives $(f_2(x^1)V^2(x^1))' = 0$, and therefore $V^2(x^1) = \frac{c_2}{f_2(x^1)}$. Consequently, $V^1(x^1)V^2(x^1) = \frac{c_1c_2}{f_2(x^1)^2}$. Integrating (3.7), we obtain

$$V^1(x^1) = \int \frac{\phi(x^1)}{f_1(x^1)} dx^1 + c_3.$$

Finally, equating the two expressions for $V^1(x^1)$,

$$\frac{c_1}{f_2(x^1)} = \int \frac{\phi(x^1)}{f_1(x^1)} dx^1 + c_3,$$

and differentiating with respect to x^1 , we obtain

$$-\frac{c_1f_2'(x^1)}{f_2(x^1)^2} = \frac{\phi(x^1)}{f_1(x^1)},$$

which implies

$$\phi(x^1) = -\frac{c_1f_1(x^1)f_2'(x^1)}{f_2(x^1)^2}.$$

The converse follows by direct substitution into the equations of Lemma 1. $\square$

Having established the necessary structural properties in the preceding propositions, we are now in a position to state a classification result for conformal vector fields on $(\mathbb{R}^2, g)$ under the following cross-dependence assumptions.

Remark 4. Under the assumptions $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$, the metric can be regarded as a special two-dimensional instance of a doubly warped product type metric. Related geometric properties of doubly warped products can be found in [18, 19].

Theorem 1. Assume that $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$ and $V^1 = V^1(x^2)$, $V^2 = V^2(x^1)$. Then $V = V^1 E_1 + V^2 E_2$ on $\mathbb{R}^2$ with the diagonal metric (3.1) is a conformal vector field if and only if the following conditions are satisfied:

- i. $V^1(x^2) = \frac{kF_1(x^2) + c_1}{f_1(x^2)},$
- ii. $V^2(x^1) = \frac{-kF_2(x^1) + c_2}{f_2(x^1)},$
- iii. $f_2^2(x^1)f_1'(x^2)V^2(x^1) = f_1^2(x^2)f_2'(x^1)V^1(x^2),$
- iv. $\phi(x^1, x^2) = -\frac{f_2(x^1)}{f_1(x^2)}f_1'(x^2)V^2(x^1) = -\frac{f_1(x^2)}{f_2(x^1)}f_2'(x^1)V^1(x^2),$

where $k, c_1, c_2 \in \mathbb{R}$ and $F_i' = f_i^2$ for $i \in \{1, 2\}$. Moreover, on any connected open subset on which $V^1 V^2$ is nowhere zero, condition (iii) is equivalent to

$$\frac{f_1'(x^2)}{f_1^2(x^2)V^1(x^2)} = \frac{f_2'(x^1)}{f_2^2(x^1)V^2(x^1)} = \mu$$

for some constant $\mu \in \mathbb{R}$.

Proof. When the cross-dependency values are substituted into Lemma 1, Eqs (3.3) and (3.4) simplify to $-\frac{f_2}{f_1}f_1'V^2 = \phi$ and $-\frac{f_1}{f_2}f_2'V^1 = \phi$, which directly yields item (iv). Equating these two alternative forms for ϕ gives $\frac{f_2(x^1)}{f_1(x^2)}f_1'(x^2)V^2(x^1) = \frac{f_1(x^2)}{f_2(x^1)}f_2'(x^1)V^1(x^2)$, which, after clearing denominators, is item (iii). On any connected open subset where $V^1 V^2$ is nowhere zero, dividing both sides of (iii) by $f_1^2(x^2)f_2^2(x^1)V^1(x^2)V^2(x^1)$ yields $\frac{f_1'(x^2)}{f_1^2(x^2)V^1(x^2)} = \frac{f_2'(x^1)}{f_2^2(x^1)V^2(x^1)}$. Since the left side depends solely on x^2 and the right side solely on x^1 , both expressions must reduce to a separation constant $\mu \in \mathbb{R}$, giving the stated equivalence. Concurrently, Eq (3.5) reduces to

$$f_1 V^{2'} + f_2 V^{1'} + \frac{f_2}{f_1} f_1' V^1 + \frac{f_1}{f_2} f_2' V^2 = 0. \quad (3.10)$$

Multiplying Eq (3.10) by $f_1 f_2$, we get $f_1^2 (f_2 V^2)' + f_2^2 (f_1 V^1)' = 0$. Dividing both sides by $f_1^2 f_2^2$ separates the variables completely. Because the two sides depend on different independent variables, they must both be equal to a constant, denoted by $k \in \mathbb{R}$:

$$\frac{(f_1 V^1)'}{f_1^2} = -\frac{(f_2 V^2)'}{f_2^2} = k \in \mathbb{R}.$$

Integrating these separated equations reveals the explicit structural forms for V^1 and V^2 as stated in items (i) and (ii), where $F_1'(x^2) = f_1^2(x^2)$, $F_2'(x^1) = f_2^2(x^1)$ and $c_1, c_2 \in \mathbb{R}$. This completes the proof. $\square$

Remark 5. When $\phi = 0$, condition (iv) forces $f_1'(x^2)V^2(x^1) = 0$ and $f_2'(x^1)V^1(x^2) = 0$; since $f_1, f_2 > 0$, this recovers the Killing-field compatibility condition $f_1' V^2 = f_2' V^1 = 0$, and the formulas in parts (i)

and (ii) of Theorem 1 reduce to the Killing-field classification given in Theorem 2.10 of [15]. Parts (iii) and (iv) provide the additional compatibility conditions that arise in the genuinely conformal case $\phi \neq 0$.

Proposition 6. Consider the vector field $V = V^1 E_1 + V^2 E_2$ and the diagonal metric (3.1) on $\mathbb{R}^2$. Assume that $f_1 = f_1(x^2)$, $f_2 = k_1$ with $k_1 > 0$ constant, $V^1 = V^1(x^2)$, and $V^2 = k_2 \in \mathbb{R} \setminus \{0\}$. Then V is a conformal vector field on $(\mathbb{R}^2, g)$ if and only if it is a Killing vector field and the functions satisfy $f_1(x^2) = c_1 > 0$ and $V^1(x^2) = c_2 \in \mathbb{R}$.

Proof. Since $f_2 = k_1$ is constant, we have $f_2'(x^1) = 0$. Therefore, by Theorem 1,

$$\phi = -\frac{f_1(x^2)}{f_2(x^1)} f_2'(x^1) V^1(x^2) = 0.$$

Hence, V is necessarily a Killing vector field. On the other hand, the alternative expression for the conformal factor given in Theorem 1 yields

$$\phi = -\frac{f_2(x^1)}{f_1(x^2)} f_1'(x^2) V^2(x^1).$$

Since $\phi = 0$, $f_1 > 0$, $f_2 = k_1 > 0$, and $V^2 = k_2 \neq 0$, it follows that $f_1'(x^2) = 0$. Consequently, $f_1(x^2) = c_1$ for some positive constant c_1 . Furthermore, Theorem 1 gives

$$\frac{(f_1 V^1)'}{f_1^2} = -\frac{(f_2 V^2)'}{f_2^2} = k.$$

Since both $f_2 = k_1$ and $V^2 = k_2$ are constant, we obtain

$$k = -\frac{(f_2 V^2)'}{f_2^2} = 0.$$

Thus, $(f_1 V^1)' = 0$. Since $f_1 = c_1$ is constant, this implies $(V^1(x^2))' = 0$, and hence $V^1(x^2) = c_2$ for some constant $c_2 \in \mathbb{R}$.

