



Research article

Long-time well-posedness of analytic solutions to the Boussinesq boundary layer equations

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Abstract: This paper addresses the well-posedness of the two-dimensional Boussinesq boundary layer system without monotonicity constraints. Compared with the classical Prandtl equations, mutual velocity-temperature nonlinear coupling generates double derivative loss in both the momentum and thermal equations, which introduces new analytical obstacles beyond those appearing in the Prandtl setting and degenerates the solution regularity. To overcome this obstacle, we adopt analytic phase modulation and Gaussian-type decaying weights, construct anisotropic weighted Sobolev spaces, and combine refined nonlinear energy estimates and fractional damping absorption techniques. Under small initial data, we establish the long-time existence and uniqueness of analytic solutions, and derive the explicit solution lifespan scaling $T_\varepsilon \sim \varepsilon^{-4/3}$.

Keywords: Boussinesq equations; boundary layer; well-posedness; analytic solutions

1. Introduction

Many geophysical and engineering buoyancy driven flows in the atmosphere and oceans can be described by the Boussinesq system [1]. This approximation holds for flows with small temperature-induced density variations, negligible compressibility, and nearly constant transport coefficients. For viscosity and thermal diffusivity of comparable small magnitudes, both viscous and thermal boundary layers develop near solid walls. Following the double-scale asymptotic ansatz underlying Prandtl's theory, one formally derives the associated Boussinesq boundary layer system [2], which applies only to laminar flow regimes and inherits all physical limitations of the original Boussinesq approximation. In the present work, we use this classical boundary layer formulation and further omit the viscous dissipation heating source term in the thermal balance equation. This simplification is physically justified for low Mach number buoyancy-driven flows with moderate temperature differences, where viscous heating constitutes a higher-order effect that is negligible compared with thermal advection and diffusion. We refer to Baranovskii [3] for mathematical investigations of stationary Boussinesq-type

systems equipped with regularized viscous dissipation source terms in the heat equation.

In the present paper, we consider the following two-dimensional Boussinesq boundary layer system defined on the half space domain $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2, y > 0\}$:

$$\begin{cases} \partial_t v + v \partial_y v + u \partial_x v - \partial_x^2 v = -\partial_y P + \theta, \\ \partial_t \theta + v \partial_y \theta + u \partial_x \theta - \partial_x^2 \theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{x=0} = v|_{x=0} = 0, \quad \partial_x \theta|_{x=0} = 0, \\ \lim_{x \rightarrow +\infty} v = V(t, y), \quad \lim_{x \rightarrow +\infty} \theta = \Theta(t, y), \\ v|_{t=0} = v_0(x, y), \quad \theta|_{t=0} = \theta_0(x, y), \end{cases} \quad (1.1)$$

where $u(t, x, y), v(t, x, y)$ represents the velocity field, $\theta(t, x, y)$ denotes the temperature field, and $P(t, y)$ stands for the pressure. The known functions $V(t, y)$, $\Theta(t, y)$, and $P(t, y)$ are the traces at the boundary $\{x = 0\}$ of the vertical velocity, temperature, and pressure of the outer flow, which satisfy the Bernoulli equations

$$\begin{cases} \partial_t V + V \partial_y V = -\partial_y P + \Theta, \\ \partial_t \Theta + V \partial_y \Theta = 0. \end{cases} \quad (1.2)$$

When the temperature field $\theta \equiv 0$, the Boussinesq boundary layer system (1.1) simplifies to the standard Prandtl system. The primary difficulty in proving the well-posedness of the classical Prandtl system stems from the loss of tangential derivatives in nonlinear terms such as $u \partial_x v$. To resolve this critical issue, two mainstream research approaches have been proposed. The first method is based on Oleinik's monotonicity condition. Using the Crocco transformation, Oleinik proved the local existence of smooth solutions [4, 5]. Instead of the Crocco transformation, Alexandre et al. [6] and Masmoudi and Wong [7] independently derived the local well-posedness of two-dimensional Prandtl equations in weighted Sobolev spaces via nonlinear energy estimates, without applying the Crocco transformation. Based on these pioneering works, Xu and Zhang [8] further extended the relevant conclusions and obtained the long-time well-posedness in high-order Sobolev spaces. Recently, Xin et al. [9, 10] established the global weak well-posedness of Prandtl equations under the favorable pressure gradient condition $\partial_x p \leq 0$.

A second distinct research line investigates the Prandtl boundary layer equations without Oleinik's monotonicity constraint on the normal velocity gradient. Without this stabilizing structural condition, nonlinear convective terms trigger the loss of tangential derivatives, which leads to ill-posedness in standard Sobolev spaces; consequently, well-posedness results for non-monotonic initial configurations are mostly constructed within Gevrey spaces, an intermediate function class between analytic functions and general smooth C^∞ functions. Gérard-Varet and Masmoudi [11] first pioneered this direction, establishing local well-posedness for two-dimensional non-monotonic Prandtl flows in the Gevrey class $G^{7/4}$ via a composite anisotropic energy strategy. Their work raised an open conjecture that the critical Gevrey threshold could be extended up to $\sigma = 2$. This conjecture was fully resolved by Li et al. [12], where they proved the local well-posedness for all Gevrey indices $\sigma \in [3/2, 2]$ under non-degenerate critical point conditions, significantly broadening the admissible regularity range of initial data. Separately, Li et al. [13] uncovered the intrinsic subelliptic property of the Prandtl system and derived explicit Gevrey smoothing effects valid for monotonicity-free setups, complementing the

existence theory with regularity propagation results. Later extensions by Li and Yang [14] further generalized the Gevrey well-posedness framework to three-dimensional (3D) Prandtl boundary layers without monotonicity restrictions, completing a systematic series of theoretical outcomes built upon Gevrey functional spaces for non-monotonic boundary layer flows.

