



Research article

Practical finite-time adaptive neural network control for fractional-order nonlinear systems

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Abstract: This paper investigates the semi-global practical finite-time adaptive neural network (NN) control problem for a class of fractional-order nonlinear systems with unknown nonlinearities. First, a novel practical finite-time stability criterion is established for fractional-order nonlinear systems. In contrast to analyses based on specific derivative properties of composite Lyapunov functions, the proposed criterion is rigorously developed through direct estimation of the fractional-order derivative of the Lyapunov function. This direct estimation framework provides a general stability analysis approach with independently adjustable convergence and residual-set parameters, thereby enhancing the flexibility of controller design. Second, an adaptive NN controller is developed based on this criterion, where NNs are employed to approximate the unknown nonlinearities. It is proven that the proposed control scheme ensures semi-global practical finite-time stability of the closed-loop system and boundedness of all closed-loop signals. Finally, numerical simulation results are provided to demonstrate the effectiveness of the proposed control scheme.

Keywords: fractional-order nonlinear systems; practical finite-time stability; adaptive neural network control; Lyapunov method

1. Introduction

In recent years, fractional-order calculus has attracted increasing attention in the field of control theory due to its remarkable capability of describing practical systems with memory and hereditary properties [1–4]. Compared with integer-order systems, fractional-order nonlinear systems (FONSs) exhibit more complicated dynamic behaviors due to the intrinsic nonlocality of fractional-order operators, which brings significant challenges to the system analysis and control design [4–7]. In

many practical applications, it is desirable that system states converge into a prescribed neighborhood of the equilibrium point within a finite time rather than merely achieving asymptotic convergence. This requirement has motivated extensive studies on practical finite-time stability and control [8–11]. Compared with asymptotic stability, practical finite-time stability provides not only finite-time convergence guarantees but also explicit estimates of the settling time, making it more suitable for practical engineering applications.

For integer-order nonlinear systems, practical finite-time control has achieved significant progress through several distinct methodological lines. Chen et al. [12] proposed an adaptive fuzzy control approach, while Xu et al. [13] investigated output-feedback tracking under finite-time constraints; however, both methods are confined to integer-order dynamics and cannot be directly extended to fractional-order systems (FOSs) due to the lack of chain rule. Li et al. [14] established a finite-time tracking criterion for multi-agent systems, and Shang et al. [15] developed fast finite-time adaptive neural control for multi-agent coordination; yet their analyses rely on integer-order Lyapunov derivatives. Along another line, dynamic surface control techniques [16] were developed to circumvent the explosion of complexity, and either prescribed-time or fixed-time strategies [17–20] provided adjustable convergence horizons; nevertheless, these results either depend on specific chain-rule properties or have not been generalized to fractional-order settings.

For FONSs, the main difficulty in the Lyapunov-based stability analysis lies in establishing appropriate estimates for the fractional-order derivative of the Lyapunov function. Unlike integer-order derivatives, fractional-order operators generally do not satisfy the conventional chain rule for composite functions. The corresponding generalized expressions usually involve infinite-series terms, which makes the direct analysis of composite Lyapunov functions considerably more complicated [21, 22]. In addition, the stability of FOSs cannot be directly characterized by classical integer-order Lyapunov theories, and several specific stability criteria have been developed to address this issue [23, 24]. Therefore, developing rigorous and practically applicable stability criteria without relying on the restrictive derivative properties of composite Lyapunov functions remains a fundamental challenge in the fractional-order control theory.

Recently, several finite-time and practical finite-time stability criteria have been developed for FOSs based on fractional-order Lyapunov methods. For instance, Aghababa [25] studied the finite-time stability of FONSs and established corresponding stability conditions. In [26, 27], finite-time control strategies were investigated for FOSs under different constraints. More recently, Cao et al. [28] proposed a practical finite-time adaptive neural network (NN) control method for FONSs. However, existing results still face several limitations regarding the generality and flexibility of stability criteria. In particular, some existing analyses rely on specific derivative properties of composite Lyapunov functions, whose applicability to fractional-order operators requires further justification. Moreover, many available criteria impose coupling constraints among power parameters and system coefficients, which may restrict the flexibility of controller design. These observations motivate the development of a more general practical finite-time stability criterion based on the direct estimation of fractional-order Lyapunov derivatives, together with an adaptive control strategy for FONSs with unknown dynamics.

Unlike the aforementioned approaches that rely on specific derivative properties of composite Lyapunov functions, this paper develops a novel practical finite-time stability criterion through a direct estimation framework of the fractional-order derivative of the Lyapunov function. The

proposed criterion distinguishes itself from existing results in three key aspects. First, it rigorously avoids the fractional-order chain rule for composite functions, whose applicability to fractional-order operators generally requires further justification and often involves complicated infinite-series expansions. Second, two power parameters and three coefficients can be independently adjusted without the coupling constraints commonly imposed in the literature, thereby providing enhanced flexibility in tuning both the convergence speed and the residual bound. It is worth noting that the independent tunability of convergence and residual parameters also appears in funnel control [29] and adaptive switched control [30], where a prescribed performance function enforces time-varying error bounds. However, funnel control typically requires the error to remain within a time-varying funnel boundary, whereas the proposed criterion achieves practical finite-time convergence through nonlinear power terms without imposing a time-varying funnel constraint. The relationship between these two frameworks deserves further investigation. Third, by integrating this criterion with adaptive NN approximation, the resulting control scheme guarantees semi-global practical finite-time stability for FONSs with unknown dynamics, where the NN approximation is only required on a prescribed compact set rather than globally.

Motivated by the above discussions, this paper investigates the practical finite-time adaptive NN control problem for a class of FONSs with unknown nonlinearities. The main contributions are summarized as follows:

- (i) A novel practical finite-time stability criterion is established for FONSs through a direct estimation framework of the fractional-order derivative of the Lyapunov function. Different from analyses that rely on specific derivative properties of composite Lyapunov functions, the proposed criterion avoids the explicit evaluation of fractional-order derivatives of composite functions. Furthermore, the developed criterion allows independently adjustable power parameters $0 < m_1 < 1 < m_2$ and coefficients $\rho_1, \rho_2 > 0$, which provides enhanced flexibility in tuning convergence characteristics and residual bounds.
- (ii) Based on the proposed stability criterion, an adaptive NN controller is developed for FONSs with unknown nonlinear dynamics. By employing NN to approximate unknown nonlinear functions, the proposed control scheme effectively addresses the difficulties caused by system uncertainties. It is rigorously proven that the closed-loop system achieves semi-global practical finite-time stability and all signals remain bounded.
- (iii) A degeneration analysis is provided to reveal the essential mechanism of finite-time convergence. It is demonstrated that when the power terms in the proposed controller and adaptive laws reduce to linear forms, the finite-time convergence property disappears and the system degenerates into a semi-globally practically asymptotically stable case. This analysis further clarifies the necessity of nonlinear power terms in achieving semi-global practical finite-time stabilization.

The remainder of this paper is organized as follows: Section 2 presents some useful preliminary results on fractional-order calculus and practical finite-time stability, followed by the problem formulation; Section 3 establishes the main results of this paper, where a novel practical finite-time stability criterion for FONSs is first proposed, and then an adaptive NN controller is designed based on the developed stability criterion with the semi-global practical finite-time stability of the closed-loop system being analyzed; Section 4 provides numerical simulation examples to demonstrate

the effectiveness and superiority of the proposed control scheme; and finally, Section 5 concludes this paper with a summary and future research directions.

2. Preliminaries and problem formulation

This section presents some fundamental results on fractional-order calculus and practical finite-time stability, and formulates the control problem under consideration.

2.1. Some useful results

Definition 2.1. [2] Consider a continuously differentiable function $f : [0, +\infty) \rightarrow \mathbb{R}$ and a fractional order $\alpha \in (0, 1)$. The Caputo fractional derivative of f at $t \geq 0$ is defined as follows

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(s)}{(t-s)^\alpha} ds, \quad (2.1)$$

where $\Gamma(\cdot)$ represents the gamma function.

Lemma 2.2. [31] For $\alpha \in (0, 1)$ and $f(t) \in C^1([0, +\infty), \mathbb{R})$, if there is a constant $t^* > 0$ such that $f(t^*) = 0$ and $f(t) < 0$ (≤ 0) for all $t \in [0, t^*)$, then $\mathcal{D}^\alpha f(t^*) > 0$ (≥ 0).