Conversely, if $f_1 = c_1$ and $V^1 = c_2$ are constant, together with $f_2 = k_1$ and $V^2 = k_2$, then the conditions of Theorem 1 are satisfied with $\phi = 0$ and $k = 0$. Therefore, V is a Killing vector field and, in particular, a conformal vector field. $\square$

3.2. ρ -Ricci vector fields for diagonal metrics in local orthogonal coordinates

In differential geometry, infinitesimal symmetries described via Lie derivatives of geometric tensors extend the classical notions of isometries and conformal transformations. Besides Killing and conformal vector fields, one may consider curvature-driven metric symmetries in which the Lie derivative of the metric is controlled by the Ricci tensor.

Let (M, g) be an n -dimensional Riemannian manifold. We say that a vector field V on (M, g) is a ρ -Ricci vector field if it satisfies

$$\frac{1}{2} \mathcal{L}_V g = \rho \operatorname{Ric}, \quad (3.11)$$

where ρ is a smooth function on M and Ric denotes the Ricci tensor of g [5].

We emphasize that condition (3.11) differs from the classical notion of Ricci conformal collineations. In the latter case, the Lie derivative acts directly on the Ricci tensor, whereas in (3.11) the Lie derivative acts on the metric and is algebraically related to the Ricci tensor. Hence, the symmetry considered here is metric-based but curvature-controlled and should not be identified with the standard Ricci conformal symmetry.

Equation (3.11) naturally interpolates between well-known geometric structures. If $\rho = 0$, then $\mathcal{L}_V g = 0$ and V is a Killing vector field. If (M, g) is an Einstein manifold, that is, $\text{Ric} = K g$ for some constant K , then

$$\mathcal{L}_V g = 2K\rho g,$$

so that V becomes a conformal vector field. Therefore, ρ -Ricci vector fields provide an intermediate symmetry class bridging Killing and conformal geometries.

Remark 6. *In dimension two, the Ricci tensor satisfies the pointwise identity*

$$\text{Ric} = Kg,$$

where K denotes the Gaussian curvature. Consequently, the ρ -Ricci vector field equation $\frac{1}{2}\mathcal{L}_V g = \rho \text{Ric}$ reduces to $\frac{1}{2}\mathcal{L}_V g = (\rho K)g$. Thus, every ρ -Ricci vector field is a conformal vector field with conformal factor $\phi = \rho K$. If ρ is allowed to be an arbitrary smooth function, however, the relation $\phi = \rho K$ does not by itself impose an additional rigidity condition. Indeed, on any open set where K is nowhere zero, one may write $\rho = \frac{\phi}{K}$. Hence, in this general setting, the relation $\phi = \rho K$ is an algebraic consequence of the two-dimensional identity $\text{Ric} = Kg$ rather than an additional geometric restriction.

Accordingly, nontrivial restrictions arise when ρ is prescribed or subject to additional assumptions. In the principal coordinate-field classification given in Theorem 2 below, we therefore assume that $\rho = \rho_0 \neq 0$ is a constant. Under this assumption, the conformal factor satisfies $\phi = \rho_0 K$, and the ρ -Ricci vector field equation imposes nontrivial differential restrictions on the metric coefficients and the components of the vector field.

We emphasize that, although the defining equation of a ρ -Ricci vector field is intrinsic, the explicit classification results obtained below are derived under specific coordinate-dependence assumptions on the metric coefficients and, where required, on the components of the vector field. These assumptions enable the resulting partial differential systems to be reduced to explicitly solvable differential equations.

Furthermore, Eq (3.11) arises naturally in the context of a generalized Ricci flow. Consider a one-parameter family of metrics $g(t)$ satisfying

$$\frac{\partial g}{\partial t} = 2\rho \text{Ric}, \quad g(0) = g. \quad (3.12)$$

Suppose that a solution of (3.12) is expressed in the self-similar form $g(t) = \sigma(t)\phi_t^* g$, where ϕ_t is generated by a time-dependent vector field V and $\sigma(t)$ is a smooth scaling function. Assuming $\sigma(0) = 1$, differentiation at $t = 0$ gives $\frac{\partial g}{\partial t}\big|_{t=0} = \sigma'(0)g + \mathcal{L}_V g$. Consequently, Eq (3.12) yields

$$\mathcal{L}_V g + \sigma'(0)g = 2\rho \text{Ric}.$$

Thus, the ρ -Ricci vector field equation

$$\mathcal{L}_V g = 2\rho \text{Ric}$$

is recovered from this self-similar representation provided that $\sigma'(0) = 0$. Therefore, under the normalization $\sigma(0) = 1$ and $\sigma'(0) = 0$, the ρ -Ricci vector field equation is recovered as the infinitesimal condition at $t = 0$ arising from this self-similar formulation.

The components of the Riemannian curvature tensor for the diagonal metric on $(\mathbb{R}^2, g)$, as introduced in Section 2, are calculated using the following expression:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where X, Y, Z are smooth vector fields on the manifold.

Having established the theoretical framework for ρ -Ricci vector fields, we now compute the components of the Riemannian curvature tensor for a diagonal metric, which will be essential for the forthcoming proposition. We provide the potentially nonzero independent components of the Riemannian curvature tensor associated with the diagonal metric g .

Proposition 7. *For the diagonal metric (3.1) on $(\mathbb{R}^2, g)$, the only potentially nonzero independent components of the curvature tensor, up to the standard curvature symmetries, are given by*

$$R(E_1, E_2)E_1 = (-E_1(I_{21}) - E_2(I_{12}) + I_{12}^2 + I_{21}^2)E_2,$$

and

$$R(E_1, E_2)E_2 = (E_1(I_{21}) + E_2(I_{12}) - I_{12}^2 - I_{21}^2)E_1.$$

Proof. The statement is a direct consequence of the curvature Eqs (2.3) and (2.4) established in Section 2. $\square$

This proposition explicitly determines the potentially nonzero independent components of the Riemannian curvature tensor for a diagonal metric, which will be instrumental in the subsequent analysis of ρ -Ricci vector fields on $(\mathbb{R}^2, g)$. The components of the Ricci curvature tensor for the diagonal metric are calculated using the standard relation

$$\text{Ric}(X, Y) = \sum_{k=1}^2 g(R(E_k, X)Y, E_k).$$

In the following proposition, we explicitly state the components of the Ricci curvature tensor with respect to the diagonal metric g .

Proposition 8. *In the context of the Riemannian manifold $(\mathbb{R}^2, g)$ equipped with a diagonal metric, the components of the Ricci curvature tensor Ric are given by*

- i. $\text{Ric}(E_1, E_1) = \text{Ric}(E_2, E_2) = E_1(I_{21}) + E_2(I_{12}) - I_{12}^2 - I_{21}^2 = K,$
- ii. $\text{Ric}(E_1, E_2) = \text{Ric}(E_2, E_1) = 0,$

where K represents the Gaussian curvature of the manifold.

