



Research article

On the sixth power mean of the generalized two-term exponential sums

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Abstract: This paper primarily focuses on deriving an exact computational formula for a specific sixth-power mean of generalized two-term exponential sums. To this end, we exploit the properties of trigonometric sums and character sums, together with the solution counts of certain congruence equations modulo p .

Keywords: congruence equation; the generalized two-term exponential sums; sixth-power mean; computational formula

1. Introduction

Let $p \geq 3$ be an odd prime, where m and n are integers. For any positive integers $r > s \geq 1$ and Dirichlet character χ modulo p , the generalized two-term exponential sums $S(m, n, r, s, \chi; p)$ are defined as

$$S(m, n, r, s, \chi; p) = \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^r + n\alpha^s),$$

where as usual, $\mathbf{e}_p(y) = \exp(2\pi iy/p) = e^{\frac{2\pi iy}{p}}$ and $i^2 = -1$.

The study of various properties of $S(m, n, r, s, \chi; p)$ is one of the most important topics in analytic number theory. These properties are closely related to many famous number theory problems, such as Waring's problem, the distribution properties of primes in arithmetic progressions, and Goldbach's problem. For general background on these classical problems, we refer to the textbook [1], while specific research contributions can be found in [2] and the references therein. Thus, the investigation of the properties of generalized two-term exponential sums holds significant theoretical importance and research value. Generally, studying such problems involves two aspects: One is investigating their

upper bounds and providing the best possible upper bound estimates; the other is computing various means and deriving an exact or asymptotic formula.

In Liu and Zhang [2], the authors studied the $2k$ -th power mean of the generalized two-term exponential sums $S(m, n, 3, 1, \chi; p)$ as follows:

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^{2k}, \quad (1.1)$$

and obtained an exact calculating formula for (1.1) with $k = 3$. That is, they proved the following two conclusions:

Theorem A. Let p be an odd prime with $3 \nmid (p-1)$, then one has

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 = p(p-1)(6p^3 - 28p^2 + 39p + 5).$$

Theorem B. For any odd prime p , one has the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha + \bar{\alpha}) \right|^6 = p(p-1)(6p^3 - 28p^2 + 39p + 5),$$

where $\bar{\alpha}$ denotes that $\alpha \cdot \bar{\alpha} \equiv 1 \pmod{p}$.

Although Theorem A is impressive, it is a pity that the work remains incomplete. Specifically, the authors did not provide any results for the case $3 \mid (p-1)$. Because this situation is relatively complicated, they did not make any progress. In fact, the major difference between the two lies in the calculation of the congruence equation

$$\sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3 \equiv \delta^3+\epsilon^3+1 \pmod{p}}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma \equiv \delta\epsilon \pmod{p}}}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1. \quad (1.2)$$

If $(3, p-1) = 1$, then the condition in (1.2) is equivalent to $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod{p}$ and $\alpha\beta\gamma \equiv \delta\epsilon \pmod{p}$. Therefore, the calculation becomes relatively simple. But if $3 \mid (p-1)$, then the situation is completely different, the calculation of (1.2) becomes much more complicated.

In this paper, we have addressed the shortcomings in [2] and completely resolved this problem. In fact, by using elementary methods and the number of solutions of the congruence equations modulo p , we have proved the following two interesting conclusions:

Theorem 1. Let p be an odd prime with $3 \mid (p-1)$, then we have the identity

$$\begin{aligned} & \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 \\ &= (p-1)(7p^4 - 29p^3 + 44p^2) - \frac{1}{p-1} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3) \right|^6. \end{aligned}$$

Theorem 2. For any odd prime p with $3 \mid (p-1)$, we have the identity

$$\frac{1}{p(p-1)} \cdot \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 = 6p^3 - 30p^2 + 45p - 2Dp - D^2 + 9D + 7,$$

where D is uniquely determined by $4p = D^2 + 27B^2$ and $D \equiv 1 \pmod{3}$.

Combining Theorems 1 and 2 and the result in [2], we may immediately deduce the following two corollaries:

Corollary 1. Let p be an odd prime with $p \equiv 1 \pmod{3}$, then we have

$$\frac{1}{p \cdot (p-1)^2} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3) \right|^6 = p^3 + p^2 - p + 2pD + D^2 - 9D - 7.$$

Corollary 2. For any odd prime p , we have the asymptotic formula

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 = 6p^5 + O(p^4).$$

Some notes: In fact, Theorem 1 establishes a close connection between the generalized two-term exponential sums and the cubic Gauss sums, which are mutually convertible via this identity. In addition, our work not only refines the conclusions in [2], but also introduces innovative research methods. We believe the new approaches presented in this paper will provide valuable insights for studying the general $2k$ power mean.