In contrast to the classical Prandtl system, the Boussinesq boundary layer equations (1.1) raise extra difficulties when analyzing well-posedness, as derivative loss takes place in both velocity equations and the nonlinear parts of the temperature equation. To resolve the double derivative loss arising in both velocity and temperature equations of the 3D Boussinesq boundary layer system, the authors of [2] extended the cancellation trick for Prandtl equations and introduced extra-good unknowns for thermal terms, then established its local well-posedness in the Gevrey-2 space without structural assumptions via energy estimates and the abstract Cauchy–Kowalewski theorem. Different from prior work [2], this paper establishes the long-time well-posedness for (1.1) and (1.2) within the analytic function framework without any structural constraints on the initial velocity profiles, and further also provides explicit lifespan scaling with respect to the small initial data norm ε . Without loss of generality, we consider the case that $V(t, x, y) = \Theta(t, x, y) = 0$, which implies that $P(t, x, y)$ is a constant, and thus the Boussinesq boundary layer equations (1.1) and (1.2) reduce to

$$\begin{cases} \partial_t v + v \partial_y v + u \partial_x v - \partial_x^2 v = \theta, \\ \partial_t \theta + v \partial_y \theta + u \partial_x \theta - \partial_x^2 \theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{x=0} = v|_{x=0} = 0, \quad \partial_x \theta|_{x=0} = 0, \\ \lim_{x \rightarrow +\infty} v = 0, \quad \lim_{x \rightarrow +\infty} \theta = 0, \\ v|_{t=0} = v_0(x, y), \quad \theta|_{t=0} = \theta_0(x, y). \end{cases} \quad (1.3)$$

While part of our analytic phase-modulation and Gaussian-weight energy setup is inspired by the previous work [15], new difficulties arise from the two-way velocity–temperature coupling in the Boussinesq boundary layer system. For the Prandtl system, derivative loss only appears within the velocity nonlinear convective terms. In our Boussinesq model, the nonlinear terms $v \partial_y \theta$ and $u \partial_x \theta$ in the temperature equation generate additional derivative loss structures. Combined with nonlinear interactions in the velocity equation, they yield the so-called double derivative loss. This coupled loss cannot be reduced to a trivial extension of the Prandtl analysis, and thus it forces us to treat cross velocity–temperature interaction terms in the weighted energy estimates, modify the closed-form ordinary differential equations (ODE) for the regularity tracking function $\rho(t)$, and jointly control velocity–temperature contributions in the absorption argument for nonlinear terms.

To overcome the double derivative loss in both the velocity and temperature equations of the two-dimensional (2D) Boussinesq boundary layer system (1.3). First, we introduce an analytic phase modifier u_Φ via Fourier multiplier transformation with the time-dependent phase function $\Phi(t, \xi) = (1 - \lambda \rho(t))(1 + |\xi|^2)^{\frac{1}{2}}$. The non-decreasing control function $\rho(t)$ dynamically tracks the analytic regularity loss over time, and the large constant parameter λ provides a positive high-order fractional dissipation term to offset the nonlinear derivative-losing interactions from $u \partial_x v$, $v \partial_y v$, $u \partial_x \theta$, $v \partial_y \theta$. Second, we equip all energy norms with a Gaussian-type decaying weight $\Psi(t, x) = \frac{1+x^2}{8\langle t \rangle}$ along the horizontal direction $x \rightarrow +\infty$. This weight satisfies a crucial differential identity $\partial_t \Psi + 2(\partial_x \Psi)^2 \leq 0$, which generates an extra self-damping term during integration by parts and suppresses the far-field growth of weighted norms, enabling uniform control of solutions defined on the half-space $\mathbb{R}_+^2$. Third, we adopt anisotropic

weighted Sobolev spaces $H^{k,\ell}$ to separate horizontal and vertical regularity, and establish a series of sharp product estimates for the phase-modified nonlinear terms $(fg)_\Phi$, which convert the nonlinear interaction bounds into linear expressions of $\dot{\rho}(t)$. This transformation allows us to absorb all nonlinear terms into the positive analytic damping term $\lambda\dot{\rho}(t)\langle D \rangle$ after selecting a sufficiently large λ .

For the uniqueness argument, we construct a composite phase function $\tilde{\Phi}$ associated with the sum of the regularity tracking functions of two distinct solutions, and exploit the subadditivity of the analytic phase exponents to perform weighted energy estimates on the difference system of two solution pairs. By combining Gronwall's inequality and vanishing initial difference data, we eliminate the residual nonlinear coupling terms and verify the uniqueness of analytic solutions on the whole existence interval.

The remainder of this paper is structured as follows. Section 2 formulates our main theorem. Section 3 establishes a priori bounds for solutions of Eq (1.3) via the energy argument. Finally, we prove the uniqueness of solution and finish the rigorous proof of Theorem 2.1 in Section 4.

2. Main results

Following the notations adopted in the previous literature [15–17], we introduce the following weighted Fourier transformation formulation for the solution decomposition:

$$u_\Phi(t, x, y) = \mathcal{F}_{\xi \rightarrow y}^{-1}(e^{\Phi(t, \xi)} \hat{u}(t, x, \xi)) \text{ with } \Phi(t, \xi) = (1 - \lambda\rho(t))(1 + |\xi|^2)^{\frac{1}{2}} \quad (2.1)$$

where λ denotes a sufficiently large positive constant to be determined subsequently. The phase function $\Phi(t, \xi) \geq 0$ is constructed to characterize the time-dependent analytic radius of the solution, and the time-varying quantity $\rho(t)$ is governed by the following ODE

$$\dot{\rho}(t) = \langle t \rangle^{\frac{1}{4}} \left(\|e^\Psi \partial_x v_\Phi(t)\|_{H^{0,1}} + \|e^\Psi \partial_x \theta_\Phi(t)\|_{H^{0,1}} \right) \text{ with } \rho(0) = 0, \quad (2.2)$$

in which $\langle t \rangle = 1 + t$.

As in [15, 18], we next define the weighted anisotropic Sobolev space norm utilized throughout this work as follows:

$$\|u\|_{H^{k,\ell}} := \sum_{n=0}^k \|\partial_x^n u\|_{L_x^2(\mathbb{R}^+; H_y^\ell)}. \quad (2.3)$$

To capture the spatial decay property of solutions in the x -direction over time, we introduce the time-dependent weight function

$$\Psi(t, x) := \frac{1 + x^2}{8\langle t \rangle},$$

which satisfies the crucial dissipation identity

$$\partial_t \Psi(t, x) + 2(\partial_x \Psi(t, x))^2 \leq 0. \quad (2.4)$$

We always assume $t < T_*$, where T_* is defined by

$$T_* \stackrel{\text{def}}{=} \sup\{t > 0 \mid \rho(t) < \frac{1}{\lambda}\}. \quad (2.5)$$

Then by virtue of (2.1), for any $t < T_*$, the following convex inequality holds:

$$\Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta), \quad \forall \xi, \eta \in \mathbb{R}^{d-1}.$$

With the notations above and preliminary definitions in hand, we now state the main well-posedness result of this paper in the following theorem.

