



Research article

Large deviation principle for two-time scale FSDEs with fractional Brownian motion

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Abstract: This paper investigates the large deviation of two-time-scale fractional stochastic systems perturbed by fractional Brownian motion. The stochastic integrals with respect to fractional Brownian motion are understood in the sense of the generalized Riemann–Stieltjes integral, and Itô integration is employed for the standard Brownian motion component. The slow variable is governed by a Caputo fractional derivative, whereas the fast variable is described by an integer-order derivative. By combining the classical weak convergence approach with the averaging principle, a large deviation principle is established for this class of systems. Finally, the theoretical findings are illustrated through a numerical example.

Keywords: large deviation principle; two-time scale; fractional stochastic systems; fractional Brownian motion; weak convergence

1. Introduction

As a theoretical tool for studying the behavior of dynamical systems, the large deviation principle (LDP) originates from the practical demands of statistical inference. It accurately characterizes the probability of tail events and rare events through an exponential decay rate, and it explains how small perturbations can lead to significant deviations in stochastic trajectories. Low probability does not imply low impact, which is one of the central issues of the theory. The pioneering work of Freidlin and Wentzell [1] on the large deviation behavior of solution orbits of stochastic differential equations (SDEs) under small noise perturbations laid the foundation for research in this field. Since then, the LDP for SDEs has gradually entered the academic arena and attracted considerable attention. Of particular importance, Budhiraja and Dupuis established a new approach based on a weak convergence framework [2] in 2000. The core of this method lies in the use of variational representations of non-negative measurable functionals of Brownian motion, and its theoretical foundation differs fundamentally from the traditional proof method in [1]. Compared to other frameworks, this approach requires milder

prerequisites and aligns more naturally with the structure of the problem. At the same time, it significantly simplifies the argumentation process, eliminating the need for tedious discretization approximations and delicate exponential probability estimates. The key steps of the proof are reduced to verifying moment estimates and their convergence. This progress has effectively promoted the development of large deviation theory and its wide application in the field of stochastic dynamical systems.

The continuous development of fractional calculus theory has made it increasingly important in the modeling of dynamical systems. Compared with integer-order operators, the inherent nonlocal property of fractional-order differential operators makes them particularly suitable for characterizing complex dynamic features such as memory effects. They are widely applied in biological populations [3], fluid dynamics [4], and other fields. The theory of fractional stochastic differential equations (FSDEs) has also been widely studied. For example, the article [5] discussed the stochastic fractional nonlinear Schrödinger equation. Recently, Huang and Luo [6] established a strong averaging principle for generalized Caputo fractional stochastic neutral differential equations driven by multiplicative fractional Brownian motion. The application of the LDP to FSDEs has been a cutting-edge topic of considerable interest in recent years. In sharp contrast to the flourishing development of integer-order SDEs, relatively little work has been done in the field of FSDEs. In 2017, Suvinthra et al. [7] established the LDP for fractional stochastic integro-differential equations using the weak convergence method. In 2022, Siva Ranjani et al. [8] proved the uniqueness of solutions for stochastic neutral fractional functional differential equations via the Picard iteration method and studied the LDP for these equations. In 2023, Alagesan Siva Ranjani et al. [9] demonstrated the LDP for a class of pantograph FSDEs. In 2025, Chen et al. [10] obtained the explicit solution of a class of Hilfer fractional McKean-Vlasov SDEs through Laplace transform, proved the LDP using the weak convergence method, and analyzed the behavior characteristics of the system under small noise perturbations through practical examples. It is worth noting that the objects studied in the above literature are limited to single-time-scale systems; scenarios involving multiple time scales have not yet been addressed.

As the theoretical framework has been progressively refined and broadened, research on SDEs has transitioned from focusing on single-time-scale problems to addressing multi-time-scale dynamics. In fact, the evolution of natural systems is rarely governed by a single characteristic rate. Two-time-scale dynamical systems exhibiting slow-fast coupling have found extensive applications across diverse domains, including climate-weather modeling [11], financial asset pricing [12], laser oscillation [13], ensemble extension problems [14], and neural activity [15]. A significant limitation of the existing LDP theory for FSDEs is that the driving noise is restricted to Brownian motion; the situation driven by fractional Brownian motion (FBM) remains almost entirely unexplored. The Hurst parameter H of FBM characterizes the regularity of sample paths: The smaller the H , the higher the roughness of the path. Ganguly et al. [16] studied the stochastic thermodynamics of active Brownian particles, derived fluctuation theorems for energy exchange, entropy generation, and so forth, and revealed the thermodynamic behavior of nonequilibrium driven active particle systems. Liang et al. [17] studied the restricted fractional Brownian motion of hyperdiffusion, showing that mean square displacement and ergodicity are broken (nonergodic), and revealing the influence of boundary constraints on the statistical properties of hyperdiffusion. When $H = 1/2$, FBM degenerates into standard Brownian motion [18]. Due to the long-range dependence and self-similarity of FBM, and its lack of Markovian and semimartingale properties, directly applying the weak convergence method based on Brownian motion faces fundamental obstacles. However, the work of Budhiraja and Song [19] effectively overcame

this theoretical bottleneck by extending the method to FBM framework. Subsequently, in 2021, Bourguin et al. [20] investigated the typical dynamics and fluctuations of two-time-scale SDEs driven by FBM. In 2022, Gailus [21] established the LDP for two-time-scale SDEs with FBM using homogenization theory and viable pairs. In 2023, Inahama et al. [22] studied the LDP for two-time-scale SDEs with mixed FBM. It is worth noting that in [21], the diffusion term corresponding to the fast variable is independent of the slow variable, whereas in [22], the fast variable depends completely on the slow variable. In the context of averaging principles for slow–fast systems driven by FBM, Pei et al. [23] recently established an averaging principle for such systems, which provides a necessary foundation for the subsequent large-deviation analysis. In 2024, Shen et al. [24] extended the LDP to multiscale distribution-dependent SDEs (DDSDEs) driven by FBM. Within the controlled rough path setting, Yang and Xu [25] proved the LDP for two-time-scale rough differential equations (RDEs) in 2024.

Inspired by the foregoing literature, the present work is devoted to establishing the LDP for the following two-time scale FSDEs with FBM:

$$\begin{cases} {}^C D_{0+}^{\beta} X_{\tau}^{(\psi, \phi)} = f(X_{\tau}^{(\psi, \phi)}, Y_{\tau}^{(\psi, \phi)}) d\tau + \psi^{\beta-\frac{1}{2}} \sigma(X_{\tau}^{(\psi, \phi)}) dW_{\tau}^H, \\ \phi dY_{\tau}^{(\psi, \phi)} = b(X_{\tau}^{(\psi, \phi)}, Y_{\tau}^{(\psi, \phi)}) d\tau + \sqrt{\phi} g(X_{\tau}^{(\psi, \phi)}, Y_{\tau}^{(\psi, \phi)}) dW_{\tau}, \end{cases} \quad (1.1)$$

where $(X_0^{(\psi, \phi)}, Y_0^{(\psi, \phi)}) = (X_0, Y_0) \in \mathfrak{R}^m \times \mathfrak{R}^n$. ${}^C D_{0+}^{\beta}$ is a Caputo derivative and $\beta \in (1/2, 1)$. Nonlinear coefficients f , σ , b and g are subject to certain structural assumptions, to be specified in the next chapter. W_{τ}^H is k_1 -dimensional FBM and $H \in (1/2, 1)$. W_{τ} is k_2 -dimensional BM, $\tau \in [0, T]$ and $T > 0$. To characterize the time-scale separation between $X^{(\psi, \phi)}$ and $Y^{(\psi, \phi)}$, the parameters ϕ, ψ satisfy $0 < \phi \ll \psi \ll 1$. Here, $X^{(\psi, \phi)} \in C^{\alpha}([0, T], \mathfrak{R}^m)$ denotes the slow process, and $Y^{(\psi, \phi)} \in C([0, T], \mathfrak{R}^n)$ denotes the fast process. $C([0, T], \mathfrak{R}^n)$ is the continuous function space. $C^{\alpha}([0, T], \mathfrak{R}^m)$ is the α -Hölder continuous function space and $1 - H < \alpha < \min\{1/2, \beta/2\}$.

This work is mainly reflected in the first exploration of the LDP for two-time scale FSDEs and optimized some of the results in existing literature. The novelty of this work lies in the first combination of three features that have not been simultaneously addressed in the existing literature: Caputo fractional derivative in the slow dynamics, two-time-scale slow–fast structure, and mixed FBM noise. The following highlights the core contribution of this article.

- Our results consider FSDEs with slow–fast reaction mechanisms. The study of the LDP for FSDEs has been extended to more complex systems.
- This article extends the research object of the LDP for FSDEs to FBM. This noise has a more general structure and a more complex form. The driving noise extends from a single type to mixed noise situations.
- This article constructs a specific example to verify the effectiveness of the theoretical framework, and further elaborates on the core values of the theory from the perspectives of robustness and practicality.
- Previous studies on bias theory have generally deviated from the consideration of the form of FSDEs, which in turn affects the accuracy of convergence rate estimation. This article draws on the equation form derived in [26] and considers the impact of fractional order β .

The structure and layout of the remaining chapters of this paper are described as follows. In Section 2, we present several function spaces pertinent to FBM, the method of weak convergence, and present the assumptions. In Section 3, we establish the well-posedness of solution to Eq (1.1) and the associated

deterministic skeleton equation. Section 4 completes the proof of the LDP. In Section 5, a concrete example is provided to illustrate the validity of the theoretical result.

2. Preliminaries

2.1. Function space framework

In this part, we first recall some fundamental function spaces that will be used throughout (see [27] for details of definitions and properties). Let $d, l \in \mathbb{N}$, and let $W_0^{\alpha, \infty} := W_0^{\alpha, \infty}([0, T], \mathfrak{R}^d)$ denote the class of all measurable functions, such that

$$\|u\|_{\alpha, \infty} := \sup_{\tau \in [0, T]} \|u\|_{\alpha, [0, \tau]} < \infty, \quad u : [0, T] \rightarrow \mathfrak{R}^d,$$

where

$$\|u\|_{\alpha, [0, \tau]} = |u(\tau)| + \int_0^\tau \frac{|u(\tau) - u(s)|}{(\tau - s)^{\alpha+1}} ds.$$

Let $W_0^{\alpha, 1} := W_0^{\alpha, 1}([0, T], \mathfrak{R}^d)$ denote the measurable functions space such that

$$\|u\|_{\alpha, 1} := \int_0^T \frac{|u(s_1)|}{s_1^\alpha} ds_1 + \int_0^T \int_0^{s_1} \frac{|u(s_1) - u(s_2)|}{(s_1 - s_2)^{\alpha+1}} ds_2 ds_1 < \infty, \quad u : [0, T] \rightarrow \mathfrak{R}^d.$$

For every $\eta \in (0, 1]$, $C^\eta := C^\eta([0, T], \mathfrak{R}^d)$ denotes the η -Hölder continuous functions space, satisfying

$$\|u\|_{\eta\text{-Hölder}} := \|u\|_\infty + \sup_{0 \leq s < \tau \leq T} \frac{|u(\tau) - u(s)|}{(\tau - s)^\eta} < \infty, \quad u : [0, T] \rightarrow \mathfrak{R}^d,$$

where

$$\|u\|_\infty = \sup_{\tau \in [0, T]} |u(\tau)|.$$

Let $W_T^{1-\alpha, \infty} := W_T^{1-\alpha, \infty}([0, T], \mathfrak{R}^d)$ denote the measurable functions space with $h : [0, T] \rightarrow \mathfrak{R}^d$, satisfying

$$\|h\|_{1-\alpha, \infty, T} := \sup_{0 \leq s_1 < \tau \leq T} \left(\frac{|h(\tau) - h(s_1)|}{(\tau - s_1)^{1-\alpha}} + \int_{s_1}^\tau \frac{|h(s_2) - h(s_1)|}{(s_2 - s_1)^{2-\alpha}} ds_2 \right) < \infty.$$

For every $\theta \in (0, \alpha)$, $C^{\alpha+\theta} \subset W_0^{\alpha, \infty} \subset C^{\alpha-\theta}$ and $C^{1-\alpha+\theta} \subset W_T^{1-\alpha, \infty} \subset C^{1-\alpha-\theta}$.

2.2. Cameron–Martin space

In this part, we recall some important definitions and conclusions about the Cameron–Martin (C–M) space for the mixed FBM. We define

$$K_{\mathcal{H}} u(\tau) = \int_0^T Z_H(\tau, r) u(r) dr, \quad u \in L^2([0, T], \mathfrak{R}^{k_1}).$$

For all $0 \leq r \leq \tau \leq T$, we set $Z_H(\tau, r) = z_H(\tau, r)1_{[0, \tau]}(r)$ with

$$z_H(\tau, r) := \frac{C_H}{\Gamma\left(H + \frac{1}{2}\right)} (\tau - r)^{H-\frac{1}{2}} G\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}; 1 - \frac{\tau}{r}\right),$$

where $C_H = \left[\frac{2H\Gamma(3/2-H)\Gamma(H+1/2)}{\Gamma(2-2H)} \right]^{1/2}$, Γ is the gamma function. G is the Gauss hypergeometric function. The operator $K_H : L^2([0, T], \mathfrak{R}^{k_1}) \rightarrow \mathcal{H}_{cm}^{H,k_1}$ constitutes a unitary isometry (see [28] for details of definitions and properties). The associated C-M space of FBM is given by

$$\mathcal{H}_{cm}^{H,k_1} = \{K_H \dot{q} : \dot{q} \in L^2([0, T], \mathfrak{R}^{k_1})\}.$$

For every $H' \in (0, H)$, the inclusion $\mathcal{H}_{cm}^{H,k_1} \subset C^{H'}([0, T], \mathfrak{R}^{k_1})$ holds. The inner product of $\mathcal{H}_{cm}^{H,k_1}$ is defined by

$$\langle q, h \rangle_{\mathcal{H}_{cm}^{H,k_1}} = \langle K_H \dot{q}, K_H \dot{h} \rangle_{\mathcal{H}_{cm}^{H,k_1}} := \langle \dot{q}, \dot{h} \rangle_{L^2}.$$

The C-M space associated with the BM is formulated as

$$\mathcal{H}_{cm}^{\frac{1}{2},k_2} := \{\varphi \in C([0, T], \mathfrak{R}^{k_2}) : \varphi_\tau = \int_0^\tau \varphi'_s ds, \quad \varphi' \in L^2([0, T], \mathfrak{R}^{k_2})\}.$$

The inner product of $\mathcal{H}_{cm}^{H,k_2}$ is defined by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_{cm}^{\frac{1}{2},k_2}}^2 := \int_0^T \langle \varphi'_{1,\tau}, \varphi'_{2,\tau} \rangle_{\mathfrak{R}^{k_2}}^2 d\tau < \infty.$$

We term the pair (W_τ^H, W_τ) the mixed FBM. Subsequently, the Cameron–Martin Hilbert subspace of the mixed FBM is defined by $\mathcal{H}_{cm} := \mathcal{H}_{cm}^{H,k_1} \oplus \mathcal{H}_{cm}^{\frac{1}{2},k_2}$. We set

$$S_N = \left\{ (\varpi, \varphi) \in \mathcal{H}_{cm} : \frac{1}{2} \|(\varpi, \varphi)\|_{\mathcal{H}_{cm}}^2 := \frac{1}{2} (\|\varpi\|_{\mathcal{H}_{cm}^{H,k_1}}^2 + \|\varphi\|_{\mathcal{H}_{cm}^{\frac{1}{2},k_2}}^2) \leq N \right\},$$

where $0 < N < \infty$. The space S_N constitutes a compact Polish space when equipped with the weak topology.

On the k -dimensional classical Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$, we denote $\mathcal{F}_\tau$ as the natural augmented filtration, $\mathcal{F}_\tau := \sigma\{(W_r^H, W_r) : 0 \leq r \leq \tau\} \vee \mathcal{N}$, where $\mathcal{N}$ stands for the collection of all $\mathbf{P}$ -negligible events. $\mathcal{F}_\tau$ fulfills the usual conditions. By the classical result, in the case where W_τ^H is given by (2.2) in [29], it is easy to see that $\mathcal{F}_\tau = \sigma\{(W_r^H, W_r) : 0 \leq r \leq \tau\} \vee \mathcal{N}$ for all τ .