Lemma 2.3. [32] If $x(t) : [0, +\infty) \rightarrow \mathbb{R}^n$ and $\mathcal{W}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuously differentiable, and $\mathcal{W}(\cdot)$ is convex on $\mathbb{R}^n$, then

$$\mathcal{D}^\alpha \mathcal{W}(x(t)) \leq \frac{\partial \mathcal{W}(x(t))}{\partial x(t)} \mathcal{D}^\alpha x(t), \quad 0 < \alpha < 1.$$

Definition 2.4. [33] Consider the FONS $\mathcal{D}^\alpha x(t) = f(t, x(t))$ with the initial condition $x(t_0) = x_0$, where $\alpha \in (0, 1)$. The origin is said to be practically finite-time stable if for any $x_0 \in \mathbb{R}^n$, there exist $\varepsilon > 0$ and $T_f(\varepsilon, x_0) < \infty$ such that $\|x(t)\| \leq \varepsilon$ for all $t \geq t_0 + T_f$.

Remark 2.5. Definition 2.4 adapts the concept of practical finite-time stability from the integer-order literature [33, 34] to FOSs. Such an extension is natural because the definition characterizes the trajectory behavior in the state space rather than properties of the differential operator itself. The existence and uniqueness of solutions to the FOS under the Caputo derivative ensure that the definition is well posed.

Lemma 2.6. [35] Suppose that $\mathbb{M} \subset \mathbb{R}^{k_1}$ is a compact set. Then, the continuous function $\mathcal{F}(X)$ defined on $\mathbb{M}$ can be approximated by radial basis function NNs (RBFNNs) $W^{*T} \Psi(X)$ with any precision $\epsilon(X)$ when the node number k_2 of NNs is sufficiently large (i.e., $\mathcal{F}(X) = W^{*T} \Psi(X) + \epsilon(X)$), where $W^* \in \mathbb{R}^{k_2}$ is the ideal weight vector, the approximation error $\epsilon(X)$ satisfies $|\epsilon(X)| < \bar{\epsilon}$ with the constant $\bar{\epsilon}$, $\Psi(X) = (\psi_1(X), \dots, \psi_{k_2}(X))^T \in \mathbb{R}^{k_2}$ represents the Gaussian-like radial basis function vector with

$$\psi_i(X) = \exp \left[-\frac{(X - \check{\omega}_i)^T (X - \check{\omega}_i)}{\delta_i^2} \right], \quad i = 1, \dots, k_2,$$

and $\check{\omega}_i = (\varpi_{i1}, \dots, \varpi_{ik_1})^T$ is the center of receptive domain and δ_i is the width of Gaussian function.

Lemma 2.7. [16] For $X = (X_1, \dots, X_{k_1})^T$ and $\bar{X} = (X_1, \dots, X_s)^T$ with $1 \leq s \leq k_1$, the following inequality holds:

$$\|\Psi(X)\| \leq \|\Psi(\bar{X})\|,$$

where $\Psi(\cdot)$ as defined in Lemma 2.6.

Lemma 2.8. Let $r = r_1/r_2$ with positive odd integers r_1, r_2 . Then, there exist constants $b_1, b_2 > 0$, depending only on r , such that for all $x_1, x_2 \in \mathbb{R}$,

$$x_1 x_2^r \leq -b_1 |x_1|^{1+r} + b_2 |x_1 + x_2|^{1+r}.$$

Proof. For $0 < r < 1$, see [15, Lemma 7]. Let $r \geq 1$ and $x = x_1 + x_2$. Since r_1, r_2 are odd, $1 + r = (r_1 + r_2)/r_2$ is an even rational, so $x^{1+r} \geq 0$.

If $x \neq 0$, then substituting $\tau = x_1/x$ and dividing by x^{1+r} yields $\tau(1 - \tau)^r \leq -b_1 \tau^{1+r} + b_2$. For any fixed $b_1 \in (0, 1)$, the function $\mathcal{G}(\tau) = \tau(1 - \tau)^r + b_1 \tau^{1+r}$ tends to $-\infty$ as $|\tau| \rightarrow \infty$ (by the sign of $b_1 - 1$) and hence has a finite maximum M ; then, choosing $b_2 > M$ gives the inequality. The case $x = 0$ reduces to $b_1 \leq 1$, which is ensured by the choice $b_1 \in (0, 1)$. $\square$

Lemma 2.9. [34] For $x_i \in \mathbb{R}, i = 1, \dots, n$ and $0 < r < 1$, the following inequality holds:

$$\left(\sum_{j=1}^n |x_j| \right)^r \leq \sum_{j=1}^n |x_j|^r \leq n^{1-r} \left(\sum_{j=1}^n |x_j| \right)^r.$$

2.2. Problem formulations

Consider the following FOS:

$$\begin{cases} \mathcal{D}^\alpha \vartheta_i = \phi_i(\vartheta) + \psi_i(\vartheta) \vartheta_{i+1}, & i = 1, \dots, n-1, \\ \mathcal{D}^\alpha \vartheta_n = \phi_n(\vartheta) + \psi_n(\vartheta) u, \\ y = \vartheta_1, \end{cases} \quad (2.2)$$

where $\alpha \in (0, 1)$, $\vartheta = (\vartheta_1, \dots, \vartheta_n)^T$ is the state variable, $\phi_i(\cdot)$ and $\psi_i(\cdot)$ are unknown continuous functions, and u and y are the control input and system output, respectively.

The control objective of this paper is to design a semi-global practical finite-time adaptive controller such that the output of system (2.2) tracks the reference signal y_d in finite time with a small error. To achieve this goal, the following standard assumptions are necessary.

Assumption 2.10. For $t \in [0, \infty)$, the function $\psi_i(\cdot)$ satisfies $0 < \bar{\psi}_i \leq |\psi_i(\cdot)| < \infty$ for some constant $\bar{\psi}_i > 0$. Without a loss of generality, we assume $\psi_i(\cdot) > 0$ and hence $0 < \bar{\psi}_i \leq \psi_i(\cdot) < \infty$.

Assumption 2.11. The reference signal y_d and its fractional-order derivative are bounded and continuous.

3. Main results

This section presents a novel practical finite-time stability criterion for FONSs and designs an adaptive NN controller based on this criterion.

3.1. Fractional-order practical finite-time stability

Different from existing analyses relying on specific derivative properties of composite Lyapunov functions, this subsection establishes a practical finite-time stability criterion for FONSs through a direct estimation of the fractional-order derivative of the Lyapunov function. Here, “direct estimation” means that an upper bound of the fractional-order derivative of a Lyapunov function candidate is obtained as a whole by means of Lemma 2.3, rather than by expanding the fractional-order derivative of a composite function via a chain rule. Because the Caputo derivative is a nonlocal operator with a memory kernel $(t - s)^{-\alpha}$, this approach respects the memory effect by avoiding any termwise expansion that would otherwise require unjustified manipulations of the history-dependent integrand. It is worth noting that the absolute continuity of the state trajectory guarantees the existence of the Caputo derivative, while continuously differentiable and convex assumptions on the Lyapunov function ensure the applicability of Lemma 2.3. These regularity requirements are standard for a fractional-order Lyapunov analysis and are kept minimal. This direct estimation approach avoids the need for explicit fractional-order differentiation of composite Lyapunov functions and provides a flexible framework to design finite-time controllers.

Theorem 3.1. *Let $x : [0, \infty) \rightarrow \mathbb{R}^n$ be absolutely continuous, and let $\mathcal{W} : \mathbb{R}^n \rightarrow \mathbb{R}$ be positive definite, continuously differentiable, and convex. Suppose that*

$$\mathcal{D}^\alpha \mathcal{W}(x(t)) \leq -\rho_1 \mathcal{W}^{m_1}(x(t)) - \rho_2 \mathcal{W}^{m_2}(x(t)) + \rho_3, \quad (3.1)$$

where $0 < \alpha < 1$, $0 < m_1 < 1 < m_2$, and $\rho_1, \rho_2, \rho_3 > 0$. Let $\mathcal{W}^* > 0$ be the unique positive root of $-\rho_1 y^{m_1} - \rho_2 y^{m_2} + \rho_3 = 0$, and set $R = \max \left\{ \mathcal{W}^*, \left(\frac{\rho_3}{\rho_2} \right)^{1/m_2} \right\}$. Then, there exists a finite time $T_f \geq 0$ such that $\mathcal{W}(x(t)) \leq R$ ($\forall t \geq T_f$) with

$$T_f = \begin{cases} 0, & \mathcal{W}(x(0)) \leq R, \\ T_1, & \mathcal{W}(x(0)) > R, \end{cases} \quad T_1 \leq \left[\frac{\Gamma(1 + \alpha)}{\rho_1(1 + m_1)R^{2m_1}} \left(\mathcal{W}^{1+m_1}(x(0)) - R^{1+m_1} \right) \right]^{\frac{1}{\alpha}}.$$

Proof. We proceed by considering two cases.