Theorem 2.1. *Let $\varepsilon > 0$ be sufficiently small and assume that the initial data u_0, θ_0 satisfy*

$$\left\| e^{\frac{1+x^2}{8}} e^{\langle D \rangle} v_0 \right\|_{H^{0,1}} + \left\| e^{\frac{1+x^2}{8}} e^{\langle D \rangle} \theta_0 \right\|_{H^{0,1}} \leq \varepsilon. \quad (2.6)$$

Then a unique solution (u, θ) to (1.3) on $[0, T_\varepsilon]$ (with $T_\varepsilon \sim \varepsilon^{-\frac{4}{3}}$) exists such that

$$(e^\Psi v_\Phi, e^\Psi \theta_\Phi)(t, \cdot) \in L^\infty(0, T_\varepsilon; H^{0,1}) \cap L^2(0, T_\varepsilon; H^{1,1}).$$

Remark 2.1. *The existence argument shares some similarities with that in [15] (see also [19]). Therefore, we only present the uniform a priori estimates and uniqueness for the solution in the following.*

3. Uniform a priori estimates

This section is dedicated to establishing the long-time uniform a priori estimates for solutions to System (1.3), which serve as the core ingredient for the long-time well-posedness theory. We borrow the analytic phase-modulation formalism from [15]. However, due to velocity-temperature nonlinear couplings and the double derivative loss coming from both the velocity and temperature equations, we need to derive new coupled energy inequalities for the joint (v_Φ, θ_Φ) unknown pair, and the ODE for the regularity indicator $\rho(t)$ now involves norms of both velocity and temperature components.

By virtue of the original System (1.3) and the phase-weighted transformation (2.1), we can derive the evolution equations satisfied by the phase-modified solutions v_Φ and θ_Φ

$$\begin{cases} \partial_t v_\Phi + \lambda \dot{\rho} \langle D \rangle v_\Phi + (v \partial_y v)_\Phi + (u \partial_x v)_\Phi - \partial_x^2 v_\Phi = \theta_\Phi, \\ \partial_t \theta_\Phi + \lambda \dot{\rho} \langle D \rangle \theta_\Phi + (v \partial_y \theta)_\Phi + (u \partial_x \theta)_\Phi - \partial_x^2 \theta_\Phi = 0, \\ u_\Phi = - \int_0^x \partial_y v_\Phi(t, z, y) dz, \\ v_\Phi|_{x=0} = v_\Phi|_{x=0} = \partial_x \theta_\Phi|_{x=0} = 0, \quad \lim_{x \rightarrow +\infty} v_\Phi = 0, \quad \lim_{x \rightarrow +\infty} \theta_\Phi = 0 \\ v_\Phi|_{t=0} = e^{\langle D \rangle} v_0(x, y), \quad \theta_\Phi|_{t=0} = e^{\langle D \rangle} \theta_0(x, y), \end{cases} \quad (3.1)$$

where $\langle D \rangle$ denotes the standard Fourier multiplier operator in the horizontal variable with symbol $(1 + |\xi|^2)^{\frac{1}{2}}$.

Lemma 3.1. *Under the assumptions of Theorem 2.1, a solution (v, θ) to (1.3) exists such that*

$$\begin{aligned} & \left(\|e^\Psi v_\Phi(t)\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi(t)\|_{H^{0,1}}^2 \right) + \int_0^t \left(\|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}^2 \right) ds \\ & \leq \left(\|e^{\frac{1+x^2}{8}} e^{\langle D \rangle} v_0\|_{H^{0,1}}^2 + \|e^{\frac{1+x^2}{8}} e^{\langle D \rangle} \theta_0\|_{H^{0,1}}^2 \right), \end{aligned}$$

for $t \in [0, T_\varepsilon]$.

Proof. We perform the standard energy method by taking the $H^{0,1}$ -inner product of the evolutionary System (3.1) with $e^{2\Psi}v_\Phi$ and $e^{2\Psi}\theta_\Phi$, respectively, which yields the coupled energy variational equations:

$$\begin{cases} (e^\Psi \partial_t v_\Phi, e^\Psi v_\Phi)_{H^{0,1}} + \lambda \dot{\rho}(\langle D \rangle e^\Psi v_\Phi, e^\Psi v_\Phi)_{H^{0,1}} \\ \quad + (e^\Psi (u \partial_x v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}} + (e^\Psi (v \partial_y v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}} \\ \quad - (e^\Psi \partial_x^2 v_\Phi, e^\Psi v_\Phi)_{H^{0,1}} = (e^\Psi \theta_\Phi, e^\Psi v_\Phi)_{H^{0,1}}, \\ (e^\Psi \partial_t \theta_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} + \lambda \dot{\rho}(\langle D \rangle e^\Psi \theta_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} \\ \quad + (e^\Psi (u \partial_x \theta)_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} + (e^\Psi (v \partial_y \theta)_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} \\ \quad - (e^\Psi \partial_x^2 \theta_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} = 0. \end{cases} \quad (3.2)$$

We now analyze and estimate each term in (3.2) term by term. For the time-derivative term, the fractional diffusion term and the Laplacian term, we derive the following identities and inequalities respectively:

$$(e^\Psi \partial_t v_\Phi, e^\Psi v_\Phi)_{H^{0,1}} = \frac{1}{2} \frac{d}{dt} \|e^\Psi v_\Phi\|_{H^{0,1}}^2 - \int_0^{+\infty} \partial_t \Psi \|e^\Psi v_\Phi\|_{H_y^1}^2 dx \quad (3.3)$$

and

$$\lambda \dot{\rho}(\langle D \rangle e^\Psi v_\Phi, e^\Psi v_\Phi)_{H^{0,1}} = \lambda \dot{\rho} \|e^\Psi v_\Phi\|_{H^{0, \frac{3}{2}}}^2, \quad (3.4)$$

$$\begin{aligned} & - (e^\Psi \partial_x^2 v_\Phi, e^\Psi v_\Phi)_{H^{0,1}} \\ & = (e^\Psi \partial_x v_\Phi, e^\Psi \partial_x v_\Phi) + (2 \partial_x \Psi e^\Psi \partial_x v_\Phi, e^\Psi v_\Phi) \\ & \geq \frac{1}{2} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}^2 - 2 \int_0^{+\infty} (\partial_x \Psi)^2 \|e^\Psi v_\Phi\|_{H_y^1}^2 dx, \end{aligned} \quad (3.5)$$

where we used the boundary conditions $v_\Phi|_{x=0} = 0$ and $v_\Phi|_{x=\infty} = 0$.

