



Research article

A curvature-dependent adapted frame generated by Frenet and Bishop normals

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Abstract: We studied a curvature-dependent adapted frame, denoted for brevity by the adaptive Bishop–Frenet (ABF) frame, on Frenet-regular intervals where the curvature is strictly positive. The name refers only to the fact that the defining vector is generated explicitly from the Frenet and Bishop normals; it does not denote an endpoint-to-endpoint interpolation. Exact normalization and a uniform nondegeneracy margin ensure that the resulting adapted frame is orthonormal on this domain. The construction neither reaches the Bishop frame as an endpoint nor extends through inflection points. The contribution is local and structural: we establish the exact normalization and its uniform lower bound, derive the two curvature coefficients and the normal-plane rotation coefficient, record the associated adapted-frame structural system, and compute the exact squared-derivative energy density. The reconstruction result included in the paper concerns general adapted orthonormal frames and is not an ABF realization theorem. No universal full-energy ordering or numerical validation is claimed. Analytical examples illustrate the resulting formulas and clarify the restricted domain of validity.

Keywords: adapted orthonormal frames; Frenet–Serret frame; Bishop frame; curvature-dependent frames; normal-plane rotation; frame energy

1. Introduction

The study of moving frames along curves in Euclidean three-space is a central topic in classical differential geometry, providing a fundamental framework for analyzing both local and global geometric properties of curves [1]. Since the seminal works of [2, 3], orthonormal frames attached to a space curve have played a crucial role in encoding geometric information through differential invariants.

The Frenet–Serret frame $\{T, N_F, B_F\}$ is the most classical and geometrically natural construction. For a smooth unit-speed curve $\gamma : I \rightarrow \mathbb{R}^3$, it is uniquely determined (up to orientation) by requiring that the principal normal N_F lies in the osculating plane generated by T and T' . The associated curvature

$\kappa = \|T'\|$ and torsion $\tau = \langle N'_F, B_F \rangle$ fully describe the intrinsic geometry of the curve, and the Frenet–Serret equations provide a complete dynamical description of the frame evolution [1].

Despite its elegance, the Frenet frame exhibits two fundamental limitations. First, it becomes undefined at inflection points where $\kappa = 0$. Second, its normal-plane rotation coefficient is the torsion τ , so its rotational energy is $\int_I \tau^2 ds$ and is generally nonzero. No claim that the Frenet frame maximizes rotational energy among all adapted frames is made.

To address these issues, [4] introduced the Bishop frame, also known as the rotation-minimizing frame (RMF). For a sufficiently regular unit-speed curve and a prescribed initial orthonormal basis of the normal plane, the Bishop frame $\{T, N_B, B_B\}$ can be continued through points of vanishing curvature and minimizes the normal-plane rotational energy among adapted orthonormal frames for which that energy is finite [4, 5]. However, this construction comes at a cost: it loses the direct geometric interpretation of the principal normal and retains an initial-phase freedom determined by the chosen normal-plane orientation [6].

The distinction between Frenet and relatively parallel frames has generated a substantial literature. Bishop-type frames avoid the division by curvature inherent in the Frenet principal normal and are therefore useful when a parallel normal-plane description is required. Their relation to Frenet data, signed curvature, and alternative curve framings has been examined in several settings [7–10]. Rotation-minimizing frames have also been studied from computational and variational perspectives. In particular, efficient approximation algorithms are available for geometric design applications, while the role of such frames in Euclidean-invariant variational systems has been analyzed systematically [5, 11].

A broad range of geometric constructions has been developed from parallel-transport and Bishop-type frames. These include characterizations of curves in Euclidean n -space, parallel curves, spherical images, acceleration decompositions, and tube-like or tubular surfaces [12–16]. Extensions to four-dimensional Euclidean space and framed curves have produced higher-dimensional parallel-frame systems and Bishop-type classifications [17–20]. These developments show that the choice of an adapted frame influences not only curve invariants but also associated surface constructions and higher-dimensional generalizations.

Parallel-frame techniques have likewise been extended to non-Euclidean ambient settings. For non-lightlike curves in Minkowski space, parallel frames, Bishop–Darboux characterizations, spherical images, and type-2 Bishop variants have been investigated in a number of works [21–26]. Related constructions include singular surfaces associated with spacelike curves, parallel curves in Minkowski space, and canal hypersurfaces generated by parallel-frame data [27–29]. Generalizations to dual space and other framed-curve settings further illustrate the continuing development of alternative moving-frame formalisms [30].

Other studies have focused on special curve classes and constructions based on Frenet, Bishop, or parallel frames, including slant helices, rectifying curves, coincident Bishop frames of W -Bertrand curves, integral and Smarandache curves, and parallel curves [31–36]. These works provide useful geometric context, but they do not by themselves furnish an endpoint interpolation between the Frenet and Bishop frames. The present construction is positioned more narrowly: it is an exactly normalized, curvature-dependent adapted frame generated from N_F and N_B on intervals where $\kappa > 0$.

In this paper, we study a two-parameter curvature-dependent adapted frame on Frenet-regular intervals, denoted for brevity by the *adaptive Bishop–Frenet (ABF) frame*. The name indicates that its

defining normal is generated explicitly from N_F and N_B ; it does not assert interpolation between the two frames or recovery of the Bishop frame as an endpoint. The curvature-dependent weighting is governed by the transition angle

$$\theta(s) = \phi(\kappa(s)) \in [0, \pi/4 - \delta], \quad 0 < \delta < \pi/4,$$

where a canonical choice is given by

$$\phi_{\lambda,\delta}(\kappa) = \left(\frac{\pi}{4} - \delta\right) e^{-\lambda\kappa}, \quad \lambda > 0.$$

This design produces a curvature-dependent Frenet–Bishop bisector regime as the positive curvature becomes small and converges toward the Frenet frame in high-curvature regions. It does not converge to the Bishop frame. Thus, the term ABF denotes a normalized curvature-dependent combination and not a genuine endpoint interpolation between the two frames. Since the definition depends explicitly on $N_F = T'/\kappa$, the construction is restricted to Frenet-regular intervals on which $\kappa > 0$; it does not define the frame at an inflection point where N_F is undefined.

The resulting ABF frame is an exactly normalized adapted orthonormal frame on the nondegenerate domain and admits explicitly computable frame coefficients and an exact squared-derivative energy density. The contribution is deliberately local and structural: it consists of the normalized curvature-dependent construction on intervals where $\kappa > 0$, the uniform nondegeneracy estimate, the explicit adapted-frame coefficients, and the exact energy-density identity. The reconstruction result recorded below applies to general adapted orthonormal frames and is explicitly distinguished from an ABF realization theorem, which would require additional compatibility conditions. No extension through inflection points, Bishop endpoint limit, universal full-energy ordering, or numerical validation is claimed.

From an applied perspective, the ABF frame provides a flexible framework for modeling Frenet-regular curve segments. By tuning the parameters λ and δ , one obtains a continuum of adapted frames ranging from a normalized Frenet–Bishop weighted combination to the Frenet frame. The Bishop frame itself is not attained by the present parameter range. Such adapted-frame constructions are relevant to geometric computations in which a controlled normal-plane orientation is required [5, 6]; curves containing inflection points require a separate extension that is not developed here. Thus, the scope of the paper is the analysis of a curvature-controlled normalized change of normal-plane orientation on Frenet-regular intervals; a construction possessing both an inflection-compatible definition and a genuine Bishop endpoint is outside the present theory.

2. Materials and methods

Let $\gamma : I \rightarrow \mathbb{R}^3$ be a C^4 unit-speed curve parametrized by arc-length s and assume throughout that

$$\kappa(s) = \|T'(s)\| > 0, \quad s \in I. \quad (2.1)$$

Thus, I is a Frenet-regular interval. The classical Frenet–Serret frame $\{T, N_F, B_F\}$ is defined by

$$T = \gamma'(s), \quad N_F = \frac{T'}{\kappa}, \quad B_F = T \times N_F, \quad (2.2)$$

where $\kappa = \|T'\|$ is the curvature. The Frenet–Serret equations are given by

$$T' = \kappa N_F, \quad N'_F = -\kappa T + \tau B_F, \quad B'_F = -\tau N_F, \quad (2.3)$$

where the torsion is defined by $\tau = \langle N'_F, B_F \rangle$.

The Bishop (rotation-minimizing) frame $\{T, N_B, B_B\}$ is defined, after prescribing an initial orthonormal basis of the normal plane, by the complete system

$$T' = \kappa_1 N_B + \kappa_2 B_B, \quad N'_B = -\kappa_1 T, \quad B'_B = -\kappa_2 T, \quad (2.4)$$

where κ_1 and κ_2 are the Bishop curvature coefficients. It is related to the Frenet frame via the angle function

$$\alpha(s) = \alpha_0 - \int_{s_0}^s \tau(u) du, \quad \alpha(s_0) = \alpha_0, \quad \alpha'(s) = -\tau(s), \quad (2.5)$$

through the transformation

$$\begin{aligned} N_B &= \cos \alpha N_F + \sin \alpha B_F, & B_B &= -\sin \alpha N_F + \cos \alpha B_F, \\ \kappa_1 &= \kappa \cos \alpha, & \kappa_2 &= -\kappa \sin \alpha. \end{aligned} \quad (2.6)$$

Here α_0 is the constant initial phase determined by the chosen orientation of the Bishop normal at s_0 . Unless an aligned initial condition is explicitly imposed, α_0 need not vanish.

