



Research article

Infinitely many sign-changing solutions for a variable-exponent quasilinear Schrödinger equation at the quartic threshold

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Abstract: We prove the existence of infinitely many sign-changing solutions for a variable-exponent modified quasilinear Schrödinger equation on a smooth bounded domain in three dimensions. The exponent is allowed to attain the quartic threshold on a nonempty null set. Consequently, the transformed nonlinearity is not uniformly superquadratic. The classical dual change of variables $v = h(u)$, where $h'(s) = (1 + s^2)^{1/2}$, converts the equation into a semilinear problem on $H_0^1(\Omega)$ with subcritical growth. An exact identity for h^{-1} provides the scaling monotonicity used in the Cerami boundedness argument and a quantitative logarithmic nonquadraticity remainder at the threshold. We establish the Cerami condition, prove the strict invariance of small neighborhoods of the positive and negative cones, and verify the hypotheses of an invariant-set symmetric minimax theorem. These steps yield an unbounded sequence of nodal critical values. A bootstrap argument and an admissible transformed test function then convert the critical points into weak solutions of the original equation. The main contribution is a nonperturbative treatment of the variable-exponent regime in which the quartic threshold is attained on a null set. The dual substitution and the invariant-set theorem are used as established tools.

Keywords: variable exponent; quasilinear Schrödinger equation; dual transformation; quartic threshold; sign-changing solution; invariant cone

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. We consider

$$\begin{cases} -\Delta u - \frac{1}{2}u\Delta(u^2) = |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

The exponent $q(x)$ is assumed to satisfy Condition (Q_0) ,

$$q \in C(\overline{\Omega}), \quad 4 \leq q \leq q^+ < 12, \quad (1.2)$$

where $q^+ = \max_{\overline{\Omega}} q(x)$, and condition (Q_1) is as follows:

$$D_0 := \{x \in \Omega : q(x) = 4\} \neq \emptyset, \quad |D_0| = 0. \quad (1.3)$$

Here and below, $|A|$ denotes the Lebesgue measure of a measurable set A .

Equation (1.1) has a variational form, but its formal energy

$$I(u) = \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 dx - \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx \quad (1.4)$$

is not naturally defined on all of $H_0^1(\Omega)$. The absence of a convenient linear energy space is a central difficulty in modified quasilinear Schrödinger equations. Modified nonlinear Schrödinger models continue to arise in diverse nonlinear-wave settings, including optical soliton, bifurcation, and fractional models [1, 2]. These works illustrate the breadth of modified Schrödinger-type structures, although they are not direct antecedents of the present variational theorem. The dual change of variables developed for scalar quasilinear Schrödinger equations transforms (1.4) into a smooth functional on $H_0^1(\Omega)$ [3, 4]. We adapt this mechanism to a variable exponent that reaches the quartic threshold.

The threshold has a transparent meaning after the transformation. If $q(x) > 4$, the transformed potential is superquadratic at x . If $q(x) = 4$, it is asymptotically quadratic. Thus, no uniform power-type Ambrosetti–Rabinowitz gap remains. Superlinear elliptic problems under weaker alternatives to classical superquadraticity assumptions were studied by Schechter and Zou [5]. The exact inverse transformation nevertheless retains two useful structures: a monotonicity relation under scalar dilation and a logarithmically divergent nonquadraticity remainder. The former is used directly to bound Cerami sequences, and the latter quantifies the residual gap at the threshold.

Elliptic equations have been studied using critical-point, genus, and invariant-set methods; see, for example, [6–9]. For background on invariant sets of descending flows and critical-point theory in partially ordered Hilbert spaces, see [10, 11]. A systematic treatment of sign-changing critical point theory is given in [12]. Chu et al. [8] treated a semilinear equation whose exponent reaches its quadratic threshold by adding a perturbation and then passing to the limit. Chu and Yu [9] considered a variable-exponent Kirchhoff problem and imposed additional structural assumptions adapted to its nonlocal term. For constant-exponent quasilinear Schrödinger equations, the dual substitution, Cerami deformation, and invariant-set machinery are already available [3, 13]. Related invariant-set approaches have also been used for supercritical quasilinear Schrödinger equations [14]. The closest available results therefore cover different parts of the present setting. The semilinear variable-exponent result of [8] includes a spatially varying exponent and infinitely many nodal solutions, but the principal operator is linear, and the threshold difficulty is handled by perturbation. The Kirchhoff result of [9] includes both a variable exponent and a nonlocal quasilinear feature, but its hypotheses impose measure and separation conditions tailored to the Kirchhoff term. The quasilinear Schrödinger multiplicity result of [13] develops the dual-transformation and Cerami-deformation mechanisms for constant-exponent or nonvariable nonlinearities, and [14] treated a supercritical quasilinear Schrödinger setting. None of these theorems directly yields infinitely many sign-changing solutions for the present variable-exponent

modified quasilinear Schrödinger equation when the exponent attains the quartic threshold on a null set, and no exponent perturbation is introduced. The issue addressed here is their compatibility with a spatially varying exponent that reaches the quartic threshold. The threshold set destroys uniform superquadraticity but, when it has measure zero, still permits the finite-dimensional descending geometry and cone estimates required by the invariant-set minimax theorem of Liu [15].

The proof proceeds as follows. Section 2 derives the transformed functional and its structural identities. Section 3 proves the Cerami condition. Section 4 establishes strict cone invariance and the associated admissible deformation. Section 5 states and verifies the precise symmetric invariant-set theorem. Section 6 supplies the regularity and testing details needed to recover the original equation.

The main result is stated below. By a weak solution of (1.1), we mean a function $u \in H_0^1(\Omega)$ with $u^2 \in H_0^1(\Omega)$ that satisfies (6.1) for every $\varphi \in C_c^\infty(\Omega)$. This is the natural energy class for (1.1), because the quasilinear part contains $(1 + u^2)|\nabla u|^2$. It therefore requires both $u \in H_0^1(\Omega)$ and $u^2 \in H_0^1(\Omega)$ for the two gradient terms in (6.1) to be meaningful.

Theorem 1.1. *Assume (1.2) and (1.3). Then, Problem (1.1) possesses infinitely many pairs of sign-changing weak solutions $\{\pm u_k\}_{k \geq 1}$. If $v_k = h(u_k)$ denotes the corresponding dual solution, then the dual critical values satisfy $J(v_k) \rightarrow +\infty$.*

The assumptions impose no quantitative rate at which $q(x)$ approaches four. In particular, the proof neither perturbs the exponent nor postulates separate pseudogradient, energy-barrier, or limit-preservation conditions. These properties are verified directly for the transformed functional. The nonemptiness of D_0 identifies the threshold case of interest. The argument also applies when D_0 is empty, provided that $|D_0| = 0$.