For the coupling source term, we apply the standard Cauchy–Schwarz inequality to obtain

$$(e^\Psi \theta_\Phi, e^\Psi v_\Phi)_{H^{0,1}} \leq \frac{1}{2} \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2 + \frac{1}{2} \|e^\Psi v_\Phi\|_{H^{0,1}}^2. \quad (3.6)$$

We next handle the nonlinear convective terms. To this end, we adopt the key nonlinear estimate established in [15]

$$\|(fg)_\Phi\|_{H_y^s} \lesssim \|f_\Phi\|_{H_y^1} \|g_\Phi\|_{H_y^s}, \quad 0 \leq s \leq 1, \quad (3.7)$$

which implies

$$\begin{aligned} & |(e^\Psi (v \partial_y v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}}| \\ & \lesssim |(e^\Psi \langle D \rangle^{\frac{1}{2}} (v \partial_y v)_\Phi, e^\Psi \langle D \rangle^{\frac{3}{2}} u_\Phi)_{L_{x,y}^2}| \\ & \lesssim \int_0^\infty e^{2\Psi} \|(v \partial_y v)_\Phi\|_{H_y^{\frac{1}{2}}} \|v_\Phi\|_{H_y^{\frac{3}{2}}} dy \\ & \lesssim \int_0^\infty e^{2\Psi} \|v_\Phi\|_{H_y^1} \|\partial_y v_\Phi\|_{H_y^{\frac{1}{2}}} \|v_\Phi\|_{H_y^{\frac{3}{2}}} dx \end{aligned}$$

$$\lesssim \|v_\Phi\|_{L_x^\infty(H_y^1)} \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2.$$

Because of $v_\Phi(t, x, y) = \int_0^x (\partial_\xi v_\Phi)(t, \xi, y) d\xi$, we have

$$\|v_\Phi\|_{L_x^\infty(H_y^1)} \lesssim \left\| \int_0^x e^{-\Psi} \|e^\Psi \partial_\xi v_\Phi\|_{H_y^1} d\xi \right\|_{L_x^\infty} \lesssim \|e^{-\Psi}\|_{L_x^2} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}. \quad (3.8)$$

Furthermore, a direct calculation based on the explicit form of $\Psi(t, x)$ gives

$$\|e^{-\Psi}\|_{L_x^2} \lesssim \langle t \rangle^{\frac{1}{4}}. \quad (3.9)$$

Combining the three inequalities above, we ultimately obtain the controlled bound for the vertical nonlinear term:

$$\begin{aligned} |(e^\Psi(v\partial_y v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}}| &\lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}} \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2 \\ &\lesssim \dot{\rho}(t) \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2. \end{aligned} \quad (3.10)$$

By virtue of Hölder's inequality together with (3.7), we get

$$\begin{aligned} |(e^\Psi(u\partial_x v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}}| &\lesssim |(e^\Psi \langle D \rangle^{\frac{1}{2}}(u\partial_x v)_\Phi, e^\Psi \langle D \rangle^{\frac{3}{2}} v_\Phi)_{L_{x,y}^2}| \\ &\lesssim \int_0^\infty e^{2\Psi} \|(u\partial_x v)_\Phi\|_{H_y^{\frac{1}{2}}} \|v_\Phi\|_{H_y^{\frac{3}{2}}} dx \\ &\lesssim \int_0^\infty e^{2\Psi} \|u_\Phi\|_{H_y^{\frac{1}{2}}} \|\partial_x v_\Phi\|_{H_y^1} \|v_\Phi\|_{H_y^{\frac{3}{2}}} dx \\ &\lesssim \|u_\Phi\|_{L_x^\infty(H_y^{\frac{1}{2}})} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}} \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}. \end{aligned}$$

In view of the divergence-free relation (3.1)₃, we derive the uniform bound of u_Φ

$$\begin{aligned} \|u_\Phi\|_{L_x^\infty(H_y^{\frac{1}{2}})} &= \left\| \int_0^x \partial_y v_\Phi \right\|_{L_x^\infty(H_y^{\frac{1}{2}})} \\ &\lesssim \left\| \int_0^x e^{-\Psi} e^\Psi \|v_\Phi\|_{H_y^{\frac{3}{2}}} dx \right\|_{L_x^\infty} \\ &\lesssim \|e^{-\Psi}\|_{L_x^2} \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}. \end{aligned} \quad (3.11)$$

Thus, we can see that

$$\begin{aligned} |(e^\Psi(u\partial_x v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}}| &\lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}} \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2 \\ &\lesssim \dot{\rho}(t) \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2. \end{aligned} \quad (3.12)$$

Gathering all the term-by-term estimates (3.3)–(3.6), (3.10), and (3.12), utilizing the dissipation property (2.4) of the weight function Ψ , and choosing the constant λ to be sufficiently large to absorb all nonlinear terms, we arrive at the energy inequality for the velocity component

$$\frac{d}{dt} \|e^\Psi v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}^2 + 2\lambda \dot{\rho}(t) \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2$$

$$\leq \|e^\Psi v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2. \quad (3.13)$$

The energy estimation procedure for the temperature component θ_Φ is completely analogous to that for v_Φ . For the estimates of the linear terms (3.2)₂, we have

$$(e^\Psi \partial_t \theta_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} = \frac{1}{2} \frac{d}{dt} \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2 - \int_0^{+\infty} \partial_t \Psi \|e^\Psi \theta_\Phi\|_{H_y^1}^2 dx$$

and

$$\begin{aligned} \lambda \dot{\rho}(\langle D \rangle e^\Psi \theta_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} &= \lambda \dot{\rho} \|e^\Psi \theta_\Phi\|_{H^{0, \frac{3}{2}}}^2, \\ &- (e^\Psi \partial_x^2 \theta_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}} \\ &= (e^\Psi \partial_x \theta_\Phi, e^\Psi \partial_x \theta_\Phi) + (2 \partial_x \Psi e^\Psi \partial_x \theta_\Phi, e^\Psi \theta_\Phi) \\ &\geq \frac{1}{2} \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}^2 - 2 \int_0^{+\infty} (\partial_x \Psi)^2 \|e^\Psi \theta_\Phi\|_{H_y^1}^2 dx, \end{aligned}$$

where we used the boundary condition $\partial_x \theta_\Phi|_{x=0} = 0$ and $\theta_\Phi|_{x=+\infty} = 0$.

For the nonlinear term $v \partial_y \theta$, using (3.7)–(3.9), we deduce

$$\begin{aligned} &|(e^\Psi (v \partial_y \theta)_\Phi, e^\Psi \theta_\Phi)_{H^{0,1}}| \\ &\lesssim |(e^\Psi \langle D \rangle^{\frac{1}{2}} (v \partial_y \theta)_\Phi, e^\Psi \langle D \rangle^{\frac{3}{2}} \theta_\Phi)_{L_{x,y}^2}| \\ &\lesssim \int_0^\infty e^{2\Psi} \|(v \partial_y \theta)_\Phi\|_{H_y^{\frac{1}{2}}} \|\theta_\Phi\|_{H_y^{\frac{3}{2}}} dy \\ &\lesssim \int_0^\infty e^{2\Psi} \|v_\Phi\|_{H_y^1} \|\partial_y \theta_\Phi\|_{H_y^{\frac{1}{2}}} \|\theta_\Phi\|_{H_y^{\frac{3}{2}}} dx \\ &\lesssim \|v_\Phi\|_{L_x^\infty(H_y^1)} \|e^\Psi \theta_\Phi\|_{H^{0, \frac{3}{2}}}^2 \\ &\lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}} \|e^\Psi v_\Phi\|_{H^{0, \frac{3}{2}}}^2 \\ &\lesssim \dot{\rho}(t) \|e^\Psi v_\Phi\|_{H^{0, \frac{3}{2}}}^2. \end{aligned}$$