Throughout this paper, the normal-plane rotation coefficient of the adapted orthonormal frame $\{T, N_\phi, B_\phi\}$ is defined by

$$\omega_\phi := \langle N'_\phi, B_\phi \rangle. \quad (2.7)$$

This coefficient is not, in general, the classical torsion of the curve; it tends to the Frenet torsion τ in the Frenet limit.

For the subsequent analysis, we use the exact normalization factor

$$\mu := \sqrt{1 + \sin(2\theta) \cos \alpha}. \quad (2.8)$$

The transition angle is given by

$$\theta(s) := \phi(\kappa(s)), \quad \theta' = \frac{d\phi}{d\kappa} \kappa', \quad (2.9)$$

and the unnormalized ABF normal vector is defined as

$$v := \cos \theta N_F + \sin \theta N_B. \quad (2.10)$$

Remark 2.1 (Domain of the construction). The lower bound imposed on μ prevents the weighted vector v from vanishing and makes the exact normalization well-defined. It does not extend the Frenet normal through a point where $\kappa = 0$. Indeed, if $N_F = T'/\kappa$ is undefined, then the numerator v is also undefined, independently of the normalization factor. Therefore, all ABF definitions and results below are understood on the Frenet-regular interval I , where $\kappa > 0$.

3. The adaptive Bishop–Frenet frame

We now define the curvature-dependent adapted frame, abbreviated as the adaptive Bishop–Frenet (ABF) frame because its defining vector contains both N_F and N_B . This terminology does not indicate a genuine interpolation: the present parameter range contains the Frenet endpoint but not the Bishop endpoint.

Let $\delta \in (0, \pi/4)$ be fixed and let $\phi : [0, \infty) \rightarrow [0, \pi/4 - \delta]$ be a smooth transition function satisfying:

- ϕ is strictly decreasing,
- $\phi(0) = \pi/4 - \delta$ as the continuous boundary value of the transition function (the curve itself is assumed to satisfy $\kappa > 0$),
- $\phi(\kappa) \rightarrow 0$ as $\kappa \rightarrow \infty$.

A canonical choice is given by the two-parameter family

$$\phi_{\lambda, \delta}(\kappa) = \left(\frac{\pi}{4} - \delta\right) e^{-\lambda \kappa}, \quad \lambda > 0. \quad (3.1)$$

Define the composed angle $\theta(s) := \phi(\kappa(s))$, and recall the unnormalized vector

$$v := \cos \theta N_F + \sin \theta N_B.$$

Since $0 \leq \theta \leq \pi/4 - \delta$,

$$\mu^2 = 1 + \sin(2\theta) \cos \alpha \geq 1 - \sin(2\theta) \geq 1 - \cos(2\delta) > 0.$$

Thus the exact normalization is everywhere nondegenerate on the domain of the Frenet frame. Define

$$N_\phi := \frac{v}{\mu}, \quad B_\phi := T \times N_\phi. \quad (3.2)$$

Lemma 3.1 (Exact orthonormality). *The triple $\{T, N_\phi, B_\phi\}$ is a positively oriented orthonormal adapted frame, and*

$$\mu(s) \geq \sqrt{1 - \cos(2\delta)} = \sqrt{2} \sin \delta > 0.$$

Proof. Because $\|v\|^2 = \mu^2$ and $v \perp T$, we have $\|N_\phi\| = 1$ and $N_\phi \perp T$. Hence $B_\phi = T \times N_\phi$ is unit and orthogonal to both T and N_ϕ . The lower bound follows from the preceding estimate. $\square$

4. ABF adapted-frame coefficients

We now derive the two curvature coefficients and the normal-plane rotation coefficient associated with the orthonormal ABF frame.

Let

$$v := \cos \theta N_F + \sin \theta N_B$$

denote the unnormalized ABF normal vector.

Proposition 4.1. With $v = \cos \theta N_F + \sin \theta N_B$ and using the Bishop evolution $N'_B = -\kappa_1 T$, one has

$$v' = (-\kappa \cos \theta - \kappa_1 \sin \theta)T - \theta' \sin \theta N_F + \theta' \cos \theta N_B + \tau \cos \theta B_F. \quad (4.1)$$

Proof. Applying the Leibniz rule yields

$$v' = -\theta' \sin \theta N_F + \cos \theta N'_F + \theta' \cos \theta N_B + \sin \theta N'_B.$$

Substituting $N'_F = -\kappa T + \tau B_F$ and $N'_B = -\kappa_1 T$, and collecting terms with respect to the basis $\{T, N_F, N_B, B_F\}$, gives the stated formula. $\square$

Lemma 4.2. Using $N_B = \cos \alpha N_F + \sin \alpha B_F$ and $B_\phi = T \times N_\phi$, one obtains

$$\mu B_\phi = (\cos \theta + \sin \theta \cos \alpha)B_F - \sin \theta \sin \alpha N_F, \quad (4.2)$$

and the inner products

$$\langle N_F, B_\phi \rangle = -\frac{\sin \theta \sin \alpha}{\mu}, \quad (4.3)$$

$$\langle N_B, B_\phi \rangle = \frac{\sin \alpha \cos \theta}{\mu}, \quad (4.4)$$

$$\langle B_F, B_\phi \rangle = \frac{\cos \theta + \sin \theta \cos \alpha}{\mu}. \quad (4.5)$$

Proof. Since $B_\phi = T \times (v/\mu)$, we write

$$\mu B_\phi = T \times v = \cos \theta (T \times N_F) + \sin \theta (T \times N_B).$$

Using $T \times N_F = B_F$ and

$$T \times N_B = \cos \alpha B_F - \sin \alpha N_F,$$

the result follows. The inner products are obtained by direct computation. $\square$

Theorem 4.3. The first ABF curvature coefficient is given by

$$\kappa_{\phi,1} = \langle T', N_\phi \rangle = \frac{\kappa(\cos \theta + \sin \theta \cos \alpha)}{\mu} \geq 0. \quad (4.6)$$

Proof. Since $T' = \kappa N_F$, we compute

$$\kappa_{\phi,1} = \left\langle \kappa N_F, \frac{v}{\mu} \right\rangle = \frac{\kappa}{\mu} (\cos \theta \langle N_F, N_F \rangle + \sin \theta \langle N_F, N_B \rangle).$$

Using $\langle N_F, N_B \rangle = \cos \alpha$, we obtain the formula.

Positivity follows from

$$\cos \theta + \sin \theta \cos \alpha \geq \cos \theta - \sin \theta = \sqrt{2} \cos \left(\theta + \frac{\pi}{4} \right) \geq 0,$$

since $\theta \in [0, \pi/4 - \delta]$. $\square$

Theorem 4.4. *The ABF normal-plane rotation coefficient is given by*

$$\omega_\phi = \langle N'_\phi, B_\phi \rangle = \frac{\theta' \sin \alpha + \tau \cos \theta (\cos \theta + \sin \theta \cos \alpha)}{\mu^2}. \quad (4.7)$$

Proof. Differentiating $N_\phi = v/\mu$ yields

$$N'_\phi = \frac{v'}{\mu} - \frac{\mu'}{\mu} N_\phi.$$

Since $\langle N_\phi, B_\phi \rangle = 0$, the second term does not contribute and

$$\omega_\phi = \frac{1}{\mu} \langle v', B_\phi \rangle.$$

Using the expression for v' and the identities from Lemma 4.2, we compute

$$\langle v', B_\phi \rangle = \frac{\theta' \sin \alpha (\sin^2 \theta + \cos^2 \theta) + \tau \cos \theta (\cos \theta + \sin \theta \cos \alpha)}{\mu}.$$

Since $\sin^2 \theta + \cos^2 \theta = 1$, the result follows. $\square$

Lemma 4.5. *With $\alpha' = -\tau$, the derivative of μ is given by*

$$\mu' = \frac{\theta' \cos(2\theta) \cos \alpha + \frac{\tau}{2} \sin(2\theta) \sin \alpha}{\mu}. \quad (4.8)$$

Proof. Differentiating $\mu^2 = 1 + \sin(2\theta) \cos \alpha$ gives

$$(\mu^2)' = 2\theta' \cos(2\theta) \cos \alpha + \tau \sin(2\theta) \sin \alpha.$$

Dividing by 2μ yields the result. $\square$

Theorem 4.6 (ABF adapted-frame system). *Define the second curvature coefficient by*

$$\kappa_{\phi,2} := \langle T', B_\phi \rangle = -\frac{\kappa \sin \theta \sin \alpha}{\mu}. \quad (4.9)$$