2. Some lemmas

This section collects several auxiliary lemmas, which are derived from a decomposition technique used later in the main proofs. We omit the standard elementary number theory facts (available in [3–5]) and state the first result as follows:

Lemma 1. Let p be a prime with $p \equiv 1 \pmod{3}$, and λ be any three-order character modulo p . Then we have the identity

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = Dp,$$

where $\tau(\chi) = \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(\alpha)$ denotes the classical Gauss sums, D is uniquely determined by $4p = D^2 + 27B^2$, and $D \equiv 1 \pmod{3}$.

Proof. See Berndt and Evans [6]. □

Lemma 2. Let p be an odd prime with $3 \mid (p-1)$, then we have the identity

$$\sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3\equiv\delta^3+\epsilon^3+1 \pmod p}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p}}^{p-1} \sum_{\substack{\gamma=1 \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1 \pmod p}}^{p-1} \sum_{\substack{\delta=1}}^{p-1} \sum_{\substack{\epsilon=1}}^{p-1} 1 = 7p^2 - 29p + 44.$$

Proof. First, we rewrite the given cubic congruence by applying the identity for the cube of a sum, so as to convert these congruence conditions into some equivalent forms. Note that the congruence conditions $\alpha^3 + \beta^3 + \gamma^3 \equiv \delta^3 + \epsilon^3 + 1 \pmod p$, $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod p$, and $\alpha\beta\gamma \equiv \delta\epsilon \pmod p$ are equivalent to $(\alpha + \beta + \gamma)^3 - \alpha^3 - \beta^3 - \gamma^3 \equiv (\delta + \epsilon + 1)^3 - (\delta^3 + \epsilon^3 + 1) \pmod p$, $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod p$, and $\alpha\beta\gamma \equiv \delta\epsilon \pmod p$. These congruence conditions are also equivalent to $\alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) \equiv \delta^2(\epsilon + 1) + \epsilon^2(\delta + 1) + (\delta + \epsilon) \pmod p$, $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod p$, and $\alpha\beta\gamma \equiv \delta\epsilon \pmod p$, they are equivalent to $\alpha^2(\delta + \epsilon + 1 - \alpha) + \beta^2(\delta + \epsilon + 1 - \beta) + \gamma^2(\delta + \epsilon + 1 - \gamma) \equiv \delta^2(\alpha + \beta + \gamma - \delta) + \epsilon^2(\alpha + \beta + \gamma - \epsilon) + (\alpha + \beta + \gamma - 1) \pmod p$, $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod p$, and $\alpha\beta\gamma \equiv \delta\epsilon \pmod p$; or they are also equivalent to $(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \delta^2 - \epsilon^2 - 1) \equiv 0 \pmod p$, $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod p$ and $\alpha\beta\gamma \equiv \delta\epsilon \pmod p$.

It is clear that $(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \delta^2 - \epsilon^2 - 1) \equiv 0 \pmod p$ is equivalent to $\alpha + \beta + \gamma \equiv 0 \pmod p$ or $\alpha^2 + \beta^2 + \gamma^2 - \delta^2 - \epsilon^2 - 1 \equiv 0 \pmod p$.

First case: If $\alpha + \beta + \gamma \equiv 0 \pmod p$, let λ be a third-order character modulo p , then note that $\lambda(-1) = 1$ and $\lambda^2 = \bar{\lambda}$. From the properties of the reduced residue system modulo p , we have