By virtue of Hölder's inequality, (3.7) together with (3.9) and (3.11), we derive

$$\begin{aligned} |(e^\Psi (u \partial_x v)_\Phi, e^\Psi v_\Phi)_{H^{0,1}}| &\lesssim |(e^\Psi \langle D \rangle^{\frac{1}{2}} (u \partial_x v)_\Phi, e^\Psi \langle D \rangle^{\frac{3}{2}} v_\Phi)_{L_{x,y}^2}| \\ &\lesssim \int_0^\infty e^{2\Psi} \|(u \partial_x v)_\Phi\|_{H_y^{\frac{1}{2}}} \|v_\Phi\|_{H_y^{\frac{3}{2}}} dx \\ &\lesssim \int_0^\infty e^{2\Psi} \|u_\Phi\|_{H_y^{\frac{1}{2}}} \|\partial_x v_\Phi\|_{H_y^1} \|v_\Phi\|_{H_y^{\frac{3}{2}}} dx \\ &\lesssim \|u_\Phi\|_{L_x^\infty(H_y^{\frac{1}{2}})} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}} \|e^\Psi v_\Phi\|_{H^{0, \frac{3}{2}}}^2 \\ &\lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}} \|e^\Psi v_\Phi\|_{H^{0, \frac{3}{2}}}^2 \\ &\lesssim \dot{\rho}(t) \|e^\Psi v_\Phi\|_{H^{0, \frac{3}{2}}}^2. \end{aligned}$$

Collecting the all estimates above and selecting λ to be large enough to eliminate all residual nonlinear terms, we obtain the corresponding energy inequality for the temperature field

$$\frac{d}{dt} \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}^2 + 2\lambda \dot{\rho}(t) \|e^\Psi \theta_\Phi\|_{H^{0,\frac{3}{2}}}^2 \leq \dot{\rho}(t) \|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2. \quad (3.14)$$

We now sum up the two energy inequalities (3.13) and (3.14). By virtue of the continuous Sobolev embedding $H^{\frac{3}{2}} \hookrightarrow H^1$ and taking λ to be sufficiently large to absorb all nonlinear cross-terms, the total energy dissipation inequality simplifies to

$$\begin{aligned} & \frac{d}{dt} (\|e^\Psi v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2) + (\|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}^2) \\ & + 2\lambda \dot{\rho}(t) (\|e^\Psi v_\Phi\|_{H^{0,\frac{3}{2}}}^2 + \|e^\Psi \theta_\Phi\|_{H^{0,\frac{3}{2}}}^2) \leq 0. \end{aligned} \quad (3.15)$$

The uniform estimate (3.15) relying on the assumption concerning the phase function

$$\Phi(t, \xi) := (1 - \lambda \rho(t))(1 + |\xi|^2)^{\frac{1}{2}}$$

must remain non-negative for all admissible t, ξ . This imposes the crucial constraint

$$\rho(t) \leq \frac{1}{2\lambda}. \quad (3.16)$$

Integrating (3.15) with respect to time over the interval $[0, t]$, combining the initial data assumption (2.6) and the definition of $\dot{\rho}(t)$, we deduce the uniform energy bound

$$\begin{aligned} & (\|e^\Psi v_\Phi(t)\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi(t)\|_{H^{0,1}}^2) + \int_0^t (\|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}^2) ds \\ & \leq (\|e^{\frac{1+x^2}{8}} e^{\langle D \rangle} v_0\|_{H^{0,1}}^2 + \|e^{\frac{1+x^2}{8}} e^{\langle D \rangle} \theta_0\|_{H^{0,1}}^2) \leq \varepsilon^2. \end{aligned} \quad (3.17)$$

We finally verify the feasibility of the constraint (3.16) via a rigorous estimate of $\rho(t)$. By the Cauchy–Schwarz inequality and the uniform energy bound (3.17), we compute

$$\begin{aligned} \rho(t) &= \int_0^t \langle s \rangle^{\frac{1}{4}} (\|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}(s) + \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}(s)) ds \\ &\leq \left(\int_0^t \langle s \rangle^{\frac{1}{2}} ds \right)^{\frac{1}{2}} \left(\int_0^t \|e^\Psi \partial_x v_\Phi\|_{H^{0,1}}^2(s) ds \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^t \langle s \rangle^{\frac{1}{2}} ds \right)^{\frac{1}{2}} \left(\int_0^t \|e^\Psi \partial_x \theta_\Phi\|_{H^{0,1}}^2(s) ds \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{2}{3}} (1+t)^{\frac{3}{4}} \varepsilon. \end{aligned}$$

A direct algebraic calculation shows that the non-negativity condition (3.16) holds, provided that the time variable satisfies

$$0 \leq t \leq \left(\frac{3}{8} \right)^{\frac{2}{3}} \lambda^{-\frac{4}{3}} \varepsilon^{-\frac{4}{3}} - 1.$$

We then have

$$\sup_{t \in [0, T_1]} \rho(t) \leq \frac{1}{2\lambda}, \text{ for } T_1 = \left(\frac{3}{8}\right)^{\frac{2}{3}} \lambda^{-\frac{4}{3}} \varepsilon^{-\frac{4}{3}} - 1$$

By virtue of (2.5), we have $T_* \geq T_1$. Combining this with (3.17), we arrive at the desired result. This finishes the existence part of Theorem 2.1.