For each $N \in (0, \infty)$, let $\mathcal{A}_b^N$ designate the collection of $\mathfrak{R}^k$ -valued processes $(\varpi_\tau, \varphi_\tau)$ that are $\mathcal{F}_\tau$ -progressively measurable on $(\Omega, \mathcal{F}, \mathbf{P})$ and fulfill the constraint

$$\int_0^T (|\varpi_\tau|_{\mathfrak{R}^{k_1}}^2 + |\varphi_\tau|_{\mathfrak{R}^{k_2}}^2) d\tau \leq 2N, \quad \mathbf{P}\text{-a.s.}$$

For arbitrary $(\varpi, \varphi) \in \mathcal{A}_b$ with $\mathcal{A}_b := \cup_{0 < N < \infty} \mathcal{A}_b^N$, the distribution of $(W^H + K_H \varpi, W + \int_0^\cdot \varphi_s ds)$ is mutually absolutely continuous to the distribution of (W^H, W) under $\mathbf{P}$. This is a direct consequence of Girsanov's theorem and the Novikov condition.

Analogously, $\bar{\mathcal{A}}_b^N$ is defined as the class of $\mathfrak{R}^k$ -valued $\mathcal{F}_\tau$ -progressively measurable processes $(\varpi_\tau, \varphi_\tau)$ on $(\Omega, \mathcal{F}, \mathbf{P})$ admitting the representation

$$(\varpi, \varphi) = \left(K_H \dot{\varpi}, \int_0^\cdot \varphi'_r dr \right), \quad \mathbf{P}\text{-a.s.} \quad (\dot{\varpi}, \varphi') \in \mathcal{A}_b^N.$$

Let $\bar{\mathcal{A}}_b := \cup_{0 < N < \infty} \bar{\mathcal{A}}_b^N$. Then, every $(\varpi, \varphi) \in \bar{\mathcal{A}}_b^N$ constitutes an S_N -valued random variable. In view of the tightness of S_N , the family of induced measures $\{\mathbf{P} \circ (\varpi, \varphi)^{-1} : (\varpi, \varphi) \in \bar{\mathcal{A}}_b^N\}$ is automatically compact. Evidently, for every $(\varpi, \varphi) \in \bar{\mathcal{A}}_b$, the law of $(W^H + \varpi, W + \varphi)$ is mutually absolutely continuous to that of (W^H, W) .

2.3. Weak convergence method

This section reviews the method framework developed by the literature [2], which provides sufficient criteria for the random variable $X^{(\psi, \phi)}$ to satisfy Laplace principle (equivalent to the LDP).

Definition 2.1 (Rate function [30]). Let $\mathcal{E} = C^\alpha([0, T], \mathfrak{X}^m)$, $I: \mathcal{E} \rightarrow [0, \infty]$ be called a rate function if I is lower semicontinuous. I is called a good rate function if for each $M < \infty$, the level set $\{Z \in \mathcal{E} : I(Z) \leq M\}$ is compact in $\mathcal{E}$.

Definition 2.2 (LDP [22]). Let $I: \mathcal{E} \rightarrow [0, \infty]$ be a rate function. The family of $\mathcal{E}$ -valued random elements is said to satisfy the LDP with rate function I provided that the following two bounds hold simultaneously:

(i) Upper bound. For each closed set $\mathbb{F} \subset \mathcal{E}$,

$$\limsup_{\psi \rightarrow 0} \psi \log \mathbf{P}(X^{(\psi, \phi)} \in \mathbb{F}) \leq -\inf_{x \in \mathbb{F}} I(x).$$

(ii) Lower bound. For each open set $\mathbb{G} \subset \mathcal{E}$,

$$\liminf_{\psi \rightarrow 0} \psi \log \mathbf{P}(X^{(\psi, \phi)} \in \mathbb{G}) \geq -\inf_{x \in \mathbb{G}} I(x).$$

Here, $\mathcal{E} = C^\alpha([0, T], \mathfrak{X}^m)$.

Definition 2.3 (Laplace principle [30]). The family of random variables $X^{(\psi, \phi)}$ is said to satisfy the Laplace principle (LP) if the following is satisfied for arbitrary bounded continuous functions F on $\mathcal{E}$:

(i) LP upper bound.

$$\limsup_{\psi \rightarrow 0} -\psi \log \mathbb{E} \left\{ \exp \left(\frac{-F(X^{(\psi, \phi)})}{\psi} \right) \right\} \leq \inf_{x \in \mathcal{E}} \{F(x) + I(x)\}.$$

(ii) LP lower bound.

$$\liminf_{\psi \rightarrow 0} -\psi \log \mathbb{E} \left\{ \exp \left(\frac{-F(X^{(\psi, \phi)})}{\psi} \right) \right\} \geq \inf_{x \in \mathcal{E}} \{F(x) + I(x)\}.$$

The weak convergence framework relies fundamentally on the LDP being equivalent to LP. To confirm the latter's applicability, one must demonstrate as follows.

Proposition 2.1. Let $0 < \phi < \psi$ with $\phi = o(\psi)$. Suppose there exists a measurable mapping $\mathcal{G}^{(\psi, \phi)} : C_0([0, T], \mathfrak{X}^k) \rightarrow \mathcal{E}$ for which the subsequent conditions are satisfied:

(i) For $j \rightarrow \infty$, suppose $(\varpi^{(j)}, \varphi^{(j)})$ and $(\varpi, \varphi) \in S_N$ such that $(\varpi^{(j)}, \varphi^{(j)}) \rightarrow (\varpi, \varphi)$ in the weak topology of $\mathcal{H}$. Then, one obtains the convergence:

$$\mathcal{G}^0(\varpi^{(j)}, \varphi^{(j)}) \rightarrow \mathcal{G}^0(\varpi, \varphi)$$

in $\mathcal{E}$ as $j \rightarrow \infty$.

(ii) For $\psi \rightarrow \infty$, suppose $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \in \bar{\mathcal{A}}_b^N$, such that $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \rightarrow (\varpi, \varphi)$. Then, one obtains the convergence:

$$\mathcal{G}^{(\psi, \phi)}(\psi^{\beta-\frac{1}{2}} W^H + \varpi^{(\psi, \phi)}, \psi^{\beta-\frac{1}{2}} W + \varphi^{(\psi, \phi)}) \xrightarrow{\text{weakly}} \mathcal{G}^0(\varpi, \varphi)$$

in $\mathcal{E}$ as $\psi \rightarrow 0$.

We say that the process $X^{(\psi, \phi)}$ satisfies the LP on $\mathcal{E}$, governed by the rate function $I: \mathcal{E} \rightarrow [0, \infty]$.

2.4. Hypothesis

To establish the main results, the following hypotheses are imposed.

(S1). $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times k_1}$ belong to C^1 . There is a positive $C > 0$ such that for all $\wp_1, \wp_2 \in \mathbb{R}^m$,

$$|\nabla \sigma(\wp_1)| \leq C, \quad |\nabla \sigma(\wp_1) - \nabla \sigma(\wp_2)| \leq C |\wp_1 - \wp_2|,$$

where ∇ refers to standard gradient on $\mathbb{R}^m$.

(S2). $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$. There is a positive $C > 0$ such that for all $(\wp_1, \varsigma_1), (\wp_2, \varsigma_2) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$|f(\wp_1, \varsigma_1) - f(\wp_2, \varsigma_2)| + |b(\wp_1, \varsigma_1) - b(\wp_2, \varsigma_2)| \\ + |g(\wp_1, \varsigma_1) - g(\wp_2, \varsigma_2)| \leq C(|\wp_1 - \wp_2| + |\varsigma_1 - \varsigma_2|),$$

and

$$\sup_{\varsigma_1 \in \mathbb{R}^n} (|f(\wp_1, \varsigma_1)| + |b(\wp_1, \varsigma_1)|) \leq C(1 + |\wp_1|).$$

(S3). $b : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k_2}$. There exist constants $C > 0$ and κ_i ($i = 1, 2$) > 0 such that for all $(\wp, \varsigma_1), (\wp, \varsigma_2) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$2 \langle \varsigma_1 - \varsigma_2, b(\wp, \varsigma_1) - b(\wp, \varsigma_2) \rangle + |g(\wp, \varsigma_1) - g(\wp, \varsigma_2)|^2 \leq -\kappa_1 |\varsigma_1 - \varsigma_2|^2, \\ 2 \langle \varsigma_1, b(\wp, \varsigma_1) \rangle + |g(\wp, \varsigma_1)|^2 \leq -\kappa_2 |\varsigma_1|^2 + C|\wp|^2 + C.$$

Hypothesis (S2) implies that there exists a positive C' such that for all $(\wp, \varsigma) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$|b(\wp, \varsigma)| \leq C'(1 + |\wp| + |\varsigma|).$$

3. Existence and uniqueness

Equation (1.1)'s well-posedness follows from the literature [31] and [32]. Then, we have the following conclusion.

Lemma 3.1. Assume f , σ , b , and g satisfy (S1), (S2). For every $\tau \geq 0$, Equation (1.1) admits a unique pathwise solution, denoted by $(X^{\psi, \phi}, Y^{\psi, \phi})$; that is,

$$\begin{cases} X_\tau^{(\psi, \phi)} = X_0 + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{f(X_s^{(\psi, \phi)}, Y_s^{(\psi, \phi)}) ds}{(\tau-s)^{1-\beta}} + \frac{\psi^{\beta-\frac{1}{2}}}{\Gamma(\beta)} \int_0^\tau \frac{\sigma(X_s^{(\psi, \phi)}) dW_s^H}{(\tau-s)^{1-\beta}}, \\ Y_\tau^{(\psi, \phi)} = Y_0 + \frac{1}{\phi} \int_0^\tau b(X_s^{(\psi, \phi)}, Y_s^{(\psi, \phi)}) ds + \frac{1}{\sqrt{\phi}} \int_0^\tau g(X_s^{(\psi, \phi)}, Y_s^{(\psi, \phi)}) dW_s. \end{cases}$$

Then, we introduce the frozen equation. When the slow variable $\wp \in \mathbb{R}^m$ is fixed, the corresponding equation for the fast variable is given by

$$d\tilde{Y}_\tau = b(\wp, \tilde{Y}_\tau) d\tau + g(\wp, \tilde{Y}_\tau) dW_\tau. \quad (3.1)$$

According to [33, Theorem 6.3.2], Eq (3.1) possesses a unique invariant probability measure $\mu_\wp$ for each fixed $\wp$. Indeed, the first inequality of (S3) with $\kappa_1 > 0$ yields the exponential contraction

$$\mathbb{E} |\tilde{Y}_\tau^{\varsigma_1} - \tilde{Y}_\tau^{\varsigma_2}|^2 \leq e^{-\kappa_1 \tau} |\varsigma_1 - \varsigma_2|^2,$$

which verifies Condition (6.1.2) in [33, Theorem 6.3.2] and guarantees uniqueness. The second inequality with $\kappa_2 > 0$ provides the Lyapunov bound

$$\limsup_{\tau \rightarrow \infty} \mathbb{E} |\tilde{Y}_\tau^s|^2 \leq \frac{C(|\varphi|^2 + 1)}{\kappa_2},$$

verifying Condition (6.1.3) and ensuring existence. We now proceed to introduce the deterministic skeleton equation:

$${}^c D_{0+}^\beta \tilde{X}_\tau = \bar{f}(\tilde{X}_\tau) d\tau + \sigma(\tilde{X}_\tau) d\varpi_\tau, \quad \tilde{X}_0 = X_0, \quad (3.2)$$

where $\bar{f}(\varphi) = \int_{\mathfrak{R}^n} f(x, y) \mu_\varphi(dy)$.

Lemma 3.2. For the deterministic skeleton equation, the coefficients $\bar{f}$ and σ are subject to Lipschitz continuity and linear growth assumptions. Given arbitrary $\varpi \in S_N$, there exists a unique solution $\tilde{X} \in W_0^{\alpha, \infty}([0, T], \mathfrak{R}^m)$ to the deterministic skeleton equation.

Proof. According to (S1), the Jensen inequality, and [27, Lemma 0.9], for any $\varphi \in \mathfrak{R}^m$ and any $s_1, s_2 \in \mathfrak{R}^n$, we can get

$$\begin{aligned} |\mathbb{E}[f(\varphi, \tilde{Y}_\tau^{s_1})] - \bar{f}(\varphi)|^2 &= \left| \int_{R^m} \mathbb{E}[f(\varphi, \tilde{Y}_\tau^{s_1}) - f(\varphi, \tilde{Y}_\tau^{s_2})] \varpi_\varphi(ds_2) \right|^2 \\ &\leq \mathcal{K} \int_{R^m} \mathbb{E} |\tilde{Y}_\tau^{s_1} - \tilde{Y}_\tau^{s_2}|^2 \varpi_\varphi(ds_2) \\ &\leq \mathcal{K} \int_{R^m} e^{-\kappa_1 \tau} |s_1 - s_2| \varpi_\varphi(ds_2). \end{aligned}$$

Using the formula (0.30) in [27, Appendix], we obtain

$$\begin{aligned} |\mathbb{E}[f(\varphi, \tilde{Y}_\tau^{s_1})] - \bar{f}(\varphi)|^2 &\leq \mathcal{K} e^{-\kappa_1 \tau} \left[|s_1|^2 + \int_{R^d} |s_2|^2 \kappa^\varphi(ds_2) \right] \\ &\leq \mathcal{K} e^{-\kappa_1 \tau} (1 + |\varphi|^2 + |s_1|^2). \end{aligned} \quad (3.3)$$

For any $\varphi_1, \varphi_2, \varphi \in \mathfrak{R}^m$ and any initial value $\varsigma \in \mathfrak{R}^n$, according to (3.3), we have

$$\begin{aligned} |\bar{f}(\varphi_1) - \bar{f}(\varphi_2)|^2 &= |\bar{f}(\varphi_1) - \mathbb{E}[f(\varphi_1, \tilde{Y}_s^{\varphi_1, \varsigma})] + \mathbb{E}[f(\varphi_1, \tilde{Y}_s^{\varphi_1, \varsigma})] \\ &\quad - \bar{f}(\varphi_2) - \mathbb{E}[f(\varphi_2, \tilde{Y}_s^{\varphi_2, \varsigma})] + \mathbb{E}[f(\varphi_2, \tilde{Y}_s^{\varphi_2, \varsigma})]|^2 \\ &\leq 3|\bar{f}(\varphi_1) - \mathbb{E}[f(\varphi_1, \tilde{Y}_s^{\varphi_1, \varsigma})]|^2 + 3|\bar{f}(\varphi_2) - \mathbb{E}[f(\varphi_2, \tilde{Y}_s^{\varphi_2, \varsigma})]|^2 \\ &\quad + 3|\mathbb{E}[f(\varphi_1, \tilde{Y}_s^{\varphi_1, \varsigma})] - \mathbb{E}[f(\varphi_2, \tilde{Y}_s^{\varphi_2, \varsigma})]|^2 \\ &\leq \mathcal{K} e^{-\kappa_1 \tau} (1 + |\varphi_1|^2 + |\varphi_2|^2 + |\varsigma|^2) \\ &\quad + \mathcal{K} |\mathbb{E}[f(\varphi_1, \tilde{Y}_s^{\varphi_1, \varsigma})] - \mathbb{E}[f(\varphi_2, \tilde{Y}_s^{\varphi_2, \varsigma})]|^2. \end{aligned}$$

By the Lipschitz continuity of f in both variables from (S2), we have

$$\begin{aligned} |\mathbb{E}[f(\varphi_1, \tilde{Y}_\tau^{\varphi_1, \varsigma})] - \mathbb{E}[f(\varphi_2, \tilde{Y}_\tau^{\varphi_2, \varsigma})]|^2 &\leq \mathbb{E} |f(\varphi_1, \tilde{Y}_\tau^{\varphi_1, \varsigma}) - f(\varphi_2, \tilde{Y}_\tau^{\varphi_2, \varsigma})|^2 \\ &\leq \mathcal{K} \mathbb{E} [|\varphi_1 - \varphi_2|^2 + |\tilde{Y}_\tau^{\varphi_1, \varsigma} - \tilde{Y}_\tau^{\varphi_2, \varsigma}|^2] \\ &\leq \mathcal{K} |\varphi_1 - \varphi_2|^2 + \mathcal{K} \mathbb{E} [|\tilde{Y}_\tau^{\varphi_1, \varsigma} - \tilde{Y}_\tau^{\varphi_2, \varsigma}|^2]. \end{aligned} \quad (3.4)$$