Then the ABF frame satisfies

$$\begin{pmatrix} T \\ N_\phi \\ B_\phi \end{pmatrix}' = A_\phi \begin{pmatrix} T \\ N_\phi \\ B_\phi \end{pmatrix}, \quad (4.10)$$

where $A_\phi \in \mathfrak{so}(3)$ is given by

$$A_\phi = \begin{pmatrix} 0 & \kappa_{\phi,1} & \kappa_{\phi,2} \\ -\kappa_{\phi,1} & 0 & \omega_\phi \\ -\kappa_{\phi,2} & -\omega_\phi & 0 \end{pmatrix}. \quad (4.11)$$

Proof. Writing $N_\phi = aN_F + bB_F$ gives $a = (\cos \theta + \sin \theta \cos \alpha)/\mu$ and $b = \sin \theta \sin \alpha/\mu$. Since $B_\phi = aB_F - bN_F$, one has $N_F = aN_\phi - bB_\phi$. Hence $T' = \kappa N_F = \kappa_{\phi,1}N_\phi + \kappa_{\phi,2}B_\phi$. The remaining two equations follow from orthonormality and the definition $\omega_\phi = \langle N'_\phi, B_\phi \rangle$. $\square$

Corollary 4.7. *The curvature of the underlying curve is recovered from the two ABF curvature coefficients by*

$$\kappa^2 = \kappa_{\phi,1}^2 + \kappa_{\phi,2}^2. \quad (4.12)$$

Remark 4.8. Thus, $\kappa_{\phi,1}$ alone is not the curvature invariant of the curve, and ω_ϕ is not generally its Frenet torsion. They are connection coefficients of the exactly normalized adapted orthonormal ABF frame.

5. Special cases and sign verification

We identify the Frenet limit and the normalized Frenet–Bishop bisector limit of the adaptive Bishop–Frenet frame.

Proposition 5.1. *The ABF frame has the following limiting behavior:*

(i) **Frenet limit.** *As $\theta \rightarrow 0$, $\theta' \rightarrow 0$, and $\mu \rightarrow 1$, one has*

$$\kappa_{\phi,1} \rightarrow \kappa, \quad \kappa_{\phi,2} \rightarrow 0, \quad \omega_\phi \rightarrow \tau.$$

(ii) **Equal-weight normalized-bisector boundary limit.** *Consider the canonical choice $\phi_{\lambda,\delta}(\kappa) = (\pi/4 - \delta)e^{-\lambda\kappa}$ with fixed $\lambda > 0$. Take first the one-sided boundary limit $\kappa \rightarrow 0^+$ and then $\delta \rightarrow 0^+$, so that*

$$\lim_{\delta \rightarrow 0^+} \lim_{\kappa \rightarrow 0^+} \phi_{\lambda,\delta}(\kappa) = \frac{\pi}{4}.$$

Assume along this limiting process that $\theta' \rightarrow 0$, that $\alpha \rightarrow \alpha_$ with $\alpha_* \neq (2k+1)\pi$, and that N_F and N_B admit one-sided limits N_F^* and N_B^* respectively. Then*

$$\lim_{\delta \rightarrow 0^+} \lim_{\kappa \rightarrow 0^+} N_\phi = \frac{N_F^* + N_B^*}{\sqrt{2(1 + \cos \alpha_*)}} = \frac{N_F^* + N_B^*}{\|N_F^* + N_B^*\|}.$$

The limiting vector is the normalized bisector of the one-sided Frenet and Bishop limits, not the Bishop normal in general. Equivalently, the corresponding frame coefficients have the boundary asymptotics

$$\begin{aligned} \kappa_{\phi,1} &= \kappa \sqrt{\frac{1 + \cos \alpha_*}{2}} + o(\kappa), & \kappa_{\phi,2} &= -\frac{\kappa \sin \alpha_*}{\sqrt{2(1 + \cos \alpha_*)}} + o(\kappa), \\ \omega_\phi &= \frac{\tau}{2} + o(1), \end{aligned}$$

provided the quantities appearing in these asymptotics possess the stated one-sided limits. This is a boundary-limit statement only. Since the ABF construction is defined on intervals where $\kappa > 0$, neither $\kappa = 0$ nor $\theta = \pi/4$ is attained within its domain. The formula does not define an ABF frame at $\kappa = 0$. It agrees with the Bishop frame only in the special aligned case $N_F^ = N_B^*$; no general Bishop endpoint or Bishop limit is claimed.*

(iii) **Planar curves.** If $\tau \equiv 0$, then $\alpha(s) \equiv \alpha_0$, where α_0 is determined by the initial Bishop orientation. Hence

$$N_B = \cos \alpha_0 N_F + \sin \alpha_0 B_F,$$

and the ABF coefficients become

$$\mu^2 = 1 + \sin(2\theta) \cos \alpha_0,$$

$$\kappa_{\phi,1} = \frac{\kappa(\cos \theta + \sin \theta \cos \alpha_0)}{\mu}, \quad \kappa_{\phi,2} = -\frac{\kappa \sin \theta \sin \alpha_0}{\mu}, \quad \omega_\phi = \frac{\theta' \sin \alpha_0}{\mu^2}.$$

Thus planarity alone does not imply $N_B = N_F$, $N_\phi = N_F$, or $\omega_\phi = 0$. Under the aligned initial-phase convention $\alpha_0 = 0 \pmod{2\pi}$, one obtains $N_B = N_F$, $N_\phi = N_F$, $\kappa_{\phi,1} = \kappa$, $\kappa_{\phi,2} = 0$, and $\omega_\phi = 0$.

The following table (Table 1) summarizes the main geometric properties of the Frenet, Bishop, and ABF frames.

Table 1. Comparison of Frenet, Bishop, and ABF frames.

Property	Frenet	Bishop	ABF
Defined at $\kappa = 0$	No	Yes	No (not by the present definition)
Frame coefficients	$(\kappa, 0, \tau)$	$(\kappa_1, \kappa_2, 0)$	$(\kappa_{\phi,1}, \kappa_{\phi,2}, \omega_\phi)$
Available on an interval where $\kappa > 0$	Yes	Yes	Yes
Normal-plane rotational energy	$\int_I \tau^2 ds$	0	$\int_I \omega_\phi^2 ds$
Full energy ordering	—	—	No universal ordering asserted
Free parameter	—	—	$\lambda > 0, 0 < \delta < \pi/4$

6. Reconstruction from adapted-frame coefficients

The following result is the standard reconstruction theorem for a general adapted orthonormal frame. It does not characterize which prescribed coefficients arise from the ABF construction.

Theorem 6.1 (Reconstruction from prescribed adapted-frame coefficients). *Let*

$$k_1, k_2, \omega \in C^0(I)$$

with $k_1^2 + k_2^2 > 0$ on I . Given an initial point $p_0 \in \mathbb{R}^3$ and an initial positively oriented orthonormal frame, there exists a C^2 unit-speed curve $\gamma : I \rightarrow \mathbb{R}^3$ and a positively oriented adapted orthonormal frame $\{T, N, B\}$ satisfying

$$T' = k_1 N + k_2 B, \quad N' = -k_1 T + \omega B, \quad B' = -k_2 T - \omega N. \quad (6.1)$$

The curve and frame are unique for the prescribed initial data. Consequently, after allowing the initial point and initial frame to vary, the curve is unique up to a direct Euclidean motion.

Proof. We adopt the row-frame convention

$$F(s) = \begin{pmatrix} T(s)^\top \\ N(s)^\top \\ B(s)^\top \end{pmatrix}. \quad (6.2)$$

With this convention, the frame equations are equivalent to $F' = AF$, where

$$A(s) = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & \omega \\ -k_2 & -\omega & 0 \end{pmatrix} \in \mathfrak{so}(3). \quad (6.3)$$

The linear system $F' = AF$ with prescribed $F(s_0) \in SO(3)$ has a unique C^1 solution. Because the frame vectors are arranged as rows, the appropriate Gram matrix is

$$G(s) := F(s)F(s)^\top. \quad (6.4)$$

It satisfies

$$G' = F'F^\top + F(F')^\top = AG + GA^\top. \quad (6.5)$$

Since $A^\top = -A$, the constant function $G(s) \equiv I$ solves this linear matrix initial-value problem with $G(s_0) = I$. Uniqueness therefore gives $G(s) = I$ for all $s \in I$. Thus the rows T, N, B remain orthonormal. Moreover, continuity of $\det F$ and $\det F(s_0) = 1$ imply $\det F(s) = 1$, so $F(s) \in SO(3)$. Define

$$\gamma(s) = p_0 + \int_{s_0}^s T(u) du. \quad (6.6)$$

Then $\gamma' = T$ and $\|T\| = 1$. Uniqueness for fixed initial data follows directly from uniqueness of $F' = AF$ and from the integral defining γ .