4. The proof of the uniqueness of solution

This section is devoted to proving the uniqueness of the solution to System (1.3) within the lifespan $[0, T_\varepsilon]$. Let (v^1, θ^1) and (v^2, θ^2) be two arbitrary solutions of (1.3) satisfying the same initial data and boundary conditions. We define the difference functions $v = v^1 - v^2$ and $\theta = \theta^1 - \theta^2$. It follows from the original System (1.3) that the difference pair (v, θ) satisfies the following homogeneous perturbation system:

$$\begin{cases} \partial_t v - \partial_x^2 v + F = \theta, \\ \partial_t \theta - \partial_x^2 \theta + G = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{x=0} = v|_{x=0} = 0, \quad \partial_x \theta|_{x=0} = 0, \\ \lim_{x \rightarrow +\infty} v = 0, \quad \lim_{x \rightarrow +\infty} \theta = 0, \\ v|_{t=0} = 0, \quad \theta|_{t=0} = 0, \end{cases} \quad (4.1)$$

where the nonlinear perturbation source terms are given by

$$\begin{cases} F = v^1 \partial_y v + v \partial_y v^2 + u^1 \partial_x v + u \partial_x v^2, \\ G = v^1 \partial_y \theta + v \partial_y \theta^2 + u^1 \partial_x \theta + u \partial_x \theta^2. \end{cases} \quad (4.2)$$

For the two solution branches, we define the corresponding time-varying modulation functions $\rho^i(t)$ ($i = 1, 2$) by

$$\dot{\rho}^i(t) = \langle t \rangle^{\frac{1}{4}} \left(\|e^\Psi \partial_x v_{\Phi^i}^i(t)\|_{H^{0,1}} + \|e^\Psi \partial_x \theta_{\Phi^i}^i(t)\|_{H^{0,1}} \right) \text{ with } \rho^i(0) = 0. \quad (4.3)$$

To perform the energy estimation for difference solutions, we introduce the composite modulation parameter and the new phase function

$$\varrho(t) \stackrel{\text{def}}{=} (\rho^1 + \rho^2)(t), \quad \widetilde{\Phi}(t, \xi) \stackrel{\text{def}}{=} \left(\frac{1}{2} - \lambda \varrho(t) \right) (1 + |\xi|^2)^{\frac{1}{2}}, \quad (4.4)$$

together with the individual phase functions for each solution

$$\Phi^i(t, \xi) \stackrel{\text{def}}{=} (1 - \lambda \rho^i(t)) (1 + |\xi|^2)^{\frac{1}{2}}, \quad \text{for } i = 1, 2.$$

We restrict our consideration to a sufficiently small time t such that the non-negativity condition

$$\frac{1}{2} - \lambda \varrho(t) \geq 0$$

holds, which guarantees the subadditivity property of the phase function

$$\widetilde{\Phi}(t, \xi) \leq \widetilde{\Phi}(t, \xi - \eta) + \widetilde{\Phi}(t, \eta) \quad \text{for } \forall \xi, \eta \in \mathbb{R}.$$

Since $\widetilde{\Phi}(t) \leq \min\{\Phi^1(t), \Phi^2(t)\}$, we have the following for $i = 1, 2$:

$$\|e^\Psi \partial_x v_{\widetilde{\Phi}}^i\|_{H^{0,1}} \lesssim \|e^\Psi \partial_x v_{\Phi^i}^i\|_{H^{0,1}} \quad (4.5)$$

Applying the phase transformation associated with $\widetilde{\Phi}$ to System (4.1), we derive the evolutionary equations for the phase-weighted difference solutions

$$\begin{cases} \partial_t v_{\widetilde{\Phi}} + \lambda \dot{\varrho} \langle D \rangle v_{\widetilde{\Phi}} - \partial_x^2 v_{\widetilde{\Phi}} + F_{\widetilde{\Phi}} = \theta_{\widetilde{\Phi}}, \\ \partial_t \theta_{\widetilde{\Phi}} + \lambda \dot{\varrho} \langle D \rangle \theta_{\widetilde{\Phi}} - \partial_x^2 \theta_{\widetilde{\Phi}} + G_{\widetilde{\Phi}} = 0. \end{cases} \quad (4.6)$$

We then implement the weighted energy method again by taking the $H^{0,1}$ -inner product of System (4.6) with $e^{2\Psi} v_{\widetilde{\Phi}}$ and $e^{2\Psi} \theta_{\widetilde{\Phi}}$, respectively, leading to the variational energy equations for the difference solutions

$$\begin{cases} (e^\Psi \partial_t v_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}} + \lambda \dot{\varrho} (\langle D \rangle e^\Psi v_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}} \\ - (e^\Psi \partial_x^2 v_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}} + (e^\Psi F_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}} = (e^\Psi \theta_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}}, \\ (e^\Psi \partial_t \theta_{\widetilde{\Phi}}, e^\Psi \theta_{\widetilde{\Phi}})_{H^{0,1}} + \lambda \dot{\varrho} (\langle D \rangle e^\Psi \theta_{\widetilde{\Phi}}, e^\Psi \theta_{\widetilde{\Phi}})_{H^{0,1}} \\ - (e^\Psi \partial_x^2 \theta_{\widetilde{\Phi}}, e^\Psi \theta_{\widetilde{\Phi}})_{H^{0,1}} + (e^\Psi G_{\widetilde{\Phi}}, e^\Psi \theta_{\widetilde{\Phi}})_{H^{0,1}} = 0. \end{cases} \quad (4.7)$$

Similar to the estimates of (3.15) and using (2.4), we obtain the total energy inequality for the difference solution pair

$$\begin{aligned} & \frac{d}{dt} (\|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,1}}^2 + \|e^\Psi \theta_{\widetilde{\Phi}}\|_{H^{0,1}}^2) + (\|e^\Psi \partial_x v_{\widetilde{\Phi}}\|_{H^{0,1}}^2 + \|e^\Psi \partial_x \theta_{\widetilde{\Phi}}\|_{H^{0,1}}^2) \\ & + 2\lambda \dot{\varrho}(t) (\|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}}^2 + \|e^\Psi \theta_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}}^2) \\ & \leq \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,1}}^2 + \|e^\Psi \theta_{\widetilde{\Phi}}\|_{H^{0,1}}^2 + (e^\Psi F_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}} + (e^\Psi G_{\widetilde{\Phi}}, e^\Psi \theta_{\widetilde{\Phi}})_{H^{0,1}}. \end{aligned} \quad (4.8)$$

We now estimate each nonlinear term on the right-hand side of (4.8). We estimate the first term by applying Hölder's inequality.

$$\begin{aligned} & |(e^\Psi (v^1 \partial_y v)_{\widetilde{\Phi}}, e^\Psi v_{\widetilde{\Phi}})_{H^{0,1}}| \\ & \lesssim |(e^\Psi \langle D \rangle^{\frac{1}{2}} (v^1 \partial_y v)_{\widetilde{\Phi}}, e^\Psi \langle D \rangle^{\frac{3}{2}} u_{\widetilde{\Phi}})_{L^2_{x,y}}| \\ & \lesssim \int_0^\infty e^{2\Psi} \|(v^1 \partial_y v)_{\widetilde{\Phi}}\|_{H_y^{\frac{1}{2}}} \|v_{\widetilde{\Phi}}\|_{H_y^{\frac{3}{2}}} dy \\ & \lesssim \int_0^\infty e^{2\Psi} \|v_{\widetilde{\Phi}}^1\|_{H_y^1} \|\partial_y v_{\widetilde{\Phi}}\|_{H_y^{\frac{1}{2}}} \|v_{\widetilde{\Phi}}\|_{H_y^{\frac{3}{2}}} dx \\ & \lesssim \|v_{\widetilde{\Phi}}^1\|_{L_x^\infty(H_y^1)} \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}} \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}} \\ & \lesssim \|e^{-\Psi}\|_{L_x^2} \|e^\Psi \partial_x v_{\widetilde{\Phi}}^1\|_{H^{0,1}} \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}} \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}} \\ & \lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_{\widetilde{\Phi}}^1\|_{H^{0,1}} \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}}^2 \\ & \lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_{\Phi^1}^1\|_{H^{0,1}} \|e^\Psi v_{\widetilde{\Phi}}\|_{H^{0,\frac{3}{2}}}^2 \end{aligned}$$

$$\lesssim \dot{\varrho}(t) \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0, \frac{3}{2}}}^2, \quad (4.9)$$

where (4.5) is used for the penultimate inequality.