For the last expectation, by [27, lemma 0.11], we have

$$\mathbb{E}|\tilde{Y}_\tau^{\wp_1, \varsigma} - \tilde{Y}_\tau^{\wp_2, \varsigma}|^2 \leq e^{-\kappa_1 \tau} |\varsigma - \varsigma|^2 + \mathcal{K}|\wp_1 - \wp_2|^2 = \mathcal{K}|\wp_1 - \wp_2|^2. \quad (3.5)$$

Substituting (3.5) into (3.4), we obtain

$$|\mathbb{E}[f(\wp_1, \tilde{Y}_\tau^{\wp_1, \varsigma})] - \mathbb{E}[f(\wp_2, \tilde{Y}_\tau^{\wp_2, \varsigma})]|^2 \leq (\mathcal{K} + \mathcal{K}^2)|\wp_1 - \wp_2|^2. \quad (3.6)$$

Substituting the above estimates into $|\tilde{f}(\wp_1) - \tilde{f}(\wp_2)|^2$, we get

$$|\tilde{f}(\wp_1) - \tilde{f}(\wp_2)|^2 \leq \mathcal{K}e^{-\kappa_1 \tau} (1 + |\wp_1|^2 + |\wp_2|^2 + |\varsigma|^2) + (\mathcal{K} + \mathcal{K}^2)|\wp_1 - \wp_2|. \quad (3.7)$$

Now, fix ς in (3.7), and let $\tau \rightarrow \infty$. Because $\kappa_1 > 0$, the exponential term vanishes, and we obtain

$$|\tilde{f}(\wp_1) - \tilde{f}(\wp_2)|^2 \leq (\mathcal{K} + \mathcal{K}^2)|\wp_1 - \wp_2|^2. \quad (3.8)$$

Thus, $\tilde{f}$ is Lipschitz continuous as $\tau \rightarrow \infty$:

$$|\tilde{f}(\wp)| \leq \int_{R^m} |f(\wp, \varsigma)| \varpi_\wp(d\varsigma) \leq \int_{R^m} (1 + |\wp| + |\varsigma|) \varpi_\wp(d\varsigma) \leq \mathcal{K}(1 + |\wp|).$$

Therefore, $\tilde{f}$ satisfy the linear growth condition. Then, the uniqueness of the solution to the deterministic equation (3.2) is ensured by [34, Theorem 4.1]. $\square$

4. Main proof

We will show that $X^{(\psi, \phi)}$ satisfies the LDP in this part. First of all, the derivation of controlled system estimates essential to the main proof is presented below. We continue to let $0 < \phi < \psi \leq 1$, $H \in (1/2, 1)$, and $1 - H < \alpha < \min\{1/2, \beta/2\}$.

4.1. Preliminary lemmas

We consider the controlled version of (1.1) for a pair of control $(X^{(\psi, \phi)}, Y^{(\psi, \phi)})$ as follows:

$$\begin{cases} {}^c D_{0+}^\beta \tilde{X}_\tau^{(\psi, \phi)} &= f(\tilde{X}_\tau^{(\psi, \phi)}, \tilde{Y}_\tau^{(\psi, \phi)}) dt + \sigma(\tilde{X}_\tau^{(\psi, \phi)}) d\varpi_\tau^{(\psi, \phi)} + \psi^{\beta-\frac{1}{2}} \sigma(\tilde{X}_\tau^{(\psi, \phi)}) dW_\tau^H, \\ \phi d\tilde{Y}_\tau^{(\psi, \phi)} &= b(\tilde{X}_\tau^{(\psi, \phi)}, \tilde{Y}_\tau^{(\psi, \phi)}) d\tau + \frac{\sqrt{\phi}}{\psi^{\beta-\frac{1}{2}}} g(\tilde{X}_\tau^{(\psi, \phi)}, \tilde{Y}_\tau^{(\psi, \phi)}) d\varphi_\tau^{(\psi, \phi)} + \sqrt{\phi} g(\tilde{X}_\tau^{(\psi, \phi)}, \tilde{Y}_\tau^{(\psi, \phi)}) dW_\tau, \end{cases} \quad (4.1)$$

with initial value $(\tilde{X}_0^{(\psi, \phi)}, \tilde{Y}_0^{(\psi, \phi)}) = (X_0, Y_0) \in \mathfrak{R}^m \times \mathfrak{R}^n$. According to Girsanov's theorem, it is easy to verify that there exists a solution $(\tilde{X}^{(\psi, \phi)}, \tilde{Y}^{(\psi, \phi)}) = \mathcal{G}^{(\psi, \phi)}(\psi^{\beta-\frac{1}{2}} W^H + \varpi^{(\psi, \phi)}, \psi^{\beta-\frac{1}{2}} W + \varphi^{(\psi, \phi)})$ of (4.1).

To establish the uniform regularity of the family of $\tilde{X}^{(\psi, \phi)}$ in the Hölder path space, thereby providing the basic moment control and norm framework for the subsequent weak convergence analysis, we first derive the following uniform estimate for the $\|\cdot\|_{\alpha, \infty}$ -norm of $\tilde{X}^{(\psi, \phi)}$.

Lemma 4.1. Suppose that (S1)–(S3) are fulfilled, and let $p \geq 1$. Then, there exists a positive constant $\mathcal{K}$, depending only on p and N , such that for all $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \in \tilde{\mathcal{A}}_b^N$, the following holds:

$$\mathbb{E}[\|\tilde{X}^{(\psi, \phi)}\|_{\alpha, \infty}^p] \leq \mathcal{K}.$$

Proof. Throughout this proof process, $\mathcal{K} > 0$ denotes a constant depending solely on p and N , possibly differing across lines. To begin, we denote $\Lambda := \Lambda_\alpha(W^H) \vee 1$, letting

$$\|f\|_{\lambda,\tau} := \sup_{0 \leq s \leq \tau} e^{-\lambda s} |f(s)|, \quad \lambda \geq 1,$$

$$\|f\|_{1,\lambda,\tau} := \sup_{0 \leq s \leq \tau} e^{-\lambda s} \int_0^s \frac{|f(s) - f(r)|}{(s-r)^{\alpha+1}} dr.$$

We now proceed to bound $\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau}$:

$$\begin{aligned} \|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau} &= \sup_{0 \leq s \leq \tau} \exp(-\lambda s) \left| X_0 + \frac{1}{\Gamma(\beta)} \left[\int_0^s \frac{f(\tilde{X}_r^{(\psi,\phi)}, \tilde{Y}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} dr \right. \right. \\ &\quad \left. \left. + \int_0^s \frac{\sigma(\tilde{X}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} d\varpi_r^\psi + \psi^{\beta-\frac{1}{2}} \int_0^s \frac{\sigma(\tilde{X}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} dW_r^H \right] \right|. \end{aligned}$$

In light of [29, Proposition 2.1], it can be derived that

$$\int_0^s \frac{\sigma(\tilde{X}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} dW_r^H \leq \Lambda_\alpha(W^H) \left\| \frac{\sigma(\tilde{X}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} \right\|_{\alpha,1}.$$

Then, under the assumptions of **(S1)**–**(S2)**, we have

$$\begin{aligned} \int_0^s \frac{\sigma(\tilde{X}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} dW_r^H &\leq \Lambda_\alpha(W^H) \times \int_0^s \frac{1}{r^\alpha} \left| \frac{\sigma(\tilde{X}_r^{(\psi,\phi)})}{(s-r)^{1-\beta}} \right| dr \\ &\quad + \Lambda_\alpha(W^H) \times \int_0^s \int_0^r \frac{|(s-r)^{\beta-1} \sigma(\tilde{X}_r^{(\psi,\phi)}) - (s-z)^{\beta-1} \sigma(\tilde{X}_z^{(\psi,\phi)})|}{(r-z)^{\alpha+1}} dz dr \\ &\leq \mathcal{K} \Lambda \left[1 + \int_0^s \frac{|\tilde{X}_r^{(\psi,\phi)}|}{r^\alpha (s-r)^{1-\beta}} dr + \int_0^s \int_0^r \frac{|\tilde{X}_r^{(\psi,\phi)} - \tilde{X}_z^{(\psi,\phi)}|}{(s-r)^{1-\beta} (r-z)^{\alpha+1}} dz dr \right] \\ &\leq \mathcal{K} \Lambda \left[1 + e^{\lambda r} \int_0^s \frac{\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau}}{r^\alpha (s-r)^{1-\beta}} dr + e^{\lambda r} \int_0^s \frac{\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}}{(s-r)^{1-\beta}} dr \right]. \end{aligned}$$

Furthermore, in view of [2, Lemma 4.1], for any $\mu > 0$, one obtains the chain of estimates $\|\varpi^\psi\|_{1-\alpha,\infty} \leq \mathcal{K} \|\varpi^\psi\|_{1-\alpha+\mu} \leq \mathcal{K} \|\varpi^\psi\|_H \leq \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}}$. Consequently,

$$\begin{aligned} \int_0^s (s-r)^{\beta-1} \sigma(\tilde{X}_r^{(\psi,\phi)}) d\varpi_r^\psi &\leq \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}} \left[1 + e^{\lambda r} \int_0^s \frac{r^{-\alpha} \|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau}}{(s-r)^{1-\beta}} dr \right] \\ &\quad + \mathcal{K} \|\varpi^\psi\|_{1-\alpha,\infty} e^{\lambda r} \int_0^s \frac{\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}}{(s-r)^{1-\beta}} dr \\ &\leq \mathcal{K} \|\varpi^\psi\|_{1-\alpha,\infty} \left[1 + e^{\lambda r} \int_0^s \frac{r^{-\alpha} \|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau}}{(s-r)^{1-\beta}} dr \right] \\ &\quad + \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}} e^{\lambda r} \int_0^s \frac{\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}}{(s-r)^{1-\beta}} dr. \end{aligned}$$

Substituting $\gamma = \lambda(s-r)$ yields

$$\int_0^s \frac{e^{-\lambda(s-r)}}{r^\alpha (s-r)^{1-\beta}} dr = \lambda^{-1} \int_0^{\lambda s} \frac{e^{-\gamma} \lambda^\alpha}{(\lambda s - \gamma)^\alpha} \left(\frac{\gamma}{\lambda} \right)^{\beta-1} d\gamma \leq \lambda^{\alpha-\beta} \sup_{z>0} \int_0^z \frac{e^{-\gamma} \gamma^{\beta-1}}{(z-\gamma)^\alpha} d\gamma \leq \mathcal{K} \lambda^{\alpha-\beta}, \quad (4.2)$$

where $q = \lambda\tau$. Then, we derive that

$$\begin{aligned}\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau} &\leq \mathcal{K}\Lambda\left(1 + \sup_{0\leq s\leq\tau} \int_0^s e^{-\lambda(s-r)} \times \left[\frac{\|\tilde{X}^{\psi,\phi}\|_{\lambda,\tau}}{r^\alpha(s-r)^{1-\beta}} + \frac{\|\tilde{X}^{\psi,\phi}\|_{1,\lambda,\tau}}{(s-r)^{1-\beta}}\right]dr\right) \\ &\quad + \mathcal{K}\|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}}\left(1 + \sup_{0\leq s\leq\tau} \int_0^s e^{-\lambda(s-r)} \times \left[\frac{\|\tilde{X}^{\psi,\phi}\|_{\lambda,\tau}}{r^\alpha(s-r)^{1-\beta}} + \frac{\|\tilde{X}^{\psi,\phi}\|_{1,\lambda,\tau}}{(s-r)^{1-\beta}}\right]dr\right) \\ &\leq \mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})(1 + \lambda^{\alpha-\beta}\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau} + \lambda^{-\beta}\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}).\end{aligned}$$

We next derive an estimate for $\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}$,

$$\begin{aligned}\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau} &= \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \frac{1}{(s-r)^{1+\alpha}} \left| X_0 + \frac{1}{\Gamma(\beta)} \left[\int_0^s \frac{f(\tilde{X}_z^{(\psi,\phi)}, \tilde{Y}_z^{(\psi,\phi)})}{(s-z)^{1-\beta}} dz \right. \right. \\ &\quad \left. \left. - \int_0^r \frac{f(\tilde{X}_z^{(\psi,\phi)}, \tilde{Y}_z^{(\psi,\phi)})}{(r-z)^{1-\beta}} dz \right] + \frac{1}{\Gamma(\beta)} \left[\int_0^s \frac{\psi^{\beta-\frac{1}{2}} \sigma(\tilde{X}_z^{(\psi,\phi)})}{(s-z)^{1-\beta}} dW_z^H \right. \right. \\ &\quad \left. \left. - \int_0^r \frac{\psi^{\beta-\frac{1}{2}} \sigma(\tilde{X}_z^{(\psi,\phi)})}{(r-z)^{1-\beta}} dW_z^H \right] + \frac{1}{\Gamma(\beta)} \left[\int_0^s \frac{\sigma(\tilde{X}_z^{(\psi,\phi)})}{(s-z)^{1-\beta}} d\varpi_z^\psi \right. \right. \\ &\quad \left. \left. - \int_0^r \frac{\sigma(\tilde{X}_z^{(\psi,\phi)})}{(r-z)^{1-\beta}} d\varpi_z^\psi \right] \right| dr.\end{aligned}$$

Let

$$M_1 := \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \frac{1}{(s-r)^{1+\alpha}} \left| \int_r^s \frac{f(\tilde{X}_z^{(\psi,\phi)}, \tilde{Y}_z^{(\psi,\phi)})}{(s-z)^{1-\beta}} dz \right| dr.$$

An application of Fubini's theorem together with hypothesis **(S2)** yields

$$\begin{aligned}M_1 &\leq \mathcal{K} \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \frac{1}{(s-r)^{1+\alpha}} \int_r^s \frac{(1 + |\tilde{X}_z^{(\psi,\phi)}|)}{(s-z)^{1-\beta}} dz dr \\ &\leq \mathcal{K} \left(1 + \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \frac{1}{(s-r)^{1+\alpha}} \int_r^s \frac{(z-r)^\alpha}{(s-z)^{1-\beta}} \frac{|\tilde{X}_z^{(\psi,\phi)}|}{(z-r)^\alpha} dz dr \right) \\ &\leq \mathcal{K} \left(1 + \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \frac{1}{(s-r)^{1+\alpha}} \int_r^s \frac{1}{(s-z)^{1-\beta}} \frac{|\tilde{X}_z^{(\psi,\phi)}|}{(z-r)^\alpha} dz dr \right) \\ &= \mathcal{K} \left(1 + \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \int_0^z \frac{1}{(s-r)^{1+\alpha}} \frac{1}{(s-z)^{1-\beta}} dr |\tilde{X}_z^{(\psi,\phi)}| dz \right) \\ &\leq \mathcal{K} \left(1 + \sup_{0\leq s\leq\tau} e^{-\lambda s} \int_0^s \frac{1}{(s-z)^{2\alpha}} \frac{|\tilde{X}_z^{(\psi,\phi)}|}{(s-z)^{1-\beta}} dz \right) \\ &\leq \mathcal{K} \left(1 + \sup_{0\leq s\leq\tau} \int_0^s e^{-\lambda(s-z)} \frac{\|\tilde{X}_z^{(\psi,\phi)}\|_{\lambda,\tau}}{(s-z)^{1-\beta+2\alpha}} dz \right).\end{aligned}$$

Line five relies on

$$\int_0^z \frac{(z-r)^{-\alpha}}{(s-r)^{1+\alpha}} dr = (s-z)^{-\alpha} \int_0^{z/(s-z)} \frac{\gamma^{-\alpha}}{(1+\gamma)^{1+\alpha}} d\gamma \leq C_\alpha (s-z)^{-2\alpha},$$

with $s = r - (\tau - r)\gamma$ and $C_\alpha = B(2\alpha, 1 - \alpha) = \int_0^\infty \frac{z^{-\alpha}}{(1+z)^{1+\alpha}} dz$, where B denotes the Beta function. Let

$$M_2 := \int_0^\tau \frac{1}{(\tau - s)^{1+\alpha}} \left| \int_s^\tau \frac{f(r)}{(\tau - r)^{1-\beta}} dW_r^H \right| ds.$$