Now let (F, p_0) and $(\tilde{F}, \tilde{p}_0)$ correspond to two positively oriented initial frames and initial points for the same coefficient functions. There is a unique constant rotation $R \in SO(3)$ such that

$$\tilde{F}(s_0) = F(s_0)R^\top. \quad (6.7)$$

Because R is constant,

$$(FR^\top)' = F'R^\top = AFR^\top. \quad (6.8)$$

Hence $FR^\top$ and $\tilde{F}$ solve the same initial-value problem, and uniqueness yields

$$\tilde{F}(s) = F(s)R^\top. \quad (6.9)$$

In particular, $\tilde{T}(s) = RT(s)$. Therefore, with $c := \tilde{p}_0 - Rp_0$,

$$\tilde{\gamma}(s) = R\gamma(s) + c. \quad (6.10)$$

Thus the reconstructed curves differ by a direct Euclidean motion. This argument concerns only the general adapted-frame reconstruction; the additional ABF compatibility requirements are stated in the following remark. $\square$

Remark 6.2 (ABF compatibility is an additional requirement). The preceding theorem reconstructs a general adapted orthonormal frame; it does not prove that the reconstructed frame is generated by the ABF formula. For an ABF realization, the prescribed coefficients must arise from functions κ , τ , α , and θ satisfying

$$k_1 = \frac{\kappa(\cos \theta + \sin \theta \cos \alpha)}{\mu}, \quad (6.11)$$

$$k_2 = -\frac{\kappa \sin \theta \sin \alpha}{\mu}, \quad (6.12)$$

$$\omega = \frac{\theta' \sin \alpha + \tau \cos \theta (\cos \theta + \sin \theta \cos \alpha)}{\mu^2}, \quad (6.13)$$

together with

$$\mu^2 = 1 + \sin(2\theta) \cos \alpha, \quad \theta = \phi(\kappa), \quad \theta' = \phi'(\kappa)\kappa', \quad \alpha' = -\tau, \quad \kappa^2 = k_1^2 + k_2^2. \quad (6.14)$$

These relations show that the three ABF coefficients cannot be prescribed independently. Necessary and sufficient conditions for an arbitrary triple (k_1, k_2, ω) to admit an ABF realization are not established here.

7. Rotational energy

We analyze the rotational and full squared-derivative energies associated with adapted normal fields and with the ABF frame. Throughout this section, $I = [a, b]$ is assumed to be a compact interval; equivalently, the statements remain valid on a general interval whenever the displayed energy integrands are integrable.

Let $E[N] := \int_I \|N'(s)\|^2 ds$ denote the squared-derivative energy of a unit normal field.

7.1. Bishop minimality

Remark 7.1. For the ABF frame, the general adapted-frame coefficients are $\kappa_N = \kappa_{\phi,1}$, $\kappa_B = \kappa_{\phi,2}$, and $\omega = \omega_\phi$. In general, $\kappa_{\phi,2}$ does not vanish; it vanishes only when $\sin \theta \sin \alpha = 0$.

For an adapted orthonormal frame $\{T, N, B\}$ along γ , the frame equations may be written in the form

$$T' = \kappa_N N + \kappa_B B, \quad (7.1)$$

$$N' = -\kappa_N T + \omega B, \quad B' = -\kappa_B T - \omega N, \quad (7.2)$$

where $\omega = \langle N', B \rangle$ measures the instantaneous rotation of the normal plane about the tangent direction T . Since N is a unit vector adapted to γ (i.e., $N \perp T$), one has $\langle N', T \rangle = -\langle N, T' \rangle$ and

$$\|N'\|^2 = \kappa_N^2 + \omega^2.$$

The coefficients of the three frames in the general adapted-frame equations (7.1) and (7.2) are summarized in Table 2. We define the *rotational part* of the energy by

$$E_{\text{rot}}[N, B] := \int_I \omega(s)^2 ds, \quad (7.3)$$

so that $E[N] = \int_I \kappa_N^2 ds + E_{\text{rot}}[N, B]$. For the Bishop frame, $\kappa_N = \kappa_1$ and $\omega_B = 0$, so $E[N_B] = \int_I \kappa_1^2 ds$. For a general adapted frame, $E[N] \geq E_{\text{rot}}[N, B]$.

Table 2. Coefficients in the general adapted-frame equations (7.1) and (7.2) for the three frames considered in this paper. The Frenet frame satisfies $\kappa_B = 0$, whereas the Bishop frame satisfies $\omega = 0$.

Frame	κ_N	κ_B	ω
Frenet	κ	0	τ
Bishop	κ_1	κ_2	0
ABF	$\kappa_{\phi,1}$	$\kappa_{\phi,2}$	ω_{ϕ}

Lemma 7.2 (Bishop minimality). *Among all adapted orthonormal frames along the same fixed unit-speed curve γ for which the rotational energy is finite, the Bishop frame minimizes that energy. More precisely, for the Bishop frame, one has*

$$\omega_B \equiv 0,$$

and hence

$$E_{\text{rot}}[N_B, B_B] = 0.$$

Consequently, for any adapted frame $\{T, N, B\}$,

$$E_{\text{rot}}[N_B, B_B] \leq E_{\text{rot}}[N, B].$$

Proof. By definition, the Bishop frame is obtained by parallel transport in the normal plane. Equivalently, its normal vectors satisfy

$$N'_B = -\kappa_1 T, \quad B'_B = -\kappa_2 T.$$

Thus neither N'_B nor B'_B has a component in the normal plane. In particular,

$$\omega_B = \langle N'_B, B_B \rangle = 0.$$

Therefore,

$$E_{\text{rot}}[N_B, B_B] = \int_I \omega_B^2 ds = 0.$$

On the other hand, for any adapted orthonormal frame $\{T, N, B\}$, the rotational energy satisfies

$$E_{\text{rot}}[N, B] = \int_I \omega(s)^2 ds \geq 0.$$

Hence

$$E_{\text{rot}}[N_B, B_B] \leq E_{\text{rot}}[N, B].$$

This proves the minimality of the Bishop frame. □

Corollary 7.3 (Rotational-energy consequence). *For the ABF frame, one has*

$$E_{\text{rot}}[N_B, B_B] \leq E_{\text{rot}}[N_\phi, B_\phi].$$

This statement concerns only the normal-plane rotational part of the energy; it does not imply an ordering of the full squared-derivative energies $E[N_B]$ and $E[N_\phi]$.

Proof. The ABF frame $\{T, N_\phi, B_\phi\}$ is an adapted orthonormal frame. Therefore, by Lemma 7.2,

$$E_{\text{rot}}[N_B, B_B] \leq E_{\text{rot}}[N_\phi, B_\phi].$$

Since $E_{\text{rot}}[N_B, B_B] = 0$ and $E_{\text{rot}}[N_\phi, B_\phi] = \int_I \omega_\phi^2 ds \geq 0$, the asserted rotational inequality follows. No conclusion about the full energies is drawn. $\square$

7.2. Explicit energy density

Proposition 7.4 (Explicit formula for $\|v'\|^2$). *Let*

$$v = \cos \theta N_F + \sin \theta N_B,$$

where $N_B = \cos \alpha N_F + \sin \alpha B_F$ and $\kappa_1 = \kappa \cos \alpha$. Then, in the orthonormal basis $\{T, N_F, B_F\}$, the squared norm of v' is

$$\begin{aligned} \|v'\|^2 &= \kappa^2(\cos \theta + \sin \theta \cos \alpha)^2 + \tau^2 \cos^2 \theta \\ &\quad + 2\tau\theta' \sin \alpha \cos^2 \theta + \theta'^2(1 - \sin(2\theta) \cos \alpha). \end{aligned} \quad (7.4)$$

Proof. From Proposition 4.1, the derivative of v is

$$v' = (-\kappa \cos \theta - \kappa_1 \sin \theta)T - \theta' \sin \theta N_F + \theta' \cos \theta N_B + \tau \cos \theta B_F. \quad (7.5)$$

Using the relation

$$N_B = \cos \alpha N_F + \sin \alpha B_F \quad \text{and} \quad \kappa_1 = \kappa \cos \alpha,$$

we rewrite v' entirely in the orthonormal Frenet basis $\{T, N_F, B_F\}$.

Step 1: Substitute N_B into v' .

$$\begin{aligned} v' &= (-\kappa \cos \theta - \kappa \cos \alpha \sin \theta)T \\ &\quad - \theta' \sin \theta N_F + \theta' \cos \theta (\cos \alpha N_F + \sin \alpha B_F) + \tau \cos \theta B_F. \end{aligned} \quad (7.6)$$

Step 2: Collect coefficients.

Thus

$$v' = v'_T T + v'_{N_F} N_F + v'_{B_F} B_F,$$

where

$$v'_T = -\kappa \cos \theta - \kappa \cos \alpha \sin \theta, \quad (7.7)$$

$$v'_{N_F} = -\theta' \sin \theta + \theta' \cos \theta \cos \alpha, \quad (7.8)$$

$$v'_{B_F} = \tau \cos \theta + \theta' \cos \theta \sin \alpha. \quad (7.9)$$

Step 3: Compute the squared norm.

