



Research article

Formal first integrals and higher variational equations

Sergi Simon*

School of Computing, Mathematics & Physics, Faculty of Technology, University of Portsmouth,
Lion Gate Building, Lion Terrace, Portsmouth PO1 3HF, UK

* **Correspondence:** Email: sergi.simon@port.ac.uk.

Abstract: The question of how algebra can be used to solve dynamical systems and characterize chaos was first posed in a fertile mathematical context by Ziglin, Morales, Ramis, and Simó using differential Galois theory. Their study was aimed at first-order, and later higher-order, variational equations of Hamiltonian systems. Recent work by this author formalized a compact yet comprehensive expression of higher-order variationals as one infinite linear system, thereby simplifying the approach. More importantly, the dual of this linear system contains all information relevant to formal first integrals, regardless of whether the original system is Hamiltonian. This applicability to formal calculation of conserved quantities is the centerpiece of this paper, following an introduction to the requisite context. Three important examples, namely, particular cases of Dixon’s system, the susceptible-infected-recovered (SIR) epidemiological model with vital dynamics, and the Van der Pol oscillator, were tackled, and explicit convergent first integrals were provided for the first two.

Keywords: integrability; first integrals; formal calculus; chaos; dynamical systems; Dixon’s system; SIR epidemiological model; van der Pol oscillator

1. Introduction

Given an arbitrary dynamical system, the formulation of its higher variational equations as a linear (infinite) system (LVE^{*}) has shown potential to make strong inroads in the study of integrability [1, 2]. The study, part of which we explain in more detail in subsection 1.1, is twofold:

- (i) On one hand, the original set of equations (LVEs) is amenable to the Ziglin-Morales-Ramis-Simó non-integrability framework whenever the system is *Hamiltonian* and when the first integral being obstructed is *meromorphic* ([3–6] and a long assorted array of references derived therefrom, including [1, 2]). In addition, there is an as of yet unfinished study in the direction of situations where the system is not Hamiltonian ([7–9]).

- (ii) On the other hand, the *dual* system (LVE^{*}) has jets of formal first integrals among its solutions ([1, 10]). The advantage of this second approach is that the original system need not be Hamiltonian, and the formal first integrals, if convergent, need not be meromorphic. The only difficulties in this case are computational in nature, namely, in the context of resummation techniques.

In the present work, we exploit the second item (ii), namely, by describing a method to automatically produce terms of formal first integrals by way of an automatic, easily recursified, sequence of quadratures. First, we recount the minimal background exposition necessary, and then we present the main results in Section 2. Finally, we apply it to three simple examples to test its accuracy and usefulness. We must emphasize that our study, which can be seen as a continuation of [1, 10], is theoretical in nature and designed to provide a framework for the construction of formal first integrals. The first two examples in Section 3 notwithstanding, we are not presenting a bona fide efficient computational method for practical applications and closed-form convergent first integrals, let alone a justification of computational efficiency; such goals will be the subject of future, less theoretical and more specialized work.

1.1. The algebraic study of integrability

1.1.1. Basics

Let $\psi(t, \cdot)$ be the flow and $\phi(t) = \psi(t, \mathbf{x})$ a particular solution of a given autonomous system

$$\dot{\mathbf{z}} = X(\mathbf{z}), \quad X: \mathbb{C}^m \rightarrow \mathbb{C}^m, \quad (1.1)$$

respectively. The **variational system** of (1.1) along ϕ has $\frac{\partial}{\partial \mathbf{z}} \psi(t, \phi)$ as a fundamental matrix:

$$\dot{Y} = A_1 Y, \quad A_1(t) := \left. \frac{\partial X}{\partial \mathbf{z}} \right|_{\mathbf{z}=\phi(t)} \in \text{Mat}_n(K), \quad (\text{VE}_\phi)$$

with $K = \mathbb{C}(\phi)$ being the smallest differential field ([12, Def. 1.1]) containing all entries of $\phi(t)$. $\frac{\partial^k}{\partial \mathbf{z}^k} \psi(t, \phi)$ are multilinear k -forms appearing in the Taylor expansion of the flow along ϕ :

$$\psi(t, \mathbf{z}) = \psi(t, \phi) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \psi(t, \phi)}{\partial \mathbf{z}^k} \{\mathbf{z} - \phi\}^k. \quad (1.2)$$

$\frac{\partial^k}{\partial \mathbf{z}^k} \psi(t, \phi)$ also satisfies an echeloned set of systems, depending on the previous $k-1$ partial derivatives and usually called **order- k variational equations** VE_ϕ^k ([1, Cor. 3], [6, pp. 859–861]). Thus we have, with (1.1) given, a *linear* system $\text{VE}_\phi =: \text{VE}_\phi^1 =: \text{LVE}_\phi^1$ and a family of *non-linear* systems $\{\text{VE}_\phi^k\}_{k \geq 2}$.

Study [1] presented an explicit *linearized* version LVE_ϕ^k , $k \geq 1$, by means of symmetric products $\odot$ of finite and infinite matrices based on already-existing definitions by Bekbaev, e.g., [11]. This was done in preparation for the Ziglin-Morales-Ramis-Simó (ZMRS) theoretical framework based on monodromy and differential Galois groups [3, 12, 13], but has other consequences as well. More specifically, our outcomes in [1] have two applications for system (1.1), Hamiltonian or not:

- The full structure of VE_ϕ^k and LVE_ϕ^k , i.e., *recovering the flow*, which underlies the ZMRS theoretical corpus in practicality, albeit as a tool rather than as a goal;

- A byproduct is the full structure of dual systems $(\text{LVE}_\phi^k)^\star$, i.e., *having formal first integrals of (1.1) among its solutions*, in ways which simplified earlier results in [10] significantly.

As said in the introduction, results in the present paper are based on the second of these applications. For examples of the first application, see [1, Sec. 6] or the bulk of [2]. See also [14, 15] for examples where the non-linearized VE_k were used.

1.2. Symmetric products, powers, and exponentials

Notation 1.2.1. The conventions listed below were already introduced in [1, 2, 10]:

- 1) *Multi-index modulus, arithmetic, order, and lexicographic order:* for $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$, $i = |\mathbf{i}| := \sum_k i_k$; addition and subtraction are defined entrywise as usual; $\mathbf{i} \leq \mathbf{j}$ means $i_k \leq j_k$ for every $k \geq 1$; $\mathbf{i} <_{\text{lex}} \mathbf{j}$ if $i_1 = j_1, \dots, i_{k-1} = j_{k-1}$ and $i_k < j_k$ for some $k \geq 1$.
- 2) Whenever such derivation is possible, we define the *lexicographically sifted differential of* $F(z_1, \dots, z_m)$ *of order* m *as the row vector*

$$F^{(m)}(\mathbf{z}) := \text{lex} \left(\frac{\partial^m F}{\partial z_1^{i_1} \dots \partial z_n^{i_n}}(\mathbf{z}) \right), \quad (1.3)$$

where $|\mathbf{i}| = m$ and entries are ordered as per $<_{\text{lex}}$ on multi-indices. For instance, for $n = 2$, the first two differentials would be

$$F^{(1)} = \left(\frac{\partial F}{\partial z_1} \quad \frac{\partial F}{\partial z_2} \quad \frac{\partial F}{\partial z_3} \right), \quad F^{(2)} = \left(\frac{\partial^2 F}{\partial z_1^2} \quad \frac{\partial^2 F}{\partial z_1 z_2} \quad \frac{\partial^2 F}{\partial z_1 z_3} \quad \frac{\partial^2 F}{\partial z_2^2} \quad \frac{\partial^2 F}{\partial z_2 z_3} \quad \frac{\partial^2 F}{\partial z_3^2} \right).$$

- 3) We define $d_{n,k} := \binom{n+k-1}{n-1}$, $D_{n,k} := \sum_{i=1}^k d_{n,i}$. It is easy to check that there are $d_{n,k}$ k -tuples of integers in $\{1, \dots, n\}$, and just as many homogeneous monomials of degree n in k variables.
- 4) Given integers $k_1, \dots, k_n \geq 0$, we define the usual multinomial coefficient as

$$\binom{k_1 + \dots + k_n}{k_1, \dots, k_n} := \binom{k_1 + \dots + k_n}{\mathbf{k}} := \frac{(k_1 + \dots + k_n)!}{k_1! k_2! \dots k_n!}.$$

For a multi-index $\mathbf{k} \in \mathbb{Z}_{\geq 0}^n$, we define $\mathbf{k}! := k_1! \dots k_n!$. For any two such $\mathbf{k}, \mathbf{j}$, we define

$$\binom{\mathbf{k}}{\mathbf{p}} := \frac{k_1! k_2! \dots k_n!}{p_1! p_2! \dots p_n! (k_1 - p_1)! (k_2 - p_2)! \dots (k_n - p_n)!} = \binom{k_1}{p_1} \binom{k_2}{p_2} \dots \binom{k_n}{p_n}, \quad (1.4)$$

and the multi-index counterpart to the multinomial, $\binom{\mathbf{k}_1 + \dots + \mathbf{k}_m}{\mathbf{k}_1, \dots, \mathbf{k}_m} := \frac{(\mathbf{k}_1 + \dots + \mathbf{k}_m)!}{\mathbf{k}_1! \mathbf{k}_2! \dots \mathbf{k}_m!}$.

The compact formulation called for by Notation 1.2.1 (3) was achieved in [1] through an operation $\odot$ that had already been defined by Bekbaev (e.g., [11, 16]) and was systematized using basic categorical properties of the tensor product. Let K be a field and V a K -vector space. Let $\text{Sym}^r V$ be the symmetric power of V . We write $\mathbf{w}_1 \odot \mathbf{w}_2$ for equivalence classes of tensor products of these vectors. Hence, the product $\odot$ operates exactly like products of homogeneous polynomials in several variables.

Notation 1.2.2. When dealing with matrix sets, we use super-indices and subindices:

- 1) The space of (i, j) -**matrices** $\text{Mat}_{m,n}^{i,j}(K)$ is defined equivalently as the set of $d_{m,i} \times d_{n,j}$ matrices with entries in K , or the vector space $\text{Hom}_K(\text{Sym}^j K^m; \text{Sym}^i K^n)$.
- 2) It is clear from the above that $\text{Mat}_n^{0,0}(K)$ is the set of all scalars $\alpha \in K$ and $\text{Mat}_n^{0,k}(K)$ (resp., $\text{Mat}_n^{k,0}(K)$) is made up of all row (resp., column) vectors with entries indexed by $d_{n,k}$ lexicographically ordered k -tuples.
- 3) Reference to K may be dropped and notation may be abridged if dimensions are repeated or trivial, e.g., $\text{Mat}_n^{i,j} := \text{Mat}_{n,n}^{i,j}$, $\text{Mat}_{m,n}^i := \text{Mat}_{m,n}^{i,i}$, $\text{Mat}_n := \text{Mat}_n^1$, etc.
- 4) $\text{Mat}^{n,m}(K)$ denotes the set of block matrices $A = (A_{i,j})_{i,j \geq 0}$ with $A_{i,j} : \text{Sym}^i K^m \rightarrow \text{Sym}^j K^n$, hence $A_{i,j} \in \text{Mat}_{d_{n,i} \times d_{m,j}}(K) = \text{Mat}_{n,m}^{i,j}(K)$:

$$A = \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} \cdots & A_{2,2} & A_{2,1} & \leftarrow A_{2,0} \\ \cdots & A_{1,2} & A_{1,1} & \leftarrow A_{1,0} \\ \cdots & A_{0,2} & A_{0,1} & \leftarrow A_{0,0} \end{pmatrix}.$$

We write $\text{Mat} := \text{Mat}^{n,n}$ if n is unambiguous. Conversely, $\text{Mat}_{n,m}^{i,j}$ is embedded in $\text{Mat}^{n,m}$ by identifying every matrix $A_{i,j}$ with an element of $\text{Mat}^{n,m}$ equal to 0 save for block $A_{i,j}$.

Definition 1.2.3 (Symmetric products of finite and infinite matrices). [1, 11]

- (i) Let $A \in \text{Mat}_{m,n}^{i_1,j_1}(K)$, $B \in \text{Mat}_{m,n}^{i_2,j_2}(K)$. Given any multi-index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$ such that $|\mathbf{k}| = k_1 + \dots + k_n = j_1 + j_2$, define $C := A \odot B \in \text{Mat}_{m,n}^{i_1+i_2,j_1+j_2}$ by

$$C(e_1^{\odot k_1} \dots e_n^{\odot k_n}) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{\mathbf{p}} \binom{\mathbf{k}}{\mathbf{p}} A(e_1^{\odot p_1} \dots e_n^{\odot p_n}) \odot B(e_1^{\odot k_1-p_1} \dots e_n^{\odot k_n-p_n}). \quad (1.5)$$

The notation is abused by removing $\odot$ to reduce space, keeping the binomials as in (1.4), and summation taking place for specific multi-indices $\mathbf{p}$, namely, those such that

$$|\mathbf{p}| = j_1 \quad \text{and} \quad 0 \leq p_i \leq k_i, \quad i = 1, \dots, n.$$

- (ii) For any $A, B \in \text{Mat}^{n,m}(K)$, define $A \odot B = C \in \text{Mat}^{n,m}(K)$ by

$$C = (C_{i,j})_{i,j \geq 0}, \quad C_{i,j} = \sum_{0 \leq i_1 \leq i, 0 \leq j_1 \leq j} \binom{j}{j_1} A_{i_1,j_1} \odot B_{i-i_1,j-j_1}. \quad (1.6)$$

As always, $\odot^k$ stands for powers built with this product.

The following is a mere exercise in induction:

Lemma 1.2.4. Defining $\bigodot_{i=1}^r A_i$ recursively by $\left(\bigodot_{i=1}^{r-1} A_i\right) \odot A_r$ with $A_i \in \text{Mat}_{m,n}^{k_i, j_i}$,

$$(A_1 \odot \cdots \odot A_r) \mathbf{e}^{\odot \mathbf{k}} = \frac{1}{\binom{j_1 + \cdots + j_r}{j_1, j_2, \dots, j_r}} \sum_{\mathbf{p}_1, \dots, \mathbf{p}_r} \binom{\mathbf{k}}{\mathbf{p}_1, \dots, \mathbf{p}_r} \bigodot_{i=1}^r A_i \mathbf{e}^{\odot \mathbf{p}_i}, \quad (1.7)$$

if $|\mathbf{k}| = j_1 + \cdots + j_r$, where the sums are obviously taken for $\mathbf{p}_1 + \cdots + \mathbf{p}_r = \mathbf{k}$ and $|\mathbf{p}_i| = j_i$, for every $i = 1, \dots, r$. $\square$

The following is straightforward and has already been seen, e.g., in [1, 11, 16], or can be easily derived therefrom:

Proposition 1.2.5. For any A, B, C , and whenever products make sense,

- a) $A \odot B = B \odot A$.
- b) $(A + B) \odot C = A \odot C + B \odot C$.
- c) $(A \odot B) \odot C = A \odot (B \odot C)$.
- d) $(\alpha A) \odot B = \alpha (A \odot B)$ for every $\alpha \in K$.
- e) If A is square and invertible, then $(A^{-1})^{\odot k} = (A^{\odot k})^{-1}$.
- f) $A \odot B = 0$ if and only if $A = 0$ or $B = 0$.
- g) If A is a square $(1, 1)$ -matrix, then $A \mathbf{v}_1 \odot A \mathbf{v}_2 \odot \cdots \odot A \mathbf{v}_m = A^{\odot m} \mathbf{v}_1 \odot \cdots \odot \mathbf{v}_m$.
- h) If $\mathbf{v}$ is a column vector, then $(A \odot B) \mathbf{v}^{\odot(p+q)} = (A \mathbf{v}^{\odot p}) \odot (B \mathbf{v}^{\odot q})$, $p, q \in \mathbb{Z}_{\geq 0}$.