2. The dual transformation

Define

$$h(s) = \int_0^s \sqrt{1 + \tau^2} d\tau = \frac{1}{2}(s\sqrt{1 + s^2} + \operatorname{arsinh} s), \quad s \in \mathbb{R}, \quad (2.1)$$

and let $f = h^{-1}$. Then, h is an odd, increasing C^∞ diffeomorphism of $\mathbb{R}$, and the same is true of f .

Lemma 2.1. *For every $t \in \mathbb{R}$, the inverse transformation satisfies*

$$f'(t) = \frac{1}{\sqrt{1 + f(t)^2}}, \quad |f(t)| \leq |t|, \quad (2.2)$$

$$\frac{1}{2} \leq \frac{tf'(t)}{f(t)} \leq 1 \quad (t \neq 0), \quad (2.3)$$

$$\frac{tf'(t)}{f(t)} - \frac{1}{2} = \frac{\operatorname{arsinh} |f(t)|}{2|f(t)|\sqrt{1 + f(t)^2}} > 0 \quad (t \neq 0), \quad (2.4)$$

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} = 1, \quad \lim_{|t| \rightarrow \infty} \frac{|f(t)|}{\sqrt{|t|}} = \sqrt{2}. \quad (2.5)$$

Moreover, there are constants $c_0, C_0 > 0$ such that

$$c_0 \min\{|t|, |t|^{1/2}\} \leq |f(t)| \leq C_0 \min\{|t|, |t|^{1/2}\}, \quad (2.6)$$

for all $t \in \mathbb{R}$.

Proof. The first identity follows by differentiating $h(f(t)) = t$. For $s > 0$, Formula (2.1) gives

$$\frac{h(s)}{s\sqrt{1+s^2}} = \frac{1}{2} + \frac{\operatorname{arsinh} s}{2s\sqrt{1+s^2}},$$

which lies in $(1/2, 1]$. Taking $s = f(t)$ proves (2.3) and (2.4); oddness covers $t < 0$. The remaining statements follow from $h(s) \sim s$ as $s \rightarrow 0$ and $h(s) \sim s|s|/2$ as $|s| \rightarrow \infty$. $\square$

Let $E = H_0^1(\Omega)$ with norm $\|v\| = (\int_{\Omega} |\nabla v|^2)^{1/2}$. Set

$$G(x, t) = \frac{|f(t)|^{q(x)}}{q(x)}, \quad g(x, t) = |f(t)|^{q(x)-2} f(t) f'(t), \quad (2.7)$$

and define

$$J(v) = \frac{1}{2} \|v\|^2 - \int_{\Omega} G(x, v) dx. \quad (2.8)$$

By (1.2), (2.6), and Sobolev embedding, $J \in C^1(E, \mathbb{R})$ is even, and

$$\langle J'(v), \varphi \rangle = \int_{\Omega} \nabla v \cdot \nabla \varphi dx - \int_{\Omega} g(x, v) \varphi dx. \quad (2.9)$$

With

$$r := \max\{4, q^+/2\} < 6, \quad (2.10)$$

there is $C > 0$ such that

$$0 \leq G(x, t) \leq C(|t|^4 + |t|^r), \quad |g(x, t)| \leq C(|t|^3 + |t|^{r-1}). \quad (2.11)$$

Critical points of J solve the transformed problem

$$\begin{cases} -\Delta v = g(x, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

Lemma 2.2. *For every $x \in \overline{\Omega}$, the function $t \mapsto g(x, t)/t$ is nondecreasing on $(0, \infty)$. Consequently,*

$$\mathcal{D}(x, t) := \frac{1}{2} g(x, t) t - G(x, t) \geq 0, \quad (2.13)$$

and $t \mapsto \mathcal{D}(x, t)$ is nondecreasing on $[0, \infty)$.

Proof. Fix x , and write $p = q(x) \geq 4$ and $s = f(t) > 0$. Logarithmic differentiation shows that the sign of $(g(x, t)/t)'$ agrees with the sign of

$$\frac{p-1}{s} - \frac{s}{1+s^2} - \frac{\sqrt{1+s^2}}{h(s)}. \quad (2.14)$$

Because

$$h(s) \geq \frac{s(1+s^2)^{3/2}}{3+2s^2},$$

Expression (2.14) is non-negative for $p \geq 4$. The displayed elementary inequality follows by differentiating the difference between its two sides, which vanishes at zero. Indeed, if

$$H(s) = h(s) - \frac{s(1+s^2)^{3/2}}{3+2s^2},$$

then $H(0) = 0$, and

$$H'(s) = \sqrt{1+s^2} \frac{2(3+s^2)}{(3+2s^2)^2} \geq 0, \quad s \geq 0.$$

This proves the auxiliary inequality used above. The assertions about $\mathcal{D}$ follow from $G_t = g$, $G(x, 0) = 0$, and

$$\partial_t \mathcal{D}(x, t) = \frac{t^2}{2} \frac{d}{dt} \left(\frac{g(x, t)}{t} \right)$$

at points of differentiability. Continuity then completes the argument. $\square$

Lemma 2.3. *Uniformly in $x \in \overline{\Omega}$,*

$$\mathcal{D}(x, t) \longrightarrow +\infty \quad \text{as } |t| \rightarrow \infty. \quad (2.15)$$

More precisely, if $s = |f(t)| \geq 1$, then

$$2\mathcal{D}(x, t) \geq \frac{s^{q(x)-1} \operatorname{arsinh} s}{2\sqrt{1+s^2}} \geq cs^2 \log(1+s) \quad (2.16)$$

for a constant $c > 0$ independent of x .