Now, let us calculate the expression

$$(\mathcal{L}_V g)(E_i, E_j) = 2\rho \text{Ric}(E_i, E_j),$$

where $\mathcal{L}_V g$ denotes the Lie derivative along the vector field V with respect to the diagonal metric g and $i, j \in \{1, 2\}$.

$$\begin{aligned}(\mathcal{L}_V g)(E_1, E_1) &= 2\rho \operatorname{Ric}(E_1, E_1), \\ E_1 V^1 - I_{12} V^2 &= \rho(E_1 I_{21} + E_2 I_{12} - I_{12}^2 - I_{21}^2),\end{aligned}$$

which leads to

$$\begin{aligned}f_1 \frac{\partial V^1}{\partial x^1} - \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^2 &= \rho \left[f_1 \frac{\partial}{\partial x^1} \left(\frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} \right) + f_2 \frac{\partial}{\partial x^2} \left(\frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} \right) \right. \\ &\quad \left. - \left(\frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} \right)^2 - \left(\frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} \right)^2 \right] = \rho K.\end{aligned}\quad (3.13)$$

Similarly, for the off-diagonal terms, we have

$$\begin{aligned}(\mathcal{L}_V g)(E_1, E_2) &= (\mathcal{L}_V g)(E_2, E_1) = 2\rho \operatorname{Ric}(E_1, E_2), \\ E_1 V^2 + E_2 V^1 + I_{12} V^1 + I_{21} V^2 &= 0,\end{aligned}$$

which implies

$$f_1 \frac{\partial V^2}{\partial x^1} + f_2 \frac{\partial V^1}{\partial x^2} + \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^1 + \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^2 = 0.$$

For the remaining diagonal term, we have

$$\begin{aligned}(\mathcal{L}_V g)(E_2, E_2) &= 2\rho \operatorname{Ric}(E_2, E_2), \\ E_2 V^2 - I_{21} V^1 &= \rho(E_1 I_{21} + E_2 I_{12} - I_{12}^2 - I_{21}^2),\end{aligned}\quad (3.14)$$

which leads to

$$\begin{aligned}f_2 \frac{\partial V^2}{\partial x^2} - \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^1 &= \rho \left[f_1 \frac{\partial}{\partial x^1} \left(\frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} \right) + f_2 \frac{\partial}{\partial x^2} \left(\frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} \right) \right. \\ &\quad \left. - \left(\frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} \right)^2 - \left(\frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} \right)^2 \right] = \rho K.\end{aligned}\quad (3.15)$$

By equating the right-hand sides of Eqs (3.13) and (3.15), we obtain the following relation:

$$f_1 \frac{\partial V^1}{\partial x^1} - f_2 \frac{\partial V^2}{\partial x^2} - \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^2 + \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^1 = 0.$$

Thus, using the above equalities and the arguments developed so far, we are now in a position to state the following lemma, which will play a fundamental role and be frequently used in the subsequent sections of this paper.

Lemma 2. *Let $V = V^1 E_1 + V^2 E_2$ be a smooth vector field on $\mathbb{R}^2$, where $V^1, V^2 \in C^\infty(\mathbb{R}^2)$, and let*

$$g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2$$

be a diagonal Riemannian metric on $\mathbb{R}^2$, where f_1, f_2 are positive smooth functions. Then V is a ρ -Ricci vector field on $(\mathbb{R}^2, g)$ if and only if the following system of partial differential equations holds:

$$\begin{aligned} \text{i.} \quad & f_1 \frac{\partial V^1}{\partial x^1} - \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^2 = \rho K, \\ \text{ii.} \quad & f_1 \frac{\partial V^2}{\partial x^1} + f_2 \frac{\partial V^1}{\partial x^2} + \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^1 + \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^2 = 0, \\ \text{iii.} \quad & f_1 \frac{\partial V^1}{\partial x^1} - f_2 \frac{\partial V^2}{\partial x^2} - \frac{f_2}{f_1} \frac{\partial f_1}{\partial x^2} V^2 + \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} V^1 = 0. \end{aligned}$$

As an immediate consequence of Lemma 2, we now consider the special case in which the vector field coincides with the orthonormal basis vector fields. This yields the following proposition.

Proposition 9. Let $V = V^1 E_1 + V^2 E_2$ be a vector field on $\mathbb{R}^2$, where V^1 and V^2 are smooth functions, and let

$$g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2$$

be a diagonal metric on $(\mathbb{R}^2, g)$. Then, for $i \in \{1, 2\}$, the orthonormal frame vector field $E_i = f_i \frac{\partial}{\partial x^i}$ is a ρ -Ricci vector field if and only if $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$.

Proof. We first consider the case $V = E_1$. Since $E_1 = f_1 \frac{\partial}{\partial x^1} = V^1 E_1 + V^2 E_2$, we have $V^1 = 1$, $V^2 = 0$. Substituting these expressions into Lemma 2 gives

$$\rho K = 0, \quad I_{21} = 0, \quad I_{12} = 0.$$

Therefore, $\frac{\partial f_1}{\partial x^2} = 0$, $\frac{\partial f_2}{\partial x^1} = 0$, and hence $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$. Similarly, for $V = E_2$, we have

$$V^1 = 0, \quad V^2 = 1.$$

Substitution into Lemma 2 again yields

$$\rho K = 0, \quad I_{12} = 0, \quad I_{21} = 0,$$

and consequently $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$.

Conversely, suppose that $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$. Then $I_{12} = I_{21} = 0$ and hence $K \equiv 0$. Direct substitution of

$$(V^1, V^2) = (1, 0)$$

or

$$(V^1, V^2) = (0, 1)$$

into Lemma 2 shows that both E_1 and E_2 are ρ -Ricci vector fields. Therefore, the assertion follows. $\square$

Having characterized the special case in which the orthonormal frame vector fields E_1 and E_2 are ρ -Ricci vector fields, we now investigate the structure of general ρ -Ricci vector fields under the separability assumption

$$f_1 = f_1(x^1), \quad f_2 = f_2(x^2).$$

The following proposition provides a complete description of the vector field components satisfying the ρ -Ricci equation in this setting.

Proposition 10. Let $V = V^1 E_1 + V^2 E_2$ be a vector field on $\mathbb{R}^2$, where V^1 and V^2 are smooth functions, and let $g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2$ be the diagonal metric on $(\mathbb{R}^2, g)$. Assume that $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$. Then V is a ρ -Ricci vector field on $(\mathbb{R}^2, g)$ if and only if the following conditions hold:

- i. $V^1 = V^1(x^2)$ and $V^2 = V^2(x^1)$,
- ii. $V^1(x^2) = -c \int \frac{1}{f_2(x^2)} dx^2 + c_1$,
- iii. $V^2(x^1) = c \int \frac{1}{f_1(x^1)} dx^1 + c_2$,

where $c, c_1, c_2 \in \mathbb{R}$.

Proof. Substituting the values $\frac{\partial f_1}{\partial x^2} = 0$ and $\frac{\partial f_2}{\partial x^1} = 0$ into Lemma 2, we obtain $I_{12} = I_{21} = 0$, and hence the Gaussian curvature satisfies $K \equiv 0$. Therefore, Lemma 2 reduces to its simplified form in which the curvature term vanishes.

It follows from Lemma 2 (i) that $\frac{\partial V^1}{\partial x^1} = 0$, which implies that V^1 depends only on the variable x^2 . Similarly, Lemma 2 (iii) yields $\frac{\partial V^2}{\partial x^2} = 0$, implying that V^2 depends only on the variable x^1 . Consequently, we obtain

$$V^1 = V^1(x^2), \quad V^2 = V^2(x^1).$$

Substituting these results into Lemma 2 (ii), we conclude that

$$f_1(x^1) \frac{\partial V^2(x^1)}{\partial x^1} = -f_2(x^2) \frac{\partial V^1(x^2)}{\partial x^2}.$$

Since the left-hand side depends only on the variable x^1 and the right-hand side depends only on the variable x^2 , this equality can hold only if both sides are equal to the same constant. Hence, for some constants $c, c_1, c_2 \in \mathbb{R}$, we obtain

$$f_1(x^1) \frac{\partial V^2(x^1)}{\partial x^1} = c \Rightarrow V^2(x^1) = c \int \frac{1}{f_1(x^1)} dx^1 + c_2,$$

and

$$f_2(x^2) \frac{\partial V^1(x^2)}{\partial x^2} = -c \Rightarrow V^1(x^2) = -c \int \frac{1}{f_2(x^2)} dx^2 + c_1.$$

Conversely, it follows from routine computations that the sufficient conditions of the proposition are satisfied, thereby completing the proof. $\square$

As a direct application of the previous proposition, we present the following example illustrating the explicit form of the vector field components for particular choices of the functions f_1 and f_2 .