Lemma 1.2.6 ([1]). (i) Given square $A, B \in \text{Mat}_n^{k,k}$ and matrices $X_i \in \text{Mat}_n^{k, j_i}$, $i = 1, 2$,

$$(A \odot B) (X_1 \odot X_2) = \frac{1}{2} (AX_1 \odot BX_2 + BX_1 \odot AX_2), \quad (1.8)$$

and in general for any square $A_1, \dots, A_m \in \text{Mat}_n^{k,k}$ and any $X_i \in \text{Mat}_n^{k, j_i}$, $i = 1, \dots, m$,

$$\left(\bigodot_{i=1}^m A_i\right) \left(\bigodot_{i=1}^m X_i\right) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \bigodot_{i=1}^m A_{\sigma(i)} X_i. \quad \square \quad (1.9)$$

(ii) Given $A \in \text{Mat}_n^{1,j}$ and $X_1, \dots, X_m$ such that $X_i \in \text{Mat}_n^{1, q_i}$, $1 \leq j \leq m$,

$$\binom{m}{j} (A \odot \text{Id}_n^{\odot m-j}) \bigodot_{i=1}^m X_i = \sum_{1 \leq i_1 < \cdots < i_j \leq m} [A(X_{i_1} \odot \cdots \odot X_{i_j})] \odot \bigodot_{s \neq i_1, \dots, i_j} X_s. \quad \square \quad (1.10)$$

(iii) Given a square matrix $A \in \text{Mat}_n^{1,1}$ and $X_1, \dots, X_m$ such that $X_i \in \text{Mat}_n^{1, j_i}$,

$$(A \odot \text{Id}_n^{\odot m-1}) \left(\bigodot_{i=1}^m X_i\right) = \frac{1}{m} \sum_{i=1}^m (AX_i) \odot (X_1 \odot \cdots \odot \widehat{X_i} \odot \cdots \odot X_m). \quad (1.11)$$

(iv) Given square matrix $X \in \text{Mat}_n^{k-1,k-1}$ and vector $\mathbf{v} \in K^n$, we have, for each $i = 1, \dots, n$,

$$(X \odot \mathbf{e}_i^T)(\mathbf{v} \odot \text{Id}_n^{\odot k-1}) = \frac{k-1}{k} X(\mathbf{v} \odot \text{Id}_n^{\odot k-2}) \odot \mathbf{e}_i^T + \frac{v_i}{k} X. \quad (1.12)$$

Lemma 1.2.7. Let (K, ∂) be a differential field.

(i) For any given $X \in \text{Mat}_n^{k_1, j_1}(K)$ and $Y \in \text{Mat}_n^{k_2, j_2}(K)$,

$$\partial(X \odot Y) = \partial(X) \odot Y + X \odot \partial(Y). \quad \square \quad (1.13)$$

(ii) If Y is a square $n \times n$ matrix having entries in K and $\partial Y = AY$, then

$$\partial \text{Sym}^k Y = k(A \odot \text{Sym}^{k-1}(\text{Id}_n)) \text{Sym}^k Y. \quad (1.14)$$

(iii) If $X \in \text{Mat}_n^{1, j_1}$ and $Y \in \text{Mat}_n^{1, j_2}$ satisfy systems $\partial X = AX + B_1$ and $\partial Y = AY + B_2$ with $A \in \text{Mat}_n^{1,1}$, $B_i \in \text{Mat}_n^{1, j_i}$, then the symmetric product $X \odot Y$ satisfies the linear system

$$\partial(X \odot Y) = 2(A \odot \text{Id}_{d_{n,k}})(X \odot Y) + (B_1 \odot Y + B_2 \odot X). \quad (1.15)$$

(iv) If $\partial X_i = AX_i + B_i$, $i = 1, \dots, m$, with $X_i, B_i \in \text{Mat}_n^{1, j_i}$, $A \in \text{Mat}_n^{1,1}$, then

$$\partial \left(\bigodot_{i=1}^m X_i \right) = m(A \odot \text{Id}_{d_{n,k}}^{\odot m-1}) \left(\bigodot_{i=1}^m X_i \right) + \sum_{i=1}^m B_i \odot \left(\bigodot_{j \neq i}^m X_j \right). \quad \square \quad (1.16)$$

Definition 1.2.8. (See also [16]) For every matrix $A \in \text{Mat}^{n,m}$, we define the formal power series

$$\exp_{\odot} A := 1 + A^{\odot 1} + \frac{1}{2} A^{\odot 2} + \dots = \sum_{i=0}^{\infty} \frac{1}{i!} A^{\odot i}.$$

Whenever $A = 0$ save for a finite distinguished submatrix $A_{j,k}$, the abuse of notation $\exp_{\odot} A_{j,k} = \exp_{\odot} A$ is customary.

Lemma 1.2.9.

(i) For every two $A, B \in \text{Mat}^{n,m}$, $\exp_{\odot}(A + B) = \exp_{\odot} A \odot \exp_{\odot} B$.

(ii) For every $Y \in \text{Mat}^{n,m}$ and any derivation $\partial : K \rightarrow K$, $\partial \exp_{\odot} Y = (\partial Y) \odot \exp_{\odot} Y$.

(iii) ([11, Cor. 3]) Given square matrices $A, B \in \text{Mat}_n^{1,1}$, $\exp_{\odot} AB = \exp_{\odot} A \exp_{\odot} B$.

(iv) In particular, for every invertible square $A \in \text{Mat}_n^{1,1}$, $\exp_{\odot} A^{-1} = (\exp_{\odot} A)^{-1}$. $\square$

For examples and properties, see [1, Sec. 3.1].

1.3. Application to power series

Lemma 1.3.1 ([1, Lem. 3.7]). If $F = F_1 \times \cdots \times F_m$ is a vector power series, adequate $M_F^{1,i} \in \text{Mat}_{m,n}^{1,i}(K)$ render

$$\boxed{F(\mathbf{x}) = M_F \exp_{\odot} X} \quad \text{where } M_F := \left(\begin{array}{ccc|c} \cdots & M_F^{1,2} & M_F^{1,1} & M_F^{1,0} \\ \cdots & 0 & 0 & 0 \end{array} \right) \in \text{Mat}^{m,n}.$$

Following Definition 1.2.8, we write $F(\mathbf{x}) = M_F \exp_{\odot} \mathbf{x}$ if it poses no clarity issue. $\square$

Thus every formal power series $F \in K[[\mathbf{x}]]$, $\mathbf{x} = (x_1, \dots, x_n)$, can be expressed in the form $M_F \exp_{\odot} \mathbf{x}$, where

$$M_F := \left(\begin{array}{ccc|c} \cdots & M_F^{1,2} & M_F^{1,1} & M_F^{1,0} \\ \cdots & 0 & 0 & 0 \end{array} \right) \in \text{Mat}^{1,n}(K), \quad X := \left(\begin{array}{c|c} 0 & \mathbf{x} \\ 0 & 0 \end{array} \right).$$

$$M_F = J_F + M_F^{1,0} := \left(\begin{array}{ccc|c} \cdots & M_F^{1,2} & M_F^{1,1} & 0 \\ \cdots & 0 & 0 & 0 \end{array} \right) + \left(\begin{array}{c|c} 0 & M_F^{1,0} \\ 0 & 0 \end{array} \right). \quad (1.17)$$

Lemma 1.3.2 ([1, Lem. 3.8]). In the above conditions, $\exp_{\odot} F(\mathbf{x}) = (\exp_{\odot} M_F)(\exp_{\odot} \mathbf{x})$. $\square$

See [1, Sec. 3.2] for more details.

1.4. Higher-order variational equations

See [1] for proofs and further elaboration on everything summarized below:

Notation 1.4.1. For every set of indices $1 \leq i_1 \leq \cdots \leq i_r$ such that $\sum_{j=1}^r i_j = k$, $c_{i_1, \dots, i_r}^k$ is defined as the amount of totally ordered partitions of a set of k elements among subsets of sizes $i_1, \dots, i_r$:

$$c_{i_1, \dots, i_r}^k = c_{i_1, \dots, i_r}^k = \frac{\binom{k}{i_1, i_2, \dots, i_r}}{n_1! \cdots n_m!}, \quad \begin{cases} (i_1, \dots, i_r) = (k_1, \dots, k_m, \dots, k_m) \\ 1 \leq k_1 < k_2 < \cdots < k_m. \end{cases} \quad (1.18)$$

Proposition 1.4.2. Let $K = \mathbb{C}(\phi)$ and A_k, Y_k be $\partial_k X(\phi)$, $\partial_z^k \psi(t, \phi)$ minus crossed terms, and let $A = J_X^\phi, Y = J_\psi^\phi$ be the derivative jets for X and ψ at ϕ , with A_k, Y_k as blocks. Then, if $c_i^k = \#\{\text{ordered } i_1, \dots, i_r\text{-partitions of } k \text{ elements}\}$,

$$\dot{\overline{Y}}_k = \sum_{j=1}^k A_j \sum_{|\mathbf{i}|=k} c_{\mathbf{i}}^k Y_{i_1} \odot Y_{i_2} \odot \cdots \odot Y_{i_j}, \quad k \geq 1. \quad (\text{VE}_{\phi}^k)$$

Lemma 1.4.3. Let $Y \in \text{Mat}(K)$ equal to zero outside of block row $1, k$, $k \geq 1$:

$$Y := \left(\begin{array}{ccc|c} \cdots & Y_3 & Y_2 & Y_1 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad Y_i \in \text{Mat}_n^{1,i}. \quad (1.19)$$

Let $Z_{r,s}$, $s, r \geq 1$, be the corresponding block in $\exp_{\odot} Y$. Then,

a) Row block r in $\exp_{\odot} Y$ is recursively obtained in terms of row blocks 1 and $r - 1$:

$$Z_{r,s} = \frac{1}{r} \sum_{j=1}^{s-r+1} \binom{s}{j} Y_j \odot Z_{r-1,s-j}. \quad (1.20)$$

In particular, $Z_{r,r} = Y_1^{\odot r}$ and $Z_{r,s} = 0_{d_{n,r}, d_{n,s}}$ whenever $r > s$.

b) Using Notation 1.4.1 and (1.18), for every $s \geq r$,

$$Z_{r,s} = \sum_{i_1 + \dots + i_r = s} c_{i_1, \dots, i_r}^s Y_{i_1} \odot Y_{i_2} \odot \dots \odot Y_{i_r}. \quad (1.21)$$

Notation 1.4.4. $K := \mathbb{C}(\phi)$, $A_i := X^{(i)}(\phi)$, $Y_i := \text{lex}\left(\frac{\partial^i}{\partial \mathbf{z}^i} \varphi(t, \phi)\right)$, and, per Lemma 1.4.3,

$$\Upsilon_1 = Y_1, \quad \Upsilon_k = \begin{pmatrix} Z_{k,k} \\ Z_{k-1,k} \\ \vdots \\ Z_{1,k} \end{pmatrix} \begin{array}{c} \boxed{\Upsilon_{k-1}} \end{array}, \quad k \geq 2, \quad (1.22)$$

which is formed by the first k block rows and columns in $\Upsilon = \exp_{\odot} Y$. Define $A, Y \in \text{Mat}(K)$ as in Lemma 1.4.3 with the above A_i, Y_i as blocks. Denote the canonical basis on K^n (meaning the set of columns of Id_n) by $\{e_1, \dots, e_n\}$.

Proposition 1.4.5 (Explicit version of LVE_{ϕ}^k). *Still following Notation 1.4.4, the infinite system*

$$\boxed{\dot{X} = A_{\text{LVE}_{\phi}} X}, \quad A_{\text{LVE}_{\phi}} := A \odot \exp_{\odot} \text{Id}_n, \quad (\text{LVE}_{\phi})$$

has $\Upsilon := \exp_{\odot} Y$ as a solution matrix. Hence, for every $k \geq 1$,

a) the lower-triangular recursive $D_{n,k} \times D_{n,k}$ form for LVE_{ϕ}^k is $\dot{Y} = A_{\text{LVE}_{\phi}^k} Y$, with its system matrix being obtained from the first k row and column blocks of $A_{\text{LVE}_{\phi}}$:

$$A_{\text{LVE}_{\phi}^k} = \begin{pmatrix} \binom{k}{k-1} A_1 \odot \text{Id}_n^{\odot k-1} \\ \binom{k}{k-2} A_2 \odot \text{Id}_n^{\odot k-2} \\ \vdots \\ \binom{k}{0} A_k \end{pmatrix} \begin{array}{c} \boxed{A_{\text{LVE}_{\phi}^{k-1}}} \end{array}; \quad (1.23)$$

b) the principal fundamental matrix for LVE_{ϕ}^k is Υ_k from $\exp_{\odot} Y$ in Notation 1.4.4.

The construction of an infinite matrix $\Upsilon_{\text{LVE}_{\phi}} = \exp_{\odot} Y$ follows in such a way that the first $d_{n,k} := \binom{n+k-1}{n-1}$ rows and columns are a principal fundamental matrix for LVE_{ϕ}^k . The definition is recursive and amenable to symbolic computation.

Example 1.4.6. For instance, for $k = 5$, we have

$$A_{\text{LVE}_\phi^5} = \begin{pmatrix} 5A_1 \odot \text{Id}_n^{\odot 4} & & & & \\ 10A_2 \odot \text{Id}_n^{\odot 3} & 4A_1 \odot \text{Id}_n^{\odot 3} & & & \\ 10A_3 \odot \text{Id}_n^{\odot 2} & 6A_2 \odot \text{Id}_n^{\odot 2} & 3A_1 \odot \text{Id}_n^{\odot 2} & & \\ 5A_4 \odot \text{Id}_n & 4A_3 \odot \text{Id}_n & 3A_2 \odot \text{Id}_n & 2A_1 \odot \text{Id}_n & \\ A_5 & A_4 & A_3 & A_2 & A_1 \end{pmatrix},$$

and the principal fundamental matrix Υ_5 is

$$\begin{pmatrix} Y_1^{\odot 5} & & & & & \\ 10Y_1^{\odot 3} \odot Y_2 & Y_1^{\odot 4} & & & & \\ 10Y_1^{\odot 2} \odot Y_3 + 15Y_1 \odot Y_2^{\odot 2} & 6Y_1^{\odot 2} \odot Y_2 & Y_1^{\odot 3} & & & \\ 10Y_2 \odot Y_3 + 5Y_1 \odot Y_4 & 4Y_1 \odot Y_3 + 3Y_2 \odot Y_2 & 3Y_1 \odot Y_2 & Y_1^{\odot 2} & & \\ Y_5 & Y_4 & Y_3 & Y_2 & Y_1 & \end{pmatrix}, \quad (1.24)$$

hence (VE_ϕ^k) for $k = 5$ is the lowest row in $A_{\text{LVE}_\phi^5}$ times the leftmost column in Υ_5 .