$$\begin{aligned} & \sum_{\substack{\alpha=1 \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p}}^{p-1} \sum_{\substack{\gamma=1}}^{p-1} \sum_{\substack{\delta=1}}^{p-1} \sum_{\substack{\epsilon=1}}^{p-1} 1 = \sum_{\substack{\alpha=1 \\ \gamma(\alpha+\beta+1)\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\substack{\beta=1}}^{p-1} \sum_{\substack{\gamma=1}}^{p-1} \sum_{\substack{\delta=1}}^{p-1} \sum_{\substack{\epsilon=1 \\ \alpha\beta\gamma^3\equiv\delta\epsilon \pmod p}}^{p-1} 1 \\ &= \sum_{\substack{\alpha=1 \\ \alpha+\beta+1\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\substack{\beta=1}}^{p-1} \sum_{\substack{\gamma=1}}^{p-1} \sum_{\substack{\delta=1}}^{p-1} \sum_{\substack{\epsilon=1 \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p}}^{p-1} (1 + \lambda(\gamma) + \bar{\lambda}(\gamma)) \\ &= \sum_{\substack{\alpha=1 \\ \alpha+\beta+1\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\substack{\beta=1}}^{p-1} \sum_{\substack{\delta=1}}^{p-1} \sum_{\substack{\epsilon=1}}^{p-1} 1 + 2 \cdot \sum_{\substack{\alpha=1 \\ \alpha+\beta+1\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\substack{\beta=1}}^{p-1} \sum_{\substack{\delta=1}}^{p-1} \sum_{\substack{\epsilon=1}}^{p-1} \lambda(\delta\epsilon\alpha\bar{\beta}) \\ &= (p-2)^2 + 2 \cdot \left| \sum_{\substack{\alpha=1 \\ \alpha+\beta+1\equiv 0 \pmod p}}^{p-1} \sum_{\substack{\beta=1}}^{p-1} \lambda(\alpha\beta) \right|^2 = (p-2)^2 + 2 \cdot \left| \sum_{\alpha=1}^{p-1} \lambda(\alpha(\alpha+1)) \right|^2 \\ &= p^2 - 4p + 4 + 2 \cdot \left| \sum_{\alpha=1}^{p-1} \lambda(\alpha(\alpha+1)) \right|^2. \end{aligned} \tag{2.1}$$

Note that $\lambda = \bar{\lambda}^2$. From the properties of the Gauss sum we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \lambda(a(a+1)) &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \bar{\lambda}(b) \sum_{a=1}^{p-1} \lambda(a) \mathbf{e}_p(b(a+1)) \\
&= \frac{\tau(\lambda)}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \bar{\lambda}^2(b) \mathbf{e}_p(b) = \frac{\tau(\lambda)}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \lambda(b) \mathbf{e}_p(b) = \frac{\tau^2(\lambda)}{\tau(\bar{\lambda})}.
\end{aligned} \tag{2.2}$$

So from (2.1) and (2.2), using $\tau(\bar{\lambda}) = p/\tau(\lambda)$, we obtain

$$\sum_{\substack{\alpha=1 \\ \alpha+\beta+\gamma \equiv \delta+\epsilon+1 \pmod p \\ \alpha\beta\gamma \equiv \delta\epsilon \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = p^2 - 4p + 4 + 2 \cdot \left| \frac{\tau^2(\lambda)}{\tau(\bar{\lambda})} \right|^2 = p^2 - 2p + 4. \tag{2.3}$$

Second case: If $\alpha^2 + \beta^2 + \gamma^2 - \delta^2 - \epsilon^2 - 1 \equiv 0 \pmod p$, then we first algebraically transform the system of simultaneous congruences into equivalent conditions. The set of admissible tuples is then partitioned into several disjoint subsets according to whether certain polynomial factors vanish modulo p . We count each disjoint subset individually and sum their contributions to obtain the intermediate counting expression below.

$$\begin{aligned}
&\sum_{\substack{\alpha=1 \\ \alpha^2+\beta^2+\gamma^2 \equiv \delta^2+\epsilon^2+1 \pmod p \\ \alpha+\beta+\gamma \equiv \delta+\epsilon+1 \pmod p \\ \alpha\beta\gamma \equiv \delta\epsilon \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\substack{\alpha=1 \\ \alpha\beta+\alpha\gamma+\beta\gamma \equiv \delta\epsilon+\delta+\epsilon \pmod p \\ \alpha+\beta+\gamma \equiv \delta+\epsilon+1 \pmod p \\ \alpha\beta\gamma \equiv \delta\epsilon \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\substack{\alpha=1 \\ \alpha\beta+(\alpha+\beta)\gamma \equiv \alpha\beta\delta\epsilon+\alpha\delta+\beta\epsilon \pmod p \\ \alpha+\beta+\gamma \equiv \alpha\delta+\beta\epsilon+1 \pmod p \\ \gamma \equiv \delta\epsilon \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&= \sum_{\substack{\alpha=1 \\ (\alpha-1)(\beta-1)(\delta\epsilon-1) \equiv 0 \pmod p \\ \alpha+\beta+\delta\epsilon \equiv \alpha\delta+\beta\epsilon+1 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = 1 + \sum_{\substack{\delta=1 \\ (\delta-1)(\epsilon-1) \equiv 0 \pmod p \\ (\delta\epsilon-1, p)=1}}^{p-1} \sum_{\epsilon=1}^{p-1} 1 + 2 \sum_{\substack{\beta=2 \\ (\delta-1)(\beta-\delta) \equiv 0 \pmod p}}^{p-1} \sum_{\delta=1}^{p-1} 1 \\
&\quad + 2 \sum_{\substack{\beta=2 \\ (\epsilon-1)(\beta-\delta) \equiv 0 \pmod p \\ (\delta\epsilon-1, p)=1}}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 + \sum_{\substack{\alpha=2 \\ \alpha+\beta \equiv \alpha\delta+\beta\epsilon \pmod p \\ \delta\epsilon \equiv 1 \pmod p}}^{p-1} \sum_{\beta=2}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&= 1 + 2(p-2) + 4(p-2) + 2(p-2)(2p-5) + \sum_{\substack{\alpha=2 \\ (\alpha+\beta)\epsilon \equiv \alpha+\beta\epsilon^2 \pmod p}}^{p-1} \sum_{\beta=2}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&= 1 + 4(p-2)(p-1) + \sum_{\substack{\alpha=2 \\ (\epsilon-1)(\epsilon\beta-\alpha) \equiv 0 \pmod p}}^{p-1} \sum_{\beta=2}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&= 1 + 4(p-2)(p-1) + 2(p-2)^2 - (p-2) = 1 + 3(p-2)(2p-3).
\end{aligned} \tag{2.4}$$