An application of Fubini's theorem together with [29, Proposition 2.1] yields

$$\begin{aligned} M_2 &\leq \Lambda_\alpha(W_r^H) \left[\int_0^\tau \frac{1}{(\tau - s)^{1+\alpha}} \int_s^\tau \frac{(\tau - r)^{\beta-1} |f(r)|}{(r - s)^\alpha} dr \right. \\ &\quad \left. + \frac{1}{(\tau - s)^{1+\alpha}} \int_s^\tau \int_s^r \frac{(\tau - r)^{\beta-1} |f(r) - (\tau - z)^{\beta-1} f(z)|}{(r - z)^{1+\alpha}} dz dr ds \right] \\ &\leq \Lambda_\alpha(W_r^H) \left[\int_0^\tau \int_0^r \frac{(r - s)^{-\alpha}}{(\tau - s)^{2-\beta+\alpha}} ds |f(r)| dr \right. \\ &\quad \left. + \int_0^\tau \int_0^r \int_0^z \frac{1}{(\tau - s)^{2-\beta+\alpha}} ds \frac{|f(r) - f(z)|}{(r - z)^{1+\alpha}} dz dr \right]. \end{aligned}$$

Letting $s = r - (\tau - r)z$, we have

$$\int_0^r \frac{(r - s)^{-\alpha}}{(\tau - s)^{2-\beta+\alpha}} ds = \frac{1}{(\tau - r)^{1-\beta+2\alpha}} \int_0^{r/(\tau-r)} \frac{z^{-\alpha}}{(1+z)^{2-\beta+\alpha}} dz,$$

and

$$\int_0^z \frac{1}{(\tau - s)^{2-\beta+\alpha}} ds \leq \frac{(\alpha - \beta + 1)^{-1}}{(\tau - z)^{1-\beta+\alpha}}.$$

It further follows that

$$M_2 \leq \Lambda_\alpha(W_r^H) \left[C_{\alpha,\beta} \int_0^\tau \frac{|f(r)|}{(\tau - r)^{1-\beta+\alpha}} dr + \int_0^\tau \int_0^r \frac{(\alpha + \beta + 1)^{-1} |f(r) - f(z)|}{(\tau - z)^{1-\beta+\alpha} (r - z)^{1+\alpha}} dz dr \right],$$

where $C_{\alpha,\beta} = B(2\alpha - 2\beta + 2, 1 - \alpha) = \int_0^\infty (1+z)^{\beta-\alpha-2} z^{-\alpha} dz$. Let

$$M_3 := \int_0^\tau \frac{1}{(\tau - s)^{1+\alpha}} \left| \int_s^\tau \frac{f(r)}{(\tau - r)^{1-\beta}} d\varpi_r^\psi \right| ds.$$

An application of Fubini's theorem together with [29, Proposition 2.1] yields

$$M_3 \leq \Lambda_\alpha(\varpi_r^\psi) \left[\int_0^\tau \int_0^r \frac{(r - s)^{-\alpha}}{(\tau - s)^{2-\beta+\alpha}} ds |f(r)| dr + \int_0^\tau \int_0^r \int_0^z \frac{1}{(\tau - s)^{2-\beta+\alpha}} ds \frac{|f(r) - f(z)|}{(r - z)^{1+\alpha}} dz dr \right].$$

Due to $\|\varpi^\psi\|_{1-\alpha,\infty} \leq \mathcal{K} \|\varpi^\psi\|_{1-\alpha+\mu} \leq \|\varpi^\psi\|_H \leq \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}}$, we have

$$M_3 \leq \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}} \left[C_{\alpha,\beta} \int_0^\tau \frac{|f(r)|}{(\tau - r)^{1-\beta+\alpha}} dr + \int_0^\tau \int_0^r \frac{(\alpha + \beta + 1)^{-1} |f(r) - f(z)|}{(\tau - z)^{1-\beta+\alpha} (r - z)^{1+\alpha}} dz dr \right].$$

Combining M_1 , M_2 , and M_3 , we have

$$\|\tilde{X}^{\psi,\phi}\|_{1,\lambda,\tau} \leq \mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}}) \left[1 + \sup_{0 \leq s \leq \tau} \int_0^s \frac{e^{-\lambda(s-r)} \|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau}}{(s-r)^{1-\beta+2\alpha}} + \frac{e^{-\lambda(s-r)} \|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}}{(s-r)^{1-\beta+\alpha}} dr \right].$$

Setting $\gamma = \lambda(\tau - r)$, yields

$$\int_0^\tau \frac{e^{-\lambda(\tau-r)}}{(\tau-r)^{1-\beta+2\alpha}} d\gamma = \frac{1}{\lambda} \int_0^{\lambda\tau} e^{-\gamma} \times \left(\frac{\gamma}{\lambda}\right)^{\beta-2\alpha-1} d\gamma \leq \lambda^{2\alpha-\beta} \sup_{s>0} \int_0^s e^{-\gamma} \times \gamma^{\beta-2\alpha-1} d\gamma \leq \mathcal{K} \lambda^{2\alpha-\beta}, \quad (4.3)$$

and

$$\int_0^\tau \frac{e^{-\lambda(\tau-r)}}{(\tau-r)^{1-\beta+\alpha}} d\gamma \leq \mathcal{K} \lambda^{\alpha-\beta}. \quad (4.4)$$

In summary, it can be concluded that

$$\|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau} \leq \mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})(1 + \lambda^{2\alpha-\beta} \|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau} + \lambda^{\alpha-\beta} \|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}).$$

Let $\lambda = [4\mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})]^{\frac{1}{\beta-\alpha}}$,

$$\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau} \leq \frac{4}{3} \mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})(1 + \lambda^{-\beta} \|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau}). \quad (4.5)$$

Proceeding as in the derivation of (4.2), one readily deduces that

$$\begin{aligned} \|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,\tau} &\leq 1.5\mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}}) + 2[\mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})]^{\beta/(\beta-\alpha)} \\ &\leq \mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})^{\beta/(\beta-\alpha)}. \end{aligned} \quad (4.6)$$

By substituting (4.6) into (4.5),

$$\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,\tau} \leq \mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})^{\beta/(\beta-\alpha)}$$

In summary, it follows that

$$\|\tilde{X}^{(\psi,\phi)}\|_{\alpha,\infty} \leq e^{\lambda T} (\|\tilde{X}^{(\psi,\phi)}\|_{\lambda,T} + \|\tilde{X}^{(\psi,\phi)}\|_{1,\lambda,T}) \leq \mathcal{K} e^{\mathcal{K}(\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})^{\beta/(\beta-\alpha)}} (\Lambda + \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}})^{\beta/(\beta-\alpha)}$$

The proof is now concluded by noting that $0 < \alpha < 1/2$ together with $(\varpi^{(\psi,\phi)}, \varphi^{(\psi,\phi)}) \in \bar{\mathcal{A}}_b^N$ and the Fernique theorem for fractional Brownian motion.

We note that all constants $\mathcal{K}$ appearing in the above estimates are independent of ψ and ϕ . Indeed, the only potential sources of ψ - or ϕ -dependence are the coefficients of the controlled noise terms in (5); however, the estimates above have absorbed these scalings into the definition of Λ and the $\mathcal{H}_{cm}$ -norm of the controls, both of which are controlled uniformly by the energy bound N and the Fernique theorem. Therefore, the final estimate

$$\mathbb{E}[\|\tilde{X}^{(\psi,\phi)}\|_{\alpha,\infty}^p] \leq \mathcal{K}$$

holds with a constant $\mathcal{K}$ that is independent of ψ and ϕ . $\square$

Remark 4.1 (Fernique theorem [35]). Let $\mathcal{E}$ be a separable Banach space and let μ be a symmetric Gaussian measure on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. Then, there exists a constant $k_1 > 0$ such that

$$\int_{\mathcal{E}} \exp(k_1 \|x\|^2) \mu(dx) < \infty.$$

Equivalently, if X is a zero-mean Gaussian random variable with values in $\mathcal{E}$, then there exists $k_2 > 0$ such that

$$\mathbb{E}[\exp(k_2 \|X\|^2)] < \infty.$$

Having obtained the global moment boundedness of the slow variable, we further need to control its temporal increment modulus. This estimate is not only a crucial condition for verifying the tightness of this process family in $C^\alpha([0, T], \mathfrak{R}^m)$ but also constitutes the necessary analytical foundation for implementing the time-discretization scheme and proving the weak convergence of the controlled slow variable.

Lemma 4.2. Suppose that (S1)–(S3) are fulfilled. Then, there is a positive constant $\mathcal{K}$ such that for all $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \in \mathcal{A}_b^N$, the following holds:

$$\mathbb{E}[|\tilde{X}_{\tau+h}^{(\psi, \phi)} - \tilde{X}_\tau^{(\psi, \phi)}|^2] \leq \mathcal{K}h^{2\beta-2\alpha}.$$

Proof. Throughout this proof, $\mathcal{K} > 0$ stands for a constant determined only by N , possibly differing across lines.

$$\begin{aligned} \tilde{X}_{\tau+h}^{(\psi, \phi)} - \tilde{X}_\tau^{(\psi, \phi)} &= \frac{1}{\Gamma(\beta)} \left[\int_0^{\tau+h} \frac{f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)})}{(\tau+h-s)^{1-\beta}} ds - \int_0^\tau \frac{f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)})}{(\tau-s)^{1-\beta}} ds \right] \\ &\quad + \frac{1}{\Gamma(\beta)} \left[\int_0^{\tau+h} \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau+h-s)^{1-\beta}} d\varpi_s^\psi - \int_0^\tau \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau-s)^{1-\beta}} d\varpi_s^\psi \right] \\ &\quad + \frac{\psi^{\beta-\frac{1}{2}}}{\Gamma(\beta)} \left[\int_0^{\tau+h} \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau+h-s)^{1-\beta}} dW_s^H - \int_0^\tau \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau-s)^{1-\beta}} dW_s^H \right]. \end{aligned}$$

According to (S1)–(S3),

$$\begin{aligned} \mathbb{E}[|\tilde{X}_{\tau+h}^{(\psi, \phi)} - \tilde{X}_\tau^{(\psi, \phi)}|] &\leq \mathcal{K} \mathbb{E} \left[\left| \int_\tau^{\tau+h} \frac{f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)})}{(\tau+h-s)^{1-\beta}} ds \right|^2 + \left| \int_\tau^{\tau+h} \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau+h-s)^{1-\beta}} d\varpi_s^\psi \right|^2 \right] \\ &\quad + \mathcal{K} \mathbb{E} \left[\psi^{2\beta-1} \left| \int_\tau^{\tau+h} \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau+h-s)^{1-\beta}} dW_s^H \right|^2 \right] \\ &:= J_1^h + J_2^h + J_3^h. \end{aligned} \quad (4.7)$$

An application of Hölder's inequality together with (S2) and Lemma 4.1 yields

$$\begin{aligned} J_1^h &\leq \mathcal{K} \mathbb{E} \left[\int_\tau^{\tau+h} \frac{1}{(\tau+h-s)^{2-2\beta}} ds \int_\tau^{\tau+h} |f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)})|^2 ds \right] \\ &\leq \mathcal{K} \frac{h^{2\beta-1}}{2\beta-1} \mathbb{E} \left[L \int_\tau^{\tau+h} (1 + |\tilde{X}_s^{(\psi, \phi)}|^2 + |\tilde{Y}_s^{(\psi, \phi)}|^2) ds \right] \leq \mathcal{K}h^{2\beta}, \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \left| \int_s^\tau \frac{\sigma(\tilde{X}_r^{(\psi, \phi)})}{(\tau-r)^{1-\beta}} d\varpi_r^\psi \right| &\leq \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H, k_1}} \left[\int_s^\tau \frac{|(\tau-r)^{\beta-1} \sigma(\tilde{X}_r^{(\psi, \phi)})|}{(r-s)^\alpha} dr \right. \\ &\quad \left. + \int_s^\tau \int_s^r \frac{|(\tau-r)^{\beta-1} \sigma(\tilde{X}_r^{(\psi, \phi)}) - (\tau-z)^{\beta-1} \sigma(\tilde{X}_z^{(\psi, \phi)})|}{(r-z)^{1+\alpha}} dz dr \right] \\ &\leq \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H, k_1}} \int_s^\tau (\tau-r)^{\beta-1} \left[(r-s)^{-\alpha} |\tilde{X}_r^{(\psi, \phi)}| + \int_s^r \frac{|\tilde{X}_r^{(\psi, \phi)} - \tilde{X}_z^{(\psi, \phi)}|}{(r-z)^{1+\alpha}} dz \right] dr \\ &\leq \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H, k_1}} (1 + \|\tilde{X}^{(\psi, \phi)}\|_{\alpha, \infty}) \times \int_s^\tau \frac{(r-s)^{-\alpha} + (r-s)^{\beta-\alpha} + 1}{(\tau-r)^{1-\beta}} dr \end{aligned}$$

$$\leq \mathcal{K} \|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}} (1 + \|\tilde{X}^{(\psi,\phi)}\|_{\alpha,\infty}) (\tau - s)^{\beta-\alpha}. \quad (4.9)$$

Thus, we can get

$$J_2^h \leq \mathcal{K} \mathbb{E}[\|\varpi^\psi\|_{\mathcal{H}_{cm}^{H,k_1}} (1 + \|\tilde{X}^{(\psi,\phi)}\|_{\alpha,\infty})^2] h^{2\beta-2\alpha} \leq \mathcal{K} h^{2\beta-2\alpha}. \quad (4.10)$$

Analogously, we have

$$\left| \int_s^\tau (\tau - r)^{\beta-1} \sigma(\tilde{X}_r^{(\psi,\phi)}) dW_r^H \right| \leq \mathcal{K} \Lambda(W^H) (1 + \|\tilde{X}^{(\psi,\phi)}\|_{\alpha,\infty}) (\tau - s)^{\beta-\alpha}. \quad (4.11)$$

Subsequently, for $0 \leq \tau \leq \tau + h \leq T$, we have

$$J_3^h \leq \mathcal{K} \mathbb{E}[\Lambda^2 (1 + \|\tilde{X}^{(\psi,\phi)}\|_{\alpha,\infty})^2] h^{2\beta-2\alpha}. \quad (4.12)$$

In summary, combining (4.7)–(4.12) with Lemma 4.1 yields

$$\mathbb{E}[|\tilde{X}_{\tau+h}^{(\psi,\phi)} - \tilde{X}_\tau^{(\psi,\phi)}|^2] \leq \mathcal{K} h^{2\beta-2\alpha}.$$

Then, we complete the proof. $\square$

Because $\tilde{Y}^{(\psi,\phi)}$ is nontrivially coupled to the slow variable through the drift f , we establish uniform moment bounds and L^2 -integrability over $[0, T]$; these estimates prevent uncontrollable blow-up and provide the requisite regularity for the Khasminskii-type averaging principle. Demonstration follows analogous steps to Lemmas 4.3 and 4.4 in [22]; the detailed proof is omitted for brevity.

Lemma 4.3. Suppose that (S1)–(S3) are fulfilled; for all $(\varpi^{(\psi,\phi)}, \varphi^{(\psi,\phi)}) \in \bar{\mathcal{A}}_b^N$, $\sup_{0 \leq s \leq \tau} |\tilde{Y}_s^{(\psi,\phi)}|$ has moments of all orders.

Lemma 4.4. Suppose that (S1)–(S3) are fulfilled, and we additionally assume $\phi/\psi \leq (\kappa_2/2)^2$. Then there is a positive constant $\mathcal{K}$, depending only on N , such that for all $(\varpi^{(\psi,\phi)}, \varphi^{(\psi,\phi)}) \in \bar{\mathcal{A}}_b^N$,

$$\int_0^T \mathbb{E}[|\tilde{Y}_\tau^{(\psi,\phi)}|^2] d\tau \leq \mathcal{K}.$$

4.2. Proof of the large deviation principle

Proof. Indeed, from (4.1), the control perturbation in the fast equation carries the coefficient

$$\frac{\sqrt{\phi}}{\psi^{\beta-\frac{1}{2}}} = \left(\frac{\phi}{\psi} \right)^{1/2} \psi^{1-\beta} \rightarrow 0,$$

as $\beta > 1/2$, and $\phi/\psi \rightarrow 0$. Thus the fast control vanishes in the limit, and the drift term b/ϕ forces the fast variable to its invariant measure $\mu_{\bar{X}}$ on the time scale of the slow dynamics. This averaging mechanism yields the skeleton equation (3.2), independent of the fast control ϕ .