1.5. First integrals and higher-order variational equations

Let $F : U \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic function and $\phi : I \subset \mathbb{C} \rightarrow U$. First, the flow $\varphi(t, z)$ of X admits, at least formally, Taylor expansion (1.2) along ϕ , which is expressible as

$$\varphi(t, \phi + \xi) = \phi + Y_1 \xi + \frac{1}{2} Y_2 \xi^{\odot 2} + \cdots = \phi + J_\phi \exp_\odot \xi, \quad (1.25)$$

where J_ϕ is the jet for flow $\varphi(t, \cdot)$ along ϕ , displayed as Y in (1.19) and defined in Notation 1.4.4—that is, the matrix such that its $\odot$ -exponential Υ is a solution matrix for (LVE_ϕ) . Second, the Taylor series of F along ϕ can be written as, cfr. [10, Lem. 2] and Notation 1.2.1,

$$F(y + \phi) = F(\phi) + \sum_{m=1}^{\infty} \frac{1}{m!} \langle F^{(m)}(\phi), \text{Sym}^m y \rangle. \quad (1.26)$$

Lemma 1.3.1 and (1.17) trivially imply that (1.26) can be expressed as $F(y + \phi) = M_F^\phi \exp_\odot y$, where

$$M_F^\phi = J_F^\phi + F^{(0)}(\phi) := \left(\begin{array}{ccc|c} \cdots & 0 & 0 & 0 \\ \cdots & F^{(2)}(\phi) & F^{(1)}(\phi) & F^{(0)}(\phi) \\ \cdots & 0 & 0 & 0 \end{array} \right) \in \text{Mat}^{1,n}(K),$$

i.e., J_F^ϕ is the jet or horizontal strip of lex-sifted partial derivatives of F at ϕ .

Let

$$\boxed{\dot{X} = A_{\text{LVE}_\phi^*} X}, \quad A_{\text{LVE}_\phi^*} := -\left(A \odot \exp_\odot \text{Id}_n\right)^T, \quad (\text{LVE}_\phi^*)$$

be the **adjoint** or **dual** variational system of (1.1) along ϕ .

The following is immediate, upon derivation of the equation $\Upsilon_k \Upsilon_k^{-1} = \text{Id}_{D_{n,k}}$.

Lemma 1.5.1. $(\Upsilon_k^{-1})^T$ is a principal fundamental matrix of $(\text{LVE}_\phi^k)^*$, $k \geq 1$, hence $\lim_k (\Upsilon_k^{-1})^T$ is a solution to (LVE_ϕ^*) .

The following was proven in [6] and recounted in [10, Lem. 7], and may now be expressed in a simple and compact fashion:

Lemma 1.5.2. *Let F and ϕ be a holomorphic first integral and a non-constant solution of (1.1), respectively. Let $V := J_F^T$ be the transposed jet of F along ϕ . Then, V is a solution of $(\text{LVE}_\phi^\star)$.*

Hence the issue, already considered in [10], is whether some converse holds and a solution of the dual system $(\text{LVE}_\phi^k)^\star$ is or can be the set of the first k terms of the formal (or perhaps even convergent) series form of a first integral. This was addressed in the aforementioned reference and was put in compact and explicit linearized form in [1], and is recounted below.

Define

$$\widehat{A} := \left(\begin{array}{ccc|c} \cdots & 0 & 0 & 0 \\ \cdots & A_2 & A_1 & A_0 \\ \cdots & 0 & 0 & 0 \end{array} \right), \quad A_i := X^{(i)}(\phi) \in \text{Mat}_n^{1,i}(K), \quad A_0 := X(\phi) = \dot{\phi}, \quad (1.27)$$

and let

$$\widehat{A}_{\text{LVE}_\phi} := \widehat{A} \odot \exp_\odot \text{Id}_n = \lim_k \widehat{A}_{\text{LVE}_\phi^k} = \lim_k \left(\begin{array}{c|c} \begin{array}{c} \binom{k}{k} X^{(0)}(\phi) \odot \text{Id}_n^{\odot k} \\ \binom{k}{k-1} X^{(1)}(\phi) \odot \text{Id}_n^{\odot k-1} \\ \binom{k}{k-2} X^{(2)}(\phi) \odot \text{Id}_n^{\odot k-2} \\ \vdots \\ \binom{k}{0} X^{(k)}(\phi) \odot \text{Id}_n^{\odot 0} \end{array} & \widehat{A}_{\text{LVE}_\phi^{k-1}} \end{array} \right). \quad (1.28)$$

The following is a compact form of [10, Th. 12] in terms of $\odot$ and infinite matrices:

Proposition 1.5.3 ([1, Prop. 6]). *Let $F, \phi, V = (\cdots | V_3 | V_2 | V_1)$ as in Lemma 1.5.2. Then $\widehat{A}_{\text{LVE}_\phi}^T V = 0$. More specifically, $\widehat{A}_{\text{LVE}_\phi^{k-1}} (V_k | \cdots | V_2 | V_1) = 0$ for every $k \geq 1$, i.e.,*

$$\sum_{j=0}^{k-1} \binom{k-1}{j} (A_j \odot \text{Id}_n^{k-1-j})^T V_{k-j} = 0, \quad \text{for every } k \geq 1. \quad \square \quad (1.29)$$

Definition 1.5.4 ([10]). *In the above hypotheses,*

- (i) *the vectors $(V_k, \dots, V_1)^T$ belonging to the intersection of $\ker \widehat{A}_{\text{LVE}_\phi^{k-1}}^T$ and the solution subspace $\text{Sol}_K(\text{LVE}_\phi^k)^\star$ are called **admissible** solutions of the order- k adjoint system;*
- (ii) *we call the sum $\sum_{k \geq 1} \frac{1}{k!} V_k z^{\odot k}$ a **formal first integral** along ϕ if every finite truncation thereof, $\sum_{k=1}^m \frac{1}{k!} V_k z^{\odot k}$, arises from an admissible solution $(V_m, \dots, V_1)$.*

For a short introduction on how changes of variables are reflected on this scheme, as well as the explicit form for the *monodromy matrix* [17] of LVE_ϕ^k along any path γ , see [1, Secs. 4 and 5].

Lemmas 1.5.1 and 1.5.2 have a converse:

Lemma 1.5.5. *Let $f = \sum_{k \geq 1} \frac{1}{k!} V_k z^{\odot k}$ be a formal first integral for a system X holomorphic on a neighborhood U of ϕ . If f is convergent in said neighborhood, it is a first integral.*

Proof. Let $V = (\cdots | V_3 | V_2 | V_1)$, where V_k are defined as in Definition 1.5.4, and assume the series $F(z + \phi) = V^T \exp_{\odot} z$ is convergent in a vicinity of ϕ . Hence, for any solution $\psi = \varphi(t, \phi + \xi) = \phi + J_{\phi} \exp_{\odot} \xi$ in a neighborhood of ϕ , we have, in virtue of Lemma 1.3.2,

$$F(\psi) = V^T \exp_{\odot} (J_{\phi} \exp_{\odot} \xi) = V^T \exp_{\odot} (J_{\phi}) \exp_{\odot} \xi.$$

Thus, by the definition of V and as a consequence of Proposition 1.4.5,

$$\begin{aligned} \overline{F(\psi)} &= \overline{V^T \exp_{\odot} (J_{\phi}) \exp_{\odot} \xi} + V^T \overline{\exp_{\odot} (J_{\phi}) \exp_{\odot} \xi} \\ &= -V^T A_{\text{LVE}_{\phi}} \exp_{\odot} (J_{\phi}) \exp_{\odot} \xi + V^T A_{\text{LVE}_{\phi}} \exp_{\odot} (J_{\phi}) \exp_{\odot} \xi = 0. \end{aligned}$$

□

Hence, blocks in $V_1, (V_2, V_1)^T, (V_3, V_2, V_1)^T, \dots$ having all entries in the base field K and satisfying both equations in Propositions 1.5.3 and 1.5.2 are jet blocks $F^{(1)}, F^{(2)}, \dots$ of a formal series that is a first integral if convergent.

Remark 1.5.6. As said at the start of Section 1, ours is a theoretical study centered on formal first integrals (per the above definition). On a wider scale, if we consider the word *formal* to be both operative and general, formal integrability is challenging in itself and usually not feasible, e.g., [18–23]; and even if formal first integrals of some sort can be found for a given system, determining bounds on valuations, or closed forms with their general expansion terms, are open problems in their own right, not to mention the additional challenges posed by convergence and resummation.

In Section 2, we show that for the particular expansions defined above, these challenges dissipate except for the last one (convergence and closed-form first integrals). Not only are these formal first integrals well-defined with explicit terms in recursive form, but their valuation can be brought down to one, i.e., their first series term is linear. Even though convergence and closed-form invariants are left for future work, in Section 3, we also show two examples of what is achieved when summation is applied to a convergent series.

2. Jet filtering methods and computation of formal first integrals

2.1. Preliminaries

Let f, g be functions of n variables. The following are immediate:

Lemma 2.1.1. (i) For every $i \geq 1$,

$$(fg)^{(i)} = \sum_{j=0}^i \binom{i}{j} f^{(j)} \odot g^{(i-j)}, \quad (2.1)$$

with the understanding that the symmetric product by a scalar is identical to the usual product, hence

$$\begin{aligned} J\{fg\} &= U \odot (U^{\odot 2})^T [J\{f\} \odot J\{g\}] = U \odot (U^{\odot 2})^T [\exp_{\odot} J\{f\} \odot \exp_{\odot} J\{g\}] \\ &= U \odot [U^T \exp_{\odot} J\{f\}] \odot [U^T \exp_{\odot} J\{g\}], \quad \text{where } U = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array} \right). \end{aligned}$$

(ii) For every $k \geq 1$,

$$(f^k)^{(i)} = k! \sum_{j=1}^k \frac{f^{j-1}}{(j-1)!} \sum_{i_1+\dots+i_{k-j+1}=i} c_{i_1,\dots,i_{k-j+1}}^i \bigodot_{u=1}^{k-j+1} f^{(i_u)}, \quad (2.2)$$

hence defining

$$E_1 = \left(\begin{array}{c|c} \mathbf{e}_1^T & 0 \\ \hline 0 & 0 \end{array} \right), \quad U = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array} \right),$$

we have

$$J\{f^k\} = k! \left\{ U \odot (U^{\odot k})^T [\exp_{\odot} J\{f\}] \right\} = k! \left\{ U \odot (U^{\odot k})^T [\exp_{\odot} J\{f\}] \right\} \quad (2.3)$$

$$= U \odot [U^T \exp_{\odot} J\{f\}]^{\odot k} = U \odot [U^T J\{f\}]^{\odot k}. \quad (2.4)$$

(iii) For every $k \geq 1$,

$$f^k (1/f^k)^{(i)} = k \sum_{j=0}^i \frac{(k+j-1)!}{k!} f^{-j} (-1)^j \sum_{i_1+\dots+i_j=i} c_{i_1,\dots,i_j}^i \bigodot_{u=1}^j f^{(i_u)}, \quad (2.5)$$

hence

$$J\{1/f\} = \frac{1}{f} U \odot \left\{ \left(\begin{array}{c|c} \vdots & \vdots \\ \hline 0 & 3!(-1/f)^3 \\ 0 & 2!(-1/f)^2 \\ 0 & 1!(-1/f) \\ \hline 0 & 1 \end{array} \right)^T \exp_{\odot} (J\{f\} - fU) \right\} \quad (2.6)$$

$$= U \odot \left(\frac{1}{f} U^T J\{f\} \right)^{\odot -1} = U \odot \left(\sum_{i=0}^{\infty} (-1)^i \left[\frac{1}{f} U^T J\{f\} - \mathbf{1} \right]^{\odot i} \right). \quad (2.7)$$

2.2. Progressive filtering and computation of formal first integrals

Let us describe a method that starts with a jet of valuation one and then proceeds to compute each jet term of that particular function immediately in terms of the previous ones.

Lemma 2.2.1. Define

$$F_1 := F_{i,1} := \text{Id}_n - \frac{1}{X_i^0} (X^0 \odot \mathbf{e}_i^T). \quad (2.8)$$

The $n-1$ non-zero rows of $F_1 Y_1^{-1}$ are linearly independent admissible solutions of degree one.