Case I: $\mathcal{W}(x(0)) \leq R$. We prove that $\mathcal{W}(x(t)) \leq R$ for all $t \geq 0$ by contradiction. Since $\mathcal{W}(x(0)) - R \leq 0$, we have $\mathcal{W}(x(0)) - R - \epsilon < 0$ for any $\epsilon > 0$. Assume there exists $t^* > 0$ such that $\mathcal{W}(x(t^*)) = R + \epsilon$ and $\mathcal{W}(x(t)) < R + \epsilon$ for $t \in [0, t^*)$. Since $\mathcal{W}(x(t^*)) \geq R \geq (\rho_3/\rho_2)^{1/m_2}$, we have $-\rho_2 \mathcal{W}^{m_2}(x(t^*)) + \rho_3 \leq 0$. Hence, one can obtain from (3.1) that

$$\mathcal{D}^\alpha \mathcal{W}(x(t^*)) \leq -\rho_1(R + \epsilon)^{m_1} - \rho_2 \mathcal{W}^{m_2}(x(t^*)) + \rho_3 = -\rho_1(R + \epsilon)^{m_1} < 0. \quad (3.2)$$

On the other hand, define $g(t) = \mathcal{W}(x(t)) - (R + \epsilon)$. Then, $g(t^*) = 0$ and $g(t) < 0$ for $t \in [0, t^*)$. By Lemma 2.2, we have $\mathcal{D}^\alpha g(t^*) > 0$ (i.e., $\mathcal{D}^\alpha \mathcal{W}(x(t^*)) > 0$), which contradicts (3.2). Therefore, the assumption is false, and $\mathcal{W}(x(t)) \leq R$ for all $t \geq 0$. Thus, $T_f = 0$ in this case.

Case II: $\mathcal{W}(x(0)) > R$. Set $\Omega_R = \{x(t) \mid \mathcal{W}(x(t)) > R\}$, which is nonempty because $\mathcal{W}(x(0)) > R$. For $x(t) \in \Omega_R$, the condition $R \geq (\rho_3/\rho_2)^{1/m_2}$ implies $-\rho_2 \mathcal{W}^{m_2}(x(t)) + \rho_3 \leq 0$. Consequently, (3.1) reduces to the following:

$$\mathcal{D}^\alpha \mathcal{W}(x(t)) \leq -\rho_1 \mathcal{W}^{m_1}(x(t)).$$

Note that the function $\mathcal{U}(x) = \mathcal{W}^{1+m_1}(x)$ is continuously differentiable and convex on $\mathbb{R}^n$ because $\mathcal{W}$ is C^1 , positive definite (hence nonnegative), convex, and the scalar map $z \mapsto z^{1+m_1}$ is convex and nondecreasing for $z \geq 0$. Consequently, Lemma 2.3 applies to $\mathcal{U}(x(t))$, and we obtain the following:

$$\mathcal{D}^\alpha \mathcal{W}^{1+m_1}(x(t)) \leq (1 + m_1) \mathcal{W}^{m_1}(x(t)) \mathcal{D}^\alpha \mathcal{W}(x(t)) \leq -(1 + m_1) \rho_1 \mathcal{W}^{2m_1}(x(t)).$$

On Ω_R , we have $\mathcal{W}(x(t)) \geq R$, so $\mathcal{W}^{2m_1}(x(t)) \geq R^{2m_1}$. Hence,

$$\mathcal{D}^\alpha \mathcal{W}^{1+m_1}(x(t)) \leq -(1 + m_1) \rho_1 R^{2m_1} < 0. \quad (3.3)$$

Applying the fractional integral of order α to both sides of (3.3) from 0 to t yields the following:

$$\mathcal{W}^{1+m_1}(x(t)) \leq \mathcal{W}^{1+m_1}(x(0)) - \frac{(1 + m_1) \rho_1 R^{2m_1}}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau = \mathcal{W}^{1+m_1}(x(0)) - \frac{\rho_1 (1 + m_1) R^{2m_1} t^\alpha}{\Gamma(1 + \alpha)}.$$

Therefore, there exists some $t \leq T_1$ such that $\mathcal{W}(x(t)) \leq R$. Then, by taking $T_f = T_1$, we have $\mathcal{W}(x(t)) \leq R$ for all $t \geq T_f$ by the invariance proved in Case I. $\square$

Remark 3.2. The practical finite-time stability criterion established in Theorem 3.1 is derived through direct estimation of the fractional-order derivative of the Lyapunov function, without resorting to the chain rule for composite functions. This approach is not only technically simpler but also aligns with the theoretical nature of fractional-order calculus, where the chain rule generally does not admit a closed-form expression and typically involves an infinite series. Consequently, the stability criterion given in Theorem 3.1 with independently adjustable powers $0 < m_1 < 1 < m_2$ and coefficients $\rho_1, \rho_2, \rho_3 > 0$ is established on a rigorous theoretical foundation. Moreover, the increased flexibility in independently choosing m_1 , m_2 , ρ_1 , ρ_2 , and ρ_3 facilitates the controller design with tunable convergence characteristics for both the reaching phase and the settling phase. Specifically, m_1 and m_2 govern the reaching and settling phases, respectively, while ρ_1, ρ_2 act as convergence gains and ρ_3 determines the residual bound R via the equation in Theorem 3.1. These parameters are decoupled: increasing ρ_1 or ρ_2 reduces R and accelerates convergence at the cost of higher control effort, and vice versa. It is worth noting that while the stability criteria with two power terms have appeared in the integer-order systems literature (for instance, in [17, 36]), the extension to FOSs with a rigorous proof avoiding the fractional-order chain rule, as presented in Theorem 3.1, appears to be new. Moreover, because the analysis directly estimates the fractional-order derivative $\mathcal{D}^\alpha \mathcal{W}(x(t))$ through Lemma 2.3 without expanding the composite function $\mathcal{W}(x(t))$, the memory effect inherent in the Caputo kernel $(t - s)^{-\alpha}$ is preserved without simplification.

Remark 3.3. The general form of Theorem 3.1 encompasses various special cases. For instance, when $m_2 = 1$, the criterion involves a linear term combined with a fractional power term. It should be emphasized that even in this special case, the rigorous proof of Theorem 3.1 fundamentally differs from existing approaches, as it avoids the fractional-order chain rule for composite functions. Moreover, when $\alpha \rightarrow 1$ and $m_2 = 1$, the criterion reduces to the integer-order practical finite-time stability result reported in [34].

3.2. Controllers design and stability analysis

In this subsection, an adaptive controller is designed for system (2.2). Then, we analyze the stability of the controlled system and establish the semi-global practical finite-time stability of the tracking error by invoking the result of Theorem 3.1.

In the subsequent backstepping design, the unknown continuous functions $\mathcal{F}_i(X_i)$ will be approximated by RBFNNs according to Lemma 2.6 on prescribed compact sets. The semi-global nature of the resulting stability guarantee and the rigorous justification that the closed-loop trajectories remain within the approximation domains are detailed in Remark 3.5.

Define the following error system:

$$\begin{cases} \zeta_1 = \vartheta_1 - y_d, \\ \zeta_i = \vartheta_i - \chi_{i-1}, \quad i = 2, \dots, n, \end{cases} \quad (3.4)$$

where y_d is the reference signal, and $\chi_i (1 \leq i \leq n-1)$ is the virtual control signal. Next, we will design the controller for system (2.2) based on the backstepping method by n steps.

Step 1. Construct the Lyapunov function as follows:

$$V_1 = \frac{1}{2}\zeta_1^2 + \frac{1}{2\kappa_1}\tilde{\xi}_1^2, \quad (3.5)$$

where $\kappa_1 > 0$ is a designed parameter, $\tilde{\xi}_1 = \hat{\xi}_1 - \xi_1$ is the estimation error of $\xi_1 = \bar{\psi}_1^{-1}\|W_1^*\|$, $\hat{\xi}_1$ denotes the estimate value of ξ_1 , and W_1^* will be defined later. The fractional-order derivative of V_1 can be calculated as

$$\begin{aligned} \mathcal{D}^\alpha V_1 &\leq \zeta_1 \mathcal{D}^\alpha \zeta_1 + \kappa_1^{-1} \tilde{\xi}_1 \mathcal{D}^\alpha \hat{\xi}_1 \\ &= \zeta_1 [\phi_1(\vartheta) + \psi_1(\vartheta)(\zeta_2 + \chi_1) - \mathcal{D}^\alpha y_d] + \kappa_1^{-1} \tilde{\xi}_1 \mathcal{D}^\alpha \hat{\xi}_1 \\ &= \zeta_1 [\mathcal{F}_1(X_1) + \psi_1(\vartheta)\chi_1] + \kappa_1^{-1} \tilde{\xi}_1 \mathcal{D}^\alpha \hat{\xi}_1, \end{aligned} \quad (3.6)$$

where $\mathcal{F}_1(X_1) = \phi_1(\vartheta) + \psi_1(\vartheta)\zeta_2 - \mathcal{D}^\alpha y_d$, with $X_1 = (\vartheta^T, \hat{\xi}_1, y_d, \mathcal{D}^\alpha y_d)^T$. By using Lemma 2.6, the continuous function $\mathcal{F}_1(X_1)$ can be approximated as $\mathcal{F}_1(X_1) = W_1^{*T}\Psi_1(X_1) + \epsilon_1(X_1)$, where W_1^* is the ideal weight vector and $|\epsilon_1(X_1)| < \tilde{\epsilon}_1$ is the approximation error with positive constant $\tilde{\epsilon}_1$. Then, it follows from Lemma 2.7 and the Young's inequality that