Example 2. If $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$ are chosen such that

$$\int \frac{1}{f_1(x^1)} dx^1 = \ln |x^1| \quad \text{and} \quad \int \frac{1}{f_2(x^2)} dx^2 = \ln |x^2|,$$

then the components V^1 and V^2 are obtained as follows:

$$V^1(x^2) = -c \ln |x^2| + c_1,$$

and

$$V^2(x^1) = c \ln |x^1| + c_2.$$

Having analyzed the structure of ρ -Ricci vector fields in the preceding propositions and determined the explicit form of the functions under suitable separability assumptions, we are now in a position to present the main classification result of this work. The following theorem provides a complete characterization of the diagonal metrics for which the coordinate vector fields $\frac{\partial}{\partial x^j}$ are ρ -Ricci vector fields.

Theorem 2. *Let*

$$g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2$$

be a diagonal Riemannian metric on $\mathbb{R}^2$, where $f_1, f_2 > 0$ are smooth functions. Fix $j \in \{1, 2\}$ and assume that both metric coefficients depend only on x^j . Moreover, assume that $\rho = \rho_0 \neq 0$ is a constant. Then the coordinate vector field $\frac{\partial}{\partial x^j}$ is a ρ -Ricci vector field if and only if the following conditions hold. For $j = 1$, there exists a positive constant c such that $f_2(x^1) = c f_1(x^1)$. Writing $f = f_1$, one of the following cases occurs:

- (i) *If $K \equiv 0$ on the interval under consideration, then f is constant.*
- (ii) *If K is nowhere zero on the interval under consideration and $c_1 \neq 0$, then*

$$f(x^1) = \sqrt{C e^{2c_1 x^1} - \frac{1}{2\rho_0 c_1}},$$

where $c_1, C \in \mathbb{R}$, $C \neq 0$ and the expression is considered on an interval on which

$$C e^{2c_1 x^1} - \frac{1}{2\rho_0 c_1} > 0.$$

- (iii) *If K is nowhere zero on the interval under consideration and $c_1 = 0$, then*

$$f(x^1) = \sqrt{C + \frac{x^1}{\rho_0}},$$

where $C \in \mathbb{R}$ and the expression is considered on an interval on which

$$C + \frac{x^1}{\rho_0} > 0.$$

For $j = 2$, the analogous result holds with x^1 and x^2 , and f_1 and f_2 interchanged. In particular, there exists a positive constant c such that $f_1(x^2) = c f_2(x^2)$, and, writing $f = f_2$, the same three cases hold with x^1 replaced by x^2 .

Proof. We first consider the case $j = 1$. Since

$$\frac{\partial}{\partial x^1} = V^1 E_1 + V^2 E_2,$$

and

$$E_1 = f_1 \frac{\partial}{\partial x^1}, \quad E_2 = f_2 \frac{\partial}{\partial x^2},$$

the linear independence of $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$ gives

$$V^1 = \frac{1}{f_1}, \quad V^2 = 0.$$

Substituting these expressions into Lemma 2, we obtain

$$f_1 \frac{\partial}{\partial x^1} \left(\frac{1}{f_1} \right) = \rho_0 K, \quad (3.16)$$

and

$$f_1 \frac{\partial}{\partial x^1} \left(\frac{1}{f_1} \right) + \frac{1}{f_2} \frac{\partial f_2}{\partial x^1} = 0. \quad (3.17)$$

Equation (3.17) yields

$$-\frac{f'_1}{f_1} + \frac{f'_2}{f_2} = 0,$$

and hence

$$\frac{d}{dx^1} (\ln f_2) = \frac{d}{dx^1} (\ln f_1).$$

Therefore, $f_2 = c f_1$, where $c > 0$, since f_1 and f_2 are positive. Set $f := f_1$, $f_2 = c f$.

Since both functions depend only on x^1 , we have $I_{12} = 0$ and

$$I_{21} = \frac{f_1}{f_2} \frac{\partial f_2}{\partial x^1} = f'.$$

According to the curvature convention $K = E_1(I_{21}) + E_2(I_{12}) - I_{12}^2 - I_{21}^2$, we obtain $K = f f'' - (f')^2$.

On the other hand, Eq (3.16) becomes

$$-\frac{f'}{f} = \rho_0 K,$$

and therefore

$$-\frac{f'}{f} = \rho_0 (f f'' - (f')^2). \quad (3.18)$$

We first consider the case $K \equiv 0$. Then (3.16) gives $-\frac{f'}{f} = 0$, and hence $f' = 0$. Thus f is constant.

Now assume that K is nowhere zero on the interval under consideration. Put

$$u = \frac{f'}{f}.$$

Since

$$u' = \frac{f f'' - (f')^2}{f^2},$$

Eq (3.18) becomes $-u = \rho_0 f^2 u'$.

Since K is nowhere zero, Eq (3.16) implies $u \neq 0$. Using

$$u' = \frac{du}{df} f' = \frac{du}{df} u f,$$

we obtain

$$-u = \rho_0 f^2 \frac{du}{df} u f.$$

Dividing by $u \neq 0$ gives

$$\frac{du}{df} = -\frac{1}{\rho_0 f^3}.$$

Integrating with respect to f , we obtain

$$u = c_1 + \frac{1}{2\rho_0 f^2}, \quad (3.19)$$

where $c_1 \in \mathbb{R}$ is an integration constant.

Since $u = f'/f$, Eq (3.19) gives

$$\frac{f'}{f} = c_1 + \frac{1}{2\rho_0 f^2},$$

or equivalently, $f' = c_1 f + \frac{1}{2\rho_0 f}$. Setting $y = f^2$, we have $y' = 2ff'$, and hence

$$y' - 2c_1 y = \frac{1}{\rho_0}. \quad (3.20)$$

If $c_1 \neq 0$, solving (3.20) gives

$$y = C e^{2c_1 x^1} - \frac{1}{2\rho_0 c_1},$$

and therefore

$$f(x^1) = \sqrt{C e^{2c_1 x^1} - \frac{1}{2\rho_0 c_1}}.$$

If $C = 0$, then f is constant, which corresponds to the case $K \equiv 0$. Hence, under the assumption that K is nowhere zero, we must have $C \neq 0$.

If $c_1 = 0$, Eq (3.20) reduces to

$$y' = \frac{1}{\rho_0},$$

so that $y = C + \frac{x^1}{\rho_0}$, and therefore

$$f(x^1) = \sqrt{C + \frac{x^1}{\rho_0}}.$$

Since the metric coefficients are positive, these expressions are understood on intervals where the corresponding quantities under the square roots are positive.

Conversely, direct substitution of the above solution families into (3.18), together with $f_2 = c f_1$, shows that the ρ -Ricci equations of Lemma 2 are satisfied. Hence $\frac{\partial}{\partial x^1}$ is a ρ -Ricci vector field.