Step 1. Let $(\varpi^{(j)}, \varphi^{(j)}) \in \mathcal{H}_{cm}$. Owing to the continuous embedding $\mathcal{H}_{cm}^{H,k_1} \subset C^{1-\alpha}([0, T], \mathfrak{R}^{k_1})$, one has $\{\varpi^{(j)}\}_{j \geq 1}$ belongs to $C^{1-\alpha}([0, T], \mathfrak{R}^{d_1})$ for all $j \geq 0$. The family $\{\tilde{X}_t^j\}_{j \geq 1}$ then solves the deterministic skeleton equation:

$${}^c D_{0+}^\beta \tilde{X}_\tau^j = \bar{f}(\tilde{X}^j) d\tau + \sigma(\tilde{X}_\tau^j) d\varpi_\tau^j, \quad \tilde{X}_0^j = X_0. \quad (4.13)$$

By [34, Theorem 4.1], (4.13) possesses a unique solution $\{\tilde{X}^j\}$, which is uniformly bounded in $C^{1-\alpha}([0, T], \mathfrak{X}^m)$. There is a positive constant $\mathcal{K}$ such that

$$\|\tilde{X}^j\|_{1-\alpha} \leq \mathcal{K}.$$

Noting that $\alpha < 1/2$, one infers the compact embedding $C^{1-\alpha}([0, T]; \mathfrak{X}^m) \hookrightarrow C^\alpha([0, T]; \mathfrak{X}^m)$. Therefore, $\{\tilde{X}^j\}_{j \geq 1}$ is relatively compact in $C^\alpha([0, T]; \mathfrak{X}^m)$. Every accumulation point $\tilde{X}$ possesses a weakly convergent subsequence in this α -Hölder space, which we continue to denote by the same symbol. It remains to verify that the limit point $\tilde{X}$ fulfills

$${}^c D_{0+}^\beta \tilde{X}_\tau = \tilde{f}(\tilde{X}_\tau) d\tau + \sigma(\tilde{X}_\tau) d\varpi_\tau, \quad \tilde{X}_0 = X_0. \quad (4.14)$$

Owing to the uniqueness of solutions to (4.14), the accumulation point $\tilde{X}$ is uniquely determined.

Every $\varpi = K_H \dot{\varpi} \in \mathcal{H}_{cm}^{H,k_1}$ is differentiable, with the derivative given by

$$\varpi'(\tau) = C_H \tau^{H-\frac{1}{2}} \left(I_{0+}^{H-\frac{1}{2}} (v^{-1} \dot{\varpi}) \right) (\tau) = \frac{C_H \tau^{H-\frac{1}{2}}}{\Gamma(H-\frac{1}{2})} \int_0^\tau (\tau-s)^{H-\frac{3}{2}} s^{\frac{1}{2}-H} \dot{\varpi}(s) ds, \quad (4.15)$$

where $v(\varpi) = \varpi^{H-\frac{1}{2}}$.

Direct calculation yields

$$\begin{aligned} |\tilde{X}_\tau^j - \tilde{X}_\tau| &\leq \left| \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\tilde{f}(\tilde{X}_s^j) - \tilde{f}(\tilde{X}_s)}{(\tau-s)^{1-\beta}} ds \right| + \left| \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\sigma(\tilde{X}_s^j) - \sigma(\tilde{X}_s)}{(\tau-s)^{1-\beta}} d\varpi_s^{(j)} \right| \\ &\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\sigma(\tilde{X}_s)(d\varpi_s^{(j)} - d\varpi_s)}{(\tau-s)^{1-\beta}} \right| \\ &:= L_1 + L_2 + L_3. \end{aligned}$$

Regarding L_1 , α -Hölder norm together with assumption **(S2)** yields

$$L_1 \leq \frac{\mathcal{K}}{\Gamma(\beta)} \int_0^\tau \frac{|\tilde{X}_s^j - \tilde{X}_s|}{(\tau-s)^{1-\beta}} ds \leq \frac{\mathcal{K}}{\Gamma(\beta)} \int_0^\tau \frac{1}{(\tau-s)^{1-\beta}} ds \sup_{\tau \in [0, T]} |\tilde{X}_\tau^j - \tilde{X}_\tau| \leq \frac{\mathcal{K} \cdot T^\beta}{\Gamma(\beta+1)} \sup_{\tau \in [0, T]} |\tilde{X}_\tau^j - \tilde{X}_\tau|.$$

Subsequent analysis focuses on quantifying L_2 . According to the differential formula (4.15), Assumption **(S1)**, and Fubini theorem, we have

$$\begin{aligned} L_2 &\leq \left| \frac{\mathcal{K}}{\Gamma(\beta)} \int_0^\tau \frac{[\sigma(\tilde{X}_s^j) - \sigma(\tilde{X}_s)] s^{1/2-H}}{(\tau-s)^{1-\beta}} \left(\int_0^s (s-r)^{H-3/2} r^{H-1/2} \dot{\varpi}_r^{(j)} dr \right) ds \right| \\ &\leq \frac{\mathcal{K}}{\Gamma(\beta)} \int_0^\tau \left(\int_r^\tau (\tau-s)^{\beta-1} s^{1/2-H} (s-r)^{H-3/2} ds \right) r^{H-1/2} |\dot{\varpi}_r^{(j)}| \times |\tilde{X}_r^j - \tilde{X}_r| dr \\ &\leq \frac{\mathcal{K}}{\Gamma(\beta)(H-1/2)} \int_0^\tau r^{1/2-H} (\tau-r)^{\beta+H-3/2} |\dot{\varpi}_r^{(j)}| \times |\tilde{X}_r^j - \tilde{X}_r| dr \\ &\leq \frac{\mathcal{K}}{\Gamma(\beta)(H-1/2)} \sup_{\tau \in [0, T]} |\tilde{X}_\tau^j - \tilde{X}_\tau| \times \int_0^\tau r^{1/2-H} (\tau-r)^{\beta+H-3/2} |\dot{\varpi}_r^{(j)}| dr \\ &\leq \frac{\mathcal{K}}{\Gamma(\beta)(H-1/2)} \sup_{\tau \in [0, T]} |\tilde{X}_\tau^j - \tilde{X}_\tau| \times \|\varpi^{(j)}\|_{\mathcal{H}_{cm}^{H,k_1}} \int_0^\tau (\tau-r)^{\beta+H-3/2} dr \\ &\leq \frac{\mathcal{K} \cdot \sqrt{2N}}{\Gamma(\beta)(H-1/2)} \sup_{\tau \in [0, T]} |\tilde{X}_\tau^j - \tilde{X}_\tau|, \end{aligned}$$

with the sixth inequality arising from the constraint $\|\varpi^{(j)}\|_{\mathcal{H}^{H,k_1}} \leq \sqrt{2N}$. Because $\{\tilde{X}^j\}_{j \geq 1} \rightarrow \tilde{X}$ in the uniform norm, The sum $L_1 + L_2$ converges to zero as $j \rightarrow \infty$. Turning to L_3 , an application of Fubini's theorem yields

$$\begin{aligned} L_3 &= \frac{\mathcal{K}}{\Gamma(\beta)} \left| \int_0^\tau \frac{\sigma(\tilde{X}_s) s^{H-1/2}}{(\tau-s)^{1-\beta}} \left[\int_0^s (s-r)^{H-3/2} r^{1/2-H} (\dot{\varpi}_r^{(j)} - \dot{\varpi}_r) dr \right] ds \right| \\ &= \frac{\mathcal{K}}{\Gamma(\beta)} \left| \int_0^\tau r^{1/2-H} (\dot{\varpi}_r^{(j)} - \dot{\varpi}_r) \left[\int_r^\tau \frac{\sigma(\tilde{X}_s) s^{H-1/2}}{(\tau-s)^{1-\beta}} (s-r)^{H-3/2} ds \right] dr \right| \\ &= \frac{\mathcal{K}}{\Gamma(\beta)} \left| \int_0^\tau \mathbf{1}_{[0,t]}(r) r^{\frac{1}{2}-H} (\dot{\varpi}_r^{(j)} - \dot{\varpi}_r) \left[\int_r^\tau \frac{\sigma(\tilde{X}_s) s^{H-1/2}}{(\tau-s)^{1-\beta}} (s-r)^{H-3/2} ds \right] dr \right|. \end{aligned}$$

The weak convergence $(\dot{\varpi}^{(j)}, \dot{\varphi}^{(j)}) \rightarrow (\dot{\varpi}, \dot{\varphi})$ in L^2 entails that $L_3 \rightarrow 0$ as $j \rightarrow \infty$.

Therefore, $\tilde{X}$ satisfies (3.2). Then, $\{\tilde{X}^j\}_{j \geq 1} \xrightarrow{\text{weakly}} \tilde{X}$ in $C^\alpha([0, T], \mathfrak{X}^m)$.

Step 2. Assume $0 < \phi < \psi$, $\phi = o(\psi)$ and we will let $\psi \rightarrow 0$. We now turn to the verification of Condition (ii) in Proposition 2.1. Let $\tau(\Delta) := \left\lfloor \frac{\tau}{\Delta} \right\rfloor \Delta$ designate the last breakpoint prior to τ . For this purpose, we begin by introducing an auxiliary process:

$$\begin{cases} {}^C D_{0+}^\beta \hat{X}_\tau^{(\psi, \phi)} = f(\tilde{X}_{\tau(\Delta)}^{(\psi, \phi)}, \hat{Y}_\tau^{(\psi, \phi)}) d\tau + \sigma(\hat{X}_\tau^{(\psi, \phi)}) d\varpi_\tau^{(\psi, \phi)}, \\ \phi d\hat{Y}_\tau^{(\psi, \phi)} = b(\tilde{X}_{\tau(\Delta)}^{(\psi, \phi)}, \hat{Y}_\tau^{(\psi, \phi)}) d\tau + \sqrt{\phi} g(\tilde{X}_{\tau(\Delta)}^{(\psi, \phi)}, \hat{Y}_\tau^{(\psi, \phi)}) dW_\tau, \end{cases}$$

with $(\tilde{X}_0^{(\psi, \phi)}, \tilde{Y}_0^{(\psi, \phi)}) = (X_0, Y_0) \in \mathfrak{X}^m \times \mathfrak{X}^n$. Note that in $\hat{X}_\tau^{(\psi, \phi)}$ is substituted in σ .

By taking same manner in Lemmas 4.1 and 4.2, we deduce that

$$\mathbb{E}[\|\hat{X}^{(\psi, \phi)}\|_{\alpha, \infty}^p] \leq \mathcal{K}, \quad (4.16)$$

and

$$\int_0^T \mathbb{E}[|\hat{Y}_\tau^{(\psi, \phi)}|^2] d\tau \leq \mathcal{K}, \quad (4.17)$$

where $\mathcal{K}$ is independent of ψ, ϕ, Δ .

Let $R > 0$ be taken large enough. The stopping time is then defined as $t_R := \inf \{ \tau \in [0, T] : \|W^H\|_{1-\alpha, \infty, \tau} \geq R \} \wedge T$. Consequently,

$$\begin{aligned} \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 \right] &\leq \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 I_{\{t_R \leq T\}} \right] \\ &\quad + \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 I_{\{t_R > T\}} \right]. \end{aligned} \quad (4.18)$$

Invoking Lemma 4.1 and using Hölder's inequality and (4.16), it follows that

$$\begin{aligned} \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 I_{\{t_R \leq T\}} \right] &\leq \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^4 \right]^{1/2} \mathbf{P}(t_R \leq T)^{1/2} \\ &\leq \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^4 \right]^{1/2} \times R^{-1} \sqrt{\mathbb{E}[\|W^H\|_{1-\alpha, \infty, T}^2]} \end{aligned}$$

$$\leq \mathcal{K}R^{-1} \sqrt{\mathbb{E} \left[\|W^H\|_{1-\alpha, \infty, T}^2 \right]}. \quad (4.19)$$

From now, we give the estimate of

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \left\| \tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)} \right\|_{\alpha, [0, \tau]}^2 I_{\{t_R > T\}} \right].$$

Let the set D be defined by $\{\|W^H\|_{1-\alpha, \infty, T} \leq R\}$. For positive parameters λ , we establish estimate for

$$A := \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \left\| \tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)} \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right].$$

It is immediate that

$$\begin{aligned} A &\leq \mathcal{K} \mathbb{E} \left[\sup_{0 \leq \tau \leq T} e^{-\lambda \tau} \left\| \int_0^\tau \frac{f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)})}{(\tau - s)^{1-\beta}} ds \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \\ &\quad + \mathcal{K} \mathbb{E} \left[\sup_{0 \leq \tau \leq T} e^{-\lambda \tau} \left\| \int_0^\tau \frac{\sigma(\tilde{X}_s^{(\psi, \phi)}) - \sigma(\tilde{X}_s)}{(\tau - s)^{1-\beta}} d\varpi_s^{(\psi, \phi)} \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \\ &\quad + \mathcal{K} \mathbb{E} \left[\psi^{2\beta-1} \sup_{0 \leq \tau \leq T} e^{-\lambda \tau} \left\| \int_0^\tau \frac{\sigma(\tilde{X}_s^{(\psi, \phi)})}{(\tau - s)^{1-\beta}} dW_s^H \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

For any measurable $f : [0, T] \rightarrow \mathfrak{X}^k$, the quantity $Q_1 =: \left\| \int_0^\tau \frac{f}{(\tau - s)^{1-\beta}} ds \right\|_{\alpha, [0, \tau]}$ satisfies

$$\begin{aligned} Q_1 &\leq \left| \int_0^\tau \frac{f(s)}{(\tau - s)^{1-\beta}} ds \right| + \int_0^\tau \frac{1}{(\tau - s)^{2-\beta+\alpha}} \int_s^\tau |f(r)| dr ds \\ &\leq \tau^\alpha \int_0^\tau \frac{|f(r)|}{(\tau - r)^{1-\beta+\alpha}} dr + \mathcal{K} \int_0^\tau \frac{|f(r)|}{(\tau - r)^{1-\beta+\alpha}} dr \\ &\leq \mathcal{K} \int_0^\tau \frac{|f(r)|}{(\tau - r)^{1-\beta+\alpha}} dr. \end{aligned} \quad (4.20)$$

Reorganizing A_1 and appealing to (S2), (4.20), and Lemma 4.1, one obtains

$$\begin{aligned} A_1 &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left\| \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)})}{(\tau - s)^{1-\beta}} ds \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \\ &\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left\| \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \tilde{X}_s^{(\psi, \phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)})}{(\tau - s)^{1-\beta}} ds \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \\ &\leq \mathcal{K} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{1}{(\tau - s)^{2-2\beta+2\alpha}} ds \right] \times \int_0^T \mathbb{E} \left[\left| f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) \right|^2 \mathbf{1}_D \right] ds \\ &\quad + \mathcal{K} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{1}{(\tau - s)^{2-2\beta+2\alpha}} ds \right] \times \int_0^T \mathbb{E} \left[\left| f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) \right|^2 \right] ds \\ &\leq \mathcal{K} \int_0^T \mathbb{E} \left[\left| \frac{1}{(T - s)^{2-2\beta+2\alpha}} f(\tilde{X}_s^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \tilde{Y}_s^{(\psi, \phi)}) \right|^2 \mathbf{1}_D \right] ds \end{aligned}$$

$$\begin{aligned}
& + \mathcal{K} \int_0^T \mathbb{E} \left[\left| \frac{1}{(T-s)^{2-2\beta+2\alpha}} f(\tilde{X}_{s(\Delta)}^{(\psi,\phi)}, \tilde{Y}_s^{(\psi,\phi)}) - f(\tilde{X}_{s(\Delta)}^{(\psi,\phi)}, \hat{Y}_s^{(\psi,\phi)}) \right|^2 \mathbf{1}_D \right] ds \\
& \leq \mathcal{K} \int_0^T \mathbb{E} \left[\frac{1}{(T-s)^{2-2\beta+2\alpha}} |\tilde{X}_s^{(\psi,\phi)} - \tilde{X}_{s(\Delta)}^{(\psi,\phi)}|^2 + |\tilde{Y}_s^{(\psi,\phi)} - \hat{Y}_s^{(\psi,\phi)}|^2 \mathbf{1}_D \right] ds \\
& \leq \mathcal{K}_\Delta,
\end{aligned} \tag{4.21}$$

where $\mathcal{K} > 0$ denotes a constant depending solely on ψ , ϕ , and Δ , possibly differing across lines. Remark that $2\alpha < 1$. The final inequality's deduction stems from Lemma 4.2 combined with the fact that

$$\int_0^T \mathbb{E} [|\tilde{Y}_\tau^{(\psi,\phi)} - \hat{Y}_\tau^{(\psi,\phi)}|^2] dt \leq \mathcal{K}_\Delta,$$

which is a slight extension of [36, Lemma 4.4].