$$\begin{aligned} \zeta_1 \mathcal{F}_1(X_1) &\leq |\zeta_1|(\|W_1^*\| \|\Psi_1(X_1)\| + \epsilon_1(X_1)) \\ &\leq |\zeta_1|(\|W_1^*\| \|\Psi_1(\bar{X}_1)\| + \tilde{\epsilon}_1) \\ &\leq \frac{\zeta_1^2}{2a_1^2} \bar{\psi}_1 \xi_1 \Psi_1^T(\bar{X}_1) \Psi_1(\bar{X}_1) + \frac{a_1^2}{2} + \frac{\zeta_1^2}{2} + \frac{\tilde{\epsilon}_1^2}{2}, \end{aligned} \quad (3.7)$$

where $\bar{X}_1 = (\vartheta_1, \hat{\xi}_1, y_d)^T$, and $a_1 > 0$ is a designed parameter. Taking (3.7) into (3.6), one has

$$\mathcal{D}^\alpha V_1 \leq \zeta_1 \psi_1(\vartheta) \chi_1 + \frac{\zeta_1^2}{2a_1^2} \bar{\psi}_1 \xi_1 \Psi_1^T(\bar{X}_1) \Psi_1(\bar{X}_1) + \frac{\zeta_1^2}{2} + \kappa_1^{-1} \tilde{\xi}_1 \mathcal{D}^\alpha \hat{\xi}_1 + \frac{a_1^2}{2} + \frac{\tilde{\epsilon}_1^2}{2}. \quad (3.8)$$

Then, we can design the virtual controller χ_1 and the adaptive law of $\hat{\xi}_1$ as follows:

$$\chi_1 = -\frac{\zeta_1}{2\bar{\psi}_1} - \frac{\zeta_1}{2a_1^2} \hat{\xi}_1 \Psi_1^T(\bar{X}_1) \Psi_1(\bar{X}_1) - \bar{\psi}_1^{-1} [\beta_{11} \zeta_1^{2m_1-1} + \beta_{12} \zeta_1^{2m_2-1}], \quad (3.9)$$

$$\mathcal{D}^\alpha \hat{\xi}_1 = \frac{\kappa_1 \zeta_1^2}{2a_1^2} \bar{\psi}_1 \Psi_1^T(\bar{X}_1) \Psi_1(\bar{X}_1) - \hat{\xi}_1^{2m_1-1} - \hat{\xi}_1^{2m_2-1}, \quad (3.10)$$

where $\frac{1}{2} < m_1 < 1 < m_2$, and $\beta_{11}, \beta_{12} > 0$ are designed parameters. Substituting (3.9) and (3.10) into (3.8), and using Assumption 2.10, we can obtain

$$\mathcal{D}^\alpha V_1 \leq -\beta_{11}\zeta_1^{2m_1} - \beta_{12}\zeta_1^{2m_2} - \kappa_1^{-1}\tilde{\xi}_1(\hat{\xi}_1^{2m_1-1} + \hat{\xi}_1^{2m_2-1}) + \frac{a_1^2}{2} + \frac{\tilde{\xi}_1^2}{2}. \quad (3.11)$$

Step i ($2 \leq i \leq n-1$). Consider the Lyapunov function

$$V_i = V_{i-1} + \frac{1}{2}\zeta_i^2 + \frac{1}{2\kappa_i}\tilde{\xi}_i^2, \quad (3.12)$$

where $\kappa_i > 0$ is a designed parameter, $\tilde{\xi}_i = \hat{\xi}_i - \xi_i$ denotes the estimation error of ξ_i with $\xi_i = \bar{\psi}_i^{-1}\|W_i^*\|$, $\hat{\xi}_i$ representing the estimate of ξ_i , and W_i^* to be defined later. The derivative of V_i is

$$\begin{aligned} \mathcal{D}^\alpha V_i &\leq \mathcal{D}^\alpha V_{i-1} + \zeta_i \mathcal{D}^\alpha \zeta_i + \kappa_i^{-1}\tilde{\xi}_i \mathcal{D}^\alpha \hat{\xi}_i \\ &\leq \mathcal{D}^\alpha V_{i-1} + \zeta_i [\mathcal{F}_i(X_i) + \psi_i(\vartheta)\chi_i] + \kappa_i^{-1}\tilde{\xi}_i \mathcal{D}^\alpha \hat{\xi}_i, \end{aligned} \quad (3.13)$$

where $\mathcal{F}_i(X_i) = \phi_i(\vartheta) + \psi_i(\vartheta)\zeta_{i+1} - \mathcal{D}^\alpha \chi_{i-1}$ with $X_i = (\vartheta^T, \hat{\xi}_1, \dots, \hat{\xi}_i, y_d, \mathcal{D}^\alpha y_d)^T$ and $\mathcal{D}^\alpha \chi_{i-1} \approx \sum_{j=1}^{i-1} \frac{\partial \chi_{i-1}}{\partial \vartheta_j} \mathcal{D}^\alpha \vartheta_j + \sum_{j=1}^{i-1} \frac{\partial \chi_{i-1}}{\partial \hat{\xi}_j} \mathcal{D}^\alpha \hat{\xi}_j + \frac{\partial \chi_{i-1}}{\partial y_d} \mathcal{D}^\alpha y_d + \tilde{\epsilon}_{i-1}$ [37, 38], and $\tilde{\epsilon}_{i-1}$ denotes the approximation error due to the fractional-order chain rule. Since $\mathcal{F}_i(X_i)$ is approximated as a whole by the NN, the error $\tilde{\epsilon}_{i-1}$ is inherently absorbed into the NN approximation error $\epsilon_i(X_i)$, which is uniformly bounded by $\tilde{\epsilon}_i$ on the compact domain. Lemma 2.6 implies that the continuous function $\mathcal{F}_i(X_i)$ can be approximated as $\mathcal{F}_i(X_i) = W_i^{*T}\Psi_i(X_i) + \epsilon_i(X_i)$ with $|\epsilon_i(X_i)| < \tilde{\epsilon}_i$. By using Young's inequality and Lemma 2.7, we have

$$\zeta_i \mathcal{F}_i(X_i) \leq |\zeta_i|(\|W_i^*\| \|\Psi_i(X_i)\| + \epsilon_i(X_i)) \leq \frac{\zeta_i^2}{2a_i^2} \bar{\psi}_i \xi_i \Psi_i^T(\bar{X}_i) \Psi_i(\bar{X}_i) + \frac{a_i^2}{2} + \frac{\zeta_i^2}{2} + \frac{\tilde{\xi}_i^2}{2}, \quad (3.14)$$

where $\bar{X}_i = (\vartheta_1, \dots, \vartheta_i, \hat{\xi}_1, \dots, \hat{\xi}_i, y_d)^T$, and $a_i > 0$ is a designed parameter. Thus, the derivative of V_i can be rewritten as follows:

$$\mathcal{D}^\alpha V_i \leq \mathcal{D}^\alpha V_{i-1} + \zeta_i \psi_i(\vartheta) \chi_i + \frac{\zeta_i^2}{2a_i^2} \bar{\psi}_i \xi_i \Psi_i^T(\bar{X}_i) \Psi_i(\bar{X}_i) + \frac{\zeta_i^2}{2} + \kappa_i^{-1}\tilde{\xi}_i \mathcal{D}^\alpha \hat{\xi}_i + \frac{a_i^2}{2} + \frac{\tilde{\xi}_i^2}{2}. \quad (3.15)$$

The virtual controller χ_i and the adaptive law of $\hat{\xi}_i$ are designed as

$$\chi_i = -\frac{\zeta_i}{2\bar{\psi}_i} - \frac{\zeta_i}{2a_i^2} \hat{\xi}_i \Psi_i^T(\bar{X}_i) \Psi_i(\bar{X}_i) - \bar{\psi}_i^{-1} [\beta_{i1}\zeta_i^{2m_1-1} + \beta_{i2}\zeta_i^{2m_2-1}], \quad (3.16)$$

$$\mathcal{D}^\alpha \hat{\xi}_i = \frac{\kappa_i \zeta_i^2}{2a_i^2} \bar{\psi}_i \Psi_i^T(\bar{X}_i) \Psi_i(\bar{X}_i) - \hat{\xi}_i^{2m_1-1} - \hat{\xi}_i^{2m_2-1}, \quad (3.17)$$

where $\beta_{i1}, \beta_{i2} > 0$ are designed parameters. Plugging (3.16) and (3.17) into $\mathcal{D}^\alpha V_i$, and according to Assumption 2.10, one has

$$\mathcal{D}^\alpha V_i \leq \sum_{j=1}^i \left[-\beta_{j1}\zeta_j^{2m_1} - \beta_{j2}\zeta_j^{2m_2} - \kappa_j^{-1}\tilde{\xi}_j(\hat{\xi}_j^{2m_1-1} + \hat{\xi}_j^{2m_2-1}) + \frac{a_j^2}{2} + \frac{\tilde{\xi}_j^2}{2} \right]. \quad (3.18)$$