Third case: If $\alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \equiv 0 \pmod p$ and $\alpha^2 + \beta^2 + \gamma^2 \equiv \delta^2 + \epsilon^2 + 1 \pmod p$, then we have

$$\begin{aligned}
 & \sum_{\substack{\alpha=1 \\ \alpha^2+\beta^2+\gamma^2\equiv\delta^2+\epsilon^2+1 \pmod p \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1\equiv 0 \pmod p \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\substack{\alpha=1 \\ (\alpha-1)(\beta-1)(\delta-1)\equiv 0 \pmod p \\ \alpha+\beta+\delta\epsilon\equiv\alpha\delta+\beta\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\substack{\delta=1 \\ 2+\delta\epsilon\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
 & + 2 \sum_{\substack{\beta=2 \\ 2+\beta\equiv\delta+\beta\bar{\delta}+1\equiv 0 \pmod p}}^{p-1} \sum_{\delta=1}^{p-1} 1 + 2 \sum_{\substack{\beta=2 \\ 1+\beta+\delta\epsilon\equiv\delta+\beta\epsilon+1\equiv 0 \pmod p \\ (\delta\epsilon-1, p)=1}}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 + \sum_{\substack{\alpha=2 \\ \alpha+\beta+1\equiv\alpha\delta+\beta\bar{\delta}+1\equiv 0 \pmod p}}^{p-1} \sum_{\beta=2}^{p-1} \sum_{\delta=1}^{p-1} 1 \\
 & = 2 + 4 + 2(p-4 + p-5) + p-4 + p-5 = 3(2p-7). \tag{2.5}
 \end{aligned}$$

Clearly, (2.3) + (2.4) – (2.5) gives the desired result. Thus, we have the identity

$$\begin{aligned}
 & \sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3\equiv\delta^3+\epsilon^3+1 \pmod p \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\substack{\alpha=1 \\ (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\delta^2-\epsilon^2-1)\equiv 0 \pmod p \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
 & = \sum_{\substack{\alpha=1 \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 + \sum_{\substack{\alpha=1 \\ \alpha^2+\beta^2+\gamma^2\equiv\delta^2+\epsilon^2+1 \pmod p \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 - \sum_{\substack{\alpha=1 \\ \alpha^2+\beta^2+\gamma^2\equiv\delta^2+\epsilon^2+1 \pmod p \\ \alpha\beta\gamma\equiv\delta\epsilon \pmod p \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1\equiv 0 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
 & = p^2 - 2p + 4 + 1 + 3(p-2)(2p-3) - 3(2p-7) = 7p^2 - 29p + 44.
 \end{aligned}$$

This proves Lemma 2. □

For a non-zero integer m , we define the complete cubic exponential sum $A(m)$ by

$$A(m) = \sum_{\alpha=0}^{p-1} \mathbf{e}_p(m\alpha^3).$$

Lemma 3. Let p be an odd prime with $3 \mid (p-1)$, then we have the identity

$$\sum_{m=1}^{p-1} (A(m) - 1)^4 = (p-1) \cdot (6p^2 - 4Dp + 12p + 1).$$

Proof. Let λ be a third-order character modulo p . Then for any integer m with $(m, p) = 1$, by noting the identities $A(m) = \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})$, $\lambda^3(m) = 1$ and the orthogonality of Dirichlet characters modulo p . From Lemma 1, we have

$$\sum_{m=1}^{p-1} A(m) = \sum_{m=1}^{p-1} (\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})) = 0. \tag{2.6}$$

$$\sum_{m=1}^{p-1} A^2(m) = \sum_{m=1}^{p-1} \left(\bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}) \right)^2 = 2p(p-1). \quad (2.7)$$

$$\sum_{m=1}^{p-1} A^3(m) = \sum_{m=1}^{p-1} (Dp + 3pA(m)) = Dp(p-1). \quad (2.8)$$

$$\sum_{m=1}^{p-1} A^4(m) = \sum_{m=1}^{p-1} (DpA(m) + 3pA^2(m)) = 6p^2(p-1). \quad (2.9)$$