Since $\{T, N_F, B_F\}$ is orthonormal,

$$\|v'\|^2 = (v'_T)^2 + (v'_{N_F})^2 + (v'_{B_F})^2.$$

Step 4: Expand each term.

$$(v'_T)^2 = \kappa^2(\cos \theta + \sin \theta \cos \alpha)^2, \quad (7.10)$$

$$(v'_{N_F})^2 = \theta'^2(\sin^2 \theta - 2 \sin \theta \cos \theta \cos \alpha + \cos^2 \theta \cos^2 \alpha), \quad (7.11)$$

$$(v'_{B_F})^2 = \tau^2 \cos^2 \theta + 2\tau\theta' \sin \alpha \cos^2 \theta + \theta'^2 \cos^2 \theta \sin^2 \alpha. \quad (7.12)$$

Step 5: Combine terms.

Adding the θ'^2 terms and using

$$\cos^2 \alpha + \sin^2 \alpha = 1,$$

we obtain

$$\begin{aligned} & \sin^2 \theta - 2 \sin \theta \cos \theta \cos \alpha + \cos^2 \theta (\cos^2 \alpha + \sin^2 \alpha) \\ &= \sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta \cos \alpha \\ &= 1 - \sin(2\theta) \cos \alpha. \end{aligned} \quad (7.13)$$

Final result.

Substituting into the expression for $\|v'\|^2$ gives

$$\begin{aligned} \|v'\|^2 &= \kappa^2(\cos \theta + \sin \theta \cos \alpha)^2 + \tau^2 \cos^2 \theta \\ &\quad + 2\tau\theta' \sin \alpha \cos^2 \theta + \theta'^2(1 - \sin(2\theta) \cos \alpha), \end{aligned} \quad (7.14)$$

which completes the proof. $\square$

Theorem 7.5 (Energy density of the ABF frame). *The squared norm of N'_ϕ is given by*

$$\begin{aligned} \|N'_\phi\|^2 &= \frac{\|v'\|^2}{\mu^2} - \left(\frac{\mu'}{\mu}\right)^2 \\ &= \frac{\kappa^2(\cos \theta + \sin \theta \cos \alpha)^2 + \tau^2 \cos^2 \theta + 2\tau\theta' \sin \alpha \cos^2 \theta + \theta'^2(1 - \sin(2\theta) \cos \alpha)}{\mu^2} \\ &\quad - \frac{(\theta' \cos(2\theta) \cos \alpha + \frac{\tau}{2} \sin(2\theta) \sin \alpha)^2}{\mu^4}. \end{aligned} \quad (7.15)$$

Proof. Since the ABF normal is exactly normalized,

$$N_\phi = \frac{v}{\mu}, \quad \mu = \|v\| > 0. \quad (7.16)$$

Differentiation gives

$$N'_\phi = \frac{v'}{\mu} - \frac{\mu'}{\mu^2}v. \quad (7.17)$$

Therefore, retaining the cross term,

$$\|N'_\phi\|^2 = \frac{\|v'\|^2}{\mu^2} - \frac{2\mu'}{\mu^3}\langle v', v \rangle + \frac{(\mu')^2}{\mu^4}\|v\|^2. \quad (7.18)$$

Because $\|v\|^2 = \mu^2$, differentiating this identity yields

$$2\langle v', v \rangle = 2\mu\mu', \quad \text{hence} \quad \langle v', v \rangle = \mu\mu'. \quad (7.19)$$

Substitution into the preceding expansion gives

$$\|N'_\phi\|^2 = \frac{\|v'\|^2}{\mu^2} - 2\frac{(\mu')^2}{\mu^2} + \frac{(\mu')^2}{\mu^2} \quad (7.20)$$

$$= \frac{\|v'\|^2}{\mu^2} - \left(\frac{\mu'}{\mu}\right)^2. \quad (7.21)$$

Finally, inserting the previously derived formulas for $\|v'\|^2$ and μ' gives the expanded expression in the theorem. $\square$

7.3. Comparison with Frenet energy

Since $N'_F = -\kappa T + \tau B_F$, one has $\|N'_F\|^2 = \kappa^2 + \tau^2$. However, the unnormalized auxiliary vector v must not be compared directly with the unit normal N_F through $\|v'\|^2$. A geometrically meaningful comparison of full squared-derivative energies must use the exactly normalized field $N_\phi = v/\mu$ and the corrected density

$$\|N'_\phi\|^2 = \frac{\|v'\|^2}{\mu^2} - \left(\frac{\mu'}{\mu}\right)^2.$$

Consequently, no universal inequality between $E[N_\phi]$ and $E[N_F]$ is asserted; any such comparison must be based on the exactly normalized density and on explicitly stated hypotheses.

Remark 7.6 (Energy comparison after exact normalization). Because exact normalization contributes the term $-(\mu'/\mu)^2$, any comparison between $E[N_\phi]$ and $E[N_F]$ must be made directly from the corrected density above. No unconditional ordering of the full squared-derivative energies is asserted without an additional verified integral condition.

Remark 7.7 (No Bishop energy limit). For fixed $\delta > 0$, the limit $\lambda \rightarrow 0$ gives $\theta \rightarrow \pi/4 - \delta$ and hence a fixed normalized Frenet–Bishop weighted combination, not the Bishop frame. Even in the joint limit $\delta \rightarrow 0^+$, the normal approaches the normalized bisector described above. Therefore, no conclusion of the form $E[N_\phi] \rightarrow E[N_B]$ is asserted.

8. Explicit examples

We illustrate the analytical behavior of the exactly normalized adaptive Bishop–Frenet (ABF) construction on representative curves. No numerical validation is claimed.

8.1. Circular helix

Consider the circular helix

$$\gamma(t) = (a \cos t, a \sin t, bt), \quad a > 0, \quad b \in \mathbb{R}, \quad c = \sqrt{a^2 + b^2}.$$

The parameter t is not the arc-length parameter. Since $\|\gamma'(t)\| = c$, arc length measured from $t = 0$ is

$$s = ct = \sqrt{a^2 + b^2} t, \quad t = \frac{s}{c}.$$

Its curvature and torsion are constant:

$$\kappa = \frac{a}{c^2}, \quad \tau = \frac{b}{c^2}.$$

In this case, $\theta' = 0$ and

$$\theta = \left(\frac{\pi}{4} - \delta\right) e^{-\lambda a/c^2}.$$

Choosing $s_0 = 0$ and the aligned initial phase $\alpha_0 = 0$ gives

$$\alpha(s) = -\tau s, \quad \alpha(t) = -\tau ct = -\frac{b}{c} t,$$

so all formulas below are written consistently in the arc-length variable s . In particular,

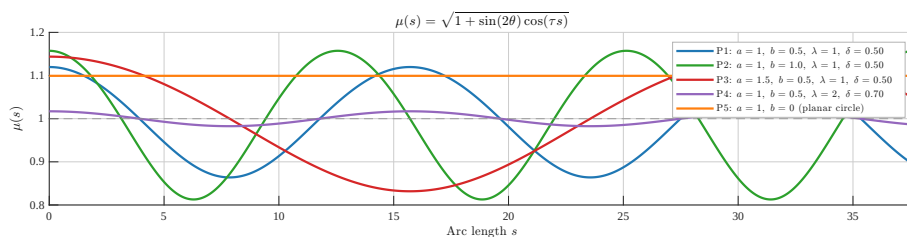
$$\mu(s) = \sqrt{1 + \sin(2\theta) \cos(\tau s)}.$$

For the helix, the corrected energy density is obtained by setting $\theta' = 0$ and $\alpha = -\tau s$ in the exact-normalization formula above. A strict energy inequality is not asserted without recomputing the resulting integral for specified (λ, δ) . For the helix, the first curvature coefficient and the normal-plane rotation coefficient are given by

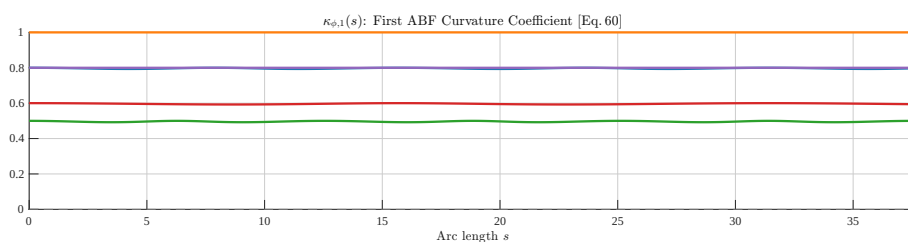
$$\kappa_{\phi,1}(s) = \frac{\kappa(\cos \theta + \sin \theta \cos(\tau s))}{\mu(s)}, \quad (8.1)$$

$$\omega_{\phi}(s) = \frac{\tau \cos \theta (\cos \theta + \sin \theta \cos(\tau s))}{\mu(s)^2}. \quad (8.2)$$

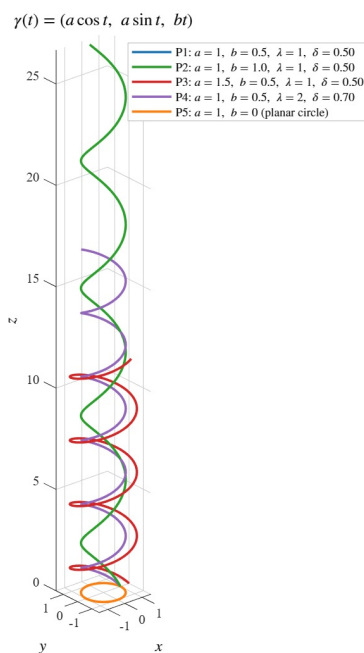
Circular Helix: Parameter Comparison of ABF Coefficients



(a) Exact normalization factor $\mu(s) = \sqrt{1 + \sin(2\theta)\cos(\tau s)}$ for the selected helix parameter sets.