The case $j = 2$ is completely analogous. In this case, $V^1 = 0$, $V^2 = \frac{1}{f_2}$, and Lemma 2 yields $f_1 = c f_2$ for some $c > 0$. Setting $f = f_2$ and repeating the above argument with x^2 in place of x^1 gives exactly the same three cases. This completes the proof. $\square$

3.3. Ricci collineations for diagonal metrics in local orthogonal coordinates

Ricci collineations constitute a fundamental class of curvature-preserving symmetries in differential geometry. A vector field V on a Riemannian manifold (M, g) is called a Ricci collineation if it satisfies

$$\mathcal{L}_V \text{Ric} = 0. \quad (3.21)$$

These vector fields generalize Killing fields by preserving only the Ricci tensor rather than the full metric, and they encode important information about the curvature structure of the manifold [6, 7].

On a two-dimensional Riemannian manifold, the Ricci tensor satisfies the Einstein identity

$$\text{Ric} = Kg, \quad (3.22)$$

where K denotes the Gaussian curvature. Substituting (3.22) into (3.21) yields the fundamental reduction

$$\mathcal{L}_V(Kg) = (\mathcal{L}_V K)g + K\mathcal{L}_V g = 0.$$

This identity clarifies a crucial geometric mechanism: In dimension two, Ricci collineations are not independent curvature symmetries but coupled curvature–metric deformations. In particular, the system degenerates depending on whether K vanishes or not.

We consider the diagonal metric on $(\mathbb{R}^2, g)$, as introduced in Section 2,

$$g = \frac{1}{f_1^2(x^1, x^2)} dx^1 \otimes dx^1 + \frac{1}{f_2^2(x^1, x^2)} dx^2 \otimes dx^2,$$

and the orthonormal frame $E_i = f_i \partial_{x^i}$, $i = 1, 2$. Let $I_{21} = E_1(\ln f_2)$, $I_{12} = E_2(\ln f_1)$, which encode the cross-directional variation of the metric coefficients.

Lemma 3. *Let $V = V^1 E_1 + V^2 E_2$ be a vector field on $(\mathbb{R}^2, g)$. Then V is a Ricci collineation if and only if*

$$V^1 E_1(K) + V^2 E_2(K) + 2K(E_1(V^1) - I_{12}V^2) = 0, \quad (3.23)$$

$$K(V^1 I_{12} + E_1(V^2) + E_2(V^1) + V^2 I_{21}) = 0, \quad (3.24)$$

and

$$V^1 E_1(K) + V^2 E_2(K) + 2K(E_2(V^2) - V^1 I_{21}) = 0. \quad (3.25)$$

Proof. Since $(\mathbb{R}^2, g)$ is two-dimensional, we have $\text{Ric} = Kg$. Therefore,

$$\mathcal{L}_V \text{Ric} = \mathcal{L}_V(Kg) = V(K)g + K\mathcal{L}_V g.$$

Using the orthonormal frame $\{E_1, E_2\}$, we have

$$(\mathcal{L}_V g)(E_1, E_1) = 2E_1(V^1) - 2I_{12}V^2,$$

$$(\mathcal{L}_V g)(E_2, E_2) = 2E_2(V^2) - 2I_{21}V^1,$$

and

$$(\mathcal{L}_V g)(E_1, E_2) = E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2.$$

Hence, evaluating $\mathcal{L}_V(Kg) = 0$ on the pairs (E_1, E_1) , (E_2, E_2) , and (E_1, E_2) , respectively, gives

$$V(K) + 2K\{E_1(V^1) - I_{12}V^2\} = 0,$$

$$V(K) + 2K\{E_2(V^2) - I_{21}V^1\} = 0,$$

and

$$K\{E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2\} = 0.$$

Thus, the assertion follows. $\square$

We now analyze the structure according to curvature regimes.

Theorem 3. *If $f_i = f_i(x^i)$, then $I_{12} = I_{21} = 0$ and consequently $K \equiv 0$. In this case, every smooth vector field V is a Ricci collineation.*

Proof. If f_i depends only on x^i , then all mixed derivatives vanish, implying that the Levi–Civita connection coefficients responsible for curvature vanish in the mixed directions. From the explicit curvature formula in two dimensions, this forces $K \equiv 0$. Substituting $K \equiv 0$ into (3.23)–(3.25) reduces the system to identities of the form $0 = 0$. Hence any V satisfies the Ricci collineation condition. $\square$

Theorem 4. *The frame fields E_1 and E_2 are Ricci collineations if and only if $K \equiv 0$ on the coordinate neighborhood under consideration.*

Proof. We first consider the vector field $V = E_1$. Then $V^1 = 1$, $V^2 = 0$, and $V(K) = E_1(K)$. Substituting these expressions into the system given in Lemma 3, we obtain $E_1(K) = 0$, $E_1(K) - 2KI_{21} = 0$, and $KI_{12} = 0$. Consequently, $E_1(K) = 0$, $KI_{12} = 0$, and $KI_{21} = 0$.

Suppose, to the contrary, that there exists a point p in the coordinate neighborhood such that $K(p) \neq 0$. By continuity, there exists an open neighborhood W of p on which K is nowhere zero. On W , the relations $KI_{12} = 0$ and $KI_{21} = 0$ imply $I_{12} = I_{21} = 0$ on W . Since $I_{12} = E_2(\ln f_1)$ and $I_{21} = E_1(\ln f_2)$, this gives $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$ on W . Thus, the metric is separable on W , and hence, by Theorem 3, $K \equiv 0$ on W , contradicting $K(p) \neq 0$. Therefore, no such point p exists, and $K \equiv 0$ on the entire coordinate neighborhood.

Similarly, for $V = E_2$, we have $V^1 = 0$, $V^2 = 1$, and $V(K) = E_2(K)$. Substitution into Lemma 3 gives $E_2(K) = 0$, $E_2(K) - 2KI_{12} = 0$, and $KI_{21} = 0$. The same continuity argument as above shows that $K \equiv 0$ on the entire coordinate neighborhood. Therefore, $K \equiv 0$ is necessary for E_1 and E_2 to be Ricci collineations.

Conversely, if $K \equiv 0$, then $\text{Ric} = Kg = 0$. Hence, $\mathcal{L}_{E_1} \text{Ric} = \mathcal{L}_{E_2} \text{Ric} = 0$. Thus, both E_1 and E_2 are Ricci collineations. Therefore, E_1 and E_2 are Ricci collineations $\iff K \equiv 0$. $\square$

3.4. Conformal Ricci collineations for diagonal metrics in local orthogonal coordinates

Conformal Ricci collineations generalize Ricci collineations by allowing the Ricci tensor to vary up to a conformal factor. A vector field V on a Riemannian manifold (M, g) is called a conformal Ricci collineation if it satisfies

$$\mathcal{L}_V \text{Ric} = 2\lambda g,$$

where λ is a smooth scalar function.

In two dimensions, the Einstein identity $\text{Ric} = Kg$ holds, where K is the Gaussian curvature. Hence,

$$\begin{aligned}\mathcal{L}_V(Kg) &= (\mathcal{L}_V K)g + K\mathcal{L}_V g \\ &= 2\lambda g.\end{aligned}$$

Using the product rule for Lie derivatives,

$$\mathcal{L}_V(Kg)(X, Y) = V(K)g(X, Y) + K(\mathcal{L}_V g)(X, Y).$$

We consider the diagonal metric on $(\mathbb{R}^2, g)$, as introduced in Section 2,

$$g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2,$$

and the orthonormal frame $E_i = f_i \frac{\partial}{\partial x^i}$, $i = 1, 2$.