Proof. Using (2.4) and $q(x) \geq 4$, we obtain

$$\begin{aligned} 2\mathcal{D}(x, t) &= s^{q(x)} \left(\frac{|t|f'(t)}{s} - \frac{2}{q(x)} \right) \\ &\geq s^{q(x)} \left(\frac{|t|f'(t)}{s} - \frac{1}{2} \right) = \frac{s^{q(x)-1} \operatorname{arsinh} s}{2\sqrt{1+s^2}}. \end{aligned}$$

The last bound follows from $q(x) \geq 4$ and $s \geq 1$. $\square$

Lemma 2.3 is retained to record the borderline information that remains exactly at the quartic threshold. It is not an independent hidden ingredient in the Cerami proof below, which uses the scaling monotonicity of $\mathcal{D}$. Rather, it shows that after the dual transformation the threshold case is not genuinely quadratic. A logarithmic nonquadraticity remainder still survives. This quantitative description clarifies why the quartic case is borderline and may be useful in variants where the monotonicity argument has to be replaced by a more direct coercivity estimate.

3. Cerami compactness

Recall that a sequence $\{v_n\} \subset E$ is a Cerami sequence at level c if

$$J(v_n) \rightarrow c, \quad (1 + \|v_n\|) \|J'(v_n)\|_{E^*} \rightarrow 0. \quad (3.1)$$

We first record a scaled compactness fact used in the boundedness argument.

Lemma 3.1. *If $w_n \rightarrow 0$ in E , then, for every fixed $R > 0$,*

$$\int_{\Omega} G(x, R w_n) dx \rightarrow 0. \quad (3.2)$$

Proof. The compact embeddings $E \hookrightarrow L^4(\Omega)$ and $E \hookrightarrow L^r(\Omega)$ give $w_n \rightarrow 0$ in both spaces. Hence, (2.11) implies

$$0 \leq \int_{\Omega} G(x, R w_n) dx \leq C_R (\|w_n\|_4^4 + \|w_n\|_r^r) \rightarrow 0.$$

□

Lemma 3.2. *Let $p > 4$, and put $G_p(t) = |f(t)|^p/p$. Then,*

$$\frac{G_p(t)}{t^2} \rightarrow +\infty \quad \text{as } |t| \rightarrow \infty.$$

More precisely, for every $\sigma \in (0, \sqrt{2})$, there exists $T_\sigma > 0$ such that $|f(t)| \geq \sigma|t|^{1/2}$ for $|t| \geq T_\sigma$, and hence,

$$\frac{G_p(t)}{t^2} \geq \frac{\sigma^p}{p} |t|^{p/2-2} \quad (|t| \geq T_\sigma).$$

Proof. The estimate follows immediately from $|f(t)|/|t|^{1/2} \rightarrow \sqrt{2}$ in (2.5). Because $p/2 - 2 > 0$, the right-hand side tends to $+\infty$. □

Proposition 3.3. *The functional J satisfies the Cerami condition: Every sequence satisfying (3.1) has a strongly convergent subsequence in E .*

Proof. We first prove boundedness. Suppose, to the contrary, that $\|v_n\| \rightarrow \infty$, and set $w_n = v_n/\|v_n\|$. Passing to a subsequence, we may assume that $w_n \rightharpoonup w$ in E , $w_n \rightarrow w$ in $L^s(\Omega)$ for $s < 6$, and $w_n(x) \rightarrow w(x)$ almost everywhere.

If $w \neq 0$, then w is nonzero on a set of positive measure. Because $|D_0| = 0$, there is a subset of positive measure on which both $w(x) \neq 0$, and $q(x) > 4$. By (2.5),

$$\frac{G(x, v_n(x))}{\|v_n\|^2} = \frac{G(x, v_n(x))}{|v_n(x)|^2} |w_n(x)|^2 \rightarrow +\infty.$$

This step is purely pointwise. At each such x , the fixed exponent $p = q(x) > 4$ is covered by Lemma 3.2, and $|v_n(x)| = \|v_n\| |w_n(x)| \rightarrow \infty$. No lower bound uniform over all points with $q(x) > 4$ is required or asserted, as $q(x)$ may approach four arbitrarily closely. Fatou's lemma contradicts

$$\frac{J(v_n)}{\|v_n\|^2} = \frac{1}{2} - \int_{\Omega} \frac{G(x, v_n)}{\|v_n\|^2} dx \rightarrow 0.$$

Hence, $w = 0$.

Let $t_n \in [0, 1]$ maximize $t \mapsto J(tv_n)$. Given $R > 0$, for n large enough $R/\|v_n\| \leq 1$, Lemma 3.1 yields

$$J(t_n v_n) \geq J(R w_n) = \frac{R^2}{2} - o(1).$$

Because R is arbitrary, $J(t_n v_n) \rightarrow +\infty$. On the other hand, $t_n \in (0, 1)$ for all sufficiently large n , and therefore,

$$\langle J'(t_n v_n), t_n v_n \rangle = 0.$$

Because $G(x, \cdot)$ and $\mathcal{D}(x, \cdot)$ are even, Lemma 2.2 also compares $\mathcal{D}(x, t_n v_n)$ and $\mathcal{D}(x, v_n)$, where $v_n < 0$. Using (3.1) and $0 < t_n < 1$, we obtain

$$\begin{aligned} J(t_n v_n) &= \int_{\Omega} \mathcal{D}(x, t_n v_n) dx \\ &\leq \int_{\Omega} \mathcal{D}(x, v_n) dx \\ &= J(v_n) - \frac{1}{2} \langle J'(v_n), v_n \rangle = O(1), \end{aligned}$$

which is a contradiction. Thus, every Cerami sequence is bounded.

Now, let $v_n \rightharpoonup v$ in E . After taking a subsequence, $v_n \rightarrow v$ in $L^s(\Omega)$ for every $s < 6$ and almost everywhere(a.e.). Because $r - 1 < 5$, use (2.11) together with Vitali's theorem to obtain

$$g(x, v_n) \longrightarrow g(x, v) \quad \text{in } L^{6/5}(\Omega).$$

Indeed, $|g(x, v_n)|^{6/5}$ is uniformly integrable because both exponents, $18/5$ and $6(r - 1)/5$, are strictly below six. Explicitly, if $0 < \alpha < 6$, and $A \subset \Omega$ is measurable, then the boundedness of $\{v_n\}$ in $L^6(\Omega)$ gives

$$\int_A |v_n|^\alpha dx \leq \|v_n\|_6^\alpha |A|^{1-\alpha/6} \leq C |A|^{1-\alpha/6}.$$