(b) First ABF curvature coefficient $\kappa_{\phi,1}(s)$ for the selected helix parameter sets.



(c) Three-dimensional circular helices corresponding to the selected values of a and b .

Figure 1. Analytical behavior of the exactly normalized ABF construction for circular helices. Panels (a) and (b) display the normalization factor and the first ABF curvature coefficient, respectively, as functions of arc length. Panel (c) shows the corresponding circular helices. These plots illustrate the analytical formulas and are not used to claim a universal full-energy ordering.

Thus, the helix example remains entirely analytical: κ and τ are the constants displayed above, while the ABF coefficients depend on the arc-length variable through the Bishop phase $\alpha(s) = -\tau s$. Figure 1 provides a visual representation of the exact analytical formulas and the corresponding circular helices for the selected admissible parameter sets. The plotted values are obtained by direct evaluation of the formulas derived above; hence, the figure is included for illustration and parameter comparison rather than as independent numerical validation. No numerical full-energy comparison or universal energy-ordering claim is made.

8.2. Planar–spatial junction

Consider the curve

$$\gamma(t) = \begin{cases} (t, t^2, 0), & t \in [-1, 0], \\ (t, t^2, t^3), & t \in [0, 1]. \end{cases}$$

The curve is C^2 and has nonzero curvature near $t = 0$, although its one-sided torsion values have a jump at the junction.

Remark 8.1 (Regularity scope of the junction example). This piecewise curve is not C^4 at $t = 0$ and therefore does not satisfy, at the junction itself, the regularity hypothesis imposed in the theoretical sections. In particular, the classical torsion is not defined at $t = 0$; only its one-sided values on $(-1, 0)$ and $(0, 1)$ are considered. Consequently, the ABF differential identities, structural equations, and energy-density formulas are applied separately on these two open subintervals and are not asserted at the junction. The purpose of the following proposition is only to examine whether the one-sided frame fields admit a common continuous extension to $t = 0$. No differentiability or smoothness of that extension is claimed.

Proposition 8.2 (Continuity at the planar–spatial junction). *For the preceding curve, the one-sided normalized Frenet binormals admit the same continuous extension at $t = 0$. Consequently, the torsion jump does not imply a discontinuity of the binormal itself. With a continuous choice of Bishop phase, the one-sided exactly normalized ABF normals likewise admit a common continuous extension to the junction. This is a continuity statement only; the extended ABF normal is not asserted to be differentiable or smooth at $t = 0$.*

Proof. For $t < 0$,

$$\gamma'(t) = (1, 2t, 0), \quad \gamma''(t) = (0, 2, 0),$$

and hence

$$\gamma'(t) \times \gamma''(t) = (0, 0, 2).$$

For $t > 0$,

$$\gamma'(t) = (1, 2t, 3t^2), \quad \gamma''(t) = (0, 2, 6t),$$

so that

$$\gamma'(t) \times \gamma''(t) = (6t^2, -6t, 2).$$

Therefore,

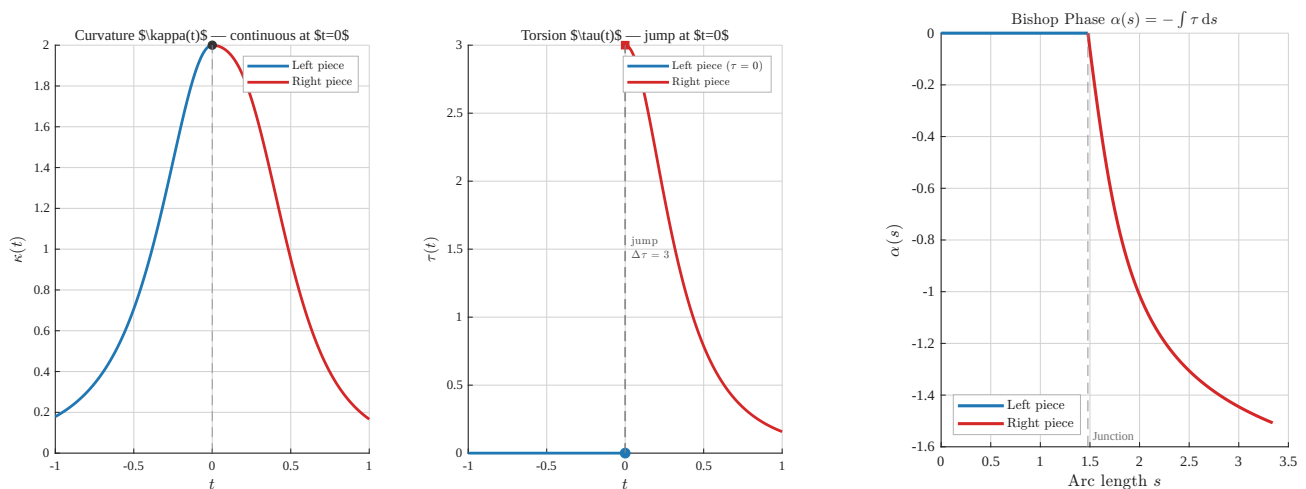
$$\lim_{t \rightarrow 0^-} \gamma'(t) \times \gamma''(t) = (0, 0, 2) = \lim_{t \rightarrow 0^+} \gamma'(t) \times \gamma''(t).$$

Since this limiting vector is nonzero, normalization is continuous and gives

$$\lim_{t \rightarrow 0^-} B_F(t) = (0, 0, 1) = \lim_{t \rightarrow 0^+} B_F(t).$$

Thus the one-sided fields B_F have a common continuous extension at the junction. A jump in torsion concerns the corresponding frame coefficient (and the differentiability of the frame), but does not by itself force the binormal vector to be discontinuous. Moreover, T and $N_F = B_F \times T$ have common continuous extensions at $t = 0$. If the Bishop phase α is chosen continuously, then the one-sided fields $N_B = \cos \alpha N_F + \sin \alpha B_F$, and hence $v = \cos \theta N_F + \sin \theta N_B$, also have common continuous extensions there. Finally, the exact-normalization bound $\mu \geq \sqrt{1 - \cos(2\delta)} > 0$ implies that the one-sided fields $N_\phi = v/\mu$ extend continuously to $t = 0$. Because the curve is only C^2 at the junction and torsion is not defined there, this argument establishes neither differentiability nor smoothness of the extended ABF normal, and none of the C^4 -based ABF differential formulas is invoked at $t = 0$. $\square$

To illustrate the one-sided geometric data used in the preceding proof, Figure 2 displays the curvature, torsion, and continuously chosen Bishop phase on the planar and spatial pieces. The curvature has a common positive limit at the junction, whereas the one-sided torsion values are different. The Bishop phase remains continuous because it is obtained by integrating the piecewise torsion.



(a) Curvature $\kappa(t)$ on the planar and spatial pieces and One-sided torsion values. The torsion has a jump at $t = 0$.

(b) Continuous Bishop phase $\alpha(s) = -\int_0^s \tau(u) du$.

Figure 2. Geometric input data for the planar-spatial junction. The curvature admits a common positive limit at the junction, whereas the one-sided torsion values are different. The Bishop phase remains continuous because it is obtained by integration of the piecewise torsion. The dashed vertical line identifies the junction.

Using the same piecewise geometric data and the parameter values $\lambda = 1$ and $\delta = 0.5$, the corresponding ABF quantities are shown in Figure 3. The normalization factor μ , the first curvature

coefficient $\kappa_{\phi,1}$, and the normal-plane rotation coefficient ω_ϕ are evaluated separately on the two open subintervals. No ABF differential formula is evaluated at the junction itself, because the curve does not satisfy the C^4 regularity assumption at $t = 0$.