Step n . Consider the following Lyapunov function:

$$V_n = V_{n-1} + \frac{1}{2}\zeta_n^2 + \frac{1}{2\kappa_n}\tilde{\xi}_n^2, \quad (3.19)$$

where $\kappa_n > 0$ is a designed parameter, $\tilde{\xi}_n = \hat{\xi}_n - \xi_n$ denotes the estimation error of $\xi_n = \bar{\psi}_n^{-1} \|W_n^*\|$, $\hat{\xi}_n$ representing the estimate of ξ_n , and W_n^* to be defined later. The fractional-order derivative of V_n can be calculated as

$$\mathcal{D}^\alpha V_n \leq \mathcal{D}^\alpha V_{n-1} + \zeta_n [\mathcal{F}_n(X_n) + \psi_n(\vartheta)u] + \kappa_n^{-1} \tilde{\xi}_n \mathcal{D}^\alpha \hat{\xi}_n, \quad (3.20)$$

where $\mathcal{F}_n(X_n) = \phi_n(\vartheta) - \mathcal{D}^\alpha \chi_{n-1}$ with $X_n = (\vartheta^T, \hat{\xi}_1, \dots, \hat{\xi}_n, y_d, \mathcal{D}^\alpha y_d)^T$. Based on Lemma 2.6, $\mathcal{F}_n(X_n)$ can be approximated as $\mathcal{F}_n(X_n) = W_n^{*T} \Psi_n(X_n) + \epsilon_n(X_n)$ with $|\epsilon_n(X_n)| < \tilde{\epsilon}_n$. Then, it follows from Young's inequality and Lemma 2.7 that

$$\zeta_n \mathcal{F}_n(X_n) \leq |\zeta_n| (\|W_n^*\| \|\Psi_n(X_n)\| + \epsilon_n(X_n)) \leq \frac{\zeta_n^2}{2a_n^2} \bar{\psi}_n \xi_n \Psi_n^T(\bar{X}_n) \Psi_n(\bar{X}_n) + \frac{a_n^2}{2} + \frac{\zeta_n^2}{2} + \frac{\tilde{\epsilon}_n^2}{2},$$

where $\bar{X}_n = (\vartheta^T, \hat{\xi}_1, \dots, \hat{\xi}_n, y_d)^T$, and $a_n > 0$ is a designed constant. Thus, the derivative of V_n is

$$\mathcal{D}^\alpha V_n \leq \mathcal{D}^\alpha V_{n-1} + \zeta_n \psi_n(\vartheta)u + \frac{\zeta_n^2}{2a_n^2} \bar{\psi}_n \xi_n \Psi_n^T(\bar{X}_n) \Psi_n(\bar{X}_n) + \frac{\zeta_n^2}{2} + \kappa_n^{-1} \tilde{\xi}_n \mathcal{D}^\alpha \hat{\xi}_n + \frac{a_n^2}{2} + \frac{\tilde{\epsilon}_n^2}{2}. \quad (3.21)$$

The final controller u and the adaptive law of $\hat{\xi}_n$ are designed as

$$u = -\frac{\zeta_n}{2\bar{\psi}_n} - \frac{\zeta_n}{2a_n^2} \hat{\xi}_n \Psi_n^T(\bar{X}_n) \Psi_n(\bar{X}_n) - \bar{\psi}_n^{-1} [\beta_{n1} \zeta_n^{2m_1-1} + \beta_{n2} \zeta_n^{2m_2-1}], \quad (3.22)$$

$$\mathcal{D}^\alpha \hat{\xi}_n = \frac{\kappa_n \zeta_n^2}{2a_n^2} \bar{\psi}_n \Psi_n^T(\bar{X}_n) \Psi_n(\bar{X}_n) - \hat{\xi}_n^{2m_1-1} - \hat{\xi}_n^{2m_2-1}, \quad (3.23)$$

where $\beta_{n1}, \beta_{n2} > 0$ are designed parameters. Then, it follows from (3.21)–(3.23) and Assumption 2.10 that

$$\mathcal{D}^\alpha V_n \leq \sum_{j=1}^n \left[-\beta_{j1} \zeta_j^{2m_1} - \beta_{j2} \zeta_j^{2m_2} - \kappa_j^{-1} \tilde{\xi}_j (\hat{\xi}_j^{2m_1-1} + \hat{\xi}_j^{2m_2-1}) + \frac{a_j^2}{2} + \frac{\tilde{\epsilon}_j^2}{2} \right]. \quad (3.24)$$

Eventually, the above results can be summarized as the following theorem.

Theorem 3.4. Consider system (2.2) under Assumptions 2.10 and 2.11. The proposed control scheme, comprised of virtual controllers (3.9) and (3.16), a final controller (3.22), and adaptive laws (3.10), (3.17), and (3.23), ensures the following: (i) all signals in the closed-loop system are bounded; and (ii) the output tracking error $\zeta_1 = \vartheta_1 - y_d$ is semi-globally practically finite-time stable within a finite settling time.

Proof. By Lemma 2.8, we can find positive constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$-\tilde{\xi}_j \hat{\xi}_j^{2m_1-1} = -\tilde{\xi}_j (\tilde{\xi}_j + \xi_j)^{2m_1-1} \leq -\lambda_1 \tilde{\xi}_j^{2m_1} + \lambda_2 \xi_j^{2m_1}, \quad (3.25)$$

$$-\tilde{\xi}_j \hat{\xi}_j^{2m_2-1} = -\tilde{\xi}_j (\tilde{\xi}_j + \xi_j)^{2m_2-1} \leq -\lambda_3 \tilde{\xi}_j^{2m_2} + \lambda_4 \xi_j^{2m_2}. \quad (3.26)$$

Let $\mathcal{W} = V_n$. We have

$$\mathcal{D}^\alpha \mathcal{W} \leq -\sum_{j=1}^n [\beta_{j1} \zeta_j^{2m_1} + \beta_{j2} \zeta_j^{2m_2} + \kappa_j^{-1} (\lambda_1 \tilde{\xi}_j^{2m_1} + \lambda_3 \tilde{\xi}_j^{2m_2})] + \rho_3, \quad (3.27)$$

where $\rho_3 = \sum_{j=1}^n [\frac{a_j^2}{2} + \frac{\tilde{\epsilon}_j^2}{2} + \kappa_j^{-1} (\lambda_2 \xi_j^{2m_1} + \lambda_4 \xi_j^{2m_2})]$. Then, utilizing Lemma 2.9 and the Jensen's inequality means that

$$\begin{aligned}
-\sum_{j=1}^n [\beta_{j1} \zeta_j^{2m_1} + \kappa_j^{-1} \lambda_1 \tilde{\xi}_j^{2m_1}] &= -\sum_{j=1}^n \left[2^{m_1} \beta_{j1} \left(\frac{1}{2} \zeta_j^2 \right)^{m_1} + \kappa_j^{-1} \lambda_1 (2\kappa_j)^{m_1} \left(\frac{1}{2\kappa_j} \tilde{\xi}_j^2 \right)^{m_1} \right] \\
&\leq -\rho_1 \sum_{j=1}^n \left[\left(\frac{1}{2} \zeta_j^2 \right)^{m_1} + \left(\frac{1}{2\kappa_j} \tilde{\xi}_j^2 \right)^{m_1} \right] \leq -\rho_1 \mathcal{W}^{m_1}, \\
-\sum_{j=1}^n [\beta_{j2} \zeta_j^{2m_2} + \kappa_j^{-1} \lambda_3 \tilde{\xi}_j^{2m_2}] &= -\sum_{j=1}^n \left[2^{m_2} \beta_{j2} \left(\frac{1}{2} \zeta_j^2 \right)^{m_2} + \kappa_j^{-1} \lambda_3 (2\kappa_j)^{m_2} \left(\frac{1}{2\kappa_j} \tilde{\xi}_j^2 \right)^{m_2} \right] \\
&\leq -\bar{\rho}_2 \sum_{j=1}^n \left[\left(\frac{1}{2} \zeta_j^2 \right)^{m_2} + \left(\frac{1}{2\kappa_j} \tilde{\xi}_j^2 \right)^{m_2} \right] \\
&\leq -\rho_2 \left[\sum_{j=1}^n \left(\frac{1}{2} \zeta_j^2 + \frac{1}{2\kappa_j} \tilde{\xi}_j^2 \right) \right]^{m_2} = -\rho_2 \mathcal{W}^{m_2},
\end{aligned}$$

where $\rho_1 = 2^{m_1} \min\{\beta_{j1}, \lambda_1 \kappa_j^{m_1-1}, j = 1, 2, \dots, n\}$, $\bar{\rho}_2 = 2^{m_2} \min\{\beta_{j2}, \lambda_3 \kappa_j^{m_2-1}, j = 1, 2, \dots, n\}$, and $\rho_2 = 2n^{1-m_2} \bar{\rho}_2$. Thus, (3.27) can be rewritten as follows:

$$\mathcal{D}^\alpha \mathcal{W} \leq -\rho_1 \mathcal{W}^{m_1} - \rho_2 \mathcal{W}^{m_2} + \rho_3. \quad (3.28)$$