Applying this estimate with $\alpha = 18/5$ and $\alpha = 6(r - 1)/5$ to the two terms in (2.11) gives uniform integrability of $|g(x, v_n)|^{6/5}$. Together with the a.e. convergence of $g(x, v_n)$, Vitali's theorem yields the stated strong convergence in $L^{6/5}(\Omega)$. Consequently, the limit v satisfies $J'(v) = 0$. Finally,

$$\begin{aligned} \|v_n - v\|^2 &= \langle J'(v_n) - J'(v), v_n - v \rangle \\ &+ \int_{\Omega} (g(x, v_n) - g(x, v))(v_n - v) dx = o(1), \end{aligned}$$

where Hölder's inequality and the boundedness of $v_n - v$ in L^6 were used in the last step. Thus, $v_n \rightarrow v$ strongly in E . $\square$

4. Invariant cones

Let

$$P = \{v \in E : v \geq 0 \text{ a.e. in } \Omega\}, \quad P_\delta = \{v \in E : \text{dist}(v, P) \leq \delta\}.$$

Thus, P_δ and $-P_\delta$ are closed cone neighborhoods. Their interiors are denoted by $P_\delta^\circ = \{v : \text{dist}(v, P) < \delta\}$ and $-P_\delta^\circ$, respectively. Define the compact operator $K : E \rightarrow E$ by

$$K(v) = (-\Delta)^{-1} g(x, v). \quad (4.1)$$

Under the Riesz identification of E with E^* , we have $J'(v) = v - K(v)$.

Lemma 4.1. *The operator K is odd, compact, and locally Lipschitz on bounded subsets of E .*

Proof. The oddness of f and the evenness of f' imply that $g(x, \cdot)$, and hence, K is odd. Let $\{v_n\}$ be bounded in E . After passing to a subsequence, $v_n \rightarrow v$ in E , $v_n \rightarrow v$ in $L^s(\Omega)$ for each $s < 6$, and $v_n(x) \rightarrow v(x)$ a.e. By (2.11) and Vitali's theorem, $g(x, v_n) \rightarrow g(x, v)$ in $L^{6/5}(\Omega)$. If $z_n = K(v_n)$, and $z = K(v)$, then

$$\|z_n - z\|^2 = \int_{\Omega} (g(x, v_n) - g(x, v))(z_n - z) dx \leq C \|g(x, v_n) - g(x, v)\|_{6/5} \|z_n - z\|,$$

and so $K(v_n) \rightarrow K(v)$ in E . This proves compactness.

For $s = |f(t)|$ and $p = q(x)$, direct differentiation gives

$$g_t(x, t) = \frac{s^{p-2}((p-1) + (p-2)s^2)}{(1+s^2)^2} \leq C(1 + |t|^{r-2}),$$

with the continuous extension at $t = 0$ and a constant independent of x . For local Lipschitz continuity, fix $R > 0$ and $\|u\|, \|v\| \leq R$. The mean-value theorem, (1.2), (2.2), and (2.6) give, for a constant C_R independent of u, v ,

$$|g(x, u) - g(x, v)| \leq C_R(1 + |u|^{r-2} + |v|^{r-2})|u - v| \quad \text{a.e. in } \Omega.$$

Because $r < 6$, the right-hand side belongs to $L^{6/5}(\Omega)$, and

$$\|g(x, u) - g(x, v)\|_{6/5} \leq C_R \|u - v\|.$$

Solving the Dirichlet equation as above yields $\|K(u) - K(v)\| \leq C_R \|u - v\|$. □

Proposition 4.2. *There exists $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0]$,*

$$K(P_\delta) \subset P_\delta^\circ, \quad K(-P_\delta) \subset -P_\delta^\circ. \quad (4.2)$$

Both inclusions are strict on the corresponding boundaries.

Proof. Write $d = \text{dist}(v, P)$. For every $w \in P$, one has $v^- \leq |v - w|$ almost everywhere. Hence, by Sobolev embedding and by taking the infimum over $w \in P$,

$$\|v^-\|_s \leq C_s d, \quad 2 \leq s \leq 6. \quad (4.3)$$

No estimate of the Dirichlet norm $\|v^-\|$ in terms of d is used. Let $z = K(v)$. Testing $-\Delta z = g(x, v)$ with z^- gives

$$\|z^-\|^2 \leq \int_{\Omega} |g(x, -v^-)| z^- dx. \quad (4.4)$$

Here, we used that $g(x, \cdot)$ is nondecreasing and odd. The monotonicity follows from Lemma 2.2, as $g(x, t)/t \geq 0$ and is nondecreasing for $t > 0$, and then oddness gives the negative half-line. Because $v \geq -v^-$ a.e., one has $g(x, v) \geq g(x, -v^-)$, and therefore, $-\int_{\Omega} g(x, v) z^- dx \leq -\int_{\Omega} g(x, -v^-) z^- dx$.

For $\eta > 0$, put

$$D_\eta = \{x \in \Omega : 4 \leq q(x) < 4 + \eta\}.$$

By continuity from above of Lebesgue measure and (1.3), $|D_\eta| \rightarrow 0$ as $\eta \rightarrow 0$. The following estimates describe the near-threshold contribution through $|D_{2\varepsilon}|^{2/3}$ and the power gain on its complement. We also give a direct contraction estimate below, which shows that cone invariance itself does not require $|D_0| = 0$. We may take $\varepsilon_0 = 1$. There is $C > 0$, independent of $\varepsilon \in (0, \varepsilon_0]$, such that for every $t \in \mathbb{R}$,

$$|g(x, t)| \leq C(|t| + |t|^{1+\varepsilon}) \quad x \in D_{2\varepsilon}, \quad (4.5)$$

$$|g(x, t)| \leq C(|t|^{1+\varepsilon} + |t|^{r-1}) \quad x \in \Omega \setminus D_{2\varepsilon}. \quad (4.6)$$

Indeed, $|g(x, t)| \leq |f(t)|^{q(x)-1} f'(t)$, and $f'(t) \leq C(1 + |f(t)|)^{-1}$. Hence, $|g(x, t)| \leq C|f(t)|^{q(x)-1}$ for $|t| \leq 1$, and $|g(x, t)| \leq C|f(t)|^{q(x)-2}$ for $|t| \geq 1$. By (2.6), if $x \in D_{2\varepsilon}$, then $q(x) \leq 4 + 2\varepsilon$, and consequently, $|g(x, t)| \leq C|t|$ for $|t| \leq 1$, and $|g(x, t)| \leq C|t|^{(q(x)-2)/2} \leq C|t|^{1+\varepsilon}$ for $|t| \geq 1$. This proves (4.5). If $x \notin D_{2\varepsilon}$, then $q(x) \geq 4 + 2\varepsilon$. For $|t| \leq 1$, we have $|g(x, t)| \leq C|t|^{q(x)-1} \leq C|t|^{1+\varepsilon}$, and for $|t| \geq 1$, we have $|g(x, t)| \leq C|t|^{(q(x)-2)/2} \leq C|t|^{r-1}$ because $q(x) \leq q^+$, and $r = \max\{4, q^+/2\}$. The constants depend only on q^+ and the constants in (2.6), so they are uniform for $0 < \varepsilon \leq \varepsilon_0$. On $D_{2\varepsilon}$, Hölder's inequality gives

$$\int_{D_{2\varepsilon}} v^- z^- dx \leq |D_{2\varepsilon}|^{2/3} \|v^-\|_6 \|z^-\|_6.$$