To estimate A_2 , we establish the preliminary estimate:

$$Q_2 := \left\| \int_0^\tau (\cdot - r)^{\beta-1} f(r) d\varpi_r^{(\psi,\phi)} \right\|_{\alpha, [0, \tau]},$$

where f denotes a measurable function. Mimicking the demonstration technique in Lemma 4.1, we derive

$$\begin{aligned}
Q_2 & \leq \mathcal{K} \|\varpi^{(\psi,\phi)}\|_{1-\alpha, \infty, T} \times \int_0^\tau \left[\frac{1}{(\tau-r)^{1-\beta+2\alpha}} + \frac{1}{r^{1-\beta+\alpha}} \right] [|f(r)| + \int_0^r \frac{|f(r) - f(z)|}{(r-z)^{1+\alpha}} dz] dr \\
& \leq \mathcal{K} \|\varpi^{(\psi,\phi)}\|_{\mathcal{H}_{cm}^{H,k_1}} \times \int_0^\tau \left[\frac{1}{(\tau-r)^{1-\beta+2\alpha}} + \frac{1}{r^{1-\beta+\alpha}} \right] [|f(r)| + \int_0^r \frac{|f(r) - f(z)|}{(r-z)^{1+\alpha}} dz] dr \\
& \leq \mathcal{K} \int_0^\tau \left[\frac{1}{(\tau-r)^{1-\beta+2\alpha}} + \frac{1}{r^{1-\beta+\alpha}} \right] \|f\|_{\alpha, [0, r]} dr.
\end{aligned}$$

Here, $\|\varpi^{(\psi,\phi)}\|_{\mathcal{H}_{cm}^{H,k_1}} \leq \sqrt{2N}$. Then, according to [37, Lemma 7.1], we know that

$$\begin{aligned}
|\sigma(\wp_1) - \sigma(\wp_2) - \sigma(\wp_3) + \sigma(\wp_4)| & \leq \mathcal{K} |\wp_1 - \wp_2 - \wp_3 + \wp_4| \\
& + \mathcal{K} |\wp_1 - \wp_3| (|\wp_1 - \wp_2| + |\wp_3 - \wp_4|).
\end{aligned}$$

Now, we estimate A_2 as follows:

$$\begin{aligned}
A_2 & \leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \int_0^\tau e^{-\lambda \tau} \left(\frac{1}{(\tau-r)^{2\alpha-\beta}} + \frac{1}{r^{1-\beta+\alpha}} \right) \right. \right. \\
& \quad \times \left. \left. \left\| \sigma(\tilde{X}_r^{(\psi,\phi)}) - \sigma(\hat{X}_r^{(\psi,\phi)}) \right\|_{\alpha, [0, r]} \mathbf{1}_D dr \right|^2 \right] \\
& \leq \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \int_0^\tau e^{-\lambda \tau} \left(\frac{1}{(\tau-r)^{1-\beta+2\alpha}} + \frac{1}{r^{1-\beta+\alpha}} \right) \right. \right. \\
& \quad \times \left. \left. \left(1 + \Delta(\tilde{X}_r^{(\psi,\phi)}) + \Delta(\hat{X}_r^{(\psi,\phi)}) \right) \left\| \tilde{X}^{(\psi,\phi)} - \hat{X}^{(\psi,\phi)} \right\|_{\alpha, [0, r]} \mathbf{1}_D dr \right|^2 \right].
\end{aligned} \tag{4.22}$$

Here, $\Delta(\tilde{X}_r^{(\psi,\phi)}) = \int_0^r \frac{|\tilde{X}_r^{(\psi,\phi)} - \tilde{X}_z^{(\psi,\phi)}|}{(r-z)^{1+\alpha}} dz$ and $\Delta(\hat{X}_r^{(\psi,\phi)}) = \int_0^r \frac{|\hat{X}_r^{(\psi,\phi)} - \hat{X}_z^{(\psi,\phi)}|}{(r-z)^{1+\alpha}} dz$. These above two parts can be dominated by $\|\tilde{X}^{(\psi,\phi)}\|_{\alpha, \infty}$ and $\|\hat{X}^{(\psi,\phi)}\|_{\alpha, \infty}$. Noting that $\|\varpi^{(\psi,\phi)}\|_{\mathcal{H}_{cm}^{H,k_1}} \leq \sqrt{2N}$, we have

$$\Delta(\tilde{X}_r^{(\psi,\phi)}) + \Delta(\hat{X}_r^{(\psi,\phi)}) \leq \mathcal{K} \|\varpi^{(\psi,\phi)}\|_{\mathcal{H}_{cm}^{H,k_1}} (1 + \|\tilde{X}^{(\psi,\phi)}\|_{\alpha, \infty} + \|\hat{X}^{(\psi,\phi)}\|_{\alpha, \infty}),$$

with $\mathcal{K} = \mathcal{K}(R, N) > 0$ depending neither on ψ, ϕ, Δ . Then, by (4.2) and (4.3), we can see that

$$A_2 \leq \mathcal{K} \lambda^{2\alpha-\beta} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right]. \quad (4.23)$$

According to Lemma 4.1, we estimate A_3 as follows

$$\begin{aligned} A_3 &\leq \psi^{2\beta-1} \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \left\| \int_0^\tau (\cdot - s)^{\beta-1} \sigma(\tilde{X}_s^{(\psi, \phi)}) dW_s^H \right\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \\ &\leq \psi^{2\beta-1} \mathcal{K}, \end{aligned} \quad (4.24)$$

with $\mathcal{K} = \mathcal{K}(R, N) > 0$ depending neither on ψ, ϕ, Δ .

According to (4.21), (4.23), and (4.24), it follows that

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \leq \mathcal{K} \Delta,$$

from which one deduces that

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}_\tau^{(\psi, \phi)} - \hat{X}_\tau^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 \mathbf{1}_D \right] \leq \mathcal{K} \Delta.$$

With aid of (4.19), we can get

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \|\tilde{X}^{(\psi, \phi)} - \hat{X}^{(\psi, \phi)}\|_{\alpha, [0, \tau]}^2 \right] \leq \mathcal{K} \Delta + \mathcal{K}' R^{-1},$$

with $\mathcal{K} = \mathcal{K}(R, N) > 0$ depending neither on ψ, ϕ, Δ . $\mathcal{K}'$ is independent of ψ, ϕ, Δ and R .

For $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \in \tilde{\mathcal{A}}_b^N$, we construct the following equation:

$${}^C D_{0+}^\beta \tilde{X}_\tau^\psi = \bar{f}(\tilde{X}_\tau^\psi) d\tau + \sigma(\tilde{X}_\tau^\psi) d\varpi_\tau^{(\psi, \phi)},$$

with $\tilde{X}_0^\psi = X_0$. In other words, $\tilde{X}^\psi = \mathcal{G}^0(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)})$. We will show that

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^\psi\|_{\alpha, [0, \tau]}^2 \right] \leq \mathcal{K} \Delta,$$

$\mathcal{K} > 0$ being constant irrespective of ψ, ϕ , and Δ . Hence, it follows that

$$\begin{aligned} &\mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^\psi\|_{\alpha, [0, \tau]}^2 \right] \\ &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \left\| \int_0^\tau \frac{f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)})}{(\tau - s)^{1-\beta}} ds \right\|_{\alpha, [0, \tau]}^2 \right] \\ &\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \left\| \int_0^\tau \frac{\bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}) - \bar{f}(\tilde{X}_s^\psi)}{(\tau - s)^{1-\beta}} ds \right\|_{\alpha, [0, \tau]}^2 \right] \\ &\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda\tau} \left\| \int_0^\tau \frac{\sigma(\hat{X}_s^{(\psi, \phi)}) - \sigma(\tilde{X}_s^\psi)}{(\tau - s)^{1-\beta}} d\varpi_s^\psi \right\|_{\alpha, [0, \tau]}^2 \right] \\ &=: H_1 + H_2 + H_3. \end{aligned}$$

The estimation of H_1 is

$$\begin{aligned}
H_1 &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \sum_{t=0}^{\lfloor \frac{\tau}{\Delta} \rfloor - 1} \int_{t\Delta}^{(t+1)\Delta} \frac{f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)})}{[(t+1)\Delta]^{1-\beta}} ds \right|^2 \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \int_{\lfloor \frac{\tau}{\Delta} \rfloor \Delta}^{\tau} \frac{f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)})}{(\tau - s)^{1-\beta}} ds \right|^2 \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left(\int_0^{\tau} \frac{| \int_s^{\tau} (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |}{(\tau - s)^{1-\beta} (\tau - s)^{1+\alpha}} ds \right)^2 \right] \\
&\leq \mathcal{K} \Delta^{2\beta-1} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \sum_{t=0}^{\lfloor \frac{\tau}{\Delta} \rfloor - 1} \int_{t\Delta}^{(t+1)\Delta} (f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)})) ds \right|^2 \right] \\
&\quad + \mathcal{K} \Delta^{2\beta-1} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \int_{\lfloor \frac{\tau}{\Delta} \rfloor \Delta}^{\tau} (f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)})) ds \right|^2 \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left(\int_0^{\tau} \frac{| \int_s^{\tau} (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |}{(\tau - s)^{1+\alpha}} ds \right)^2 \right] \\
&=: \sum_{i=1}^3 H_1^i.
\end{aligned}$$

The term H_1^1 is estimated as follows:

$$\begin{aligned}
H_1^1 &\leq \mathcal{K} \Delta^{2\beta-1} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \frac{\tau}{\Delta} \right| \sum_{t=0}^{\lfloor \frac{\tau}{\Delta} \rfloor - 1} \int_{t\Delta}^{(t+1)\Delta} (f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)})) ds \right|^2 \right] \\
&\leq \mathcal{K} \Delta^{2\beta-1} \frac{T^2}{\Delta^2} \max_{0 \leq t \leq \lfloor \frac{T}{\Delta} \rfloor - 1} \mathbb{E} \left[\left| \int_{t\Delta}^{(t+1)\Delta} (f(\tilde{X}_{t\Delta}^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)})) ds \right|^2 \right] \\
&\leq \mathcal{K} \Delta^{2\beta-3} \phi^2 \max_{0 \leq t \leq \lfloor \frac{T}{\Delta} \rfloor - 1} \int_0^{\frac{\Delta}{\phi}} \int_{\xi}^{\frac{\Delta}{\phi}} \mathcal{J}_{\tau}(s, \xi) ds d\xi \\
&\leq \mathcal{K} \Delta^{2\beta-3} \phi^2 \max_{0 \leq t \leq \lfloor \frac{T}{\Delta} \rfloor - 1} \int_0^{\frac{\Delta}{\phi}} \int_{\xi}^{\frac{\Delta}{\phi}} e^{-\frac{\kappa_1}{2}(s-\xi)} ds d\xi. \tag{4.25}
\end{aligned}$$

The estimates in the third and fourth lines follow from [27, Appendix B], with κ_1 as in **(S3)**, $0 \leq \xi \leq s \leq \Delta/\phi$, and

$$\mathcal{J}_{\tau}(s, \xi) = \mathbb{E}[\langle f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_{s\psi+t\Delta}^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)}), f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_{\xi\psi+t\Delta}^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)}) \rangle]. \tag{4.26}$$

We also compute H_1^2 as follows:

$$\begin{aligned}
H_1^2 &\leq \mathcal{K} \Delta^{2\beta-1} \mathbb{E} \left[\left| \int_{\lfloor \frac{\tau}{\Delta} \rfloor \Delta}^{\tau} (f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)})) ds \right|^2 \right] \\
&\leq \mathcal{K} \Delta^{2\beta-1} \int_{\lfloor \frac{\tau}{\Delta} \rfloor \Delta}^{\tau} \mathbb{E} [|f(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}, \hat{Y}_s^{(\psi, \phi)})|^2 - |\bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)})|^2] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{K}_\Delta^{2\beta-1} \int_{\lfloor \frac{\tau}{\Delta} \rfloor \Delta}^{\tau} \mathbb{E}[|1 + \tilde{X}_{s(\Delta)}^{(\psi, \phi)}|^2 + |\hat{Y}_s^{(\psi, \phi)}|^2] ds \\
&\leq \mathcal{K}_\Delta^{2\beta}
\end{aligned} \tag{4.27}$$

Regarding H_1^3 , by Hölder's inequality, one obtains

$$\begin{aligned}
H_1^3 &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{ds}{(\tau - s)^{\frac{1}{2} + \alpha}} \int_0^\tau \frac{| \int_s^\tau (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} ds \right] \\
&\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{| \int_s^\tau (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_{\iota^c} ds \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{| \int_s^\tau (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_\iota ds \right] \\
&=: H_1^{31} + H_1^{32},
\end{aligned} \tag{4.28}$$

where $\iota := \{\tau < (\lfloor \frac{s}{\Delta} \rfloor + 2)\Delta\}$, $\iota^c := \{\tau \geq (\lfloor \frac{s}{\Delta} \rfloor + 2)\Delta\}$. Turning to H_1^{31} , Assumption (S3) yields

$$\begin{aligned}
H_1^{31} &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{| \int_s^{\lfloor \frac{s}{\Delta} \rfloor \Delta + 1} (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_{\iota^c} ds \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{| \int_{\lfloor \frac{s}{\Delta} \rfloor \Delta}^s (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_{\iota^c} ds \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{(\lfloor \frac{\tau}{\Delta} \rfloor - \lfloor \frac{s}{\Delta} \rfloor - 1)}{(\tau - s)^{\frac{3}{2} + \alpha}} \times \sum_{t=\lfloor \frac{s}{\Delta} \rfloor + 1}^{\lfloor \frac{\tau}{\Delta} \rfloor - 1} \left| \int_{t\Delta}^{(t+1)\Delta} (f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)})) dr \right|^2 \mathbf{1}_{\iota^c} ds \right] \\
&\leq \mathcal{K} \sup_{\tau \in [0, T]} \left[\int_0^\tau \frac{1}{(\tau - s)^{\frac{3}{2} + \alpha}} ((\lfloor \frac{s}{\Delta} \rfloor + 1)\Delta - s) \mathbf{1}_{\iota^c} ds \right] + \mathcal{K} \sup_{\tau \in [0, T]} \left[\int_0^\tau \frac{1}{(\tau - s)^{\frac{3}{2} + \alpha}} (\tau - \lfloor \frac{t}{\Delta} \rfloor \Delta) \mathbf{1}_{\iota^c} ds \right] \\
&\quad + \mathcal{K}_\Delta^{-1} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{1}{(\tau - s)^{\frac{3}{2} + \alpha}} \times \sum_{t=\lfloor \frac{s}{\Delta} \rfloor + 1}^{\lfloor \frac{\tau}{\Delta} \rfloor - 1} \left| \int_{t\Delta}^{(t+1)\Delta} (f(\tilde{X}_{t\Delta}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{t\Delta}^{(\psi, \phi)})) dr \right|^2 \mathbf{1}_{\iota^c} ds \right] \\
&\leq \mathcal{K}_\Delta + \mathcal{K} \frac{\phi^2}{\Delta^2} \max_{0 \leq t \leq \lfloor \frac{\tau}{\Delta} \rfloor - 1} \int_0^{\frac{\Delta}{\phi}} \int_\xi^{\frac{\Delta}{\phi}} \mathcal{J}_t(s, \xi) ds d\xi \\
&\leq \mathcal{K}_\Delta + \mathcal{K} \frac{\phi^2}{\Delta^2} \max_{0 \leq t \leq \lfloor \frac{\tau}{\Delta} \rfloor - 1} \int_0^{\frac{\Delta}{\phi}} \int_\xi^{\frac{\Delta}{\phi}} e^{\frac{\kappa_1}{2}(s-\xi)} ds d\xi.
\end{aligned} \tag{4.29}$$

Turning to H_1^{32} , an application of (S2), Lemma 4.1 and the condition $\tau - s \leq 2\Delta$ yield

$$\begin{aligned}
H_1^{32} &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [\Delta, T]} \int_0^{\tau(\Delta) - \Delta} \frac{| \int_s^\tau (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_\iota ds \right] \\
&\quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [\Delta, T]} \int_{\tau(\Delta) - \Delta}^\tau \frac{| \int_s^\tau (f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})) dr |^2}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_\iota ds \right]
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, \Delta]} \int_0^\tau \frac{|f(\tilde{X}_{r(\Delta)}^{(\psi, \phi)}, \hat{Y}_r^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{r(\Delta)}^{(\psi, \phi)})| dr}{(\tau - s)^{\frac{3}{2} + \alpha}} \mathbf{1}_\tau ds \right] \\
& \leq \mathcal{K} \Delta \sup_{\tau \in [\Delta, T]} \left(\int_0^{\tau(\Delta) - \Delta} (\tau - s)^{-\frac{3}{2} - \alpha} ds \right) + \mathcal{K} \Delta \sup_{\tau \in [\Delta, T]} \left(\int_{\tau(\Delta) - \Delta}^\tau (\tau - s)^{\frac{3}{2} - \alpha} \mathbf{1}_\tau ds \right) \\
& \quad + \mathcal{K} \Delta \sup_{\tau \in [0, \Delta]} \left(\int_0^\tau (\tau - s)^{\frac{3}{2} - \alpha} \mathbf{1}_\tau ds \right) \\
& \leq \mathcal{K} \Delta.
\end{aligned} \tag{4.30}$$