According to Theorem 3.1, system (2.2) accomplishes practical finite-time tracking control to the reference signal y_d , and ζ_i and the estimation errors $\tilde{\xi}_i$ are all bounded. Since each ξ_i is constant, $\hat{\xi}_i$ are bounded for $i = 1, \dots, n$. The boundedness of y_d implies that $\vartheta_1 = \zeta_1 + y_d$ is bounded. Consequently, the virtual controller χ_1 defined in (3.9) is bounded. Progressively, the boundedness of the state variables ϑ_i ($i = 2, \dots, n$), virtual controllers χ_1 ($i = 2, \dots, n-1$), and final controller u can be analyzed. $\square$

Remark 3.5. Theorem 3.4 establishes semi-global practical finite-time stability because the NN approximation in Lemma 2.6 is only valid on a compact set. The validity of the approximation throughout the closed-loop operation is guaranteed as follows. For any prescribed compact set Ω_0 of initial conditions, let $c_0 = \max_{X(0) \in \Omega_0} V_n(X(0))$, let R be the residual bound in Theorem 3.1 determined by ρ_1, ρ_2, ρ_3 , and define the level set

$$\Omega_V = \{X \mid V_n(X) \leq \bar{R}\}, \quad \bar{R} := \max\{c_0, R\}.$$

Because the Lyapunov function V_n is radially unbounded, Ω_V is compact. Moreover, every trajectory starting from Ω_0 remains in Ω_V for all $t \geq 0$. Indeed, Theorem 3.1 guarantees the following uniform estimate, which directly follows from its proof: if $V_n(X(0)) \leq R$, then the invariance argument in Case I of the proof yields $V_n(X(t)) \leq R$ for all $t \geq 0$; and if $V_n(X(0)) > R$, then as long as the trajectory stays in $\{V_n > R\}$, the estimate (3.3) gives $\mathcal{D}^\alpha V_n^{1+m_1}(X(t)) \leq 0$, which, by fractional integration, implies $V_n(X(t)) \leq V_n(X(0))$ until the trajectory enters $\{V_n \leq R\}$, after which it remains therein by the same invariance argument. Hence, $V_n(X(t)) \leq \max\{V_n(X(0)), R\} \leq \bar{R}$ for all $t \geq 0$. Then, one chooses a sufficiently large compact domain Ω_{X_i} containing the projection of Ω_V onto the X_i -space and a sufficiently large node number k_2 such that Lemma 2.6 holds with uniform error bound $\bar{\epsilon}_i$ on Ω_{X_i} , which ensures that the trajectories remain inside the NN approximation domain for all time, thereby avoiding circular reasoning.

Remark 3.6. It should be emphasized that Theorem 3.4 guarantees the boundedness of the estimation errors $\tilde{\xi}_i$ (i.e., stability of the adaptation loop), rather than their asymptotic convergence to zero. The parameter estimates $\hat{\xi}_i$ converge to a compact set around the true values within finite time, which is sufficient for practical finite-time tracking control. This is standard in adaptive NN control due to the presence of approximation errors $\epsilon_i(X_i)$ and the lack of persistent excitation.

Remark 3.7. Letting $\alpha \rightarrow 1$ in Theorem 3.1 recovers the classical integer-order semi-global practical finite-time stability criterion, and the control scheme in Theorem 3.4 reduces accordingly to the integer-order case. This specialization serves a dual purpose: it validates that the proposed fractional-order framework is consistent with the classical theory in the limit, and it provides a ready-to-use design for systems governed by integer-order dynamics.

Remark 3.8. It is worth noting that the exponents m_1, m_2 ($\frac{1}{2} < m_1 < 1 < m_2$) in the virtual controllers (3.9) and (3.16), the final controller (3.22), and the adaptive laws (3.10), (3.17), and (3.23) are essential to achieve semi-global practical finite-time stability. If one sets $m_1 = m_2 = 1$ in these control and update laws, then the fractional-order derivative of the Lyapunov function $\mathcal{W}$ in (3.27) reduces to

$$\mathcal{D}^\alpha \mathcal{W} \leq -(\rho_1 + \rho_2) \mathcal{W} + \rho_3,$$

which is the standard form for semi-global practical asymptotic stability analysis. In this degenerate case, the closed-loop system can only guarantee that the tracking error converges to a small neighborhood of the origin as $t \rightarrow \infty$, rather than within a finite time. Consequently, the proposed control scheme reduces to a conventional adaptive backstepping design, and the finite-time convergence property is lost.

Remark 3.9. Several practical aspects of the proposed controller are clarified as follows.

- **State availability:** The backstepping design requires all states $\vartheta_1, \dots, \vartheta_n$ to be available for feedback.
- **Fractional-order computation:** The control input u in (3.22) and the virtual controllers χ_i in (3.9) and (3.16) only involve algebraic operations on measurable states and estimated parameters, and are therefore real-time computable. The fractional-order derivatives appearing in the adaptive laws (3.10), (3.17), and (3.23) are implemented numerically in discrete time via the predictor–corrector scheme with a fixed sampling period.
- **Computational burden:** The main computational cost per step is the evaluation of the Gaussian RBF vector $\Psi_i(\bar{X}_i)$ (dimension k_2) and the scalar parameter updates $\hat{\xi}_i$. For typical node numbers $k_2 \leq 20$, this cost is modest and compatible with standard embedded platforms.
- **Memory requirement:** Due to the nonlocal nature of the Caputo derivative, the adaptive laws require storing historical data of Ψ_i and $\hat{\xi}_i$, with memory cost growing linearly with the number of simulation steps. In long-horizon applications, the short-memory principle can be adopted to limit this growth.

4. Numerical simulations

In this section, numerical simulation examples are presented to demonstrate the effectiveness and verify the theoretical results of the proposed control scheme.

To facilitate numerical simulations, we provide the following guideline for parameter selection. In adaptive NN control for systems with unknown nonlinearities, the controller parameters cannot be computed from explicit formulas because the closed-loop coefficients ρ_1, ρ_2, ρ_3 in Theorem 3.1 depend on unknown ideal weights and approximation errors. Nevertheless, the parameters can be systematically selected according to the following monotonic relationships and iterative tuning steps:

- Step 1 *Convergence-phase shaping*: Choose $m_1 \in (\frac{1}{2}, 1)$ and $m_2 > 1$ to determine the power-law structure. A smaller m_1 enhances the reaching-phase convergence rate when $|\zeta_i| > 1$, while a larger m_2 strengthens the settling-phase damping when $|\zeta_i| < 1$.
- Step 2 *Gain calibration*: Select β_{i1} and β_{i2} to adjust the convergence speed. Increasing β_{i1} reduces the reaching time, and increasing β_{i2} tightens the residual set; both, however, increase the control magnitude.
- Step 3 *Approximation-accuracy tuning*: Choose a_i to balance Young's inequality slack. A smaller a_i reduces the residual contribution from NN approximation errors but raises the control gain.
- Step 4 *Adaptation-rate setting*: Select κ_i to govern the adaptation speed. A larger κ_i accelerates the parameter estimation and tends to reduce the steady-state residual.
- Step 5 *A posteriori verification*: Simulate the closed-loop response and iteratively adjust $\beta_{i1}, \beta_{i2}, a_i, \kappa_i$ until the empirical settling time and residual bound meet the application requirements.

This procedure reflects the standard practice in adaptive NN control: the structural relationship between parameters and performance is rigorously established by Theorem 3.1, while the exact numerical values are validated through simulations due to the presence of unknown dynamics.

Example 4.1. Consider the FONS as follows:

$$\begin{cases} \mathcal{D}^{0.9}\vartheta_1(t) = 2\vartheta_2 + 0.25 - \sin(\vartheta_1\vartheta_3), \\ \mathcal{D}^{0.9}\vartheta_2(t) = \vartheta_3 - \vartheta_2^2\vartheta_3e^{-\vartheta_2^2}, \\ \mathcal{D}^{0.9}\vartheta_3(t) = u + \vartheta_3\cos(\vartheta_1\vartheta_2), \\ y = \vartheta_1, \end{cases} \quad (4.1)$$

with the initial values $\vartheta_1(0) = 0.2, \vartheta_2(0) = -0.1$, and $\vartheta_3(0) = -0.2$. The reference signal $y_d = \sin(t) + \cos(1.4t)$.