Let the structure functions be defined by $[E_1, E_2] = -I_{12}E_1 + I_{21}E_2$.

Lemma 4. *Let $V = V^1 E_1 + V^2 E_2$. Then*

$$\begin{aligned}(\mathcal{L}_V g)(E_1, E_1) &= 2E_1(V^1) - 2V^2 I_{12}, \\ (\mathcal{L}_V g)(E_2, E_2) &= 2E_2(V^2) - 2V^1 I_{21},\end{aligned}\tag{3.26}$$

$$(\mathcal{L}_V g)(E_1, E_2) = E_1(V^2) + E_2(V^1) + V^1 I_{12} + V^2 I_{21}.\tag{3.27}$$

Proof. From the definition

$$(\mathcal{L}_V g)(X, Y) = V(g(X, Y)) - g([V, X], Y) - g(X, [V, Y])$$

and using

$$[V, E_i] = \sum_{j=1}^2 (V^j [E_j, E_i] - E_i(V^j) E_j),$$

together with $[E_1, E_2] = -I_{12}E_1 + I_{21}E_2$, the result follows by direct computation. $\square$

Lemma 5. *A vector field $V = V^1 E_1 + V^2 E_2$ is a conformal Ricci collineation if and only if*

$$V^1 E_1(K) + V^2 E_2(K) + 2K(E_1(V^1) - I_{12}V^2) = 2\lambda,$$

$$K(V^1 I_{12} + E_1(V^2) + E_2(V^1) + V^2 I_{21}) = 0,$$

and

$$V^1 E_1(K) + V^2 E_2(K) + 2K(E_2(V^2) - V^1 I_{21}) = 2\lambda.$$

Proof. Since $(\mathbb{R}^2, g)$ is two-dimensional, we have $\text{Ric} = Kg$. Therefore, $\mathcal{L}_V \text{Ric} = \mathcal{L}_V(Kg) = V(K)g + K\mathcal{L}_V g$. By the definition of a conformal Ricci collineation, $\mathcal{L}_V \text{Ric} = 2\lambda g$. Hence, $V(K)g + K\mathcal{L}_V g = 2\lambda g$.

Using the orthonormal frame $\{E_1, E_2\}$ and the structure equation $[E_1, E_2] = -I_{12}E_1 + I_{21}E_2$, we have

$$(\mathcal{L}_V g)(E_1, E_1) = 2E_1(V^1) - 2I_{12}V^2,$$

$$(\mathcal{L}_V g)(E_2, E_2) = 2E_2(V^2) - 2I_{21}V^1,$$

and

$$(\mathcal{L}_V g)(E_1, E_2) = E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2.$$

Evaluating

$$V(K)g + K\mathcal{L}_V g = 2\lambda g$$

on (E_1, E_1) gives

$$V(K) + 2K\{E_1(V^1) - I_{12}V^2\} = 2\lambda.$$

Similarly, evaluating it on (E_2, E_2) gives

$$V(K) + 2K\{E_2(V^2) - I_{21}V^1\} = 2\lambda.$$

Finally, since $g(E_1, E_2) = 0$, the mixed component yields

$$K\{E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2\} = 0.$$

Thus, the proof is complete. $\square$

Theorem 5. Assume that $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$ and let $V = V^1E_1 + V^2E_2$ be a vector field on $\mathbb{R}^2$, defined on a connected open region where K is nowhere zero. Then V is a conformal Ricci collineation if and only if

$$E_1(V^1) - E_2(V^2) + I_{21}V^1 - I_{12}V^2 = 0, \quad (3.28)$$

$$E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2 = 0, \quad (3.29)$$

and the conformal factor λ is given by

$$\lambda = \frac{1}{2}V(K) + K\{E_1(V^1) - I_{12}V^2\}. \quad (3.30)$$

Here, $V(K) = V^1E_1(K) + V^2E_2(K)$.

Proof. Since K is nowhere zero on the region under consideration, the system in Lemma 5 may be divided by K . Subtracting the third equation from the first equation of Lemma 5, we obtain

$$E_1(V^1) - E_2(V^2) + I_{21}V^1 - I_{12}V^2 = 0,$$

which is (3.28). The second equation of Lemma 5 gives

$$E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2 = 0,$$

which is (3.29).

Conversely, assume that (3.28) and (3.29) hold and define λ by (3.30). Then

$$V(K) + 2K\{E_1(V^1) - I_{12}V^2\} = 2\lambda.$$

Using (3.28), we also have

$$E_1(V^1) - I_{12}V^2 = E_2(V^2) - I_{21}V^1,$$

and hence

$$V(K) + 2K \{E_2(V^2) - I_{21}V^1\} = 2\lambda.$$

Finally, Eq (3.29) is precisely the second equation of Lemma 5. Therefore, all equations in Lemma 5 are satisfied, and consequently

$$\mathcal{L}_V \text{Ric} = 2\lambda g.$$

Thus, V is a conformal Ricci collineation. $\square$

Theorem 6. Assume that K is nowhere zero on the connected open region under consideration and suppose the cross-dependence condition $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$ holds. Let $V = V^1 E_1 + V^2 E_2$ be a conformal Ricci collineation. If $V^1 = V^2 =: W$, then $E_1(W) + I_{21}W = 0$, $E_2(W) + I_{12}W = 0$. Consequently, on every connected open set where $W \neq 0$, $W = \frac{c}{f_1 f_2}$, where c is a nonzero constant. If $W \equiv 0$, then $V \equiv 0$, and the associated conformal factor necessarily satisfies $\lambda = 0$; in this case, no further restriction on f_1 and f_2 follows from the condition $V^1 = V^2$ alone.

Proof. Since $V^1 = V^2 =: W$, Eqs (3.28) and (3.29) established in Theorem 5 become

$$E_1(W) - E_2(W) + (I_{21} - I_{12})W = 0,$$

and

$$E_1(W) + E_2(W) + (I_{12} + I_{21})W = 0.$$

Adding and subtracting these equations, respectively, we obtain

$$E_1(W) + I_{21}W = 0,$$

and

$$E_2(W) + I_{12}W = 0.$$

Under the cross-dependence assumption, $I_{21} = E_1(\ln f_2)$, $I_{12} = E_2(\ln f_1)$. Therefore,

$$E_1(W) + WE_1(\ln f_2) = 0,$$

and

$$E_2(W) + WE_2(\ln f_1) = 0.$$

Equivalently, $E_1(Wf_2) = 0$, $E_2(Wf_1) = 0$. Since $E_1(f_1) = 0$ and $E_2(f_2) = 0$, these relations give

$$E_1(Wf_1f_2) = 0, \quad E_2(Wf_1f_2) = 0.$$

Hence Wf_1f_2 is constant on each connected component. Thus, on any connected open set where $W \neq 0$, $W = \frac{c}{f_1 f_2}$, where $c \neq 0$ is a constant. If $W \equiv 0$, then $V = V^1 E_1 + V^2 E_2 \equiv 0$, so $\mathcal{L}_V \text{Ric} = 0 = 2\lambda g$ forces $\lambda = 0$, and the condition $V^1 = V^2$ imposes no additional restriction on f_1 and f_2 . This completes the proof. $\square$

Theorem 7. Let $(\mathbb{R}^2, g)$ be as in (3.1), with $f_i = f_i(x^i)$, $i = 1, 2$. Then $K \equiv 0$. Moreover, the frame field $E_i = f_i \frac{\partial}{\partial x^i}$ is a conformal Ricci collineation if and only if $\lambda = 0$. In particular, E_i is a Ricci collineation.