Proof. Let $F_{i,1} := \text{Id}_n - \frac{1}{X_i(\phi(0))} (X(\phi(0)) \odot \mathbf{e}_i^T)$. We have $\widehat{A}_{\text{LVE}_{\phi}^0} = A_0^T = X(\phi)^T$ and $(Y_1^{-1})^T = (Y_1^{-1})^T$, the fundamental matrix for VE_1^* . Thus checking that (1.29) holds equates to checking whether $A_0^T (Y_1^{-1}) F_{i,1}^T = \mathbf{0}_n^T$, i.e., that $F_{i,1} Y_1^{-1} A_0$ equals a column of zeros. This is easy to prove by noticing that the basic chain rule and (1.1) entail $\dot{A}_0 = A_1 A_0$ on one hand, and on the other, Lemma 1.5.1 for $k=1$ implies $\frac{d}{dt} Y^{-1} = -Y^{-1} A_1$. Both of these facts put together imply

$$\frac{d}{dt} (Y_1^{-1} A_0) = \overleftarrow{Y_1^{-1}} X + Y_1^{-1} \overleftarrow{X} = -Y^{-1} A_1 X + Y^{-1} A_1 X = 0,$$

and thus $\frac{d}{dt}(F_{i,1}Y_1^{-1}A_0) = F_{i,1}\frac{d}{dt}(Y_1^{-1}A_0) = 0$ as well, which means $F_{i,1}Y_1^{-1}A_0$ is a constant vector. Setting $t = t_0$ and using the fact that Y_1 is a principal fundamental matrix (i.e., $Y_1(t_0) = \text{Id}$) means that for $t = t_0$, abridging notation $X^0 = (X_1^0, \dots, X_n^0) = X(\phi(t_0))$,

$$\begin{aligned} F_{i,1}Y_1(t_0)^{-1}A_0 &= F_{i,1}A_0 = X(\phi(t_0)) - \frac{1}{X_i(\phi(t_0))} (X(\phi(t_0)) \odot e_i^T) X(\phi(t_0)) \\ &= \begin{pmatrix} X_1^0 \\ X_2^0 \\ \vdots \\ X_i^0 \\ \vdots \\ X_{n-1}^0 \\ X_n^0 \end{pmatrix} - \frac{1}{X_i^0} \begin{pmatrix} \boxed{\phantom{0_{n \times (i-1)}}} & \begin{pmatrix} X_1^0 \\ X_2^0 \\ \vdots \\ X_i^0 \\ \vdots \\ X_{n-1}^0 \\ X_n^0 \end{pmatrix} & \boxed{\phantom{0_{n \times (n-i)}}} \end{pmatrix} \begin{pmatrix} X_1^0 \\ X_2^0 \\ \vdots \\ X_i^0 \\ \vdots \\ X_{n-1}^0 \\ X_n^0 \end{pmatrix} = \mathbf{0}_n. \end{aligned}$$

Still for $k = 1$, we need to check that the $n - 1$ non-zero columns of $(F_1Y^{-1})^T$ are linearly independent. This is a consequence of the fact that

$$F_1^T = \begin{pmatrix} \boxed{\text{Id}_{i-1}} & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -\frac{X_1}{X_i} & \dots & -\frac{X_{i-1}}{X_i} & 0 & -\frac{X_{i+1}}{X_i} & \dots & -\frac{X_n}{X_i} \\ 0 & \boxed{\text{Id}_{n-i}} \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

thus in $(Y^{-1})^T F_1^T$, column i equals $\mathbf{0}_n$ and its remaining $n - 1$ columns are the output of subtracting multiples of column i in $(Y^{-1})^T$ from its remaining (already independent) $n - 1$ columns. $\square$

Let $\phi(t)$ be a particular solution of (1.1) with a parametric expression that can be represented with one differential equation involving one single function $x(t)$, e.g., (3.4), (3.11), and (3.19) in Section 3.

Theorem 2.2.2. *In the above hypotheses, let f_1 be one of the columns of $(Y_1^{-1})^T F_1^T$. Define $X_k := (Y_1^{\odot k})^T$. Then each term generated by the recursion*

$$f_k := -X_k^{-1} \int X_k \left\{ \sum_{j=2}^k \binom{k}{j} (A_j \odot \text{Id}_n^{\odot k-j})^T f_{k-j+1} \right\} dx, \quad (2.9)$$

is the degree- k term of the expansion of a formal first integral

$$f(z) = f_1 z + \frac{1}{2} f_2 z^{\odot 2} + \frac{1}{3!} f_3 z^{\odot 3} + \dots,$$

provided that the limits or constants of integration at each stage (2.9) are such that

$$f_k (A_0 \odot \text{Id}_n^{\odot k-1}) + \binom{k-1}{1} f_{k-1} (A_1 \odot \text{Id}_n^{\odot k-2}) + \dots + \binom{k-1}{k-2} f_2 (A_{k-2} \odot \text{Id}_n) + f_1 A_{k-1} = 0. \quad (2.10)$$

Proof. Most of the work has been done already by Proposition 1.5.3, Lemma 2.2.1, and the properties intrinsic to $\odot$. First, $X_k^{-1} = \left[(Y_1^{-1})^{\odot k} \right]^T$. Second, condition (2.10) is nothing but a repetition of (1.29). In order to check the consequences of defining (2.9), all it takes is to realize that the first $d_{n,k} \times d_{n,k}$ equations of $(\text{LVE}_\phi^\star)$ read

$$\overline{\left[(Y_1^{-1})^{\odot k} \right]^T} = - (A_1 \odot \text{Id}_n^{\odot k-1})^T \left[(Y_1^{-1})^{\odot k} \right]^T,$$

which means $(Y_1^{\odot k})^T$ is a principal fundamental matrix thereof. The homogeneous part of the equations satisfied by f_k displays the same matrix, thus (2.9) is just a plain variation of constants applied to what is said in Lemma 1.5.2. $\square$

2.3. Filtering with a single infinite matrix product

The above result provides a recursive method to compute formal first integrals of arbitrary autonomous systems, thereby summing up the immediate aims of the current paper. The question arises, however, as to whether a *single* infinite filter matrix $\Phi = \exp_\odot F$ exists, amenable to recursive computation but expressible in compact form and *computed without any further quadratures*, which multiplied by the fundamental matrix of the dual systems (which is also computable recursively, as seen above) yields an already-trimmed matrix comprised of linearly independent admissible solutions in a single swipe.

Building such an infinite quadrature-free filter matrix beyond $k = 1$ has a value that is more theoretical than practical in nature at this stage, since the computational inviability of ever-growing matrices and the effort of discarding the cross-products $g_{j_1}^{(i_1)} \odot \cdots \odot g_{j_m}^{(i_m)}$ get in the way of approximating jets properly. So the question arises: Why consider one such matrix? The answer resides in the recent trends in categorification of dynamical systems (e.g., [24]) and the potential to transport the techniques known in reductions of linear differential systems and flat meromorphic connections to arbitrary dynamical systems.

In this section we prove such a matrix exists for meromorphic and holomorphic systems.

2.3.1. Statement of the main result on infinite matrix filtering

Let $\phi(t)$ be a non-trivial solution for (1.1) and t_0 such that $X_i(\phi(t_0)) \neq 0$ for some i . Let Υ_k be the principal fundamental matrix for (LVE_ϕ^k) such that $\Upsilon_k(t_0) = \text{Id}$ and $F_1 := F_{i,1}$ as in (2.8).

Theorem 2.3.1. *In the above conditions, assume X is holomorphic on a neighborhood U of $\phi(t_0)$ and Y_1 and Y_1^{-1} are holomorphic on a neighborhood V of t_0 . For every $k \geq 1$, define the constant matrices*

$$F_k = -\frac{1}{X_i^0} \left[\sum_{j=0}^{k-2} \binom{k-1}{j} F_{j+1} (A_{k-j-1}^0 \odot \text{Id}^{\odot j}) \right] U_k, \quad (2.11)$$

where we use the superscript 0 to denote values at $t = t_0$,

$$U_k = \left[\sum_{j=0}^{k-1} \binom{k}{j+1} (-1)^j (\text{Id} - F_1)^{\odot j} \odot \text{Id}^{\odot k-1-j} \right] \odot \mathbf{e}_i^T = \left[\bigodot_{j=1}^{k-1} (\text{Id} - \zeta_k^j F_1) \right] \odot \mathbf{e}_i^T, \quad (2.12)$$

and $\zeta_k = \exp\left(\frac{2\pi i}{k}\right)$ is the primitive root of unity. Let Φ_k be the lower square right $D_{n,k}$ -block of

$$\Phi = \exp_{\odot} F, \quad \text{where } F = (\cdots | F_3 | F_2 | F_1).$$

Then, the last $n - 1$ non-zero columns of $(\Upsilon_k^{-1})^T (\Phi_k)^T$ are linearly independent admissible solutions $g_1^{(\cdot)}, g_2^{(\cdot)}, \dots, g_{n-1}^{(\cdot)} \in \text{Sol}(\text{LVE}_{\phi}^k)^{\star} \cap \ker \widehat{A}_{\text{LVE}_{\phi}^{k-1}}^T$, i.e., truncated jets of functionally independent formal first integrals $g_1, \dots, g_{n-1}$ of valuation 1, defined on a neighborhood of $\phi(t_0)$.

Remarks 2.3.2.

- 1) We already know all the columns of $(\Upsilon_k^{-1})^T (\Phi_k)^T$ belong to $\text{Sol}(\text{LVE}_{\phi}^k)^{\star}$ because $\frac{d}{dt}(\Upsilon^{-1})^T = A_{\text{LVE}^{\star}}(\Upsilon^{-1})^T$, where $\Upsilon = \exp_{\odot} Y$, and the above finite-order products are nothing but truncations of the right-product of $(\Upsilon^{-1})^T$ by a constant matrix. We also know that the non-zero columns in $(Y_1^{-1})^T F_1^T$ are linearly independent, and the independence of the full non-zero columns in higher orders follows.

Thus all that is needed to establish Theorem 2.3.1 is to prove that these truncations are also annihilated by left-multiplication by $\widehat{A}_{\text{LVE}_{\phi}^{k-1}}^T$. This is done in subsection 2.3.2.

- 2) In light of the above Remark, Lemma 2.2.1 basically entails that the entire theorem holds true at level $k = 1$ and, as seen in subsection 2.2, that was all we needed to compute independent jets of formal first integrals, independently of Theorem 2.3.1.

2.3.2. Proof of Theorem 2.3.1

As per Remark 2.3.2 item (1), let us now prove that matrices defined by (2.11) and (2.12) are indeed lower blocks of a filter matrix Φ reducing $(\Upsilon^{-1})^T$ to the desired form (namely, one where its lower right $D_{n,k} \times D_{n,k}$ blocks belong to $\ker \widehat{A}_{\text{LVE}_{\phi}^{k-1}}^T$). In other words, we have that

$$F \cdot \Upsilon^{-1} \cdot \widehat{A}_{\text{LVE}_{\phi}} = F \cdot \Upsilon^{-1} \cdot (\widehat{A} \odot \exp_{\odot} \text{Id}_n) = 0, \quad (2.13)$$

Our first step is proving the following.

Proposition 2.3.3. Identity (2.13) is true at t_0 , i.e., $F \cdot \widehat{A}_{\text{LVE}_{\phi}} \Big|_{t=t_0} = 0$.

Proof. Obviously, we are using the fact that $\Upsilon|_{t=t_0} = \exp_{\odot} \text{Id}$, i.e., an infinite identity matrix. All occurrences of A_i from here onward in this proof refer to constant matrices $A_i^0 = X^{(i)}(\phi(t_0))$. The case where $k = 1$ has already been tackled paragraphs earlier; so in general, we need to prove that

$$F_k(A_0 \odot \text{Id}^{\odot k-1}) + \binom{k-1}{1} F_{k-1}(A_1 \odot \text{Id}^{\odot k-2}) + \cdots + \binom{k-1}{k-2} F_2(A_{k-2} \odot \text{Id}) + F_1(A_{k-1}) = 0, \quad (2.14)$$

for every $k \geq 1$.

First of all, bilinearity from Proposition 1.2.5 and a simple induction argument imply, for every set of $n \times n$ matrices $M_1, \dots, M_m$,

$$\bigodot_{i=1}^m (\text{Id} + M_i) = \text{Id}_n^{\odot m} + \sum_{j=1}^m \text{Id}_n^{\odot m-1} \odot M_j + \sum_{1 \leq j_1 < j_2 \leq m} \text{Id}_n^{\odot m-2} \odot M_{j_1} \odot M_{j_2} + \cdots + M_1 \odot \cdots \odot M_m,$$

and thus, setting $m = k - 1$ and $M_i = -\zeta_k^i F_1$, very basic properties of cyclotomic polynomials imply

$$U_k = \left[\bigodot_{i=1}^{k-1} (\text{Id} - \zeta_k^i F_i) \right] \odot \mathbf{e}_i^T = \left(\sum_{j=0}^{k-1} \text{Id}_n^{\odot j} \odot F_1^{\odot k-1-j} \right) \odot \mathbf{e}_i^T. \quad (2.15)$$

We claim that

$$U_k (A_0 \odot \text{Id}_n^{\odot k-1}) = X_i \text{Id}_n^{\odot k-1}, \quad \text{for every } k \geq 1. \quad (2.16)$$

Indeed, we have, using (1.12) and the rest of the properties of $\odot$ (all of which can be traced back to Lemma 1.2.4 or the universal property of $\odot$, [1, Sec. 2.1]),

$$\begin{aligned} U_k (A_0 \odot \text{Id}_n^{\odot k-1}) &= \left[\left(\sum_{j=0}^{k-1} \text{Id}_n^{\odot j} \odot F_1^{\odot k-1-j} \right) \odot \mathbf{e}_i^T \right] (A_0 \odot \text{Id}_n^{\odot k-1}) \\ &= \frac{k-1}{k} \left(\sum_{j=0}^{k-1} \text{Id}_n^{\odot j} \odot F_1^{\odot k-1-j} \right) (A_0 \odot \text{Id}_n^{\odot k-2}) \odot \mathbf{e}_i^T + \frac{X_i}{k} \left(\sum_{j=0}^{k-1} \text{Id}_n^{\odot j} \odot F_1^{\odot k-j} \right) \\ &= \frac{k-1}{k} \sum_{j=0}^{k-2} \frac{k-1-j}{k-1} F_1^{\odot j} \odot A_0 \odot \text{Id}_n^{\odot k-2-j} \odot \mathbf{e}_i^T + \frac{X_i}{k} \left(\sum_{j=0}^{k-1} \text{Id}_n^{\odot j} \odot F_1^{\odot k-j} \right) \\ &= \frac{1}{k} \sum_{j=0}^{k-2} (k-1-j) F_1^{\odot j} \odot A_0 \odot \text{Id}_n^{\odot k-2-j} \odot \mathbf{e}_i^T + \frac{X_i}{k} \left(\sum_{j=0}^{k-1} \text{Id}_n^{\odot j} \odot F_1^{\odot k-j} \right) \\ &= X_i \text{Id}_n^{\odot k-1}. \end{aligned}$$

The rest follows from (2.16):

$$\begin{aligned} F_k (A_0 \odot \text{Id}_n^{\odot k-1}) &= -\frac{1}{X_i} \left[\sum_{j=0}^{k-2} \binom{k-1}{j} F_{j+1} (A_{k-1-j} \odot \text{Id}_n^{\odot j}) \right] U_k (A_0 \odot \text{Id}_n^{\odot k-1}) \\ &= -\frac{1}{X_i} \left[\sum_{j=0}^{k-2} \binom{k-1}{j} F_{j+1} (A_{k-1-j} \odot \text{Id}_n^{\odot j}) \right] X_i \text{Id}_n^{\odot k-1} \\ &= -\sum_{j=0}^{k-2} \binom{k-1}{j} F_{j+1} (A_{k-1-j} \odot \text{Id}_n^{\odot j}), \end{aligned}$$

which proves (2.14) true. $\square$

Double induction, along with properties of $\odot$, easily proves that

$$A_0 \odot A_u \odot \text{Id}^{\odot v} = (A_0 \odot \text{Id}^{\odot v+1}) (A_u \odot \text{Id}^{\odot v}), \quad u, v \geq 0,$$

and just as easily establishes the following consequence thereof:

Lemma 2.3.4. *Using the redundant notation*

$$A_0 = \left(\begin{array}{cc|c} \cdots & 0 & 0 \\ \cdots & 0 & A_0 \\ \cdots & 0 & 0 \end{array} \right),$$

and in the hypotheses of (1.27) and (1.28),

$$\begin{aligned}\overline{\widehat{A} \odot \exp_{\odot} \text{Id}} &= (A \odot \exp_{\odot} \text{Id})(A_0 \odot \exp_{\odot} \text{Id}) - A_0 \odot A \odot \exp_{\odot} \text{Id} \\ &= (A \odot \exp_{\odot} \text{Id})(A_0 \odot \exp_{\odot} \text{Id}) - (A_0 \odot \exp_{\odot} \text{Id})(A \odot \exp_{\odot} \text{Id}).\end{aligned}$$