Combining (4.25)–(4.30), we get

$$\begin{aligned}
H_1 & \leq \mathcal{K} \Delta + \mathcal{K} \frac{\phi^2}{\Delta^2} \max_{0 \leq t \leq \lfloor \frac{T}{\Delta} \rfloor - 1} \int_0^{\frac{\Delta}{\phi}} \int_\xi^{\frac{\Delta}{\phi}} \mathcal{J}_t(s, \xi) ds d\xi \\
& \leq \mathcal{K} \frac{\phi^2}{\Delta^2} \max_{0 \leq t \leq \lfloor \frac{T}{\Delta} \rfloor - 1} \int_0^{\frac{\Delta}{\phi}} \int_\xi^{\frac{\Delta}{\phi}} e^{-\frac{\kappa_1}{2}(s - \xi)} ds d\xi + \mathcal{K} \phi \\
& \leq \mathcal{K} \frac{\phi^2}{\Delta^2} \left(\kappa_1 \frac{\Delta}{\phi} - \kappa_1^2 + \kappa_1^2 e^{-\frac{\kappa_1}{2} \frac{\Delta}{\phi}} \right) + \mathcal{K} \Delta \\
& \leq \mathcal{K} (\phi \Delta^{-1} + \Delta).
\end{aligned} \tag{4.31}$$

Next, we estimate H_2 :

$$\begin{aligned}
H_2 & \leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \left\| \int_0^\tau \frac{\bar{f}(\tilde{X}_{s(\phi)}^{(\psi, \phi)}) - \bar{f}(\tilde{X}_s^{(\psi, \phi)})}{(\tau - s)^{1 - \beta}} ds \right\|_{\alpha, [0, \tau]}^2 \right] \\
& \quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \left\| \int_0^\tau \frac{\bar{f}(\tilde{X}_s^{(\psi, \phi)}) - \bar{f}(\hat{X}_s^{(\psi, \phi)})}{(\tau - s)^{1 - \beta}} ds \right\|_{\alpha, [0, \tau]}^2 \right] \\
& \quad + \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \left\| \int_0^\tau \frac{\bar{f}(\hat{X}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_s^{(\psi)})}{(\tau - s)^{1 - \beta}} ds \right\|_{\alpha, [0, \tau]}^2 \right] \\
& =: H_2^1 + H_2^2 + H_2^3.
\end{aligned} \tag{4.32}$$

Taking same manner to (4.21), with Assumption (S2), Lemma 4.1, (4.25), and (4.27), we have

$$\begin{aligned}
\sum_{i=1}^2 H_2^i & \leq \mathcal{K} \int_0^T \frac{1}{(T - s)^{2 - 2\beta + 2\alpha}} \mathbb{E} \left[|\bar{f}(\tilde{X}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)})|^2 \right] ds \\
& \quad + \mathcal{K} \int_0^T \frac{1}{(T - s)^{2 - 2\beta + 2\alpha}} \mathbb{E} \left[|\bar{f}(\tilde{X}_{s(\Delta)}^{(\psi, \phi)}) - \bar{f}(\hat{X}_s^{(\psi, \phi)})|^2 \right] ds \\
& \leq \mathcal{K} \int_0^T \frac{1}{(T - s)^{2 - 2\beta + 2\alpha}} \mathbb{E} \left[|\tilde{X}_s^{(\psi, \phi)} - \tilde{X}_{s(\Delta)}^{(\psi, \phi)}|^2 + |\hat{X}_s^{(\psi, \phi)} - \tilde{X}_s^{(\psi, \phi)}|^2 \right] ds \\
& \leq \mathcal{K} \Delta + \mathcal{K}' R^{-1},
\end{aligned} \tag{4.33}$$

with $\mathcal{K} = \mathcal{K}(R, N) > 0$ depending neither on ψ, ϕ, Δ . $\mathcal{K}'$ is independent of ψ, ϕ, Δ , and R . For the term H_2^3 , with Assumption (S2), we have

$$H_2^3 \leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \int_0^\tau \frac{|\bar{f}(\hat{X}_s^{(\psi, \phi)}) - \bar{f}(\tilde{X}_s^{(\psi)})|^2}{(\tau - s)^{1 - \beta + \alpha}} ds \right]$$

$$\begin{aligned}
&\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \int_0^\tau \frac{e^{-\lambda(\tau-s)} e^{-\lambda s}}{(\tau-s)^{1-\beta+\alpha}} |\hat{X}_s^{(\psi, \phi)} - \tilde{X}_s^{(\psi)}|^2 ds \right] \\
&\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, \tau]}^2 \right] \sup_{\tau \in [0, T]} \int_0^\tau \frac{e^{-\lambda(\tau-r)}}{(\tau-r)^{1-\beta+\alpha}} dr \\
&\leq \mathcal{K} \lambda^{\alpha-\beta} \mathbb{E} \left[\sup_{\tau \in [0, T]} e^{-\lambda \tau} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, \tau]}^2 \right]. \tag{4.34}
\end{aligned}$$

Taking the manner in (4.23), we get

$$\begin{aligned}
H_3 &\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \int_0^\tau \frac{e^{-\lambda \tau}}{(\tau-r)^{1-\beta+2\alpha}} + \frac{e^{-\lambda \tau}}{r^{1-\beta+\alpha}} \|\sigma(\hat{X}_r^{(\psi, \phi)}) - \sigma(\tilde{X}_r^{(\psi)})\|_{\alpha, [0, r]} dr \right|^2 \right] \\
&\leq \mathcal{K} \mathbb{E} \left[\sup_{\tau \in [0, T]} \left| \int_0^\tau \frac{e^{-\lambda \tau} \left(1 + \Delta(\hat{X}_r^{(\psi, \phi)}) + \Delta(\tilde{X}_r^{(\psi)}) \right) \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, r]}}{(\tau-r)^{1-\beta+2\alpha}} \right. \right. \\
&\quad \left. \left. + \frac{e^{-\lambda \tau} \left(1 + \Delta(\hat{X}_r^{(\psi, \phi)}) + \Delta(\tilde{X}_r^{(\psi)}) \right) \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, r]}}{r^{1-\beta+\alpha}} dr \right|^2 \right].
\end{aligned}$$

Here, $\Delta(\hat{X}_r^{(\psi, \phi)}) = \int_0^r \frac{|\hat{X}_r^{(\psi, \phi)} - \hat{X}_z^{(\psi, \phi)}|}{(r-z)^{1+\alpha}} dz$, $\Delta(\tilde{X}_r^{(\psi)}) = \int_0^r \frac{|\tilde{X}_r^{(\psi)} - \tilde{X}_z^{(\psi)}|}{(r-z)^{1+\alpha}} dz$. The two quantities above are bounded by $\|\hat{X}^{(\psi, \phi)}\|_{\alpha, \infty}$ and $\|\tilde{X}^{(\psi)}\|_{\alpha, \infty}$, respectively, each of which is in turn controlled by a constant $\mathcal{K} > 0$ independent of ψ, ϕ , and Δ .

Then, we have

$$H_3 \leq \mathcal{K} \lambda^{2\alpha-\beta} \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\hat{X}_s^{(\psi, \phi)} - \tilde{X}_s^{(\psi)}\|_{\alpha}^2 \right]. \tag{4.35}$$

Combining (4.31)–(4.35), we get

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, \tau]}^2 \right] \leq \mathcal{K} \lambda^{2\alpha-\beta} \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, \tau]}^2 \right] + \mathcal{K} (\phi \Delta^{-1} + \Delta) + \mathcal{K}' R^{-1}.$$

By choosing λ sufficiently large so that $\mathcal{K} \lambda^{2\alpha-\beta} < 1$, it follows that

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \|\hat{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha, [0, \tau]}^2 \right] \leq \mathcal{K} (\phi \Delta^{-1} + \Delta) + \mathcal{K}' R^{-1},$$

with $\mathcal{K} = \mathcal{K}(R, N) > 0$ depending neither on ψ, ϕ, Δ . $\mathcal{K}'$ is independent of ψ, ϕ, Δ and R .

From the above estimates together with the continuous inclusions $C^{\alpha+\theta} \subset W_0^{\alpha, \infty} \subset C^{\alpha-\theta}$, it follows that

$$\lim_{\psi \rightarrow 0} \mathbb{E} [\|\tilde{X}^{(\psi, \phi)} - \tilde{X}^{(\psi)}\|_{\alpha\text{-Hölder}}^2] = 0.$$

Specifically, our approach is as follows: We set $\Delta = \psi$, and for every fixed large R , we take $\lim_{\psi \rightarrow 0} \sup$. Subsequently, we allow R to approach infinity. It should be noted that $\mathcal{K}'$ is independent of R . If $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \in \bar{\mathcal{A}}_b^N$, $(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)}) \xrightarrow{\text{weakly}} (\varpi, \varphi)$ as $\psi \rightarrow 0$, then $\{\tilde{X}^{(\psi)}\} = \{\mathcal{G}^0(\varpi^{(\psi, \phi)}, \varphi^{(\psi, \phi)})\}$ weakly converges to $\tilde{X} = \mathcal{G}^0(\varpi, \varphi)$ in $C^\alpha([0, T], \mathfrak{R}^m)$ as $\psi \rightarrow 0$. Subsequently, applying Portmanteau's theorem [38] to any continuous bounded function $Q : C^\alpha([0, T]; \mathfrak{R}^m) \rightarrow \mathfrak{R}$, it follows that as $\psi \rightarrow 0$,

$$|\mathbb{E}[Q(\tilde{X}^{(\psi, \phi)})] - \mathbb{E}[Q(\tilde{X})]| \leq |\mathbb{E}[Q(\tilde{X}^{(\psi, \phi)})] - \mathbb{E}[Q(\tilde{X}^{(\psi)})]| + |\mathbb{E}[Q(\tilde{X}^{(\psi)})] - \mathbb{E}[Q(\tilde{X})]|$$

$$\leq L_Q \mathbb{E}[\|\tilde{X}^{(\psi,\phi)} - \tilde{X}^{(\psi)}\|_{\alpha\text{-Hölder}}^2]^{\frac{1}{2}} + |\mathbb{E}[Q(\tilde{X}^{(\psi)})] - \mathbb{E}[Q(\tilde{X})]| \rightarrow 0,$$

with Q bounded and Lipschitz continuous, and L_Q denoting its Lipschitz constant. Then, we complete the proof of Proposition 2.1.

Step 3. Following the argument in [22, Theorem 3.1] and using Proposition 2.1, one obtains the LP. Setting $\mathcal{E} := C^\alpha([0, T]; \mathbb{R}^m)$, $\{X^{(\psi,\phi)}\}$ satisfies the LP:

$$\limsup_{\psi \rightarrow 0} -\psi^{2\beta-1} \log \mathbb{E} \left\{ \exp \left(\frac{-F(X^{(\psi,\phi)})}{\psi^{2\beta-1}} \right) \right\} \leq \inf_{x \in \mathcal{E}} [F(x) + I(x)] \quad (4.36)$$

and

$$\liminf_{\psi \rightarrow 0} -\psi^{2\beta-1} \log \mathbb{E} \left\{ \exp \left(\frac{-F(X^{(\psi,\phi)})}{\psi^{2\beta-1}} \right) \right\} \geq \inf_{x \in \mathcal{E}} [F(x) + I(x)] \quad (4.37)$$

Due to the equivalence between LP (4.36) and (4.37) and the LDP, we have obtained the following main result. $\square$

Theorem 4.5. Let H and β belong to $(1/2, 1)$ and $1 - H < \alpha < \min\{1/2, \beta/2\}$. Assume that hypotheses (S1)–(S3) are satisfied, and let $\phi = o(\psi)$ as $\psi \rightarrow 0$. Then, the random variable $X^{(\psi,\phi)}$ of Eq (1.1) satisfies the LDP on the $\mathcal{E}$ with the good rate function $I : \mathcal{E} \rightarrow [0, \infty]$:

$$\begin{aligned} I(\zeta) &= \inf \left\{ \frac{1}{2} \|\varpi\|_{\mathcal{H}_{cm}^{H,k_1}}^2 : \varpi \in \mathcal{H}_{cm}^{H,k_1} \text{ such that } \zeta = \mathcal{G}^0(\varpi, 0) \right\} \\ &= \inf \left\{ \frac{1}{2} \|(\varpi, \varphi)\|_{\mathcal{H}_{cm}}^2 : (\varpi, \varphi) \in \mathcal{H}_{cm} \text{ such that } \zeta = \mathcal{G}^0(\varpi, \varphi) \right\}, \quad \zeta \in \mathcal{E}. \end{aligned}$$

The equivalence of these two expressions follows from the fact that the skeleton map $\mathcal{G}^0$ is independent of ϕ (see Section 4.2). Hence, for any fixed ϖ , the infimum over ϕ is attained at $\phi = 0$, as $\|\phi\|_{\mathcal{H}_{cm}^{\frac{1}{2}, k_2}}^2 \geq 0$ with equality only at $\phi = 0$.

5. Example

This section presents a numerical example to illustrate the main result. Without loss of generality, we restrict attention to the one-dimensional setting and consider the following FSDEs:

$$\begin{cases} {}^c D_{0+}^{0.8} X_\tau^{(\psi,\phi)} = (-X_\tau^{(\psi,\phi)} + \sin Y_\tau^{(\psi,\phi)}) d\tau + \psi^{0.3} \cdot \frac{1}{2} \arctan X_\tau^{(\psi,\phi)} dW_\tau^H, \\ \phi dY_\tau^{(\psi,\phi)} = (X_\tau^{(\psi,\phi)} - Y_\tau^{(\psi,\phi)}) d\tau + \sqrt{\phi} dW_\tau. \end{cases} \quad (5.1)$$

The coefficients in Eq (5.1) are easily seen to satisfy hypotheses (S1)–(S3). By freezing the slow variable X_τ , we obtain the frozen equation as follows:

$$d\tilde{Y}_\tau = (X - \tilde{Y}_\tau) d\tau + dW_\tau. \quad (5.2)$$

Clearly, this is the Ornstein–Uhlenbeck (OU) process, where the invariant measure μ_X follows a Gaussian distribution with mean X and variance $1/2$. Therefore, leveraging the properties of the Gaussian distribution, we have

$$\bar{f}(X) = \int_{\mathbb{R}} f(X, Y) \mu_X(dY) = -X + \mathbb{E}_{Y \sim \mathcal{N}(X, \frac{1}{2})} [\sin Y].$$

To evaluate the expectation, we use the characteristic function of the Gaussian distribution:

$$\mathbb{E}[e^{iY}] = e^{iX - \frac{1}{2} \cdot \frac{1}{2}} = e^{iX - \frac{1}{4}},$$

where $i = \sqrt{-1}$. Taking the imaginary part yields

$$\mathbb{E}[\sin Y] = \text{Im}(\mathbb{E}[e^{iY}]) = \text{Im}(e^{iX - \frac{1}{4}}) = e^{-\frac{1}{4}} \sin X.$$

Consequently,

$$\bar{f}(X) = -X + e^{-\frac{1}{4}} \sin X.$$

Thus, the deterministic skeleton equation associated to (5.1) is

$${}^C D_{0+}^\beta \tilde{X}_\tau = -\tilde{X}_\tau + e^{-\frac{1}{4}} \sin \tilde{X}_\tau + \frac{1}{2} \arctan \tilde{X}_\tau d\varpi_\tau, \quad \tilde{X}_0 = X_0. \quad (5.3)$$

According to Theorem 4.5, the solution of (5.1) satisfies the LDP on $C^\alpha([0, T])$. The corresponding rate function is

$$I(\zeta) = \inf \left\{ \frac{1}{2} \|\varpi\|_{\mathcal{H}_{cm}^{H, k_1}}^2 : \varpi \in \mathcal{H}_{cm}^{H, k_1} \text{ such that } \zeta = \mathcal{G}^0(\varpi) \right\},$$

where $\mathcal{G}^0(\varpi)$ is the solution of (5.3).