Combining (2.6)–(2.9), we deduce the identity

$$\begin{aligned} & \sum_{m=1}^{p-1} (A(m) - 1)^4 \\ &= \sum_{m=1}^{p-1} (A^4(m) - 4A^3(m) + 6A^2(m) - 4A(m) + 1) \\ &= 6p^2(p-1) - 4Dp(p-1) + 12p(p-1) + (p-1) \\ &= (p-1) \cdot (6p^2 - 4Dp + 12p + 1). \end{aligned}$$

This proves Lemma 3. □

Lemma 4. Let p be an odd prime with $3 \mid (p-1)$, then we have the identity

$$\sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = p^3 - 7p^2 + 33p + 4D - 19.$$

$\delta^3 \epsilon^3 \equiv \alpha(\delta^3 - 1) + \beta^3(\epsilon^3 - 1) + 1 \pmod{p}$

Proof. Note that the congruence equation $x^3 \equiv 1 \pmod{p}$ has three solutions. So from Lemma 2 and the properties of the congruence modulo p , we have

$$\begin{aligned} & \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = 3(p-1) \sum_{\beta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\ & \quad \delta^3 \epsilon^3 \equiv \alpha(\delta^3 - 1) + \beta^3(\epsilon^3 - 1) + 1 \pmod{p} \quad \epsilon^3 \equiv \beta^3(\epsilon^3 - 1) + 1 \pmod{p} \\ & + \sum_{\delta=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\alpha=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\ & \quad p \nmid (\delta^3 - 1) \quad \delta^3 \epsilon^3 \equiv \alpha(\delta^3 - 1) + \beta^3(\epsilon^3 - 1) + 1 \pmod{p} \\ &= 9(p-1)(2p-5) + \sum_{\delta=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\alpha=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\ & \quad p \nmid (\delta^3 - 1) \quad \delta^3 \epsilon^3 \equiv \alpha + \beta^3(\epsilon^3 - 1) + 1 \pmod{p} \end{aligned}$$

$$\begin{aligned}
&= 9(p-1)(2p-5) + \sum_{\delta=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\alpha=0}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&\quad p \nmid (\delta^3-1) \delta^3 \epsilon^3 \equiv \alpha + \beta^3 (\epsilon^3-1) + 1 \pmod{p} \\
&\quad - \sum_{\delta=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&\quad p \nmid (\delta^3-1) \delta^3 \epsilon^3 \equiv \beta^3 (\epsilon^3-1) + 1 \pmod{p} \\
&= 9(p-1)(2p-5) + (p-1)^2(p-4) + 9(2p-5) \\
&\quad - \sum_{\delta=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&\quad \delta^3 \epsilon^3 \equiv \beta^3 (\epsilon^3-1) + 1 \pmod{p} \\
&= p^3 - 6p^2 + 36p - 4 - \frac{1}{p-1} \sum_{\delta=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \sum_{\phi=1}^{p-1} 1 \\
&\quad \delta^3 \equiv \epsilon^3 - \beta^3 + \phi^3 \pmod{p} \\
&= p^3 - 6p^2 + 36p - 4 - \frac{1}{p(p-1)} \sum_{m=0}^{p-1} (A(m) - 1)^4 \\
&= p^3 - 6p^2 + 36p - 4 - (p^2 + 3p - 4D + 15) \\
&= p^3 - 7p^2 + 33p + 4D - 19.
\end{aligned}$$

This proves Lemma 4. $\square$

Lemma 5. Let p be an odd prime with $3 \mid (p-1)$, then for any three-order character λ modulo p , we have the identity

$$\chi_2(-1) \cdot \lambda(2) \cdot \tau(\bar{\lambda}\chi_2) = \frac{\tau^2(\lambda)}{\tau(\chi_2)},$$

where χ_2 denotes the Legendre symbol modulo p and λ denotes any three-order character modulo p .