In other words,

$$\binom{i+j}{i} \overline{A_j \odot \text{Id}^i} = \binom{i+j+1}{i} (A_{j+1} \odot \text{Id}^{\odot i})(A_0 \odot \text{Id}^{\odot(i+j)}) - \binom{i+j}{i-1} (A_{j+1} \odot A_0 \odot \text{Id}^{\odot(i-1)}),$$

for every $i, j \geq 0$. $\square$

Let us study the successive derivatives of $C = F \cdot \Upsilon^{-1} \cdot \widehat{A}_{\text{LVE}_{\phi}}$. We know $\overline{\Upsilon^{-1}} = -\Upsilon^{-1} A_{\text{LVE}_{\phi}}$ from Lemma 1.5.1. Using Lemma 2.3.4,

$$\begin{aligned}\dot{C} &= F \left[-\Upsilon^{-1} A_{\text{LVE}_{\phi}} \widehat{A}_{\text{LVE}_{\phi}} + \Upsilon^{-1} A_{\text{LVE}_{\phi}} (A_0 \odot \exp_{\odot} \text{Id}) - \Upsilon^{-1} A_0 \odot A \odot \exp_{\odot} \text{Id} \right] \\ &= F \Upsilon^{-1} \left[-A_{\text{LVE}_{\phi}} \widehat{A}_{\text{LVE}_{\phi}} + A_{\text{LVE}_{\phi}} (A_0 \odot \exp_{\odot} \text{Id}) - A_0 \odot A \odot \exp_{\odot} \text{Id} \right] \\ &= -F \Upsilon^{-1} \left[A_{\text{LVE}_{\phi}}^2 + A_0 \odot A \odot \exp_{\odot} \text{Id} \right] = -(\exp_{\odot} F) \Upsilon^{-1} \left[A_{\text{LVE}_{\phi}}^2 + A_0 \odot A_{\text{LVE}_{\phi}} \right] \\ &= -F \Upsilon^{-1} \left[A_{\text{LVE}_{\phi}}^2 + (A_0 \odot \exp_{\odot} \text{Id}) A_{\text{LVE}_{\phi}} \right] \\ &= -F \Upsilon^{-1} \left[\widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}} \right], \\ \ddot{C} &= -F \Upsilon^{-1} \left[\overline{\widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}}} + \widehat{A}_{\text{LVE}_{\phi}} \overline{A_{\text{LVE}_{\phi}}} \right] \\ &= -F \Upsilon^{-1} \left[-A_{\text{LVE}_{\phi}} \widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}} + \overline{\widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}}} + \widehat{A}_{\text{LVE}_{\phi}} \overline{A_{\text{LVE}_{\phi}}} \right] \\ &= -F \Upsilon^{-1} \left[-A_{\text{LVE}_{\phi}} \widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}} + (A_{\text{LVE}_{\phi}} (A_0 \odot \exp_{\odot} \text{Id}) - (A_0 \odot \exp_{\odot} \text{Id}) A_{\text{LVE}_{\phi}}) A_{\text{LVE}_{\phi}} + \widehat{A}_{\text{LVE}_{\phi}} \overline{A_{\text{LVE}_{\phi}}} \right] \\ &= -F \Upsilon^{-1} \left[-A_{\text{LVE}_{\phi}}^2 A_{\text{LVE}_{\phi}} - (A_0 \odot \exp_{\odot} \text{Id}) A_{\text{LVE}_{\phi}}^2 + \widehat{A}_{\text{LVE}_{\phi}} \overline{A_{\text{LVE}_{\phi}}} \right] \\ &= F \Upsilon^{-1} \left[\widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}}^2 - \widehat{A}_{\text{LVE}_{\phi}} \overline{A_{\text{LVE}_{\phi}}} \right].\end{aligned}$$

This leads to the following statement, which can be easily proved by induction: for every $i \geq 1$,

$$\frac{d^i C}{dt^i} = \frac{d^i}{dt^i} (F \cdot \Upsilon^{-1} \cdot \widehat{A}_{\text{LVE}_{\phi}}) = F \cdot \Upsilon^{-1} \cdot \widehat{A}_{\text{LVE}_{\phi}} \cdot M_i, \quad \text{for some } M_i \in \text{Mat}(K). \quad (2.17)$$

The case where $i = 0$ is true with $M_0 = \exp_{\odot} \text{Id}$ by definition, cases $i = 1$ and $i = 2$ are true with $M_1 = -A_{\text{LVE}_{\phi}}$ and $M_2 = A_{\text{LVE}_{\phi}}^2 - \overline{A_{\text{LVE}_{\phi}}}$ as seen above, and if we assume (2.17) to be true for an arbitrary i , then the proof for $i + 1$ works along the lines of case $i = 2$ by way of Lemma 2.3.4:

$$\begin{aligned}\frac{d^{i+1} C}{dt^{i+1}} &= \frac{d}{dt} (F \Upsilon^{-1} \widehat{A}_{\text{LVE}_{\phi}} M_i) = F \Upsilon^{-1} \left[-A_{\text{LVE}_{\phi}} \widehat{A}_{\text{LVE}_{\phi}} M_i + \overline{\widehat{A}_{\text{LVE}_{\phi}} M_i} + \widehat{A}_{\text{LVE}_{\phi}} \overline{M_i} \right] \\ &= F \Upsilon^{-1} \left[-A_{\text{LVE}_{\phi}} \widehat{A}_{\text{LVE}_{\phi}} M_i + (A_{\text{LVE}_{\phi}} (A_0 \odot \exp_{\odot} \text{Id}) - (A_0 \odot \exp_{\odot} \text{Id}) A_{\text{LVE}_{\phi}}) M_i + \widehat{A}_{\text{LVE}_{\phi}} \overline{M_i} \right] \\ &= F \Upsilon^{-1} \left[-\widehat{A}_{\text{LVE}_{\phi}} A_{\text{LVE}_{\phi}} M_i + \widehat{A}_{\text{LVE}_{\phi}} \overline{M_i} \right] = F \Upsilon^{-1} \widehat{A}_{\text{LVE}_{\phi}} \left[-A_{\text{LVE}_{\phi}} M_i + \overline{M_i} \right],\end{aligned}$$

which explicitly defines the sought-after matrices recursively as

$$M_{i+1} = -A_{\text{LVE}_{\phi}} M_i + \overline{M_i}, \quad i \geq 0.$$

Thus (2.17) is true for every $i \geq 1$ and all we need to do to finish the proof of Theorem 2.3.1 is ascertain that all derivatives of C at $t = t_0$ are identically equal to the zero infinite matrix. This is true on account of the fact that if we take (2.17) at $t = t_0$, we obtain

$$\left. \frac{d^{i+1}C}{dt^{i+1}} \right|_{t=t_0} = F \cdot \widehat{A}_{\text{LVE}_\phi} \Big|_{t=t_0} \cdot M_i(t_0) = 0 \cdot M_i(t_0) = 0,$$

in virtue of Proposition 2.3.3. Thus the matrix $C = F\Upsilon^{-1}\widehat{A}_{\text{LVE}_\phi}$ of functions holomorphic on the open neighborhood $I := \phi^{-1}(U) \cap V$ of t_0 has all of its derivatives with respect to t equal to zero in t_0 . This means $C \equiv 0$ in a neighborhood J of t_0 , and thus on that neighborhood $(\Upsilon^{-1})^T F^T \in \ker \widehat{A}_{\text{LVE}_\phi}^T$, which is the last step we needed to prove the columns of $(\Upsilon^{-1})^T F^T$ are admissible solutions defined on a neighborhood of $\{\phi(t) : t \in I \cap J\}$. Remarks 2.3.2 and Lemma 2.2.1 finish the proof. $\square$

2.3.3. Further comments

The following corresponds to single-row blocks (i.e., one formal first integral at a time) and is a valid precursor to future studies on how to maintain the filtered admissible structure even after performing changes of variables, but this is not immediately necessary to our current aims:

Corollary 2.3.5. *Let g be a formal first integral of (1.1) and $g_i^{(k)}$ its k^{th} lexicographic derivative, for every $k \geq 1$. Let $g_k := g^{(k)}(\phi(t_0))$ be its value at the original point and define the matrix (constant, just like F_k):*

$$M_k = \begin{pmatrix} g_1^{\odot k} & & & \\ c_{1,\dots,1,2} g_1^{\odot k-1} \odot g_2 & \ddots & & \\ \vdots & \ddots & g_1^{\odot 2} & \\ g_k & \cdots & g_2 & g_1 \end{pmatrix} \in \text{Mat}_{1,n}^{n,n}$$

as the first $k \times D_{n,k}$ block of $M := \exp_{\odot}(\cdots g_k \mid g_{k-1} \mid \cdots \mid g_2 \mid g_1)$. Then

$$(\Upsilon_k^{-1})^T \Phi_k^T M_k^T = \begin{pmatrix} \frac{1}{k!} (g^{(k)})^{(k)} & \cdots & \frac{1}{2} (g^{(2)})^{(k)} & g^{(k)} \\ & \ddots & \vdots & \vdots \\ & & \frac{1}{2} (g^{(2)})^{(2)} & g^{(2)} \\ & & & g^{(1)} \end{pmatrix}.$$

In other words: the columns of the infinite matrix $(\Upsilon^{-1})^T \Phi^T M^T$ are the series terms of the respective powers of g . $\square$

Example 2.3.6. We have seen two methods of obtaining every new term of a given formal first integral: the global filter provided by Theorem 2.3.1 and the progressive quadrature method in Theorem 2.2.2. Let us remain focused on the former and assume, for instance, $i = 1$ in Lemma 2.2.1. F_1 in (2.8) transforms the transpose of the fundamental matrix of the dual system $\Upsilon_1^{-1} = Y_1^{-1}$ into a matrix of the form

$$\left(\begin{array}{c} \mathbf{0} \\ \hline G_1 \end{array} \right) := \left(\begin{array}{c} \mathbf{0} \\ g_1^{(1)} \\ g_2^{(1)} \\ \vdots \\ g_{n-1}^{(1)} \end{array} \right) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ g_{1,1}^{(1)} & g_{1,2}^{(1)} & \cdots & g_{1,n-1}^{(1)} & g_{1,n}^{(1)} \\ g_{2,1}^{(1)} & g_{2,2}^{(1)} & \cdots & g_{2,n-1}^{(1)} & g_{2,n}^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{n-1,1}^{(1)} & g_{n-1,2}^{(1)} & \cdots & g_{n-1,n-1}^{(1)} & g_{n-1,n}^{(1)} \end{pmatrix}, \quad (2.18)$$

where each of these row vectors $g_1^{(1)}, \dots, g_{n-1}^{(1)}$ is the gradient of a formal first integral and all $n - 1$ of them are linearly independent. The next step would be to integrate the second variational system (from here onward, only quadratures are needed):

$$Y_2(t) = Y_1 \int_0^t Y_1^{-1}(\tau) A_2(\tau) Y_1^{\odot 2}(\tau) d\tau,$$

invert the fundamental matrix for (LVE^2) ,

$$\Upsilon_2 = \begin{pmatrix} Y_1^{\odot 2} & 0 \\ Y_2 & Y_1 \end{pmatrix}, \quad \text{thus} \quad \Psi_2 := (\Upsilon_2^{-1})^T,$$

and multiply it by the second-order filter matrix

$$\Phi_2 = \begin{pmatrix} F_1^{\odot 2} & 0 \\ F_2 & F_1 \end{pmatrix},$$

where F_2 is obtained from (2.11) and (2.12). The result we obtain is

$$\Phi_2(\Psi_2)^T = \Phi_2 \Upsilon_2^{-1} = \begin{pmatrix} G_1^{\odot 2} & 0 \\ G_2 & G_1 \end{pmatrix} = \begin{pmatrix} \begin{array}{c} \mathbf{0}_n \\ g_1^{(1)} \\ \vdots \\ g_{n-1}^{(1)} \end{array} \odot 2 & \\ \hline \mathbf{0}_{d_{n,2}} & \mathbf{0}_n \\ g_1^{(2)} & g_1^{(1)} \\ \vdots & \vdots \\ g_{n-1}^{(2)} & g_{n-1}^{(1)} \end{pmatrix}. \quad (2.19)$$

Take for instance the first non-zero row of the lower block $(G_2 | G_1)$, i.e., the one highlighted in gray. Imagine we perform the computations for (LVE_ϕ^3) and multiply Υ_3^{-1} by the filter matrix Φ_3 . We obtain a matrix having the one in (2.19) as a lower right $D_{n,k} \times D_{n,k}$ block:

$$\Phi_3 \Upsilon_3^{-1} = \begin{pmatrix} F_1^{\odot 3} & 0 & 0 \\ 3F_1 \odot F_2 & F_1^{\odot} & 0 \\ F_3 & F_2 & F_1 \end{pmatrix} \Upsilon_3^{-1} = \begin{pmatrix} G_1^{\odot 3} & 0 & 0 \\ 3G_1 \odot G_2 & G_1^{\odot} & 0 \\ G_3 & G_2 & G_1 \end{pmatrix} \quad (2.20)$$

$$= \left(\begin{array}{c} \begin{array}{c} \mathbf{0}_n \\ g_1^{(1)} \\ \vdots \\ g_{n-1}^{(1)} \end{array} \odot^3 \\ \begin{array}{c} \mathbf{0}_n \\ g_1^{(1)} \\ \vdots \\ g_{n-1}^{(1)} \end{array} \odot \begin{array}{c} \mathbf{0}_{d_{n,2}} \\ g_1^{(2)} \\ \vdots \\ g_{n-1}^{(2)} \end{array} \odot^2 \end{array} \right).$$

$\mathbf{0}_{d_{n,3}}$	$\mathbf{0}_{d_{n,2}}$	$\mathbf{0}_n$
$g_1^{(3)}$	$g_1^{(2)}$	$g_1^{(1)}$
$\vdots$	$\vdots$	$\vdots$
$g_{n-1}^{(n-1)}$	$g_{n-1}^{(2)}$	$g_{n-1}^{(1)}$

Thus we have an increasing row, highlighted in a dark gray color $\cdots | g_1^{(3)} | g_1^{(2)} | g_1^{(1)}$. These vectors are precisely the terms of a formal first integral $g_1(z)$ of (1.1) along solution ϕ :

$$g_1(z + \phi) = g_1(\phi) + g_1^{(1)}z + \frac{1}{2}g_1^{(2)}(\phi)z^{\odot 2} + \frac{1}{6}g_1^{(3)}(\phi)z^{\odot 3} + \dots, \quad (2.21)$$

where $g_1^{(i)} = g_1^{(i)}(\phi)$, i.e., the implementation of (1.3) in 1.2.1 (1.2.1) to $z = \phi$. The same applies, *mutatis mutandis*, to the rest of the rows of the infinite block $\dots G_3 | G_2 | G_1$, i.e., replacing $g_1^{(i)}$ with $g_j^{(i)}$ for some other $j = 2, \dots, n-1$, which leads to another formal first integral g_j in lieu of g_1 in (2.21). Needless to say, the upper terms $3G_1 \odot G_2$, etc., in (2.20) and higher-order counterparts contain all the cross-products appearing in derivatives of products of the formal first integrals, e.g., $g_1^{(1)} \odot g_3^{(2)}$ would be a row in $G_1 \odot G_2$ (it is easy to check which one by way of lexicographic ordering and we invite the reader to do this themselves, using the examples in Section 3).