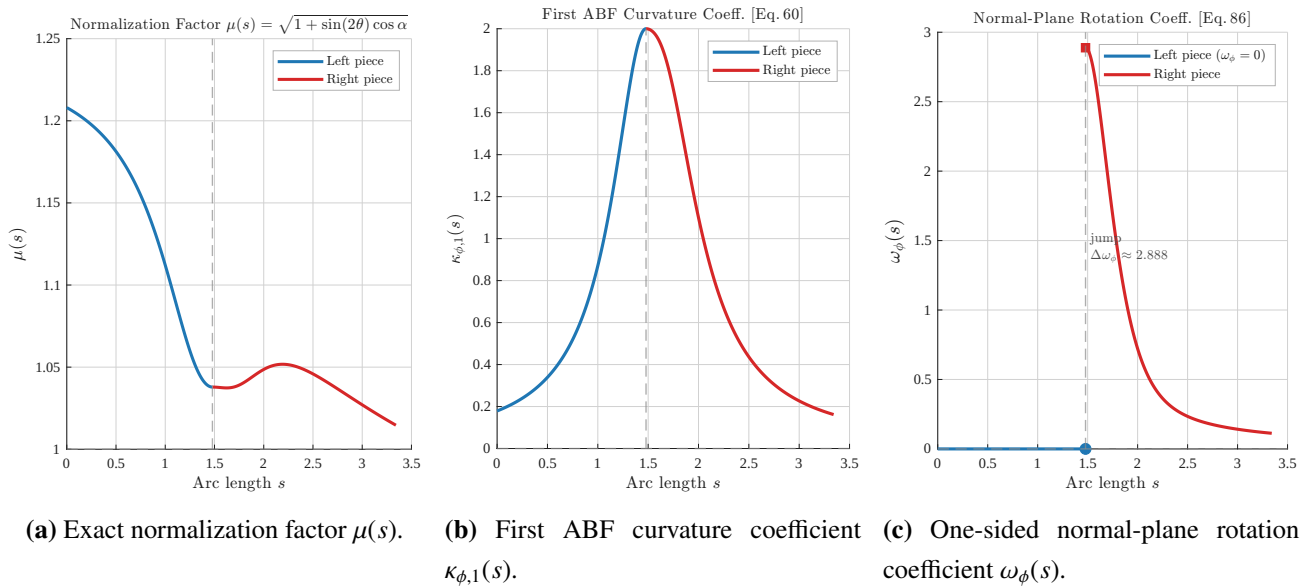


Figure 3. Behavior of the exactly normalized ABF quantities on the two sides of the planar-spatial junction for $\lambda = 1$, $\delta = 0.5$, and the aligned initial Bishop phase on the planar piece. The normalization factor and the first curvature coefficient admit compatible one-sided behavior, whereas ω_ϕ has different one-sided values because it depends on the torsion and derivative data. The plots are direct evaluations of the piecewise analytical formulas and are included for illustration rather than as independent numerical validation.

8.3. Trefoil knot

We use the standard parametrization

$$\gamma(t) = ((2 + \cos 3t) \cos 2t, (2 + \cos 3t) \sin 2t, \sin 3t), \quad t \in [0, 2\pi]. \quad (8.3)$$

The parameter t is not assumed to be arc length. Curvature and torsion are computed directly from this regular parametrization by the parameter-invariant formulas

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}, \quad (8.4)$$

$$\tau(t) = \frac{\det(\gamma'(t), \gamma''(t), \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}. \quad (8.5)$$

For this parametrization, one has

$$\|\gamma'(t)\|^2 = 25 + 16 \cos(3t) + 4 \cos^2(3t) \geq 13, \quad (8.6)$$

and, writing $c_3 = \cos(3t)$ and $e_\varphi(t) = (-\sin(2t), \cos(2t), 0)$, its component in the rotating azimuthal direction is

$$\langle \gamma'(t) \times \gamma''(t), e_\varphi(t) \rangle = -(12c_3^2 + 24c_3 + 27) = -(12(c_3 + 1)^2 + 15). \quad (8.7)$$

Hence $\gamma'(t) \neq 0$ and $\gamma'(t) \times \gamma''(t) \neq 0$ for every $t \in [0, 2\pi]$. In particular, $\kappa(t) > 0$ on the whole interval, so the Frenet and ABF quantities used below are well-defined. The corresponding arc-length variable is defined by

$$s(t) = \int_0^t \|\gamma'(u)\| du, \quad s(0) = 0, \quad (8.8)$$

and derivatives entering the ABF formulas are understood through

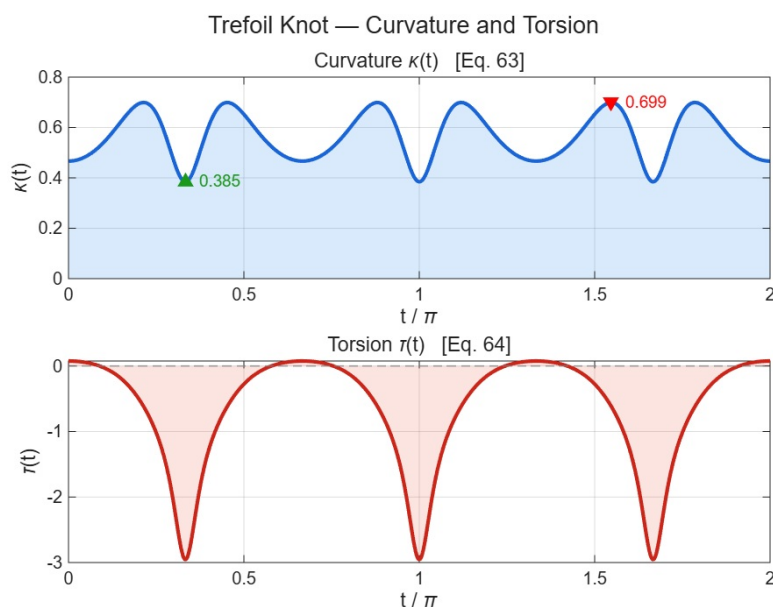
$$\frac{d}{ds} = \frac{1}{\|\gamma'(t)\|} \frac{d}{dt}. \quad (8.9)$$

With the initial phase convention $\alpha(0) = \alpha_0 = 0$, the Bishop phase is

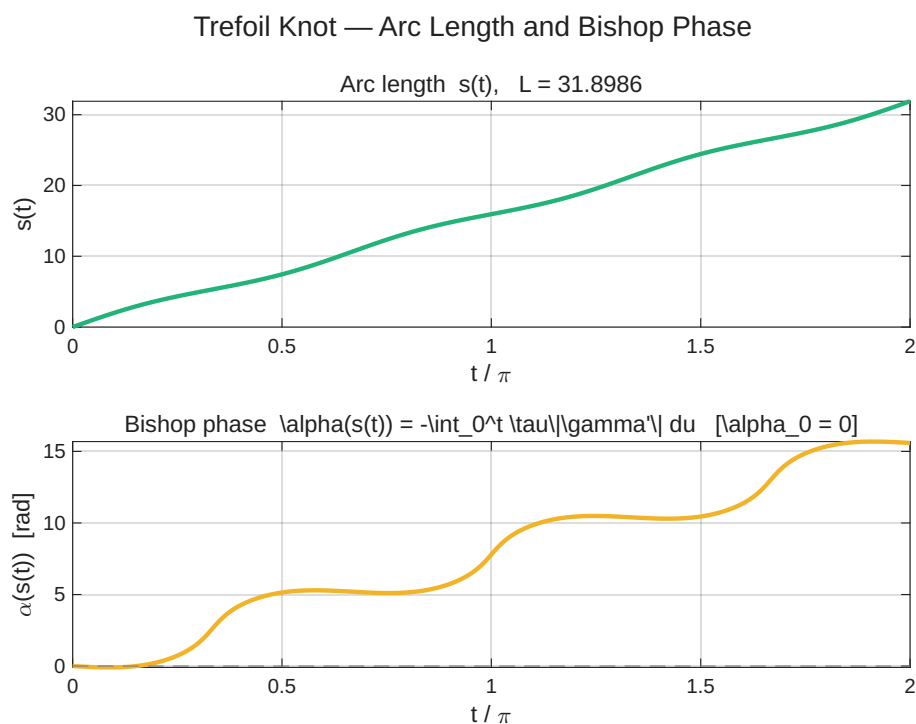
$$\alpha(s(t)) = - \int_0^{s(t)} \tau(\sigma) d\sigma = - \int_0^t \tau(u) \|\gamma'(u)\| du. \quad (8.10)$$

The transition angle is then $\theta(t) = \phi_{\lambda,\delta}(\kappa(t))$, and the exactly normalized ABF quantities are obtained from the analytical formulas established above after composition with $s(t)$.

To illustrate the geometric quantities entering the construction, Figures 4–6 display their direct computational evaluation for the parameter values $\lambda = 1$ and $\delta = 0.5$. Figure 4 shows the curvature, torsion, arc-length transformation, and accumulated Bishop phase. Figure 5 displays the Frenet frame along the knot, while Figure 6 shows the exactly normalized ABF coefficients as functions of arc length.



(a) Curvature $\kappa(t)$ and torsion $\tau(t)$ computed from the parameter-invariant formulas (8.4) and (8.5).