For the power $1 + \varepsilon$, we use $\|(v^-)^{1+\varepsilon}\|_{6/5} = \|v^-\|_{6(1+\varepsilon)/5}^{1+\varepsilon}$; reducing ε_0 if necessary makes this exponent at most six. If the exponent is below two, boundedness of Ω gives

$$\|v^-\|_s \leq |\Omega|^{1/s-1/2} \|v^-\|_2 \leq C_s d, \quad 1 \leq s < 2.$$

These constants can be bounded uniformly for the exponents used here. The same calculation applies to $r - 1 < 5$. Combining these estimates with (4.3) yields

$$\text{dist}(K(v), P) \leq \|z^-\| \leq C(|D_{2\varepsilon}|^{2/3} d + d^{1+\varepsilon} + d^{r-1}). \quad (4.7)$$

For the contraction, one can use the global growth estimate (2.11) directly in (4.4) to obtain

$$\|z^-\|^2 \leq C \left(\|v^-\|_{18/5}^3 + \|v^-\|_{6(r-1)/5}^{r-1} \right) \|z^-\|_6 \leq C(d^3 + d^{r-1}) \|z^-\|.$$

Both exponents lie in $[2, 6)$ because $4 \leq r < 6$. Thus,

$$\text{dist}(K(v), P) \leq \|z^-\| \leq C(d^3 + d^{r-1}). \quad (4.8)$$

Choose $\delta_0 > 0$ so that $C((2\delta_0)^2 + (2\delta_0)^{r-2}) \leq 1/2$. Then, $\text{dist}(K(v), P) \leq d/2$ for $d \leq 2\delta_0$. In particular, this distance is less than δ whenever $d \leq \delta \leq \delta_0$. Oddness gives the second inclusion. Unlike the small-measure term in (4.7), (4.8) uses only (1.2). $\square$

Decrease δ_0 if necessary so that the positive barrier in Lemma 5.4 holds for $0 < \delta \leq \delta_0$. That lemma uses only (2.11) and (4.3). In the following lemma, the closed neighborhoods are preserved by the flow, and their open interiors are used in the abstract minimax theorem.

Lemma 4.3. Fix $0 < \delta \leq \delta_0$, set $U = P_\delta^\circ$, $Q = -P_\delta^\circ$, $W = U \cup Q$, and $c_* := \inf_{\partial U \cap \partial Q} J > 0$. For every $c \geq c_*$ there are $\epsilon_c > 0$ and a symmetric open neighborhood N of $K_c \setminus W$ with finite genus, where $K_c = \{v : J(v) = c, J'(v) = 0\}$, such that for every $0 < \epsilon < \epsilon_c$, there is an odd continuous map $\eta : E \rightarrow E$ with

$$\begin{aligned} \eta(P_\delta) &\subset P_\delta, & \eta(-P_\delta) &\subset -P_\delta, & \eta(U) &\subset U, & \eta(Q) &\subset Q, \\ \eta|_{J^{c-2\epsilon}} &= \text{id}, & \eta(J^{c+\epsilon} \setminus (N \cup W)) &\subset J^{c-\epsilon}. \end{aligned}$$

Here, $J^a = \{v : J(v) \leq a\}$, and $N = \emptyset$ if $K_c \setminus W = \emptyset$.

Proof. First observe that any locally Lipschitz field of the form $a(v)(K(v) - v)$, with $a \geq 0$, preserves both closed cone neighborhoods. Indeed, let π_P be the metric projection onto the closed convex cone P , and write $d = \text{dist}(v, P)$. The projection inequality and the choice of δ_0 give, for $d \leq 2\delta_0$,

$$\langle v - \pi_P v, K(v) - v \rangle_E \leq d \text{dist}(K(v), P) - d^2 \leq -\frac{1}{2}d^2.$$

Along a trajectory while $d \leq 2\delta_0$, differentiability of the squared distance gives $(d^2)' \leq -a(v)d^2 \leq 0$. A trajectory starting with $d \leq \delta \leq \delta_0$ therefore cannot increase its distance to P or reach $2\delta_0$. This proves invariance of both P_δ and its open interior. The same argument applies to $-P_\delta$.

The Cerami condition makes K_c compact. Proposition 4.2 excludes critical points on $\partial P_\delta \cup \partial(-P_\delta)$, as a critical point satisfies $v = K(v)$. Consequently $K^{\text{in}} := K_c \cap W$ and $K^{\text{out}} := K_c \setminus W$ are disjoint compact symmetric sets, and $0 \notin K_c$. Suppose first that $K_c \neq \emptyset$. For a set S , write $\mathcal{U}_b(S) = \{v : \text{dist}(v, S) < b\}$, with $\mathcal{U}_b(\emptyset) = \emptyset$. Choose $\rho > 0$ sufficiently small that

$$\mathcal{U}_{3\rho}(K^{\text{in}}) \subset W, \quad \overline{\mathcal{U}_{3\rho}(K^{\text{out}})} \cap \overline{W} = \emptyset,$$

and $N := \mathcal{U}_{3\rho}(K^{\text{out}})$ has finite genus. For completeness, a compact symmetric subset of $E \setminus \{0\}$ admits finitely many continuous linear functionals with no common zero on that set. Their joint map is odd and remains nonzero in a sufficiently small neighborhood, proving this finite-genus assertion. If $K^{\text{out}} = \emptyset$, take $N = \emptyset$.