3. Examples

3.1. Dixon's system

The following system,

$$\dot{u} = \frac{uv}{u^2 + v^2} - \alpha u, \quad \dot{v} = \frac{v^2}{u^2 + v^2} - \beta v + \beta - 1, \quad (3.1)$$

where $\alpha, \beta > 0$, arose in [25] as a decoupled two-dimensional fragment of a three-dimensional dynamical model of the magnetic field of neutron stars. Several references (e.g., [26–28]) discussed the ostensible chaotic behavior of the dynamics of (3.1). The work in [29], on the other hand, purported the non-chaotic character of this system based on a version of the Poincaré-Bendixson theorem that precludes the need for compact sets comprising finitely many sets of homoclinic/heteroclinic connections. Given the existing and seldom-discussed gap between the concepts *non-chaotic* and *integrable* (regardless of which ad-hoc definition of chaos is used), we can afford to eschew the polemic altogether and fix our attention on what can be said about (3.1) using the tools described in the previous section.

We focus on the case where $\alpha = \beta > 0$. The case where $\alpha \neq \beta$, as well as the original three-dimensional model ([27, Eqs (3)–(5)]), will be tackled in future work.

$\alpha = 1$ is immediate to solve and needs no further consideration. Let us focus on $\alpha \neq 1$. Previous attempts at simplifying higher-order LVE^k suggest a change of variables:

$$(u, v) = \left(\frac{\sin y}{x}, \frac{\cos y}{x} \right), \quad (3.2)$$

which transforms (3.1) for $\alpha = \beta$ into

$$\dot{x} = \alpha x(1 - x \cos y), \quad \dot{y} = -(\alpha - 1)x \sin y. \quad (3.3)$$

We use the particular solution

$$\phi(t) = (x(t), 0) \quad \text{such that } \dot{x} = -\alpha(-1 + x)x. \quad (3.4)$$

Along Γ , the higher variational complex (1.23) and its dual (LVE _{ϕ} ^{*}) feature the following blocks for $k \geq 0$:

$$A_k = \begin{cases} X(x, y) = \begin{pmatrix} \alpha x(1 - x \cos y) \\ -(\alpha - 1)x \sin y \end{pmatrix}, & k = 0, \\ \begin{pmatrix} -\alpha(2x - 1) & 0 \\ 0 & -(\alpha - 1)x \end{pmatrix}, & k = 1, \\ \begin{pmatrix} 0_{2 \times (k-2)} & (-1)^{k/2} 2\alpha & 0 & (-1)^{k/2+1} \alpha x^2 \\ & 0 & (-1)^{k/2}(-1 + \alpha) & 0 \end{pmatrix}, & k \text{ is even,} \\ \begin{pmatrix} 0_{2 \times (k-1)} & (-1)^{\frac{k+1}{2}} 2\alpha x & 0 \\ & 0 & (-1)^{\frac{k+1}{2}}(-1 + \alpha)x \end{pmatrix}, & k \text{ is odd.} \end{cases} \quad (3.5)$$

For instance, a principal fundamental matrix of (VE _{ϕ}) is

$$\Upsilon_1 = Y_1 = \begin{pmatrix} \frac{(x-1)x}{(x_0-1)x_0} & 0 \\ 0 & (1-x_0)^{\frac{1}{\alpha}-1}(1-x)^{\frac{1}{\alpha}-1} \end{pmatrix},$$

and the first-order filter matrix is very simple in this case:

$$F_1 = \text{Id}_2 - \frac{1}{X_1^0} (X^0 \odot e_1^T) = \begin{pmatrix} 0 & 0 \\ -\frac{X_2(x_0, y_0)}{X_1(x_0, y_0)} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $x_0 = x(0)$ and $y_0 = y(0)$. So, the higher-order matrices in $\Phi = \exp_{\odot} F$ are obtained from (2.11), some of which are shown below:

$$F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1-\alpha}{\alpha(x_0-1)} & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(\alpha-1)(2\alpha-1)}{\alpha^2(x_0-1)^2} & 0 & 0 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{(\alpha-1)(2\alpha-1)(3\alpha-1)}{\alpha^3(x_0-1)^3} & 0 & -\frac{(\alpha-1)(2x_0+1)}{\alpha(x_0-1)^2} \end{pmatrix} \cdots$$

Returning to $k = 1$, Lemma 2.2.1 ensures that the second column of the filtered matrix

$$(Y_1^{-1})^T F_1^T = \begin{pmatrix} 0 & 0 \\ 0 & (1-x_0)^{1-\frac{1}{\alpha}}(1-x)^{\frac{1}{\alpha}-1} \end{pmatrix}$$

is an admissible solution of degree 1. We can normalize this to eliminate appearances of x_0 :

$$(\Phi_1^{-1})^T F_1^T P^T := \begin{pmatrix} 0 & 0 \\ 0 & (1-x_0)^{1-\frac{1}{\alpha}}(1-x)^{\frac{1}{\alpha}-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-x_0)^{\frac{1}{\alpha}-1} \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & (1-x)^{\frac{1}{\alpha}-1} \end{pmatrix},$$

where its second column is the gradient $f^{(1)}$ of a formal first integral (this is exactly what appears as a row $g_1^{(1)}$ in (2.18)).

The higher-order trimming (in which we discard redundant symmetric products $f^{(i_1)} \odot \dots \odot f^{(i_k)}$, i.e., derivatives of powers of f) is clear once we write down the dual system of equations and filter out telescopically using Lemma 2.1.1, and the general form of each new term $f^{(k)}$ of our formal first integral f is expressible as the following (bear in mind that $x = x(t)$ here): $f^{(k)}(x) = \left(\frac{d}{dx} f^{(k-1)} \mid g_k(x) \right)$, where

$$\sum_{j=0}^{\frac{k-1}{2}} \left((-1)^{j+1} (\alpha-1) \binom{k}{-1-2j+k} g_{k-2j}(x) + (-1)^{j+1} \alpha \binom{k}{-2j+k} x g'_{k-2j}(x) \right) + \alpha g'_k(x) = 0,$$

which means $g_k(x) = 0$ for every even $k \geq 2$, e.g.,

$$\begin{aligned} f_1 &= \begin{pmatrix} 0 & (1-x)^{\frac{1}{\alpha}-1} \end{pmatrix}, \\ f_2 &= \begin{pmatrix} 0 & \frac{(\alpha-1)(1-x)^{\frac{1}{\alpha}-2}}{\alpha} & 0 \end{pmatrix}, \\ f_3 &= \begin{pmatrix} 0 & \left(\frac{1}{\alpha}-2\right)\left(\frac{1}{\alpha}-1\right)(1-x)^{\frac{1}{\alpha}-3} & 0 & \frac{(1-x)^{\frac{1}{\alpha}-2}(2\alpha+(\alpha-2)x-1)}{\alpha-2} \end{pmatrix}, \\ f_4 &= \begin{pmatrix} \frac{d}{dx} f_3 \mid 0 \end{pmatrix}, \\ f_5 &= \begin{pmatrix} \frac{d}{dx} f_4 \mid -\frac{(1-x)^{\frac{1}{\alpha}-3}(-(\alpha-2)x((\alpha-2)(3\alpha-4)x+\alpha(39\alpha-55)+14)+\alpha(2(53-24\alpha)\alpha-73)+16)}{(\alpha-2)^2(3\alpha-4)} \end{pmatrix}. \end{aligned}$$

This can be achieved via (2.9) in Theorem 2.2.2 (the easiest option even for low values of k) or the filter matrix described in subsection 2.3 to compute each f_k in terms of the preceding $f_{k-1}, \dots, f_1$. We invite the reader to apply either method and realize that the above terms appear with only minor attention paid at each step.

A tedious scrutiny of the general form of these vectors shows that they are precisely limit cases of the higher-order Taylor terms (along $(x(t), 0) \in \Gamma$) of the following function:

$$f = \left(x \sin^{\frac{\alpha}{1-\alpha}}(y) - \left(\frac{\alpha \cos y}{\alpha-1} \right) {}_2F_1 \left(\frac{1}{2}, \frac{\alpha}{2(\alpha-1)} + 1; \frac{3}{2}; \cos^2 y \right) \right)^{\frac{1}{\alpha}-1},$$

where ${}_2F_1$ is the **Gaussian hypergeometric function** [30]. Removing the outer power yields a first integral of (3.3) as well, as can be easily checked:

$$f = x \sin^{\frac{\alpha}{1-\alpha}}(y) - \left(\frac{\alpha \cos y}{\alpha-1} \right) {}_2F_1 \left(\frac{1}{2}, \frac{\alpha}{2(\alpha-1)} + 1; \frac{3}{2}; \cos^2 y \right).$$

Undoing the transformation (3.2), we obtain:

Theorem 3.1.1. *The following function*

$$F = u^{-\frac{\alpha}{\alpha-1}} (u^2 + v^2)^{\frac{1}{2(\alpha-1)}} - \frac{\alpha v {}_2F_1\left(\frac{1}{2}, \frac{\alpha}{2(\alpha-1)} + 1; \frac{3}{2}; \frac{v^2}{u^2 + v^2}\right)}{(\alpha-1)\sqrt{u^2 + v^2}} \quad (3.6)$$

is a first integral of the original special Cummings, Dixon, and Kaus (CDK) system. $\square$

It is worth noting that the form taken by this first integral matches general expressions studied in recent work [19, 20, 31]; see item (viii) in Section 4 for more details.

3.2. The SIR model with vital dynamics

The **SIR epidemiological model with vital dynamics** [32, 33] is given by

$$\begin{cases} \dot{S} &= \mu(n - S) - \frac{\beta SI}{n}, \\ \dot{I} &= \frac{\beta SI}{n} - I(\gamma + \mu), \\ \dot{R} &= \gamma I - \mu R. \end{cases} \quad (3.7)$$

The system was first introduced in [33] strictly for the modeling of infectious diseases, but has since been applied to a number of situations allowing dynamic compartmentalization, e.g., marketing [34]. The study of its integrability has been the subject of several works ([35, 36]) and the search for a first integral is still an open problem. We address specific examples, leaving the general case for future work.

First, let us start with a very simple example with an already-known solution: $\beta = \gamma$ and $\mu = 0$ with the change of variable

$$S = n(1 + xy), \quad I = -ny, \quad R = \gamma n z. \quad (3.8)$$

We obtain a system of equations $\frac{d}{dt}(x, y, z) = (\gamma - \gamma(x-1)xy, \gamma xy^2, -y)$, and applying Theorem 2.3.1 and Corollary 2.3.5 (and if the matrix is used, Lemma 2.1.1), the general terms of the jets telegraph a very identifiable structure: that of two first integrals for the transformed system, $g^* = \frac{\log(xy+1)}{\gamma} + z$, $f^* = -xy + \log(xy+1) + y$, which transform back by (3.8) into $g = \frac{n \log(\frac{S}{n}) + R}{\gamma n}$, $f = -\frac{S+I}{n} + \log\left(\frac{S}{n}\right) + 1$. This takes us to the already-known first integral $S + I + R - n$ (thus, $S + I + R$) and the remaining one, $\frac{n \log(\frac{S}{n}) + R}{\gamma n}$ (alternatively, $\frac{S}{n} e^{\frac{R}{n}}$). This is all in keeping with what is already known about the integrability of the system for $\mu = 0$ (see, e.g., [35, Sec. 1]). The reader can fill in the blanks for filter matrices or progressive filtering (Theorem 2.2.2) and formal Taylor terms here; we add more detail about these in the less-trivial case below.

Next, we focus on the case in which $\gamma = 0$ and $\beta, \mu \neq 0$. To the best of our knowledge, the integrability of this case is an open problem in earlier references. The change of variables

$$(S, I, R) = \left(y^2 \left(x^{-\frac{\beta}{\mu}} + 1\right) + n, -x^{-\frac{\beta}{\mu}} y^2, y^2 z\right), \quad (3.9)$$

itself found by simplifying LVE_ϕ^3 , provides us with the new system

$$\dot{x} = -\frac{\mu x \left[y^2 \left(x^{-\frac{\beta}{\mu}} + 1\right) + n\right]}{n}, \quad \dot{y} = -\frac{1}{2}\mu y, \quad \dot{z} = 0. \quad (3.10)$$

This system admits an invariant plane $y = z = 0$ containing the particular solution

$$\phi(t) = (\widehat{x}(t), 0, 0), \quad \text{where } \frac{d}{dt}\widehat{x} = -\mu\widehat{x}. \quad (3.11)$$

This leads to the variational system $\dot{Y} = (A \odot \exp_{\odot} \text{Id}_n) Y$ along ϕ , where

$$A_{\text{LVE}_{\phi}} = A \odot \exp_{\odot} \text{Id}_n = \begin{pmatrix} \ddots & & & & \\ \cdots & 4A_1 \odot \text{Id}_n^{\odot 3} & & & \\ \cdots & 6A_2 \odot \text{Id}_n^{\odot 2} & 3A_1 \odot \text{Id}_n^{\odot 2} & & \\ \cdots & 4A_3 \odot \text{Id}_n & 3A_2 \odot \text{Id}_n & 2A_1 \odot \text{Id}_n & \\ \cdots & A_4 & A_3 & A_2 & A_1 \end{pmatrix},$$

and A_i defined per (1.27) appear as follows:

$$\begin{aligned} A_0 &= \begin{pmatrix} -\mu\widehat{x}(t) \\ 0 \\ 0 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\frac{\mu}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0 & 0 & \frac{2\mu\widehat{x}(t)\left(-\widehat{x}(t)^{-\frac{\beta}{\mu}}-1\right)}{n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 0 & 0 & 0 & \frac{2\left((\beta-\mu)\widehat{x}(t)^{-\frac{\beta}{\mu}}-\mu\right)}{n} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ A_k &= \begin{pmatrix} 0 & 0 & 0 & \frac{2\left((\beta-\mu)\widehat{x}(t)^{-\frac{\beta}{\mu}}-\mu\right)}{n} & \boxed{0_{3 \times d_{3,k-4}}} \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{pmatrix}, \quad k \geq 4. \end{aligned}$$