Proof. First from the properties of the Gauss sum, we have

$$\begin{aligned}
\sum_{\alpha=1}^{p-1} \lambda(\alpha(\alpha-1)) &= \frac{1}{\tau(\bar{\lambda})} \sum_{\beta=1}^{p-1} \bar{\lambda}(\beta) \sum_{\alpha=1}^{p-1} \lambda(\alpha) \mathbf{e}_p(\alpha\beta - \beta) \\
&= \frac{\tau(\lambda)}{\tau(\bar{\lambda})} \sum_{\beta=1}^{p-1} \lambda(\beta) \mathbf{e}_p(-\beta) = \frac{\tau^2(\lambda)}{\tau(\bar{\lambda})}.
\end{aligned} \tag{2.10}$$

On the other hand, we have

$$\begin{aligned}
\sum_{\alpha=1}^{p-1} \lambda(\alpha(\alpha-1)) &= \lambda(2) \sum_{\alpha=0}^{p-1} \lambda(4\alpha^2 - 4\alpha) = \lambda(2) \sum_{\alpha=0}^{p-1} \lambda((2\alpha-1)^2 - 1) \\
&= \lambda(2) \sum_{\alpha=0}^{p-1} \lambda(\alpha^2 - 1) = \frac{\lambda(2)}{\tau(\bar{\lambda})} \sum_{\beta=1}^{p-1} \bar{\lambda}(\beta) \sum_{\alpha=0}^{p-1} \mathbf{e}_p(\beta\alpha^2 - \beta)
\end{aligned}$$

$$= \frac{\lambda(2)\tau(\chi_2)}{\tau(\bar{\lambda})} \sum_{\beta=1}^{p-1} \bar{\lambda}(\beta)\chi_2(\beta)\mathbf{e}_p(-\beta) = \frac{\chi_2(-1)\lambda(2) \cdot \tau(\chi_2) \cdot \tau(\bar{\lambda}\chi_2)}{\tau(\bar{\lambda})}. \quad (2.11)$$

Combining (2.10) and (2.11), we have the identity

$$\chi_2(-1) \cdot \lambda(2) \cdot \tau(\bar{\lambda}\chi_2) = \frac{\tau^2(\lambda)}{\tau(\chi_2)}.$$

This proves Lemma 5. □

Lemma 6. Let p be an odd prime with $3 \mid (p-1)$, then for any three-order character λ modulo p , we have the identity

$$\sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha-1)\lambda(\beta-1)\lambda(\alpha\beta-1)\bar{\lambda}(\alpha\beta) = \frac{\tau^6(\bar{\lambda})}{p^2} - \frac{\tau^3(\lambda)}{p}.$$

Proof. Note that $\lambda(-1) = 1$ and $\lambda^3 = \chi_0$, where χ_0 denotes the principal character modulo p . Combining this with Eq (2.2). From the properties of the complete residue system and reduced residue system modulo p , we have the identity

$$\begin{aligned} & \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha-1)\lambda(\beta-1)\lambda(\alpha\beta-1)\bar{\lambda}(\alpha\beta) \\ &= \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(1-\bar{\alpha})\lambda(\beta-1)\lambda(\beta-\bar{\alpha})\lambda(\alpha)\bar{\lambda}(\beta) \\ &= \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha-1)\lambda(\beta-1)\lambda(\alpha-\beta)\bar{\lambda}(\alpha\beta) \\ &= \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha)\lambda(\beta)\lambda(\alpha-\beta)\bar{\lambda}(\alpha\beta+\alpha+\beta+1) \\ &= \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha\beta)\lambda(\beta)\lambda(\alpha\beta-\beta)\bar{\lambda}(\alpha\beta^2+\alpha\beta+\beta+1) \\ &= \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha)\lambda(\alpha-1)\bar{\lambda}(\alpha)\bar{\lambda}(\beta^2+\beta(\bar{\alpha}+1)+\bar{\alpha}) \\ &= \lambda(4) \sum_{\alpha=1}^{p-1} \sum_{\beta=0}^{p-1} \lambda(\alpha-1)\bar{\lambda}((2\beta+\bar{\alpha}+1)^2-(\bar{\alpha}-1)^2) - \sum_{\alpha=1}^{p-1} \lambda(\alpha)\lambda(\alpha-1) \\ &= \lambda(4) \sum_{\alpha=1}^{p-1} \lambda(\alpha-1) \sum_{\beta=0}^{p-1} \bar{\lambda}(\beta^2-(\bar{\alpha}-1)^2) - \frac{\tau^3(\lambda)}{p}. \end{aligned} \quad (2.12)$$