There exist $\epsilon_0 > 0$ and $\alpha > 0$ such that

$$(1 + \|v\|)\|J'(v)\| \geq \alpha \quad \text{if } |J(v) - c| \leq 2\epsilon_0, \quad \text{dist}(v, K_c) \geq \rho/2. \quad (4.9)$$

Otherwise, one could choose v_n with $J(v_n) \rightarrow c$, $(1 + \|v_n\|)\|J'(v_n)\| \rightarrow 0$, and $\text{dist}(v_n, K_c) \geq \rho/2$, contradicting Cerami compactness. Notice that (4.9) also applies inside W ; only a neighborhood of the full critical set is removed.

Put $B = 1 + \max_{z \in K_c} \|z\| + 3\rho$, and choose

$$0 < \epsilon_c < \min\{\epsilon_0, c/4, \alpha\rho/(4B)\}.$$

For $0 < \epsilon < \epsilon_c$, define the even locally Lipschitz cutoffs

$$\theta(v) = \min\{1, \max\{0, \text{dist}(v, K_c)/\rho - 1\}\}, \quad \chi_\epsilon(v) = \min\{1, \max\{0, 2 - |J(v) - c|/\epsilon\}\}.$$

The energy cutoff is locally Lipschitz because $J \in C^1(E)$. Set

$$V_\epsilon(v) = \begin{cases} \chi_\epsilon(v)\theta(v)(1 + \|v\|)\frac{v - K(v)}{\|v - K(v)\|}, & J'(v) \neq 0, \\ 0, & J'(v) = 0. \end{cases} \quad (4.10)$$

This is an odd locally Lipschitz field on all of E . Indeed, every critical point with $|J(v) - c| \leq 2\epsilon_0$ belongs to $\mathcal{U}_{\rho/2}(K_c)$ by (4.9), where θ vanishes in a neighborhood. Near any other critical point, the energy cutoff vanishes. Away from critical points, Lemma 4.1 gives local Lipschitz continuity directly. In particular, the zero extension in (4.10) also handles critical points inside W .

Because $\|V_\epsilon(v)\| \leq 1 + \|v\|$, the initial value problem

$$\partial_t \eta_\epsilon(t, v) = -V_\epsilon(\eta_\epsilon(t, v)), \quad \eta_\epsilon(0, v) = v \quad (4.11)$$

has a unique solution for all $t \geq 0$, continuous in (t, v) . The growth bound prevents finite-time blow-up. More explicitly, $1 + \|\eta_\epsilon(t, v)\| \leq (1 + \|v\|)e^t$; hence, the derivative is bounded on each finite time interval, and the trajectory extends at any finite endpoint. Oddness and uniqueness give odd dependence on v . The preceding projection argument proves all four invariance statements, because $-V_\epsilon$ is a non-negative scalar multiple of $K(v) - v$. Moreover,

$$\frac{d}{dt} J(\eta_\epsilon(t, v)) = -\chi_\epsilon \theta (1 + \|\eta_\epsilon\|) \|J'(\eta_\epsilon)\| \leq 0.$$

We next justify the finite-time lowering. Take $v \in J^{c+\epsilon} \setminus (N \cup W)$ with $J(v) > c - \epsilon$. Then, $\text{dist}(v, K_c) \geq 3\rho$. As long as the energy has not reached $c - \epsilon$, the trajectory cannot reach $\mathcal{U}_{2\rho}(K_c)$. To see this, suppose that it first reaches distance 2ρ , and consider the segment since its last crossing of distance 3ρ . On this segment, the energy lies in $[c - \epsilon, c + \epsilon]$, both cutoffs equal one, and $1 + \|\eta_\epsilon\| \leq B$. Therefore, the energy lost on the segment is

$$\int \|J'(\eta_\epsilon)\| \|\partial_t \eta_\epsilon\| dt \geq \frac{\alpha}{B} \int \|\partial_t \eta_\epsilon\| dt \geq \frac{\alpha\rho}{B} > 2\epsilon.$$

Here, the last length bound follows because distance to K_c is 1-Lipschitz. This contradicts the width 2ϵ of the energy strip. Thus, until the target energy is reached, both cutoffs remain one, and (4.9) gives $dJ/dt \leq -\alpha$. At time $T = 3\epsilon/\alpha$, the energy is therefore at most $c - \epsilon$. Monotonicity covers starting points already below that level. The map $\eta(v) = \eta_\epsilon(T, v)$ has the required lowering property and fixes $J^{c-2\epsilon}$ because $\chi_\epsilon = 0$ there.

If $K_c = \emptyset$, Cerami compactness gives $(1 + \|v\|)\|J'(v)\| \geq \alpha > 0$ throughout a sufficiently small energy strip about c . Take $N = \emptyset$, omit the spatial cutoff by setting $\theta \equiv 1$, and use the same energy cutoff and field. There are no critical points in its support. The global flow and cone invariance follow as above, and $dJ/dt \leq -\alpha$ throughout the target strip proves the same conclusion for $T = 3\epsilon/\alpha$. This also completes the empty-critical-set case. $\square$

5. Nodal minimax geometry

We use the sign involution $\mathcal{G} = -\text{id}$ throughout. The following is the corresponding Banach-space formulation of the invariant-set theorem of [15], as restated in the preprint version of [16]. Its sets are open; Lemma 4.3 also preserves their closed neighborhoods. For a symmetric set $S \subset X \setminus \{0\}$, $\gamma(S)$ is the least integer k for which there is an odd continuous map $S \rightarrow \mathbb{R}^k \setminus \{0\}$, or infinity if none exists; $\gamma(\emptyset) = 0$.

Definition 5.1. Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$ even, and $U \subset X$ open. Put $Q = -U$, $W = U \cup Q$, $\Phi^a = \{u : \Phi(u) \leq a\}$, and $K_c = \{u : \Phi(u) = c, \Phi'(u) = 0\}$. We call U $(-\text{id})$ -admissible at level c if there exist $\epsilon_c > 0$ and a symmetric open neighborhood N of $K_c \setminus W$ with $\gamma(N) < \infty$, such that for every $0 < \epsilon < \epsilon_c$, there is an odd continuous map $\eta : X \rightarrow X$ with

$$\eta(U) \subset U, \quad \eta(Q) \subset Q, \quad \eta|_{\Phi^{c-2\epsilon}} = \text{id},$$

$$\eta(\Phi^{c+\epsilon} \setminus (N \cup W)) \subset \Phi^{c-\epsilon}.$$

The requirement includes the case $K_c \setminus W = \emptyset$; in that case, we require the deformation with $N = \emptyset$.