(b) Arc-length transformation $s(t)$ and accumulated Bishop phase $\alpha(s(t))$ with $\alpha(0) = 0$.

Figure 4. Geometric input data for the trefoil knot. The positive curvature confirms that the Frenet and ABF quantities are well-defined on the entire parameter interval. The lower panel illustrates the conversion from the original parameter t to arc length and the phase accumulation determined by torsion.

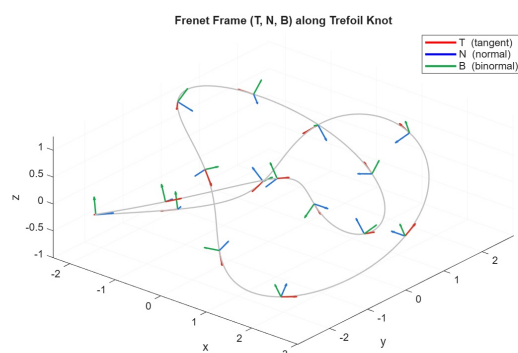


Figure 5. The Frenet frame $\{T, N_F, B_F\}$ sampled along the trefoil knot. The arrows show the tangent, principal normal, and binormal directions at representative points of the curve.

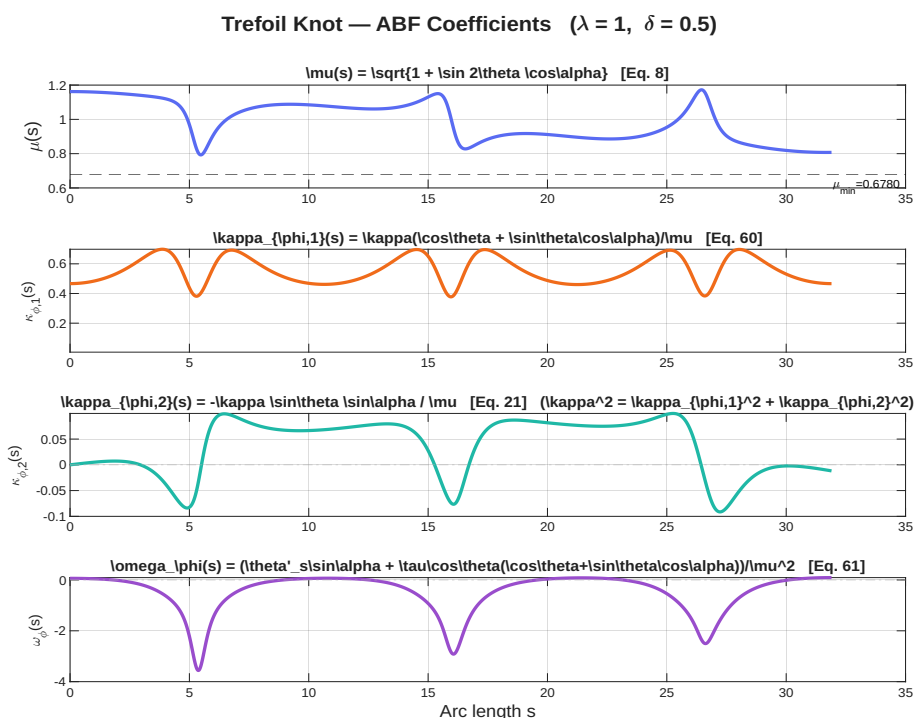


Figure 6. Direct computational evaluation of the exactly normalized ABF quantities for the Trefoil knot with $\lambda = 1$, $\delta = 0.5$, and $\alpha(0) = 0$. From top to bottom, the panels show the normalization factor $\mu(s)$, the curvature coefficients $\kappa_{\phi,1}(s)$ and $\kappa_{\phi,2}(s)$, and the normal-plane rotation coefficient $\omega_\phi(s)$. The strictly positive computed value of $\mu(s)$ is consistent with the theoretical nondegeneracy bound.

These plots are obtained by direct evaluation of the analytical formulas, with numerical quadrature used only for the parameter-to-arc-length transformation and the accumulated Bishop phase. They illustrate the dependence of the frame quantities on the geometry of the trefoil knot and are not presented as independent numerical validation, a convergence study, or evidence for a universal energy ordering. No numerical energy comparison is made.

9. Discussion

In this work, we studied a curvature-dependent adapted frame generated by the Frenet and Bishop normals on Frenet-regular intervals satisfying $\kappa > 0$. The abbreviation ABF records the two normal fields used in the definition and does not denote an endpoint-to-endpoint interpolation. Exact normalization and the nondegeneracy margin make the construction orthonormal on this domain, but they neither extend it through inflection points nor produce the Bishop frame as an endpoint.

A key point of the construction is exact normalization. The restriction $0 \leq \theta \leq \pi/4 - \delta$ yields $\mu \geq \sqrt{1 - \cos(2\delta)} > 0$, so $N_\phi = v/\mu$ and $B_\phi = T \times N_\phi$ form a genuine orthonormal adapted frame on every interval where $\kappa > 0$. This estimate controls only the normalization denominator; it does not define the numerator at a point where N_F is unavailable.

From an analytical perspective, the explicit computation of the energy density constitutes a central result. By expressing all quantities in the orthonormal basis $\{T, N_F, B_F\}$ and substituting $N_B = \cos \alpha N_F + \sin \alpha B_F$, the squared norm $\|v'\|^2$ is derived in fully expanded form. This permits case-specific energy comparisons, but no universal ordering of the full energies is claimed.

The Bishop frame minimizes the normal-plane rotational energy because its rotation coefficient vanishes. The ABF rotational contribution is $\int_I \omega_\phi^2 ds$. Since the ABF family does not attain the Bishop frame in the limit $\theta \rightarrow \pi/4$, Bishop convergence and Bishop energy convergence are not asserted.

The examples describe the exactly normalized ABF construction for a circular helix, the one-sided behavior at a planar-spatial junction, and the analytical setup for a trefoil knot. They are not presented as numerical validation.

Open problems. Several directions for future research remain:

- (1) For a fixed δ and a compact parameter interval, determination of an optimal parameter $\lambda = \lambda(\gamma, \delta)$ minimizing the energy functional $E[N_\phi]$; alternatively, joint optimization of (λ, δ) under the nondegeneracy constraint.
- (2) Extension of the ABF framework to curves constrained on surfaces, where additional geometric structures (e.g., normal curvature) come into play.
- (3) Generalization to higher-dimensional Euclidean spaces $\mathbb{R}^n$, including appropriate analogues of rotation-minimizing frames.
- (4) Development of a genuinely inflection-point-compatible construction formulated without explicit dependence on the Frenet normal at $\kappa = 0$, together with a separate construction that attains the Bishop frame as a genuine endpoint. Establishing both properties would require a definition and analysis beyond the present ABF family.

These questions are outside the scope of the present local construction. In particular, a frame that is simultaneously well-defined at inflection points and possesses a genuine Bishop endpoint remains a separate problem requiring a new definition and a separate theoretical analysis.

10. Conclusions

In this work, we address the tension between geometric interpretability and normal-plane rotational optimality in the theory of moving frames along space curves. The classical Frenet–Serret

frame provides a complete geometric characterization of a curve via curvature and torsion, but becomes undefined at inflection points and has normal-plane rotational energy $\int_I \tau^2 ds$. In contrast, for a sufficiently regular unit-speed curve and a prescribed initial normal-plane basis, the Bishop (rotation-minimizing) frame can be continued through points of vanishing curvature and minimizes the normal-plane rotational energy, yet lacks the direct principal-normal interpretation and retains an initial-phase freedom. No maximality claim is made for the Frenet rotational energy.

We study a curvature-dependent adapted frame governed by $\theta(s) = \phi(\kappa(s))$. The abbreviation ABF refers only to the Frenet and Bishop normals appearing in its defining vector. The construction uses exact normalization and $0 \leq \theta \leq \pi/4 - \delta$, and is restricted to Frenet-regular intervals satisfying $\kappa > 0$. It is a normalized Frenet–Bishop weighted combination rather than an endpoint-to-endpoint interpolation, and it is not an extension through inflection points.

We derive the three ABF adapted-frame coefficients $(\kappa_{\phi,1}, \kappa_{\phi,2}, \omega_{\phi})$ and the corresponding skew-symmetric structural system. We also record the standard reconstruction theorem for arbitrary adapted-frame coefficients, while explicitly noting that it is not an ABF realization theorem and that ABF coefficients must satisfy additional compatibility relations.

The exact-normalization calculation yields the corrected identity $\|N'_{\phi}\|^2 = \|v'\|^2/\mu^2 - (\mu'/\mu)^2$. Accordingly, no universal ordering of the full squared-derivative energies is asserted without a separately verified integral condition.

The examples illustrate the exactly normalized ABF construction on curve segments satisfying the stated Frenet-regularity assumptions.

Overall, the results provide a local structural analysis of a curvature-controlled normal-plane orientation on Frenet-regular curve segments. A genuinely inflection-compatible frame with a true Bishop endpoint would require a different definition and remains a separate problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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