Lower blocks of the quadrature-free filter matrix are built as proposed in subsection 2.3, and we will write the first few cases:

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 & \boxed{0_{3 \times 4}} \\ 0 & -\frac{1}{2\widehat{x}_0} & \\ 0 & 0 & \end{pmatrix}, F_3 = \begin{pmatrix} 0 & 0 & \boxed{0_{3 \times 8}} \\ 0 & \frac{3}{4\widehat{x}_0^2} & \\ 0 & 0 & \end{pmatrix}, F_4 = \begin{pmatrix} 0 & 0 & \boxed{0_{3 \times 4}} & 0 & \boxed{0_{3 \times 8}} \\ 0 & -\frac{15}{8\widehat{x}_0^3} & & \frac{3\left(\widehat{x}_0^{-\frac{\beta}{\mu}}+1\right)}{n\widehat{x}_0} & \\ 0 & 0 & & 0 & \end{pmatrix}.$$

The filtered matrix has two linearly independent admissible solutions as its nonzero rows:

$$F_1 Y_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{\widehat{x}_0}}{\sqrt{\widehat{x}(t)}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which, (optionally) multiplied by

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\widehat{x}_0}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.12)$$

removes occurrences of the initial condition $\widehat{x}_0$:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\widehat{x}(t)}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ f^{(1)} \\ g^{(1)} \end{pmatrix}.$$

Let us write things down for $k = 3$. Once we have solved the direct variational system (LVE_T^3) and obtained its fundamental matrix Υ_3 , the principal fundamental matrix of the dual system is nothing but the transpose of its inverse; we can either keep said transpose $(\Upsilon_3^{-1})^T$ and right-multiply it by proposed filter Φ_3^T (in which case, formal first-integral jets appear as columns) or left-multiply the original Υ_3^{-1} by Φ_3 , in which case, jets appear as rows in the bottom block:

$$\begin{aligned} \Phi_3 \Upsilon_3^{-1} &= \begin{pmatrix} F_1^{\odot 3} & 0 & 0 \\ 3F_1 \odot F_2 & F_1^{\odot 2} & 0 \\ F_3 & F_2 & F_1 \end{pmatrix} \begin{pmatrix} Y_1^{\odot 3} & 0 & 0 \\ 3Y_1 \odot Y_2 & Y_1^{\odot 2} & 0 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \text{0}_{6 \times 19} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\widehat{x}_0^{3/2}}{\widehat{x}^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\widehat{x}_0}{\widehat{x}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\widehat{x}_0}}{\sqrt{\widehat{x}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\widehat{x}_0}{\widehat{x}^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\widehat{x}_0}{\widehat{x}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{\widehat{x}_0}}{2\widehat{x}^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\widehat{x}_0}}{\sqrt{\widehat{x}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3\sqrt{\widehat{x}_0}}{4\widehat{x}^{5/2}} & 0 & 0 & 0 & 0 & a_{18,7} & 0 & 0 & 0 & 0 & -\frac{\sqrt{\widehat{x}_0}}{2\widehat{x}^{3/2}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\widehat{x}_0}}{\sqrt{\widehat{x}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \end{pmatrix}, \end{aligned}$$

where $\widehat{x}_0 = \widehat{x}(0)$ and

$$a_{18,7} = \frac{\widehat{x}(t)^{-\frac{\beta}{\mu}-\frac{3}{2}} \left(3\mu \widehat{x}(t)^{\beta/\mu} \widehat{x}_0^{\frac{3}{2}-\frac{\beta}{\mu}} + 3(\mu-\beta) \sqrt{\widehat{x}_0} (\widehat{x}_0 - \widehat{x}(t)) \widehat{x}(t)^{\beta/\mu} - 3\mu \sqrt{\widehat{x}_0} \widehat{x}(t) \right)}{(\beta-\mu)n}.$$

The bottom two rows are already admissible jets $\star^{(3)} \mid \star^{(2)} \mid \star^{(1)}$, but we can trim them further by

multiplying this matrix by

$$\Pi_3 = \begin{pmatrix} P_1^{\odot 3} & 0 & 0 \\ 3P_1 \odot P_2 & P_1^{\odot 2} & 0 \\ P_3 & P_2 & P_1 \end{pmatrix}^T,$$

where P_1 is as in (3.12) and P_2, P_3 are successively chosen so $\widehat{x}_0$ vanishes from the matrix (in this case, all terms in P_2, P_3 are equal to zero except for $(P_3)_{2,7} = \frac{3\left(\mu\left(-\widehat{x}_0^{-\frac{\beta}{\mu}} - 1\right) + \beta\right)}{n\sqrt{\widehat{x}_0}(\beta-\mu)}$). This equates to weeding out copies of successive symmetric products of $f^{(i)}$, and/or $g^{(j)}$, and harks back to Lemma 2.1.1 and Conjecture 2.3.5. The resulting matrix retains its neat echeloned structure:

$$\Pi_3 \Phi_3 \Upsilon_3^{-1} = \begin{pmatrix} G_1^{\odot 3} & 0 & 0 \\ 3G_1 \odot G_2 & G_1^{\odot 2} & 0 \\ G_3 & G_2 & G_1 \end{pmatrix},$$

where lower block $G_3 \mid G_2 \mid G_1$ has independent formal first-integral jets as its non-zero rows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4\widehat{x}^{5/2}} & 0 & 0 & 0 & 0 & \frac{3\left(-\mu\widehat{x}^{-\frac{\beta}{\mu}} + \beta - \mu\right)}{(\beta-\mu)n\sqrt{\widehat{x}}} & 0 & 0 & 0 & 0 & -\frac{1}{2\widehat{x}^{3/2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\widehat{x}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.13)$$

In other words, the following are degree-three asymptotic expressions of formal first integrals ($\mathbf{z} = (x, y, z)$, $\phi = (\widehat{x}, 0, 0)$ is the particular solution as usual):

$$\begin{aligned} f(\mathbf{z} + \phi) &= f^{(1)}(\phi) \mathbf{z} + \frac{1}{2} f^{(2)}(\phi) \mathbf{z}^{\odot 2} + \frac{1}{3!} f^{(3)}(\phi) \mathbf{z}^{\odot 3} + O(\mathbf{z}^{\odot 4}) \\ &= \frac{y}{\sqrt{\widehat{x}}} - \frac{xy}{2\widehat{x}^{3/2}} + \frac{1}{6} \left(\frac{3y^3 \left(\frac{\mu\widehat{x}^{1-\frac{\beta}{\mu}}}{\mu-\beta} + \widehat{x} \right)}{n\widehat{x}^{3/2}} + \frac{9x^2y}{4\widehat{x}^{5/2}} \right) + O(\mathbf{z}^{\odot 4}), \\ g(\mathbf{z} + \phi) &= g^{(1)}(\phi) \mathbf{z} + \frac{1}{2} g^{(2)}(\phi) \mathbf{z}^{\odot 2} + \frac{1}{3!} g^{(3)}(\phi) \mathbf{z}^{\odot 3} + O(\mathbf{z}^{\odot 4}) = \mathbf{z} + O(\mathbf{z}^{\odot 4}). \end{aligned}$$

Again, Theorem 2.2.2 does the exact same job faster in this case, e.g.,

$$\begin{aligned} f^{(2)} &= -\left[\left(Y_1^{-1} \right)^{\odot 2} \right]^T \left[\int_{\widehat{x}_0}^{\widehat{x}} \left(Y_1^{\odot 2} \right)^T f^{(1)} A_2 dx - \left(0, \frac{1}{2\widehat{x}_0^{3/2}}, 0, 0, 0, 0 \right) \right], \\ f^{(3)} &= -\left[\left(Y_1^{-1} \right)^{\odot 3} \right]^T \left[\int_{\widehat{x}_0}^{\widehat{x}} \left(Y_1^{\odot 3} \right)^T \left[3f^{(2)} (A_2 \odot \text{Id}_3) + f^{(1)} A_3 \right] dx - \left(0, -\frac{3}{4\widehat{x}_0^{5/2}}, 0, 0, 0, 0, 0, 0, 0 \right) \right], \end{aligned}$$

and the reader can check that the jet terms are exactly the same as those in (3.13). Either method may sound like a cumbersome process but it only requires special attention in its first stages if savvy simplifications are found, as was the case for Dixon's system. Later stages of the bottom row remain zero and accordingly $\boxed{g = z}$ is easily checked to be a first integral (formal and convergent) for (3.10).

For f , and much like in §3.1, pattern checks along with simple resummation yield a first integral for system (3.10),

$$f = \frac{ye^{\frac{y^2}{2n}}}{\left[e^{\frac{uy^2}{n}} \left(\frac{uy^2}{n} \right)^u \Gamma\left(1 - u, \frac{uy^2}{n}\right) + x^u \right]^{\frac{1}{2u}}},$$

where $u = \frac{\beta}{\mu}$ and $\Gamma(\cdot, \cdot)$ is the **incomplete gamma function** [30]. Undoing transformation (3.9), we obtain something that is now immediate to prove on its own:

Theorem 3.2.1. *The following two functions,*

$$\begin{aligned} F &= \frac{e^{\frac{-n+S+I}{2n}} \sqrt{-n+S+I}}{\left(\frac{-n+S+I}{I} - \left(\frac{\beta}{\mu n} \right)^{\beta/\mu} e^{\frac{\beta(-n+S+I)}{\mu n}} (-n+S+I)^{\beta/\mu} \Gamma\left(1 - \frac{\beta}{\mu}, \frac{\beta(-n+S+I)}{\mu n}\right) \right)^{\frac{\mu}{2\beta}}}, \\ G &= \frac{R}{-n+S+I}, \end{aligned} \quad (3.14)$$

are two functionally independent first integrals of the SIR model with $\gamma = 0$ (3.7). $\square$

The same comment at the end of subsection 3.1 applies to these first integrals; see item (viii) in Section 4.

It is worth noting, once again, that the only easy goal we can aspire to in most cases is that of formal first integrals with a compact identifiable general term but without a straightforward convergent expression, and resummation and deep-pattern-recognition techniques need to be adapted to each problem. This is also expected to be the subject of future research in the short term. Given that formal integrability is already a complex goal in itself, as said in Remark 1.5.6, we devote the next section to a short example thereof.

3.3. The van der Pol oscillator

This well-known system is usually presented as an epitome of the proverbial “simple yet chaotic” dynamical system [28]

$$\ddot{u} = \mu(1 - u^2)\dot{u} - u. \quad (3.15)$$

The value $\mu = 2$ appears to have a significant, yet still unclarified, role in potential qualitative simplifications in variable transformations. Thus let us fix this value for the present paper:

$$\dot{u} = v, \quad \dot{v} = 2(1 - u^2)v - u. \quad (3.16)$$

We perform the change of variables

$$(u(t), v(t)) = \left(\frac{x(t)y(t)}{\sqrt{2}}, \left(\frac{x(t)}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) y(t) \right) \quad (3.17)$$

that simplify the corresponding jet, and our transformed system is

$$\dot{x} = -(x-1)x^3y^2 - 1, \quad \dot{y} = (x-1)x^2y^3 + y. \quad (3.18)$$

The dual infinite variational system $(VE_\Gamma)^*$ along the particular solution

$$\Gamma = \{(x(t), 0) : t \in \mathbb{C}\}, \quad \text{where } \dot{x}(t) = -1, \quad (3.19)$$

has the following very simple matrices:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0 & -2(x(t)-1)x(t)^3 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 0 & 0 & 2(3-4x(t))x(t)^2 & 0 \\ 0 & 0 & 0 & 6(x(t)-1)x(t)^2 \end{pmatrix}, \\ A_4 &= \begin{pmatrix} 0 & 0 & 12(1-2x(t))x(t) & 0 & 0 \\ 0 & 0 & 0 & 6x(t)(3x(t)-2) & 0 \end{pmatrix}, \\ A_5 &= \begin{pmatrix} 0 & 0 & 12-48x(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 36x(t)-12 & 0 & 0 \end{pmatrix}, \\ A_6 &= \begin{pmatrix} 0 & 0 & -48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 \end{pmatrix}, \\ A_k &= 0_{2 \times d_{2,k}}, \quad k \geq 7. \end{aligned}$$

The fact that the ordinary differential equation (ODE) in (3.19) is so trivial to solve ($x(t) = -t + C$) adds no simplicity to what follows below. The fundamental matrix of VE_1 and the first filter matrix are equally simple:

$$Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{x(0)-x(t)} \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and the generic form of the admissible solutions after applying Theorem 2.2.2 is the following: $f_k = \left(\frac{d}{dx}f_{k-1} \mid g_k(x)\right)$, where g_k is a solution to

$$-(k-1)k(x-1)x^3 g'_{k-2}(x) - g'_k(x) + (k-2)(k-1)k(x-1)x^2 g_{k-2}(x) + k g_k(x) = 0. \quad (3.20)$$

Moreover, $g_k(x) = 0$ if k is even, and $g_1(x) = e^x$. Equation (3.20) has a solution (which of course only merits being written if k is odd):

$$g_k(x) = e^{kx} \int (k-1)k(\xi-1)\xi^2 e^{-k\xi} ((k-2)g_{k-2}(\xi) - \xi g'_{k-2}(\xi)) d\xi.$$

The formal first integral of (3.18) takes the form

$$F = \sum_{k=1}^{\infty} \frac{1}{k!} g_k(x) y^k = \sum_{i=0}^{\infty} (-1)^i e^{(2i+1)x} y^{2i+1} G_{2i+1}; \quad (3.21)$$

the first terms of $G_\star$ are easy to discern:

$$G_1 = 1,$$

$$G_3 = \int e^{-2x}(x-1)^2 x^2 dx,$$

$$G_5 = \int (x-1)^3 e^{-4x} x^5 dx + 3 \int (x-1)^2 e^{-2x} x^2 \left(\int (x-1)^2 e^{-2x} x^2 dx \right) dx;$$

and the general term is

$$G_{2i+1} = \sum_{m=1}^i \sum_{j=1}^{\binom{i-1}{m-1}} D_{i-1,m-1,j} \Phi((i-m) \mathbf{C}_{m,i-m-1,j} + \mathbf{1}_m, 2(i-m) \mathbf{C}_{m,i-m-1,j} + \mathbf{1}_m, 3(i-m) \mathbf{C}_{m,i-m-1,j} + \mathbf{1}_m),$$

where we define the following constant vectors: $\mathbf{1}_n := (1, \dots, 1) \in \mathbb{Z}^n$, and $C_{\star, \star, \star}$ are the columns of the symmetric product

$$\text{Id}_k \odot \mathbf{1}_{d_{k,l}}^T = (C_{k,l,1} \mid C_{k,l,2} \mid \dots \mid C_{k,l,d_{k,l}+1}) \in \text{Mat}_{k,d_{k,l}+1}(\mathbb{Z})$$

(obviously $C_{k,l,j} = 0$ for $l < 0$). Φ is defined recursively as follows: for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^m$,

$$\Phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \begin{cases} \int (x-1)^{a_1+1} e^{-(b_1+1)x} x^{c_1+1} dx, & m = 1, \\ \int (x-1)^{a_1+1} e^{-(b_1+1)x} x^{c_1+1} \Phi((a_2, \dots, a_m), (b_2, \dots, b_m), (c_2, \dots, c_m)) dx, & m > 1, \end{cases}$$

and $D_{n,m,j} = 1$ if $m = 0$, and for $m > 1$, it is the j^{th} term in the vector $\mathbf{D}_{n,m}$ obtained from discarding terms containing repeating factors from the reverse-order version of $\mathbf{i}_n^T \odot \dots \odot \mathbf{i}_n^T$, where $\mathbf{i}_n$ is the reverse-order sequence of the first n odd integers > 1 , e.g.,

$$\mathbf{D}_{3,2} = (3 \cdot 5, 3 \cdot 7, 5 \cdot 7) = (D_{3,2,j})_{j=1,2,3}, \quad \mathbf{D}_{4,3} = (3 \cdot 5 \cdot 7, 3 \cdot 5 \cdot 9, 3 \cdot 7 \cdot 9, 5 \cdot 7 \cdot 9).$$

Studies of possible convergence and, if applicable, functional classification regarding formal first integral (3.21) are to be undertaken in the near future.