To achieve a semi-global practical finite-time control of system (4.1), the parameters are tuned with the following qualitative objectives: (i) fast initial convergence of the tracking error; (ii) a satisfactorily small steady-state residual; and (iii) moderate control amplitude. Accordingly, the control parameters are selected as follows: $a_1 = 2, a_2 = 1.2, a_3 = 2.1, \beta_{11} = 2.8, \beta_{21} = 6.4, \beta_{31} = 15.5, \beta_{12} = 3.1, \beta_{22} = 5.1, \beta_{32} = 15.4, \tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\psi}_3 = 1, \kappa_1 = 16.45, \kappa_2 = 9.4, \kappa_3 = 10.81, m_1 = \frac{19}{23}$, and $m_2 = \frac{25}{21}$. The initial values of the parameter adaptive laws are $(\hat{\xi}_1(0), \hat{\xi}_2(0), \hat{\xi}_3(0)) = (0.1, 0.1, 0.1)$. The parameters of the radial basis function NNs are $\tilde{\omega}_i = (-3, -2, -1, 0, 1, 2, 3)^T$ and $\delta_i = 0.5$ ($i = 1, \dots, k_2$).

Numerical simulation results are shown in Figures 1–6. Figure 1 depicts the trajectories of the reference signal y_d and the system output $y = \vartheta_1$. Figure 2 presents the trajectory of the tracking error

$\zeta_1 = \vartheta_1 - y_d$. It can be observed from Figures 1 and 2 that the system output y can track the reference signal y_d with a small bounded error under the proposed practical finite-time controller. Moreover, from Figure 2, the tracking error exhibits a fast initial decay and settles to a small bounded residual ($|\zeta_1| \leq 0.3$ after $t \approx 1.5$), which confirms the effectiveness of the parameter tuning strategy. It should be noted that this residual magnitude is empirically obtained from the simulation, as the exact theoretical bound R in Theorem 3.1 cannot be computed a priori due to the presence of unknown ideal NN weights and approximation errors in ρ_3 . Figures 3 and 4 illustrate the trajectories of the system state variables ϑ_2 and ϑ_3 , respectively. Figures 5 and 6 show the control input u and the estimated values of the parameters ξ_1, ξ_2, ξ_3 , respectively. All these figures demonstrate that all signals in the closed-loop system remain bounded.

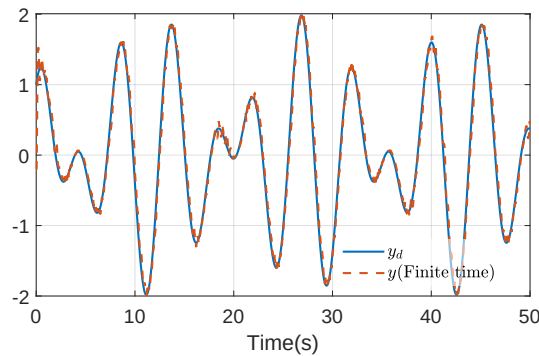


Figure 1. The reference signal y_d and system output $y = \vartheta_1$.

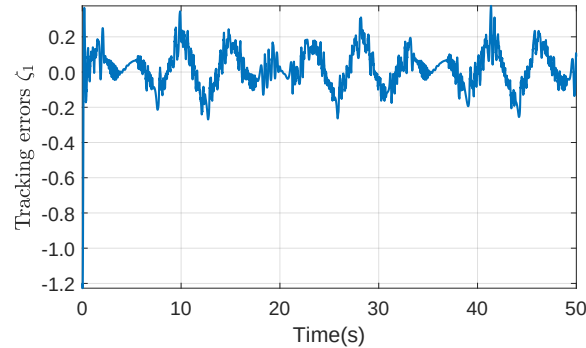


Figure 2. The tracking error $\zeta_1 = \vartheta_1 - y_d$.

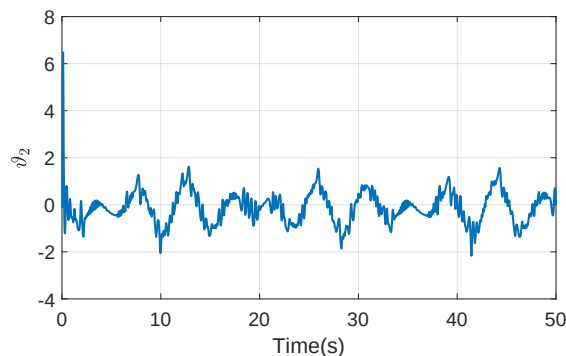


Figure 3. The system state variable ϑ_2 .

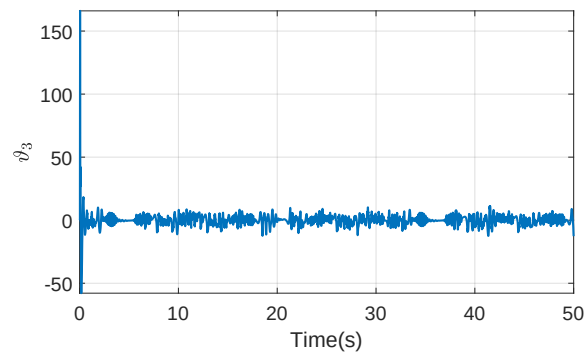


Figure 4. The system state variable ϑ_3 .

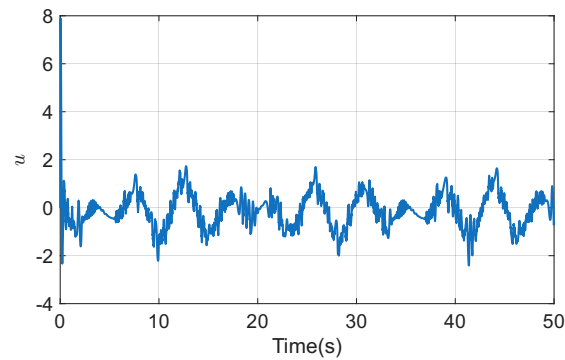


Figure 5. The control input u .

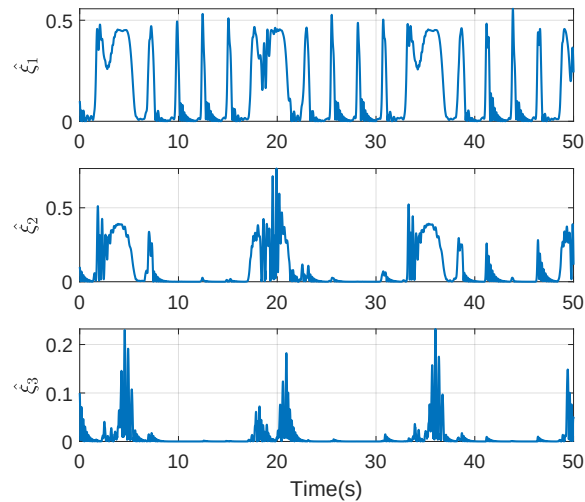


Figure 6. The estimated values of ξ_1 , ξ_2 , and ξ_3 .

Example 4.2. Consider the fractional-order Chua's system [39] as follows:

$$\begin{cases} \mathcal{D}^{0.95}\vartheta_1 = \nu_1(\vartheta_2 - \vartheta_1 - \nu_5\vartheta_1 - \mathcal{G}(\vartheta_1)), \\ \mathcal{D}^{0.95}\vartheta_2 = \vartheta_1 - \vartheta_2 + \vartheta_3, \\ \mathcal{D}^{0.95}\vartheta_3 = -\nu_2\vartheta_2 - \nu_3\vartheta_3 + u, \\ y = \vartheta_1, \end{cases} \quad (4.2)$$

where $\mathcal{G}(\vartheta_1) = \frac{1}{2}(\nu_4 - \nu_5)(|\vartheta_1 + 1| - |\vartheta_1 - 1|)$, $\nu_1 = 10.725$, $\nu_2 = 10.593$, $\nu_3 = 0.268$, $\nu_4 = -1.1726$, and $\nu_5 = -0.7872$. The initial value of system (4.2) is $(\vartheta_1(0), \vartheta_2(0), \vartheta_3(0)) = (0.2, -0.1, -0.2)$, and the reference signal is $y_d = 1.4 \sin(t) + \cos(1.4t)$.

To achieve practical finite-time control of system (4.2), the control parameters are selected as follows: $a_1 = 0.5$, $a_2 = a_3 = 2$, $\beta_{11} = 12.6$, $\beta_{21} = 21.3$, $\beta_{31} = 51.5$, $\beta_{12} = 18$, $\beta_{22} = 20.3$, $\beta_{32} = 35.3$, $\bar{\psi}_1 = 10$, $\bar{\psi}_2 = \bar{\psi}_3 = 1$, $\kappa_1 = 7.6$, $\kappa_2 = 8.36$, $\kappa_3 = 7.6$, $m_1 = \frac{17}{21}$, and $m_2 = \frac{25}{21}$. The initial values of the parameter adaptive laws and the parameters of the radial basis function NNs are identical to those in Example 4.1.