To handle the nonlinear cross-terms involving mixed solution components, we establish a key embedding relation between different phase-weighted spaces. By the definition of fractional Sobolev norms and the phase monotonicity, we have

$$\begin{aligned} \|e^\Psi \partial_y v_{\tilde{\Phi}}^i\|_{H_y^{0,1}}^2 &= \int_{\mathbb{R}} \langle \xi \rangle^2 e^{2\Psi} e^{2\tilde{\Phi}} |\partial_y \widehat{v^i}|^2 d\xi \\ &\lesssim \int_{\mathbb{R}} \langle \xi \rangle^2 e^{2\Psi} e^{2\Phi^i} e^{-|\xi|} |\partial_y \widehat{v^i}|^2 d\xi \\ &\lesssim \int_{\mathbb{R}} e^{2\Psi} e^{2\Phi^i} |\partial_y \widehat{v^i}|^2 d\xi \\ &= \|e^\Psi v_{\Phi^i}^i\|_{H_y^{0,1}}^2 \end{aligned}$$

which implies

$$\|e^\Psi v_{\tilde{\Phi}}^i\|_{H_y^{0,2}} \lesssim \|e^\Psi v_{\Phi^i}^i\|_{H_y^{0,1}}, \quad i = 1, 2. \quad (4.10)$$

This inequality justifies the introduction of the composite phase function $\tilde{\Phi}(t, \xi)$ in (4.4) and enables the control of nonlinear cross-terms.

Using (3.8), (3.9), and (4.10), we can conclude that

$$\begin{aligned} &|(e^\Psi (v \partial_y v^2)_{\tilde{\Phi}}, e^\Psi v_{\tilde{\Phi}})_{H^{0,1}}| \\ &\lesssim |(e^\Psi \langle D \rangle (v \partial_y v^2)_{\tilde{\Phi}}, e^\Psi \langle D \rangle u_{\tilde{\Phi}})_{L_{x,y}^2}| \\ &\lesssim \int_0^\infty e^{2\Psi} \|(v \partial_y v^2)_{\tilde{\Phi}}\|_{H_y^1} \|v_{\tilde{\Phi}}\|_{H_y^1} dx \\ &\lesssim \int_0^\infty e^{2\Psi} \|v_{\tilde{\Phi}}\|_{H_y^1} \|\partial_y v_{\tilde{\Phi}}^2\|_{H_y^1} \|v_{\tilde{\Phi}}\|_{H_y^1} dx \\ &\lesssim \|\partial_y v_{\tilde{\Phi}}^2\|_{L_x^\infty(H_y^1)} \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0,1}} \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0,1}} \\ &\lesssim \|e^{-\Psi}\|_{L_x^2} \|e^\Psi \partial_x v_{\tilde{\Phi}}^2\|_{H^{0,2}} \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0,1}}^2 \\ &\lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_x v_{\tilde{\Phi}}^2\|_{H^{0,1}} \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0,1}}^2 \\ &\lesssim \dot{\varrho}(t) \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0, \frac{3}{2}}}^2. \end{aligned} \quad (4.11)$$

For the estimate of the nonlinear term $u^1 \partial_x v$

$$\begin{aligned} |(e^\Psi (u^1 \partial_x v)_{\tilde{\Phi}}, e^\Psi v_{\tilde{\Phi}})_{H^{0,1}}| &\lesssim |(e^\Psi \langle D \rangle^1 (u^1 \partial_x v)_{\tilde{\Phi}}, e^\Psi \langle D \rangle^1 v_{\tilde{\Phi}})_{L_{x,y}^2}| \\ &\lesssim \int_0^\infty e^{2\Psi} \|(u^1 \partial_x v)_{\tilde{\Phi}}\|_{H_y^1} \|v_{\tilde{\Phi}}\|_{H_y^1} dx \\ &\lesssim \int_0^\infty e^{2\Psi} \|u_{\tilde{\Phi}}^1\|_{H_y^1} \|\partial_x v_{\tilde{\Phi}}\|_{H_y^1} \|v_{\tilde{\Phi}}\|_{H_y^1} dx \\ &\lesssim \|u_{\tilde{\Phi}}^1\|_{L_x^\infty(H_y^1)} \|e^\Psi \partial_x v_{\tilde{\Phi}}\|_{H^{0,1}} \|e^\Psi v_{\tilde{\Phi}}\|_{H^{0,1}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \|e^{-\Psi}\|_{L_x^2} \|e^{\Psi} v_{\Phi}^1\|_{H^{0,2}} \|e^{\Psi} \partial_x v_{\Phi}\|_{H^{0,1}} \|e^{\Psi} v_{\Phi}\|_{H^{0,1}} \\
&\lesssim \langle t \rangle^{\frac{1}{4}} \|e^{\Psi} v_{\Phi}^1\|_{H^{0,1}} \|e^{\Psi} \partial_x v_{\Phi}\|_{H^{0,1}} \|e^{\Psi} v_{\Phi}\|_{H^{0,1}}
\end{aligned} \tag{4.12}$$

where the penultimate inequality follows from (3.9) and the last inequality follows from (3.11) together with (4.10).