4. Conclusions, future work, and further comments

Given any autonomous finite-dimensional system, and a particular solution ϕ thereof, Theorem 2.3.1 provides a recursive method to produce formal first integrals up to any order along ϕ for the system. Furthermore, if ϕ is of a simple, invariant plane form (something usually attainable by adequate variable changes), Theorem 2.2.2 provides an even easier method to the same end, weeding out redundant terms and allowing for a closed form for the *general* term of the formal series, regardless of whether or not ϕ is explicitly known as a function. As said in previous sections, the main goal of this study is formal integrability; the first two systems shown in §3 should be seen as examples of what happens when the gap is bridged between formal expansions and convergent, closed form functions, as expected from Lemma 1.5.5, but do not constitute a global statement on any methods to find closed forms in general—such a statement will be sought in future work.

Proving that Theorem 2.3.1 is still true if we drop the local analyticity assumptions on X and Υ^{-1} can be considered to be a minor-scope open problem, although, as said above, Theorem 2.2.2 renders such a task unnecessary when it comes to tackling particular dynamical systems, especially if computational ease is a factor and redundant terms arising from products and powers of formal first integrals are to be discarded.

Ever since it first took form in [10], the use of linearized higher variational equations to characterize, obstruct, or redefine integrability is proving itself to be a welcome addition to the domain of dynamical systems. This contribution arrives at a time in which other attempts (e.g., [37, 38]) at finding methods to *integrate* dynamical systems rather than proving them non-integrable are gradually gaining traction.

As said above, our current study is restricted to formal expansions and is fundamentally theoretical in nature, but a number of facts are germane to further practical studies:

- (i) higher-variational equations show a (still unexplored) potential to find adequate changes of variables that simplify the system into tractability and, possibly, integrability (this is most exemplified in the first two sample systems in Section 3).
- (ii) Linked to the above, a *Baker-Campbell-Hausdorff* [39] type of formula could, hypothetically, be applied to variable transformations, using the decomposition of a transformed fundamental matrix into a product of matrices to our advantage.
- (iii) Results in Section 2 above are potentially amenable to machine learning [40].
- (iv) This automatic quadrature-algebraic method may find its way into the study of generalized, multivariate *Padé approximants* [41, 42] of first integrals, to be tackled in an immediate future.
- (v) The autonomous system (1.1) need not be Hamiltonian, thus allowing any arbitrary system to be considered and losing the serious constraint posed by symplectic transformations, since variable transformations can now be chosen freely.
- (vi) (v) overrides the obstacles posed by the direct study of variational systems, which only has an implementation to Hamiltonian systems and requires first integrals to be meromorphic.
- (vii) In those situations in which Lemma 1.5.5 applies and a closed-form solution is possible, *Darboux integrability* (see, e.g., [43]) seems to be generalized to a wider set of ansätze, as already hinted at by the form taken in (3.6) or (3.14).
- (viii) Related to the above is *Liouville integrability* [44], which is fundamentally an extension to first integrals of Maximovič [45] and linear differential [12] definitions of integrability. Recent studies [19, 20, 31] used Liouvillian integrability as a stepping stone and found an avenue to generalize it by characterizing first integrals which, while not necessarily Liouvillian, can be expressed as functions of rational or polynomial invariants of the given system. The explicit first integrals (3.6) and (3.14) appear to fall right into this taxon. A thorough study of similar examples shows promise for future work, again provided Lemma 1.5.5 applies.
- (ix) There is some potential (mentioned at the start of subsection 2.3 and already opened in [1]) for categorical functoriality or translation between the study of general dynamical systems and that of linear operators.

Use of AI tools declaration

The author declares that Artificial Intelligence (AI) tools have not been used in the creation of this article. All software used in this article was programmed entirely by the author and can be seen in reference [46].

Conflict of interest

The author declares there is no conflict of interest.

References

1. S. Simon, Linearised higher variational equations, *Discrete Contin. Dyn. Syst.*, **34** (2014), 4827–4854. <https://doi.org/10.3934/dcds.2014.34.4827>
2. S. Simon, Conditions and evidence for non-integrability in the Friedmann–Robertson–Walker Hamiltonian, *J. Nonlinear Math. Phys.*, **21** (2014), 1–16. <https://doi.org/10.1080/14029251.2014.894710>
3. J. J. Morales-Ruiz, *Differential Galois theory and non-integrability of Hamiltonian systems*, Progress in Mathematics, Birkhäuser Verlag, Basel, 1999. https://doi.org/10.1007/978-3-0348-0723-4_4
4. J. J. Morales-Ruiz, J. P. Ramis, Galoisian obstructions to integrability of Hamiltonian systems I, *Methods Appl. Anal.*, **8** (2001), 33–96. <https://doi.org/10.4310/MAA.2001.v8.n1.a3>
5. J. J. Morales-Ruiz, J. P. Ramis, A note on the non-integrability of some Hamiltonian systems with a homogeneous potential, *Methods Appl. Anal.*, **8** (2001), 113–120. <https://doi.org/10.4310/MAA.2001.v8.n1.a5>
6. J. J. Morales-Ruiz, J. P. Ramis, C. Simó, Integrability of Hamiltonian systems and differential Galois groups of higher variational equations, *Ann. Sci. École Norm. Sup.*, **40** (2007), 845–884. <https://doi.org/10.1016/j.ansens.2007.09.002>
7. K. Huang, S. Shi, W. Li, Meromorphic non-integrability of several 3D dynamical systems, *Entropy*, **19** (2017), 211. <https://doi.org/10.3390/e19050211>
8. W. Li, S. Shi, Galoisian obstruction to the integrability of general dynamical systems, *J. Differ. Equations*, **252** (2012), 5518–5534. <https://doi.org/10.1016/j.jde.2012.01.004>
9. W. Li, S. Shi, Corrigendum to “Galoisian obstruction to the integrability of general dynamical systems”[J. Differ. Equ. 252 (10)(2012) 5518–5534], *J. Differ. Equations*, **262** (2017), 1253–1256. <https://doi.org/10.1016/j.jde.2016.10.007>
10. A. Aparicio-Monforte, M. Barkatou, S. Simon, J. A. Weil, Formal first integrals along solutions of differential systems I, in *ISSAC 2011 Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, (2011), 19–26. <https://doi.org/10.1145/1993886.1993896>
11. U. Bekbaev, A matrix representation of composition of polynomial maps, preprint, arXiv:0901.3179. <https://doi.org/10.48550/arXiv.0901.3179>
12. M. van der Put, M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2003. <https://doi.org/10.1007/978-3-642-55750-7>
13. S. L. Ziglin, Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics I, *Funktsional. Anal. I Prilozhen.*, **16** (1982), 30–41. <https://doi.org/10.1007/BF01081586>

14. R. Martínez, C. Simó, Non-integrability of the degenerate cases of the swinging Atwood's machine using higher order variational equations, *Discrete Contin. Dyn. Syst.*, **29** (2011), 1–24.
15. R. Martínez, C. Simó, Non-integrability of Hamiltonian systems through high order variational equations: Summary of results and examples, *Regul. Chaotic Dyn.*, **14** (2009), 323–348. <https://doi.org/10.1134/S1560354709030010>
16. U. Bekbaev, Matrix representations for symmetric and antisymmetric multi-linear maps, preprint, arXiv:1010.2579. <https://doi.org/10.48550/arXiv.1010.2579>
17. H. Żoładek, *The Monodromy Group*, Basel: Birkhäuser Basel, 2006.
18. I. A. García, J. Giné, A. L. Rodero, Existence and nonexistence of Puiseux inverse integrating factors in analytic monodromic singularities, *Stud. Appl. Math.*, **153** (2024), e12724. <https://doi.org/10.1111/sapm.12724>
19. J. Giné, D. Sinelshchikov, Integrability of oscillators and transcendental invariant curves, *Qual. Theory Dyn. Syst.*, **24** (2025), 26. <https://doi.org/10.1007/s12346-024-01182-x>
20. J. Giné, D. Sinelshchikov, A new mechanism for producing degenerate centers in polynomial differential systems, *Nonlinear Anal.*, **264** (2026), 113981. <https://doi.org/10.1016/j.na.2025.113981>
21. R. Kurth, Formal first integrals of ordinary differential equations, *Rend. Circ. Mat. Palermo*, **20** (1971), 43–52. <https://doi.org/10.1007/BF02849466>
22. H. Poincaré, Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, *Rend. Circ. Matem.*, **5** (1891), 161–191. <https://doi.org/10.1007/BF03015693>
23. Z. Xu, W. Li, S. Shi, Higher order criterion for the nonexistence of formal first integral for nonlinear systems, *Electron. J. Differ. Equations*, **274** (2017) 11.
24. H. Chen, F. Girelli, Integrability from categorification, preprint, arXiv:2307.03831. <https://doi.org/10.48550/arXiv.2307.03831>
25. F. Cummings, D. Dixon, P. Kaus, Dynamical model of the magnetic field of neutron stars, *Astrophys. J.*, **386** (1992), 215–221. <https://doi.org/10.1086/171007>
26. J. Alvarez-Ramirez, J. Delgado-Fernandez, G. Espinosa-Paredes, The origin of a continuous two-dimensional “chaotic” dynamics, *Int. J. Bifurcation Chaos*, **15** (2005), 3023–3029. <https://doi.org/10.1142/S021812740501385X>
27. D. D. Dixon, F. W. Cummings, P. E. Kaus, Continuous “chaotic” dynamics in two dimensions, *Phys. D*, **65** (1993), 109–116. [https://doi.org/10.1016/0167-2789\(93\)90008-O](https://doi.org/10.1016/0167-2789(93)90008-O)
28. J. C. Sprott, *Elegant Chaos: Algebraically Simple Chaotic Flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010. <https://doi.org/10.1142/7183>
29. W. M. Seiler, M. Seiß, No chaos in Dixon's system, *Int. J. Bifurcation Chaos*, **31** (2021), 2150044. <https://doi.org/10.1142/S0218127421500449>
30. M. Abramowitz, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U.S. Department of Commerce, National Bureau of Standards, 1964.
31. J. Giné, D. Sinelshchikov, On the geometric and analytical properties of the anharmonic oscillator, *Commun. Nonlinear Sci. Numer. Simul.*, **131** (2024), 107875. <https://doi.org/10.1016/j.cnsns.2024.107875>

32. D. J. Daley, J. Gani, *Epidemic Modeling: An Introduction*, Cambridge University Press, Cambridge, 2005.
33. W. O. Kermack, A. G. McKendrick, A contribution to the mathematical theory of epidemics, in *Proceedings of the Royal Society of London. Series A*, **115** (1927), 700–721. <https://doi.org/10.1098/rspa.1927.0118>
34. H. S. Rodrigues, M. J. Fonseca, Can information be spread as a virus? Viral marketing as epidemiological model, *Math. Methods Appl. Sci.*, **39** (2016), 4780–4786. <https://doi.org/10.1002/mma.3783>
35. D. Chen, On the integrability of the SIR epidemic model with vital dynamics, *Adv. Math. Phys.*, **2020** (2020), 5869275. <https://doi.org/10.1155/2020/5869275>
36. T. Harko, F. S. N. Lobo, M. K. Mak, Exact analytical solutions of the susceptible-infected-recovered (SIR) epidemic model and of the SIR model with equal death and birth rates, *Appl. Math. Comput.*, **236**, (2014), 184–194. <https://doi.org/10.1016/j.amc.2014.03.030>
37. A. Mitsopoulos, M. Tsampanlis, Higher order first integrals of autonomous dynamical systems, *J. Geom. Phys.*, **170** (2021), 104383. <https://doi.org/10.1016/j.geomphys.2021.104383>
38. A. Mitsopoulos, M. Tsampanlis, A. Ukpog, Higher-Order First Integrals of Autonomous Non-Riemannian Dynamical Systems, *Symmetry*, **15** (2023), 222. <https://doi.org/10.3390/sym15010222>
39. E. B. Dynkin, Evaluation of the coefficients of the Campbell-Hausdorff formula, *Dokl. Akad. Nauk SSSR*, **57** (1947).
40. G. Montavon, S. Lapuschkin, A. Binder, W. Samek, K. R. Müller, Explaining nonlinear classification decisions with deep Taylor decomposition, *Pattern Recognit.*, **65** (2017), 211–222. <https://doi.org/10.1016/j.patcog.2016.11.008>
41. A. A. M. Cuyt, A review of multivariate Padé approximation theory, *J. Comput. Appl. Math.*, **12** (1985), 221–232. [https://doi.org/10.1016/0377-0427\(85\)90019-6](https://doi.org/10.1016/0377-0427(85)90019-6)
42. P. Guillaume, A. Huard, Multivariate Padé approximation, *J. Comput. Appl. Math.*, **121** (2000), 197–219. [https://doi.org/10.1016/S0377-0427\(00\)00337-X](https://doi.org/10.1016/S0377-0427(00)00337-X)
43. F. Dumortier, J. Llibre, J. C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer Berlin Heidelberg, 2006.
44. M. F. Singer, Liouvillian first integrals of differential equations, *Trans. Am. Math. Soc.*, **333** (1992), 673–688. <https://doi.org/10.1090/S0002-9947-1992-1062869-X>
45. K. M. Schmidt, On the integrability of differential equations by quadratures according to Maximovič, *J. Comput. Appl. Math.*, **171** (2004), 393–410. <https://doi.org/10.1016/j.cam.2004.01.021>
46. S. Simon, *Tools for Symmetric Products of Finite and Infinite Matrices*, GitHub repository, 2025. Available from: <https://github.com/Sergi-Simon/Tools-for-symmetric-products-of-finite-and-infinite-matrices>.