As pointed out in Remark 3.8, the proposed controller degenerates into a non-finite-time controller when $m_1 = m_2 = 1$. Therefore, numerical simulations are also conducted for the case where $m_1 = m_2 = 1$ with the remaining parameters unchanged in this example. The numerical simulation results of system (4.2) under both the practical finite-time controller and the non-finite-time controller are shown in Figures 7–12. Figures 7 and 8 present the trajectories of the system output $y = \vartheta_1$ and the tracking error $\zeta_1 = \vartheta_1 - y_d$ under the two controllers, respectively. It can be seen that both controllers can effectively track the reference signal y_d , while the tracking error under the practical finite-time controller exhibits a smaller fluctuation range near the initial time. Figure 8 further shows that the finite-time controller achieves a steady-state residual of approximately $|\zeta_1| \leq 0.2$. These quantitative observations confirm that the proposed finite-time scheme yields superior steady-state tracking accuracy. The trajectory of the control input u is shown in Figure 9. It is observed that the practical finite-time controller exhibits smaller control fluctuations during the initial transient phase compared with the non-finite-time controller, while both controllers require comparable steady-state efforts after the initial period. This behavior can be attributed to the specific shape of the nonlinear power terms: when the tracking error is relatively large, the fractional power $2m_1 - 1 < 1$ yields a different control profile from linear feedback, and the faster convergence of the finite-time scheme reduces the duration of significant control transients. It should be noted that finite-time control generally involves a trade-off between the convergence speed and the control energy. Here, the observed reduction in initial fluctuation is a consequence of the specific parameter tuning and does not contradict this well-known trade-off. Figures 10 and 11 illustrate the trajectories of the state variables ϑ_2 and ϑ_3 under the practical finite-time controller, respectively. Figure 12 shows the estimated values of the unknown parameters ξ_1, ξ_2 , and ξ_3 under the practical finite-time controller. The above results demonstrate that, compared with the non-finite-time controller, the proposed practical finite-time controller achieves superior transient performances in terms of both the tracking accuracy and the control energy consumption.

Remark 4.1. For a fair comparison, the non-finite-time controller is obtained by setting $m_1 = m_2 = 1$ in the same control structure (3.22) and adaptive laws (3.23), with all other parameters identical to the finite-time case. Thus, both controllers have the same number of NN nodes, the same adaptive parameters, and the same backstepping structure; the only difference is the power-law terms versus linear terms.

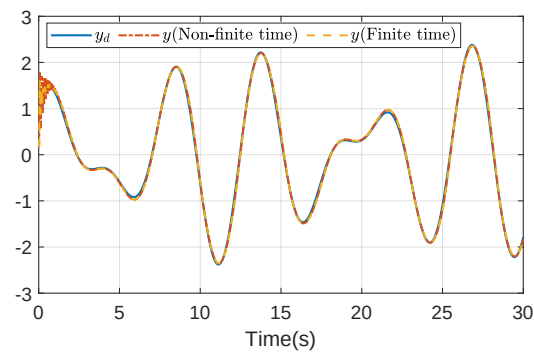


Figure 7. The reference signal y_d and system output $y = \vartheta_1$.

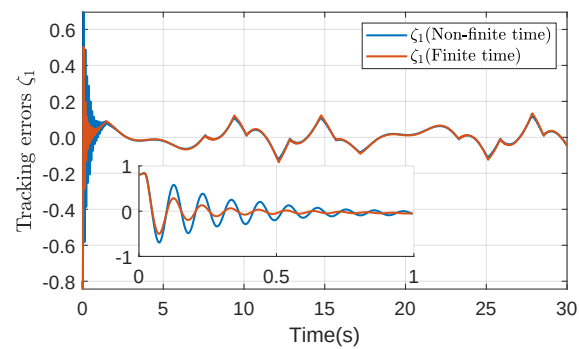


Figure 8. The tracking error $\zeta_1 = \vartheta_1 - y_d$.

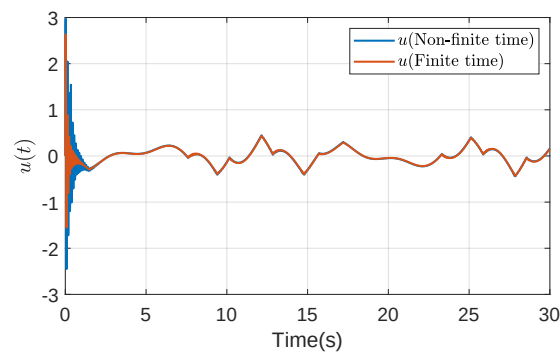


Figure 9. The control input u .

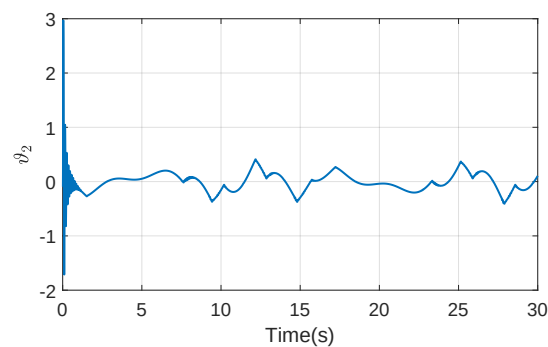


Figure 10. The system state variable ϑ_2 under the practical finite-time controller.

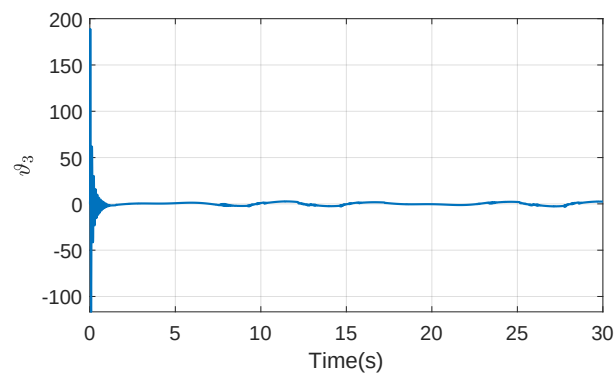


Figure 11. The system state variable ϑ_3 under the practical finite-time controller.

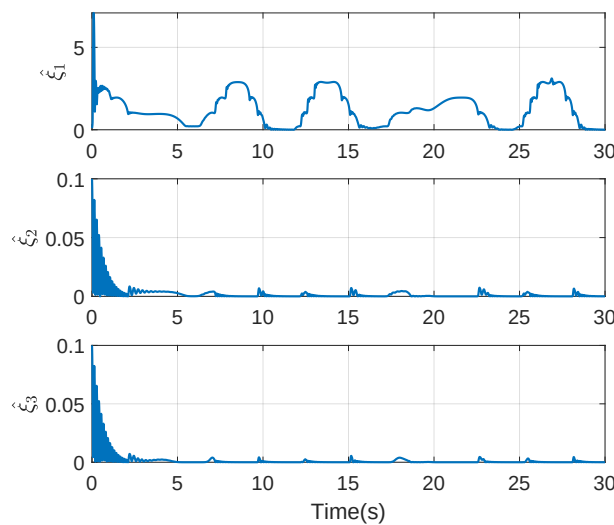


Figure 12. The estimated values of ξ_1 , ξ_2 , and ξ_3 under the practical finite-time controller.

5. Conclusions

In this paper, the semi-global practical finite-time adaptive NN control problem was investigated for a class of FONSSs with unknown nonlinear dynamics. A novel practical finite-time stability criterion was established, whose strict proof does not rely on the fractional-order chain rule for composite functions. Based on this criterion, an adaptive NN controller was designed by the NN approximation approach, thereby guaranteeing that all signals of the closed-loop system are bounded and the tracking error converges to a small neighborhood of the origin within a finite time. It is worth noting that although NNs are employed to learn unknown nonlinearities, the proposed scheme is fundamentally a model-based adaptive control design: the strict-feedback structure is explicitly preserved, and the Lyapunov-based stability proof relies on this available structural information. Consequently, the proposed method differs from fully model-free data-driven approaches (such as [40]), where the absence of the system structure poses fundamentally different challenges for the stability analysis. This distinction aligns with the general understanding emphasized in recent literature on control and optimization techniques, which highlights that stability guarantees are

inherently tied to the available model or structural knowledge. Future research directions include extending the proposed stability criterion to FONSSs with time delays, incorporating event-triggered control strategies to reduce the communication burden, investigating discrete-time implementation of the proposed continuous-time controller with a rigorous sampled-data stability analysis, and applying the developed results to real physical systems such as fractional-order robotic manipulators.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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