Theorem 5.2 (Invariant-set symmetric minimax theorem [15, 16]). *Let X be a Banach space, and let $\Phi \in C^1(X, \mathbb{R})$ be even. Let $U \subset X$ be open, and set $Q = -U$, $M = U \cap Q$, $\Sigma = \partial U \cap \partial Q$, $W = U \cup Q$, and $c_* = \inf_{\Sigma} \Phi$. Assume that U is $(-id)$ -admissible at every level $c \geq c_*$. For every $n \geq 1$, suppose there is a continuous map $\phi_n : B_n \rightarrow X$, where $B_n = \{t \in \mathbb{R}^n : |t| \leq 1\}$, satisfying*

$$1) \quad \phi_n(0) = 0 \in M \text{ and } \phi_n(-t) = -\phi_n(t);$$

$$2) \quad \phi_n(\partial B_n) \cap M = \emptyset;$$

$$3) \quad \sup_{\{0\} \cup \phi_n(\partial B_n)} \Phi < c_*.$$

For $j \in \mathbb{N}$, define

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus W} \Phi(u),$$

where

$$\Gamma_j = \left\{ \varphi(B_n \setminus Y) : \varphi \in G_n, n \geq j, Y \subset B_n \text{ relatively open, } -Y = Y, 0 \notin \bar{Y}, \gamma(\bar{Y}) \leq n - j \right\},$$

$$G_n = \left\{ \varphi \in C(B_n, X) : \varphi(-t) = -\varphi(t), \varphi(0) \in M, \varphi|_{\partial B_n} = \phi_n|_{\partial B_n} \right\}.$$

Then, for $j \geq 2$, $c_j \geq c_*$, $K_{c_j} \setminus W \neq \emptyset$, and $c_j \rightarrow +\infty$.

For the present application, $X = E$, $\Phi = J$, $U = P_\delta^\circ$, $Q = -P_\delta^\circ$, and $\text{Fix}_{-id} = \{0\}$. Lemma 4.3 proves the full deformation property using Cerami compactness; no substitution of Cerami for an unverified Palais–Smale hypothesis is needed. Lemmas 5.3 and 5.4 provide the finite-dimensional descent and positive barrier.

Lemma 5.3. *For every finite-dimensional subspace $Y \subset E$,*

$$J(v) \longrightarrow -\infty \quad \text{as } \|v\| \rightarrow \infty, \quad v \in Y. \quad (5.1)$$

Proof. It remains only to prove uniformity. Suppose that uniformity fails. Then, there are $v_n \in Y$, $\|v_n\| = 1$, and $t_n \rightarrow \infty$, for which $J(t_n v_n)/t_n^2$ is bounded below. After passing to a subsequence, $v_n \rightarrow v$ in Y , and hence in E and almost everywhere, with $\|v\| = 1$. The set $\{v \neq 0\} \setminus D_0$ has positive measure. Indeed, $v \neq 0$ in $H_0^1(\Omega)$ implies $|\{v \neq 0\}| > 0$, whereas the standing assumption gives $|D_0| = 0$. Therefore, removing D_0 cannot eliminate the support of v . This is the step in which the null-measure threshold assumption restores the finite-dimensional descending geometry. At each point of this set, $q(x) > 4$, $t_n |v_n(x)| \rightarrow \infty$, and (2.5) gives

$$\frac{G(x, t_n v_n(x))}{t_n^2} \longrightarrow +\infty.$$

Fatou's lemma therefore implies $\int_{\Omega} G(x, t_n v_n)/t_n^2 dx \rightarrow \infty$, contradicting $J(t_n v_n)/t_n^2 = 1/2 - \int_{\Omega} G(x, t_n v_n)/t_n^2 dx$. This proves (5.1) uniformly on the unit sphere of Y . $\square$

Lemma 5.4. *For all sufficiently small $\delta > 0$,*

$$c_* := \inf_{\partial P_\delta \cap \partial(-P_\delta)} J > 0.$$

Proof. If $v \in \partial P_\delta \cap \partial(-P_\delta)$, then $\|v\| \geq \delta$. Applying (4.3) to v and to $-v$ gives $\|v^-\|_s + \|v^+\|_s \leq C_s \delta$ for $2 \leq s \leq 6$. Hence,

$$J(v) \geq \frac{1}{2}\|v\|^2 - C(\|v\|_4^4 + \|v\|_r^r) \geq \frac{1}{2}\delta^2 - C(\delta^4 + \delta^r) > 0$$

after decreasing δ . $\square$

Proposition 5.5. *Problem (2.12) has infinitely many pairs $\{\pm v_k\}$ of sign-changing weak solutions with $J(v_k) \rightarrow +\infty$.*

Proof. Apply Theorem 5.2 with $X = E$, $\Phi = J$, $U = P_\delta^\circ$, and $Q = -P_\delta^\circ$, where δ satisfies Proposition 4.2 and Lemma 5.4. Lemma 4.3 verifies admissibility and $c_* > 0$. Let $Y_n = \text{span}\{e_1, \dots, e_n\}$ for Dirichlet eigenfunctions e_j , identify $\mathbb{R}^n$ with Y_n , and set $\phi_n(t) = R_n t$. This map is odd, and $\phi_n(0) = 0 \in M$. By (4.3), $v \in M$ implies $\|v\|_2 \leq C\delta$. On Y_n , equivalence of norms gives $\|v\|_2 \geq a_n \|v\|$ for some $a_n > 0$. Hence, $M \cap Y_n$ is bounded, and R_n may be chosen so large that $\phi_n(\partial B_n) \cap M = \emptyset$. Increasing R_n if necessary, Lemma 5.3 gives $\sup_{\phi_n(\partial B_n)} J < 0$. Because $\text{Fix}_{-\text{id}} = \{0\}$, and $J(0) = 0 < c_*$, all three hypotheses of Theorem 5.2 are satisfied. We therefore obtain critical points $v_k \notin U \cup Q$ with $J(v_k) \rightarrow \infty$. There are no critical points on the cone-neighborhood boundaries by Proposition 4.2; thus, $v_k \notin P_\delta \cup (-P_\delta)$, as well. Each such v_k is neither non-negative nor nonpositive, and is therefore sign-changing. $\square$

6. Return to the original equation

Lemma 6.1. *If v is a weak solution of (2.12), and $u = f(v)$, then $u \in H_0^1(\Omega)$, $u^2 \in H_0^1(\Omega)$, and u is a weak solution of (1.1). More precisely,*

$$\int_{\Omega} (1 + u^2) \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} u |\nabla u|^2 \varphi \, dx = \int_{\Omega} |u|^{q(x)-2} u \varphi \, dx, \quad (6.1)$$

for every $\varphi \in C_c^\infty(\Omega)$.