Proof. Since $f_i = f_i(x^i)$, we have $I_{12} = I_{21} = 0$. Hence, by Theorem 3, $K \equiv 0$. Consequently, $\text{Ric} = Kg = 0$.

Let $V = E_i$. Then $V^i = 1$, $V^j = 0$, $j \neq i$. Substituting $V = E_i$ into the system in Lemma 5 gives $E_i(K) = 2\lambda$. Since $K \equiv 0$, we have $E_i(K) = 0$, and therefore $\lambda = 0$.

Conversely, if $\lambda = 0$, then

$$\mathcal{L}_{E_i} \text{Ric} = \mathcal{L}_{E_i}(0) = 0 = 2\lambda g.$$

Thus, E_i is a conformal Ricci collineation. Hence the assertion follows. $\square$

3.5. Ricci solitons for diagonal metrics in local orthogonal coordinates

Ricci solitons are self-similar solutions of the Ricci flow and play a central role in geometric analysis. A vector field V on a Riemannian manifold (M, g) is called a Ricci soliton if it satisfies

$$\mathcal{L}_V g + 2 \text{Ric} = 2\lambda g,$$

where $\lambda \in \mathbb{R}$ is the soliton constant. In dimension two, the identity $\text{Ric} = Kg$ implies that the soliton equation reduces to

$$\mathcal{L}_V g = 2(\lambda - K)g,$$

showing that Ricci solitons in dimension two correspond to curvature-dependent conformal deformations of the metric.

Let $\{E_1, E_2\}$ be a local orthonormal frame field associated with the diagonal metric on $(\mathbb{R}^2, g)$, as introduced in Section 2,

$$g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2,$$

and let $V = V^1 E_1 + V^2 E_2$. Denote the structural functions by

$$\nabla_{E_i} E_j = \sum_{k=1}^2 \Gamma_{ij}^k E_k, \quad I_{12} := g(\nabla_{E_1} E_1, E_2), \quad I_{21} := g(\nabla_{E_2} E_2, E_1).$$

Using the metric compatibility of the Levi-Civita connection and the orthonormality of $\{E_1, E_2\}$, we have

$$\nabla_{E_1} E_1 = I_{12} E_2, \quad \nabla_{E_1} E_2 = -I_{12} E_1,$$

and

$$\nabla_{E_2} E_2 = I_{21} E_1, \quad \nabla_{E_2} E_1 = -I_{21} E_2.$$

Then the Lie derivative of the metric satisfies

$$\begin{aligned} (\mathcal{L}_V g)(E_1, E_1) &= 2E_1(V^1) - 2V^2 I_{12}, \\ (\mathcal{L}_V g)(E_2, E_2) &= 2E_2(V^2) - 2V^1 I_{21}, \\ (\mathcal{L}_V g)(E_1, E_2) &= E_1(V^2) + E_2(V^1) + V^1 I_{12} + V^2 I_{21}. \end{aligned}$$

Using $\mathcal{L}_V g = 2(\lambda - K)g$, we obtain the Ricci soliton system:

$$\begin{aligned} E_1(V^1) - V^2 I_{12} + K &= \lambda, \\ E_2(V^2) - V^1 I_{21} + K &= \lambda, \\ E_1(V^2) + E_2(V^1) + V^1 I_{12} + V^2 I_{21} &= 0. \end{aligned}$$

Theorem 8. Let $(\mathbb{R}^2, g)$ be as in (3.1), with

$$g = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2,$$

where f_1 and f_2 are positive smooth functions. Let $V = V^1 E_1 + V^2 E_2$ be a smooth vector field and let $\lambda \in \mathbb{R}$. Consider the Ricci soliton equation

$$\mathcal{L}_V g + 2 \operatorname{Ric} = 2\lambda g.$$

(i) If $f_1 = f_1(x^1)$, $f_2 = f_2(x^2)$, then $I_{12} = I_{21} = 0$, $K \equiv 0$. Consequently, V is a Ricci soliton if and only if

$$E_1(V^1) = \lambda, \quad E_2(V^2) = \lambda, \quad E_1(V^2) + E_2(V^1) = 0.$$

In particular, the metric is flat and locally Euclidean, and the Ricci soliton equation reduces precisely to the homothetic vector field equation $\mathcal{L}_V g = 2\lambda g$.

(ii) If $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$ and $V^1 = V^1(x^2)$, $V^2 = V^2(x^1)$, then the off-diagonal equation can be separated. More precisely, there exists a constant $c_3 \in \mathbb{R}$ such that

$$\frac{E_1(V^2) + I_{21} V^2}{f_1 f_2} = c_3, \quad \frac{E_2(V^1) + I_{12} V^1}{f_1 f_2} = -c_3.$$

Equivalently,

$$\frac{dV^2}{dx^1} + \frac{f_2'}{f_2} V^2 = c_3 f_2,$$

and

$$\frac{dV^1}{dx^2} + \frac{f_1'}{f_1} V^1 = -c_3 f_1.$$

Hence,

$$V^2(x^1) = \frac{c_1 + c_3 \int f_2^2(x^1) dx^1}{f_2(x^1)},$$

and

$$V^1(x^2) = \frac{c_2 - c_3 \int f_1^2(x^2) dx^2}{f_1(x^2)},$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

The remaining diagonal equations are

$$K - \left(c_1 + c_3 \int f_2^2(x^1) dx^1 \right) \frac{f_1'}{f_1} = \lambda,$$

and

$$K - \left(c_2 - c_3 \int f_1^2(x^2) dx^2 \right) \frac{f_2'}{f_2} = \lambda.$$

Therefore, these equations together with the expressions above for V^1 and V^2 give necessary and sufficient conditions for a Ricci soliton in the cross-dependence case.

Proof. For the Ricci soliton equation

$$\mathcal{L}_V g = 2(\lambda - K)g,$$

the components with respect to the orthonormal frame $\{E_1, E_2\}$ yield the system

$$E_1(V^1) - I_{12}V^2 + K = \lambda,$$

$$E_2(V^2) - I_{21}V^1 + K = \lambda,$$

and

$$E_1(V^2) + E_2(V^1) + I_{12}V^1 + I_{21}V^2 = 0.$$

In case (i), the assumptions $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$ imply $I_{12} = I_{21} = 0$. Moreover, the corresponding Gaussian curvature satisfies $K \equiv 0$. Therefore, the above system reduces to $E_1(V^1) = \lambda$, $E_2(V^2) = \lambda$, and $E_1(V^2) + E_2(V^1) = 0$. This proves (i).