For any integer c with $(c, p) = 1$, let χ_2 denote the Legendre symbol modulo p , then we have

$$\begin{aligned}
\sum_{\beta=0}^{p-1} \bar{\lambda}(\beta^2 + \gamma) &= \sum_{\beta=0}^{p-1} (1 + \chi_2(\beta)) \bar{\lambda}(\beta + \gamma) \\
&= \sum_{\beta=0}^{p-1} \chi_2(\beta) \bar{\lambda}(\beta + \gamma) = \frac{1}{\tau(\lambda)} \sum_{\alpha=1}^{p-1} \lambda(\alpha) \sum_{\beta=1}^{p-1} \chi_2(\beta) \mathbf{e}_p(\alpha\beta + \alpha\gamma) \\
&= \frac{\tau(\chi_2)}{\tau(\lambda)} \sum_{\alpha=1}^{p-1} \lambda(\alpha) \chi_2(\alpha) \mathbf{e}_p(\alpha\gamma) = \bar{\lambda}(\gamma) \chi_2(\gamma) \cdot \frac{\tau(\chi_2) \tau(\lambda \chi_2)}{\tau(\lambda)}.
\end{aligned} \tag{2.13}$$

Applying Lemma 5 and (2.13), we have

$$\begin{aligned}
&\lambda(4) \cdot \sum_{\alpha=1}^{p-1} \lambda(\alpha - 1) \sum_{\beta=0}^{p-1} \bar{\lambda}(\beta^2 - (\bar{\alpha} - 1)^2) \\
&= \frac{\lambda(4) \cdot \tau(\chi_2) \tau(\lambda \chi_2)}{\tau(\lambda)} \sum_{\alpha=2}^{p-1} \lambda(\alpha - 1) \lambda(\bar{\alpha} - 1) \chi_2(-1) \\
&= \frac{\lambda(4) \cdot \chi_2(-1) \tau(\chi_2) \tau(\lambda \chi_2)}{\tau(\lambda)} \cdot \frac{\tau^3(\bar{\lambda})}{p} = \frac{\tau^6(\bar{\lambda})}{p^2}.
\end{aligned} \tag{2.14}$$

Combining (2.12) and (2.14), we have the identity

$$\sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda(\alpha - 1) \lambda(\beta - 1) \lambda(\alpha\beta - 1) \bar{\lambda}(\alpha\beta) = \frac{\tau^6(\bar{\lambda})}{p^2} - \frac{\tau^3(\lambda)}{p}.$$

This proves Lemma 6. □

Lemma 7. Let p be an odd prime with $3 \mid (p - 1)$, then we have the identity

$$\sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3 \equiv \delta^3+\epsilon^3+1 \pmod p}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma \equiv \delta\epsilon \pmod p}}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = p^3 + p^2 - p + 2pD + D^2 - 9D - 7,$$

where D is the same as defined in Lemma 1.

Proof. Let λ be a third-order character modulo p . Then from the properties of the reduced residue system and third-order character modulo p , we have

$$\begin{aligned}
&\sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3 \equiv \delta^3+\epsilon^3+1 \pmod p}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma \equiv \delta\epsilon \pmod p}}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\substack{\alpha=1 \\ \gamma^3 \equiv \alpha^3(\delta^3-1) + \beta^3(\epsilon^3-1) + 1 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
&= \sum_{\substack{\alpha=1 \\ \delta^3\epsilon^3 \equiv \alpha(\delta^3-1) + \beta^3(\epsilon^3-1) + 1 \pmod p}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} (1 + \lambda(\alpha) + \bar{\lambda}(\alpha)).
\end{aligned} \tag{2.15}$$

From Lemma 4, we have

$$\sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = p^3 - 7p^2 + 33p + 4D - 19. \quad (2.16)$$

$\delta^3 \epsilon^3 \equiv \alpha(\delta^3 - 1) + \beta^3(\epsilon^3 - 1) + 1 \pmod{p}$