Proof. Put $m = r - 1 < 5$. Starting from $v \in L^6(\Omega)$, (2.11) gives $g(x, v) \in L^{6/m}(\Omega)$. Suppose that $v \in L^a(\Omega)$ for some $a \geq 6$. Then, $g(x, v) \in L^{a/m}(\Omega)$. Because Ω is assumed smooth, the global Dirichlet $W^{2,p}$ estimates for the Laplacian apply for $1 < p < \infty$; it would be enough here to assume a $C^{1,1}$ boundary [17, 18]. If $1 < a/m < 3/2$, the Dirichlet $W^{2,a/m}$ estimate and the Sobolev embedding in dimension three give

$$v \in L^{a'}(\Omega), \quad a' = \frac{3a}{3m - 2a}.$$

The map $F(a) = 3a/(3m - 2a)$ satisfies $F(a) > a$ exactly when $a > 3(m - 1)/2$. Because $m < 5$, the initial value $a_0 = 6$ satisfies $a_0 > 3(m - 1)/2$. Therefore, the sequence $a_{k+1} = F(a_k)$ is strictly increasing as long as $a_k/m < 3/2$. If it remained below $3m/2$ forever, it would converge to $L \leq 3m/2$. The case $L = 3m/2$ is impossible because then, $F(a_k) \rightarrow +\infty$, contradicting boundedness. If $L < 3m/2$, continuity would give $F(L) = L$, whereas the only positive fixed point is $3(m - 1)/2 < a_0$. Thus, after finitely many steps, either $a_k/m > 3/2$, or $a_k/m = 3/2$.

If $a_k/m > 3/2$, then $g(x, v) \in L^p(\Omega)$ for some $p > 3/2$, and $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. If the endpoint $a_k/m = 3/2$ occurs, then $g(x, v) \in L^{3/2}(\Omega)$, and $v \in W^{2,3/2}(\Omega) \hookrightarrow L^s(\Omega)$ for every finite $s < \infty$. Choosing $s > 3m/2$ and repeating the previous step gives $g(x, v) \in L^{s/m}(\Omega)$ with $s/m > 3/2$, whence again $v \in L^\infty(\Omega)$. This proves the endpoint case without using the false embedding $W^{2,3/2} \hookrightarrow L^\infty$.

Consequently, $u = f(v) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, as $|f'| \leq 1$, and $u^2 \in H_0^1(\Omega)$ because $\nabla(u^2) = 2u\nabla u \in L^2$, and the trace vanishes.

In fact, $g(x, v) \in L^\infty(\Omega)$, so global regularity gives $v \in W^{2,p}(\Omega)$ for every finite $p > 1$. Taking $p > 3$ first gives a bounded gradient, and composition with the smooth function f then yields $u \in W^{2,p}(\Omega)$ for every finite $p > 1$. Thus, the original equation also holds almost everywhere.

For $\varphi \in C_c^\infty(\Omega)$, the function $\psi = \sqrt{1+u^2} \varphi$ belongs to $H_0^1(\Omega)$ and is therefore an admissible test function in the weak formulation of (2.12). Because $\nabla v = \sqrt{1+u^2} \nabla u$, direct calculation gives

$$\nabla v \cdot \nabla \psi = (1+u^2) \nabla u \cdot \nabla \varphi + u |\nabla u|^2 \varphi.$$

Moreover, by (2.7),

$$g(x, v) \sqrt{1+u^2} = |u|^{q(x)-2} u.$$

Substitution proves (6.1). □

Proof of Theorem 1.1. Let v_k be given by Proposition 5.5, and set $u_k = f(v_k)$. Lemma 6.1 shows that u_k solves (1.1). Because f is odd and strictly increasing, it preserves positive and negative sets. Hence, every u_k is sign-changing, and the solutions occur in pairs. The divergence of $J(v_k)$ makes these pairs distinct. □

7. Discussion

The dual transformation changes the analytical meaning of the quartic threshold. In the original variable, the power $q = 4$ is balanced with the quasilinear energy $u^2 |\nabla u|^2$. In the dual variable, because $f(t) \sim \sqrt{2|t|} \operatorname{sgn} t$, the transformed potential behaves as $|t|^{q(x)/2}$. It is therefore asymptotically quadratic on D_0 and superquadratic outside D_0 . This explains both the restriction $q^+ < 12$ in dimension three and the central role of the assumption $|D_0| = 0$.

The null-set assumption ensures that a nonzero normalized weak limit meets a positive-measure region where $q(x) > 4$ in the Cerami argument, and it supplies the same property in the finite-dimensional descending geometry. Although $|D_\eta| \rightarrow 0$ also controls the near-threshold term in (4.7), the direct estimate (4.8) proves cone contraction under (1.2) alone. Thus, zero measure is a sufficient condition used in the present compactness and descending-geometry proofs, not a necessary condition for cone invariance. If D_0 has positive measure, those two proofs need further information about the region where $q > 4$; we do not claim that the theorem must then fail.

Lemma 2.3 quantifies residual nonquadraticity, but, does not itself provide the required descending geometry. At $q = 4$, $G_4(t)/t^2 \rightarrow 1$, even though $\mathcal{D}_4(t) \rightarrow \infty$. For example, when $q \equiv 4$, along a fixed nonzero w , one has $J(tw)/t^2 \rightarrow 1/2\|w\|^2 - \|w\|_2^2$, which is positive in sufficiently high Dirichlet eigenfunction directions. This illustrates why the logarithmic remainder alone does not yield descent on every finite-dimensional subspace. In the present proof, compactness uses scaling monotonicity and the null-set condition, cone invariance follows from global growth, and multiplicity uses the fully specified invariant-set deformation and descending geometry. The scalar change of variables does not directly extend to general anisotropic quasilinear operators.

8. Conclusions

The classical dual transformation reduces the variable-exponent modified quasilinear Schrödinger equation to a semilinear problem on $H_0^1(\Omega)$. Its inverse supplies the scaling monotonicity needed for Cerami compactness and a quantitative nonquadraticity remainder at the quartic threshold, and the nullity of the threshold set restores the finite-dimensional descending geometry. After verifying cone admissibility and each hypothesis of the invariant-set symmetric minimax theorem, we obtain infinitely many sign-changing solution pairs without perturbing the exponent.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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