For case (ii), we have $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$ and $V^1 = V^1(x^2)$, $V^2 = V^2(x^1)$. Using $I_{12} = \frac{f_2 f_1'}{f_1}$, $I_{21} = \frac{f_1 f_2'}{f_2}$, the off-diagonal equation becomes

$$f_1 \left(\frac{dV^2}{dx^1} + \frac{f_2'}{f_2} V^2 \right) + f_2 \left(\frac{dV^1}{dx^2} + \frac{f_1'}{f_1} V^1 \right) = 0.$$

After division by $f_1 f_2$, we obtain

$$\frac{1}{f_2} \left(\frac{dV^2}{dx^1} + \frac{f_2'}{f_2} V^2 \right) + \frac{1}{f_1} \left(\frac{dV^1}{dx^2} + \frac{f_1'}{f_1} V^1 \right) = 0.$$

The first term depends only on x^1 , whereas the second term depends only on x^2 . Hence both terms must be equal to constants with opposite signs. Thus, for some $c_3 \in \mathbb{R}$,

$$\frac{1}{f_2} \left(\frac{dV^2}{dx^1} + \frac{f_2'}{f_2} V^2 \right) = c_3,$$

and

$$\frac{1}{f_1} \left(\frac{dV^1}{dx^2} + \frac{f_1'}{f_1} V^1 \right) = -c_3.$$

Equivalently,

$$\frac{dV^2}{dx^1} + \frac{f_2'}{f_2} V^2 = c_3 f_2,$$

and

$$\frac{dV^1}{dx^2} + \frac{f_1'}{f_1} V^1 = -c_3 f_1.$$

Multiplying the first equation by f_2 and the second one by f_1 gives

$$\frac{d}{dx^1}(f_2 V^2) = c_3 f_2^2,$$

and

$$\frac{d}{dx^2}(f_1 V^1) = -c_3 f_1^2.$$

Therefore,

$$V^2(x^1) = \frac{c_1 + c_3 \int f_2^2(x^1) dx^1}{f_2(x^1)},$$

and

$$V^1(x^2) = \frac{c_2 - c_3 \int f_1^2(x^2) dx^2}{f_1(x^2)}.$$

Finally, since V^1 depends only on x^2 and V^2 only on x^1 , we have

$$E_1(V^1) = 0, \quad E_2(V^2) = 0.$$

Substitution into the two diagonal equations yields

$$K - \left(c_1 + c_3 \int f_2^2(x^1) dx^1 \right) \frac{f_1'}{f_1} = \lambda,$$

and

$$K - \left(c_2 - c_3 \int f_1^2(x^2) dx^2 \right) \frac{f_2'}{f_2} = \lambda.$$

Thus, the stated conditions are necessary and sufficient for the existence of the considered Ricci soliton. $\square$

Example 3. Consider again the diagonal metric

$$g = \frac{1}{(x^1)^2} dx^1 \otimes dx^1 + \frac{1}{(x^2)^2} dx^2 \otimes dx^2$$

on $D = \{(x^1, x^2) \in \mathbb{R}^2 : x^1 > 0, x^2 > 0\}$. Then $f_1(x^1) = x^1$, $f_2(x^2) = x^2$, so that $I_{12} = I_{21} = 0$. By the Gaussian curvature formula, $K \equiv 0$, and consequently $\text{Ric} = 0$.

Let $V = \lambda \ln x^1 E_1 + \lambda \ln x^2 E_2$, where $\lambda \in \mathbb{R}$ is a constant. Thus, $V^1 = \lambda \ln x^1$, $V^2 = \lambda \ln x^2$. We have $E_1(V^1) = \lambda$, $E_2(V^2) = \lambda$, and $E_1(V^2) + E_2(V^1) = 0$. Therefore, all three equations of the Ricci soliton system in Theorem 8 (i) are satisfied.

Indeed, $\mathcal{L}_V g = 2\lambda g$ and, since $\text{Ric} = 0$, $\mathcal{L}_V g + 2\text{Ric} = 2\lambda g$. Hence (D, g, V, λ) defines a Ricci soliton. Moreover, if $u = \ln x^1$, $v = \ln x^2$, then $g = du^2 + dv^2$ and $V = \lambda u \frac{\partial}{\partial u} + \lambda v \frac{\partial}{\partial v}$. Thus, since the map $(x^1, x^2) \mapsto (u, v) = (\ln x^1, \ln x^2)$ is a bijection from D onto all of $\mathbb{R}^2$, this example corresponds to the standard homothetic Ricci soliton structure on the Euclidean plane expressed in the original diagonal coordinates.

4. Conclusions

In this paper, we developed a unified framework for the study of curvature-related infinitesimal symmetries and Ricci-type structures on two-dimensional Riemannian manifolds represented locally in orthogonal coordinates by a diagonal metric. Using a local orthonormal frame adapted to the metric coefficients, we derived differential systems characterizing conformal vector fields, ρ -Ricci vector fields, Ricci collineations, conformal Ricci collineations, and Ricci solitons. The two-dimensional

pointwise identity $\text{Ric} = Kg$, where K denotes the Gaussian curvature, provides a common geometric basis for these structures and allows their defining equations to be expressed in terms of the Gaussian curvature, the metric coefficients, the associated structure functions, and the components of the corresponding vector fields.

The resulting differential systems exhibit distinct geometric behavior depending on the symmetry condition under consideration. For ρ -Ricci vector fields, the relation between the conformal factor and the Gaussian curvature becomes restrictive when ρ is prescribed. Accordingly, in the principal coordinate-field classification, ρ was assumed to be a nonzero constant. Under suitable separability and cross-dependence assumptions on the metric coefficients, the corresponding equations reduce to explicitly solvable differential systems and yield concrete restrictions on the admissible metric functions and vector fields. These results distinguish between the flat, locally Euclidean regime and connected open regions where the Gaussian curvature is nowhere zero.

For Ricci collineations, the condition $\mathcal{L}_V \text{Ric} = 0$ leads to curvature restrictions, most notably the necessity of $K \equiv 0$ in several distinguished cases considered in the paper. Conformal Ricci collineations introduce an additional scalar degree of freedom through the conformal factor, while their governing equations still impose differential constraints involving the Gaussian curvature, the structure functions, and the components of the vector field.

For Ricci solitons, the soliton equation was reduced to a system of differential equations with respect to the adapted orthonormal frame, from which necessary and sufficient conditions were obtained under the coordinate-dependence assumptions considered. These conditions provide explicit families for the components of the potential vector field together with differential restrictions on the metric coefficients and illustrate how the infinitesimal deformation of the metric is coupled to its Gaussian curvature. In particular, in the flat separable case, the metric is locally Euclidean, and the Ricci soliton equation reduces to the homothetic vector field equation $\mathcal{L}_V g = 2\lambda g$. The examples presented in the paper complement the theoretical results by exhibiting explicit non-Killing conformal vector fields and Ricci soliton structures.

Although the geometric equations defining the symmetry classes studied here are intrinsic, the explicit classification results rely on specific coordinate-dependence assumptions on the metric coefficients and, in some cases, on the components of the vector field. Therefore, the results should be understood as explicit classifications within the local orthogonal coordinate setting and under the specific coordinate-dependence assumptions imposed in the paper, rather than as a coordinate-independent classification of all two-dimensional Riemannian metrics.

In particular, the use of the general local orthogonal representation, rather than imposing isothermal coordinates from the outset, preserves the independent coordinate behavior of the two metric coefficients underlying the separability and cross-dependence assumptions considered above. Accordingly, the classifications obtained here are local in nature and do not require the existence of a global orthogonal coordinate system on the underlying manifold.

This distinction also suggests several directions for further investigation. In particular, it would be interesting to determine to what extent the present approach can be extended to broader classes of metrics and to higher-dimensional settings, where the identity $\text{Ric} = Kg$ is no longer available and additional curvature tensors may contribute to the classification of infinitesimal symmetries. Related problems concerning geometric soliton structures on tangent and cotangent bundles also provide a natural direction for future research.

Use of AI tools declaration

The authors used Artificial Intelligence (AI) solely for English-language editing, including grammar, spelling, punctuation, and linguistic corrections. AI tools were not used to generate or modify any mathematical or scientific content. All AI-assisted language revisions were carefully reviewed and approved by the authors.

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Conflict of interest

The authors declare there is no conflict of interest.

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