For the estimate of the nonlinear term $u \partial_x v^2$,

$$\begin{aligned}
|(e^{\Psi}(u \partial_x v^2)_{\Phi}, e^{\Psi} v_{\Phi})_{H^{0,1}}| &\lesssim |(e^{\Psi} \langle D \rangle^{\frac{1}{2}}(u \partial_x v^2)_{\Phi}, e^{\Psi} \langle D \rangle^{\frac{3}{2}} v_{\Phi})_{L_{x,y}^2}| \\
&\lesssim \int_0^{\infty} e^{2\Psi} \|(u \partial_x v^2)_{\Phi}\|_{H_y^{\frac{1}{2}}} \|v_{\Phi}\|_{H_y^{\frac{3}{2}}} dx \\
&\lesssim \int_0^{\infty} e^{2\Psi} \|u_{\Phi}^1\|_{H_y^{\frac{1}{2}}} \|\partial_x v_{\Phi}\|_{H_y^1} \|v_{\Phi}\|_{H_y^{\frac{3}{2}}} dx \\
&\lesssim \|u_{\Phi}\|_{L_x^{\infty}(H_y^{\frac{1}{2}})} \|e^{\Psi} \partial_x v_{\Phi}^2\|_{H^{0,1}} \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}} \\
&\lesssim \|e^{-\Psi}\|_{L_x^2} \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}} \|e^{\Psi} \partial_x v_{\Phi}^2\|_{H^{0,1}} \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}} \\
&\lesssim \langle t \rangle^{\frac{1}{4}} \|e^{\Psi} \partial_x v_{\Phi}^2\|_{H^{0,1}} \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}}^2 \\
&\lesssim \dot{\rho}(t) \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}}^2.
\end{aligned} \tag{4.13}$$

By the analogous arguments of (4.9), (4.11), and (4.13), we obtain the following estimates for all remaining nonlinear terms:

$$\left\{ \begin{array}{l} |(e^{\Psi}(v^1 \partial_y \theta)_{\Phi}, e^{\Psi} \theta_{\Phi})_{H^{0,1}}| \lesssim \langle t \rangle^{\frac{1}{4}} \|e^{\Psi} \partial_x v_{\Phi}^1\|_{H^{0,1}} \|e^{\Psi} \theta_{\Phi}\|_{H^{0,\frac{3}{2}}}^2 \\ \qquad \qquad \qquad \lesssim \dot{\rho}(t) \|e^{\Psi} \theta_{\Phi}\|_{H^{0,\frac{3}{2}}}^2, \\ |(e^{\Psi}(v \partial_y \theta^2)_{\Phi}, e^{\Psi} \theta_{\Phi})_{H^{0,1}}| \lesssim \dot{\rho}(t) \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}} \|e^{\Psi} \theta_{\Phi}\|_{H^{0,\frac{3}{2}}}, \\ |(e^{\Psi}(u^1 \partial_x \theta)_{\Phi}, e^{\Psi} \theta_{\Phi})_{H^{0,1}}| \lesssim \langle t \rangle^{\frac{1}{4}} \|e^{\Psi} v_{\Phi}^1\|_{H^{0,1}} \|e^{\Psi} \partial_x \theta_{\Phi}\|_{H^{0,1}} \|e^{\Psi} \theta_{\Phi}\|_{H^{0,1}}, \\ |(e^{\Psi}(u \partial_x \theta^2)_{\Phi}, e^{\Psi} \theta_{\Phi})_{H^{0,1}}| \lesssim \dot{\rho}(t) \|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}} \|e^{\Psi} \theta_{\Phi}\|_{H^{0,\frac{3}{2}}}. \end{array} \right. \tag{4.14}$$

Substituting (4.9) and (4.11)–(4.14) into (4.8), choosing λ to be sufficiently large to absorb all high-order controlled terms, and applying Young's inequality, we obtain the simplified differential inequality for the total energy of difference solutions

$$\begin{aligned}
&\frac{d}{dt} (\|e^{\Psi} v_{\Phi}\|_{H^{0,1}}^2 + \|e^{\Psi} \theta_{\Phi}\|_{H^{0,1}}^2) + (\|e^{\Psi} \partial_x v_{\Phi}\|_{H^{0,1}}^2 + \|e^{\Psi} \partial_x \theta_{\Phi}\|_{H^{0,1}}^2) \\
&\quad + 2\lambda \dot{\rho}(t) (\|e^{\Psi} v_{\Phi}\|_{H^{0,\frac{3}{2}}}^2 + \|e^{\Psi} \theta_{\Phi}\|_{H^{0,\frac{3}{2}}}^2) \\
&\lesssim \langle t \rangle^{\frac{1}{2}} \|e^{\Psi} v_{\Phi}^1\|_{H^{0,1}}^2 (\|e^{\Psi} v_{\Phi}\|_{H^{0,1}}^2 + \|e^{\Psi} \theta_{\Phi}\|_{H^{0,1}}^2),
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{d}{dt} (\|e^{\Psi} v_{\Phi}\|_{H^{0,1}}^2 + \|e^{\Psi} \theta_{\Phi}\|_{H^{0,1}}^2) \\
&\lesssim \langle t \rangle^{\frac{1}{2}} \|e^{\Psi} v_{\Phi}^1\|_{H^{0,1}}^2 (\|e^{\Psi} v_{\Phi}\|_{H^{0,1}}^2 + \|e^{\Psi} \theta_{\Phi}\|_{H^{0,1}}^2),
\end{aligned}$$

which, combined with Gronwall's inequality, leads to

$$(\|e^\Psi v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2) \lesssim M(0) \exp \left\{ \int_0^t \langle s \rangle^{\frac{1}{2}} H(s) ds \right\}, \quad (4.15)$$

where the initial energy and time-dependent coefficient are defined by

$$M(0) = (\|e^\Psi v_\Phi(0)\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi(0)\|_{H^{0,1}}^2)$$

and

$$H(s) = \|e^\Psi v_{\Phi^1}^1\|_{H^{0,1}}^2.$$

Recall that the difference solutions satisfy zero initial data, i.e., $M(0) = 0$. This immediately implies that

$$\|e^\Psi v_\Phi\|_{H^{0,1}}^2 + \|e^\Psi \theta_\Phi\|_{H^{0,1}}^2 = 0.$$

for all sufficiently small $t > 0$. By a standard time-continuous extension argument, this trivial identity holds for the entire existence interval $[0, T_\varepsilon]$, which yields $v \equiv 0$ and $\theta \equiv 0$.

Therefore, the solution to System (1.3) is unique within the maximal lifespan T_ε . The proof of Theorem 2.1 is thus completed.

5. Conclusions

This paper establishes the long-time well-posedness of analytic solutions to the 2D Boussinesq boundary layer equations without monotonicity constraints. We resolve the double derivative loss from coupled velocity–temperature nonlinearities via analytic phase modulation, Gaussian decaying weights, and anisotropic weighted energy estimates, and derive the explicit solution lifespan scaling $T_\varepsilon \sim \varepsilon^{-4/3}$ under small analytic initial data. This scaling quantitatively reveals that the maximal existence time of analytic solutions decays as the amplitude of initial perturbations increases, and provides a theoretical basis for the reliable time window in high-order numerical simulations of perturbed buoyancy-driven boundary layer flows. The results remove the structural assumptions required by classical boundary layer theories and offer mathematical support for thermal convection flow analysis in atmospheric and engineering thermal problems. Future work can extend this theory to 3D cases and lift the small-data limitation.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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