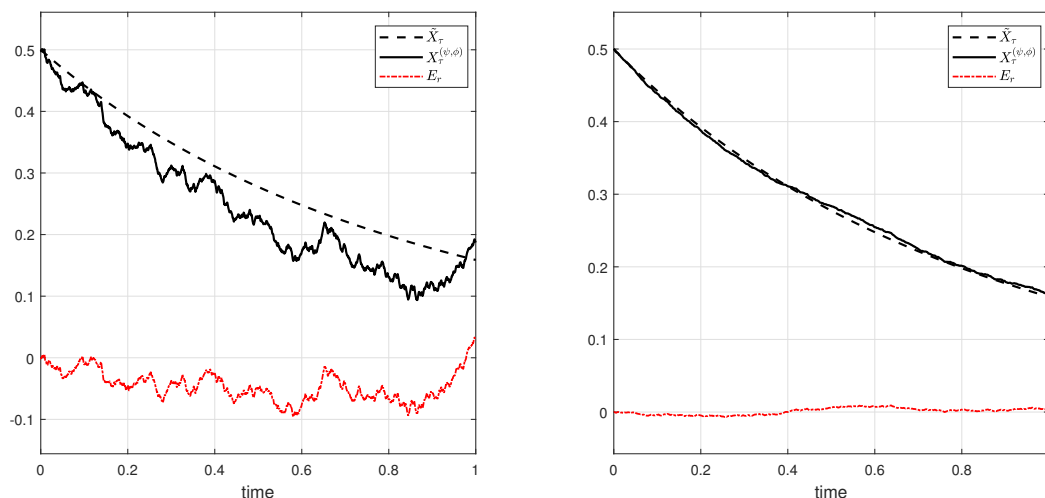


Figure 1. Comparison of $X_\tau^{(\psi, \phi)}$ and $\tilde{X}$ under different noise intensity, where $\psi=0.1$ in the left figure, $\psi=0.01$ in the right figure, and the Hurst index $H = 0.7$.

The LDP is to study the probability characteristics of extreme events, especially the paths of events and the exponential decay of probabilities. By defining an action functional to describe the probability of system trajectory deviations from deterministic paths, it transforms the calculation of this probability into a variational problem. The action functional $I(\varpi)$ is defined on the system's possible paths ϖ , measuring the degree of deviation of ϖ from the deterministic path. For any deviation ϖ ,

$$\mathbb{P}(X^{(\psi, \phi)} \approx \varpi) \sim \exp\left(-\frac{I(\varpi)}{\psi^{2\beta-1}}\right),$$

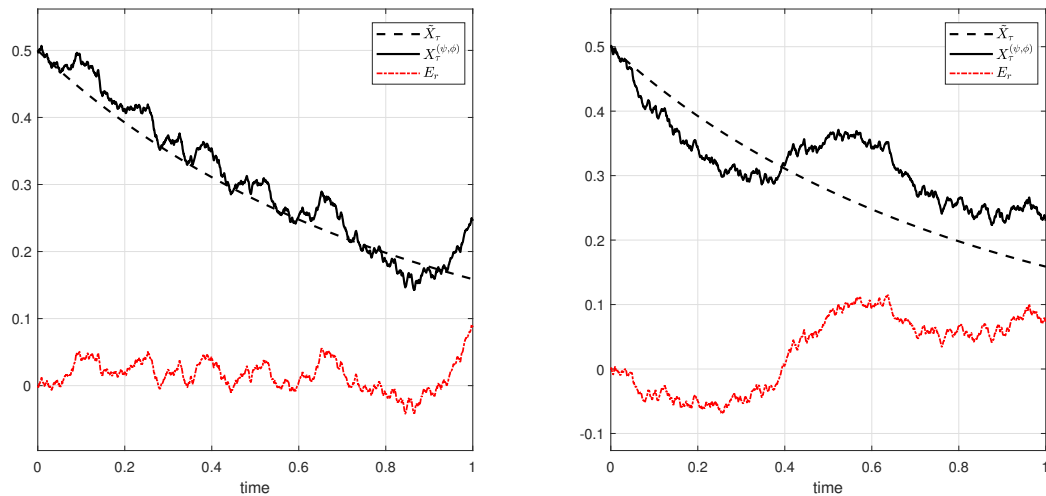


Figure 2. Comparison of $X_{\tau}^{(\psi, \phi)}$ and $\tilde{X}$ under different Hurst index, where $H = 0.8$ in the left figure, $H = 0.6$ in the right figure, and the noise intensity $\psi = 0.1$.

where I is a fixed rate function independent of ψ . Thus, as $\psi \rightarrow 0$, the probability of any non-zero deviation decays exponentially, and trajectories concentrate near the skeleton. The smaller ψ , the stronger this concentration effect. Obviously, when $I(\varpi) = 0$, the corresponding path is deterministic. We now present a numerical illustration of the theoretical result. Specifically, we compare sample paths of the slow process $X_{\tau}^{(\psi, \phi)}$ for two different noise intensities, $\psi = 0.1$ and $\psi = 0.01$, with the deterministic skeleton solution $\tilde{X}_t$ obtained from (5.3). As shown in Figure 1, when $\psi = 0.01$ (right panel), the stochastic trajectory is visibly closer to the deterministic skeleton than when $\psi = 0.1$ (left panel). This is consistent with the LDP prediction that, as $\psi \rightarrow 0$, the probability of large deviations from the skeleton decays exponentially, and typical trajectories concentrate near the skeleton. When we keep the value of $\psi = 0.1$, we can observe from Figure 2 that as the Hurst parameter H decreases, the degree of fitting between the original system and the skeleton system deteriorates. By taking a smaller value of ψ can a sufficiently good fitting effect be achieved. This indicates that the value of the Hurst parameter affects the fitting effect of the LDP. Therefore, in complex stochastic systems driven by FBM, good fitting results occur when the noise intensity is low, and the Hurst parameter is large.

The numerical simulation in Figures 1 and 2 is performed on $[0, T]$ with $T = 1$, using initial conditions $X_0 = 0.5$, $Y_0 = 0$, and a uniform time step $\Delta\tau = 10^{-3}$. The Caputo derivative is discretized using the L1 scheme [39]. The fractional Brownian motion with H is generated via the Cholesky decomposition method based on the covariance function $\mathbb{E}[W_{\tau}^H W_s^H] = (\tau^{2H} + s^{2H} - |\tau - s|^{2H})/2$.

6. Conclusions

In this paper, we have made theoretical and modeling progress on the LDP for FSDEs. We now return to the main questions raised in the Introduction and summarize how they have been addressed. The main mathematical difficulties introduced by the fractional derivative and the slow-fast structure are that the nonlocal Caputo derivative requires a fractional-kernel-compatible averaging argument for the controlled fast process, and the mixed noise introduces coupled controls that must be handled

simultaneously in the weak-convergence framework. These challenges are overcome through the estimates in Lemmas 4.1–4.4 and the convergence proof in Section 4.2. First, unlike previous work, we have fully incorporated the influence of fractional derivatives on the asymptotic behavior of deviations and readjusted the formulation of FSDEs, thereby providing a more accurate analytical framework for long-memory stochastic systems. Second, we have extended the research scope of the LDP for FSDEs from BM to FBM. This extension fills a gap in the existing literature by systematically addressing the singularity of noise types and the characterization of non-Markovian features, which were previously not well-covered. Third, we have integrated the coupling dynamics of fast and slow variables into our analytical framework. This makes the established mathematical model more closely connected to complex physical phenomena, such as those observed in climate systems and multiscale material processes. Despite these contributions, our current framework is limited to Gaussian noise settings. Looking forward, two important directions for future work are the inclusion of non-Gaussian noise scenarios and the development of distribution-dependent fractional systems, which would further generalize the theory and broaden its applicability.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. M. I. Freidlin, A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer, New York, 1998. <https://doi.org/10.1007/978-1-4612-0611-8>
2. A. Budhiraja, P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, *Ann. Probab.*, **26** (1998), 1641–1659. <https://doi.org/10.1214/aop/1022855876>
3. K. Balachandran, J. Y. Park, J. J. Trujillo, Controllability of nonlinear fractional dynamical systems, *Nonlinear Anal. Theory Methods Appl.*, **75** (2012), 1919–1926. <https://doi.org/10.1016/j.na.2011.09.042>
4. X. Q. Fang, H. W. Ma, C. S. Zhu, Nonlocal multi-fields coupling response of a piezoelectric semiconductor nanofiber under shear force, *Mech. Adv. Mater. Struct.*, **31** (2024), 2452–2459. <https://doi.org/10.1080/15376494.2022.2158503>

5. M. A. E. Abdelrahman, M. A. Sohaly, Y. F. Alharbi, Fundamental stochastic solutions for the conformable fractional NLSE with spatiotemporal dispersion via exponential distribution, *Phys. Scr.*, **96** (2021), 125223. <https://doi.org/10.1088/1402-4896/ac119c>
6. R. Huang, D. Luo, Strong averaging principle for generalized Caputo fractional stochastic neutral differential equations driven by multiplicative fractional Brownian motion, *Chaos, Solitons Fractals*, **201** (2025), 117179. <https://doi.org/10.1016/j.chaos.2025.117179>
7. M. Suvinthra, K. Balachandran, R. M. Lizzy, Large deviations for stochastic fractional integrodifferential equations, *AIMS Math.*, **2** (2017), 348–364. <https://doi.org/10.3934/Math.2017.2.348>
8. A. S. Ranjani, M. Suvinthra, K. Balachandran, Y. K. Ma, Analysis of stochastic neutral fractional functional differential equations, *Boundary Value Probl.*, **2022** (2022), 49. <https://doi.org/10.1186/s13661-022-01628-8>
9. A. S. Ranjani, M. Suvinthra, Large deviations for stochastic fractional pantograph differential equation, *Int. J. Dyn. Control*, **12** (2024), 136–147. <https://doi.org/10.1007/s40435-023-01339-7>
10. J. Chen, H. Gu, Y. Yan, L. Liu, Large deviation principle for Hilfer fractional stochastic McKean-Vlasov differential equations, *Fractal Fract.*, **9** (2025), 544. <https://doi.org/10.3390/fractalfract9080544>
11. Y. Kifer, Averaging and climate models, in *Stochastic Climate Models, Progress in Probability*, Birkhäuser, Basel, **49** (2001). https://doi.org/10.1007/978-3-0348-8287-3_7
12. O. E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics, *J. R. Stat. Soc. B*, **63** (2001), 167–241. <https://doi.org/10.1111/1467-9868.00282>
13. K. Al-Naimee, F. Marino, M. Ciszak, R. Meucci, F. T. Arecchi, Chaotic spiking and incomplete homoclinic scenarios in semiconductor lasers with optoelectronic feedback, *New J. Phys.*, **11** (2009), 073022. <https://doi.org/10.1088/1367-2630/11/7/073022>
14. M. Krupa, N. Popović, N. Kopell, H. G. Rotstein, Mixed-mode oscillations in a three time-scale model for the dopaminergic neuron, *Chaos*, **18** (2008), 015106. <https://doi.org/10.1063/1.2779859>
15. E. Pirozzi, Some fractional stochastic models for neuronal activity with different time-scales and correlated inputs, *Fractal Fract.*, **8** (2024), 57. <https://doi.org/10.3390/fractalfract8010057>
16. C. Ganguly, D. Chaudhuri, Stochastic thermodynamics of active Brownian particles, *Phys. Rev. E*, **88** (2013), 032102. <https://doi.org/10.1103/PhysRevE.88.032102>
17. Y. Liang, W. Wang, R. Metzler, A. G. Cherstvy, Nonergodicity of confined superdiffusive fractional Brownian motion, *Phys. Rev. E*, **108** (2023), L052101. <https://doi.org/10.1103/physreve.108.l052101>
18. Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Springer, Berlin, 2008. <https://doi.org/10.1007/978-3-540-75873-0>
19. A. Budhiraja, X. Song, *Large Deviation Principles for Functionals of Fractional Brownian Motions*, in *Frontiers of Statistics and Data Science*, Springer, Singapore, (2025), 101–131.
20. S. Bourguin, S. Gailus, K. Spiliopoulos, Typical dynamics and fluctuation analysis of slow–fast systems driven by fractional Brownian motion, *Stochastics Dyn.*, **21** (2021), 2150030. <https://doi.org/10.1142/s0219493721500301>

21. S. Gailus, I. Gasteratos, Large deviations of slow–fast systems driven by fractional Brownian motion, *Electron. J. Probab.*, **30** (2025), 1–56. <https://doi.org/10.1214/25-ejp1340>
22. Y. Inahama, Y. Xu, X. Yang, Large deviation principle for slow–fast system with mixed fractional Brownian motion, preprint, arXiv:2303.06626. <https://doi.org/10.48550/arXiv.2303.06626>
23. B. Pei, B. Schmalfuss, Y. Xu, Almost sure averaging for evolution equations driven by fractional Brownian motions, *SIAM J. Appl. Dyn. Syst.*, **23** (2024), 2807–2852. <https://doi.org/10.1137/23m1554448>
24. G. Shen, H. Zhou, J. L. Wu, Large deviation principle for multi-scale distribution-dependent stochastic differential equations driven by fractional Brownian motions, *J. Evol. Equations*, **24** (2024), 35. <https://doi.org/10.1007/s00028-024-00960-z>
25. X. Yang, Y. Xu, Large deviation principle for slow–fast rough differential equations via controlled rough paths, *Proc. R. Soc. Edinburgh Sect. A: Math.*, (2025), 1–31. <https://doi.org/10.1017/prm.2025.4>
26. J. Liu, H. Zhang, J. Wang, C. Jin, J. Li, W. Xu, A note on averaging principles for fractional stochastic differential equations, *Fractal Fract.*, **8** (2024), 216. <https://doi.org/10.3390/fractalfract8040216>
27. B. Pei, Y. Inahama, Y. Xu, Averaging principles for mixed fast-slow systems driven by fractional Brownian motion, *Kyoto J. Math.*, **63** (2023), 721–748. <https://doi.org/10.1215/21562261-2023-0001>
28. L. Decreusefond, A. S. Üstünel, Stochastic analysis of the fractional Brownian motion, *Potential Anal.*, **10** (1999), 177–214. <https://doi.org/10.1023/A:1008634027843>
29. X. Yang, Y. Inahama, Y. Xu, Moderate deviations for two-time scale systems with mixed fractional Brownian motion, *Appl. Math. Optim.*, **90** (2024), 18. <https://doi.org/10.1007/s00245-024-10159-w>
30. P. Dupuis, R. S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, John Wiley & Sons, New York, 1997. <https://doi.org/10.1002/9781118165904>
31. J. Guerra, D. Nualart, Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, *Stochastic Anal. Appl.*, **26** (2008), 1053–1075. <https://doi.org/10.1080/07362990802286483>
32. R. Wang, Y. Xu, B. Pei, Stochastic averaging for a type of fractional differential equations with multiplicative fractional Brownian motion, *Chaos*, **32** (2022), 093108. <https://doi.org/10.1063/5.0131433>
33. G. D. Prato, J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996. <https://doi.org/10.1017/CBO9780511662829>
34. J. Han, L. Yan, A time fractional functional differential equation driven by the fractional Brownian motion, *J. Appl. Anal. Comput.*, **9** (2019), 547–567. <https://doi.org/10.11948/2156-907x.20180068>
35. G. D. Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd edition, Cambridge University Press, Cambridge, 2014. <https://doi.org/10.1017/CBO9781107295513>
36. X. Sun, R. Wang, L. Xu, X. Yang, Large deviation for two-time-scale stochastic Burgers equation, *Stochastics Dyn.*, **21** (2021), 2150023. <https://doi.org/10.1142/s0219493721500234>
37. A. Rascanu, Differential equations driven by fractional Brownian motion, *Collect. Math.*, **53** (2002), 55–81.

-
38. A. Klenke, *Probability Theory: A Comprehensive Course*, 2nd edition, Springer London, 2014. <https://doi.org/10.1007/978-1-4471-5361-0>
39. Z. Z. Sun, X. N. Wu, A fully discrete difference scheme for a diffusion-wave system, *Appl. Numer. Math.*, **56** (2006), 193–209. <https://doi.org/10.1016/j.apnum.2005.03.003>



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