$$\begin{aligned} & \sum_{\delta=1}^{p-1} \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \lambda(\alpha) \\ & \quad p \nmid (\delta^3 - 1) \quad \delta^3 \epsilon^3 \equiv \alpha(\delta^3 - 1) + \beta^3(\epsilon^3 - 1) \pmod{p} \\ &= \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta^3(\epsilon^3 - 1) - 1) \bar{\lambda}(\delta^3 - 1) \\ &= \sum_{\substack{\epsilon=1 \\ p \nmid (\epsilon^3 - 1)}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta^3(\epsilon^3 - 1) - 1) \bar{\lambda}(\delta^3 - 1) \\ & \quad + 3(p-1)(p-4) \\ &= \sum_{\substack{\epsilon=1 \\ p \nmid (\epsilon^3 - 1)}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta(\epsilon^3 - 1) - 1) \bar{\lambda}(\delta^3 - 1) (1 + \lambda(\beta) + \bar{\lambda}(\beta)) \\ & \quad + 3(p-1)(p-4) \\ &= - \sum_{\substack{\epsilon=1 \\ p \nmid (\epsilon^3 - 1)}}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - 1) \bar{\lambda}(\delta^3 - 1) + 3(p-1)(p-4) \\ & \quad + \sum_{\substack{\epsilon=1 \\ p \nmid (\epsilon^3 - 1)}}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta(\epsilon^3 - 1) - 1) \bar{\lambda}(\delta^3 - 1) (\lambda(\beta) + \bar{\lambda}(\beta)) \\ &= - \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - 1) \bar{\lambda}(\delta^3 - 1) + 3p(p-4) \\ & \quad + \sum_{\epsilon=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta - 1) \bar{\lambda}(\delta^3 - 1) \bar{\lambda}(\epsilon^3 - 1) \lambda(\beta) \\ & \quad + \sum_{\epsilon=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta - 1) \bar{\lambda}(\delta^3 - 1) \lambda(\epsilon^3 - 1) \bar{\lambda}(\beta). \end{aligned} \quad (2.17)$$

From the properties of the Gauss sums and Lemma 1, we have

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - 1) \bar{\lambda}(\delta^3 - 1) = \left| \sum_{\delta=1}^{p-1} \lambda(\delta^3 - 1) \right|^2 \\
&= \left| \sum_{\delta=1}^{p-1} \lambda(\delta - 1) (1 + \lambda(\delta) + \bar{\lambda}(\delta)) \right|^2 \\
&= \left| -2 + \frac{1}{\tau(\bar{\lambda})} \sum_{\beta=1}^{p-1} \bar{\lambda}(\beta) \sum_{\delta=1}^{p-1} \lambda(\delta) \mathbf{e}_p(\beta(\delta - 1)) \right|^2 \\
&= \left| -2 + \frac{\tau^2(\lambda)}{\tau(\bar{\lambda})} \right|^2 = \left| -2 + \frac{\tau^3(\lambda)}{p} \right|^2 \\
&= 4 + p - \frac{2}{p} \cdot (\tau^3(\lambda) + \tau^3(\bar{\lambda})) = p - 2D + 4.
\end{aligned} \tag{2.18}$$

We establish the following auxiliary identities. Detailed but lengthy algebraic manipulations are omitted for brevity.

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta - 1) \bar{\lambda}(\delta^3 - 1) \bar{\lambda}(\epsilon^3 - 1) \lambda(\beta) \\
&= \frac{\tau^3(\lambda)}{p} \cdot \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta^3 - 1) \bar{\lambda}(\epsilon^3 - 1) \bar{\lambda}(\delta^3 \epsilon^3 - 1) \\
&= \frac{\tau^3(\lambda)}{p} \cdot \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta - 1) \bar{\lambda}(\epsilon - 1) \bar{\lambda}(\delta\epsilon - 1) (1 + \lambda(\delta) + \bar{\lambda}(\delta)) \\
&\quad \times (1 + \lambda(\epsilon) + \bar{\lambda}(\epsilon)).
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta - 1) \bar{\lambda}(\epsilon - 1) \bar{\lambda}(\delta\epsilon - 1) = \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta) \bar{\lambda}(\epsilon) \bar{\lambda}(\delta\epsilon + \delta + \epsilon) + 1 \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta\epsilon^2) \bar{\lambda}(\delta\epsilon^2 + \delta\epsilon + e) + 1 = \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta) \bar{\lambda}(\delta\epsilon + \delta + 1) + 1 \\
&= \sum_{\epsilon=0}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta) \bar{\lambda}(\delta\epsilon + \delta + 1) + 1 - \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta(\delta + 1)) = 1 - \frac{\tau^3(\bar{\lambda})}{p}.
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \lambda(\epsilon) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \lambda(\delta) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon) \bar{\lambda}(\delta\epsilon+\delta-1) \lambda(\delta) - \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \lambda(\delta) \\
&= 1 + \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \lambda(\delta) \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon) \bar{\lambda}(\delta\epsilon+\delta-1) \\
&= 1 + \frac{1}{\tau(\lambda)} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \lambda(\delta) \sum_{\alpha=1}^{p-1} \lambda(\alpha) \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon) \mathbf{e}_p(\alpha\delta\epsilon + \alpha(\delta-1)) \\
&= 1 + \frac{\tau(\bar{\lambda})}{\tau(\lambda)} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\delta) \sum_{\alpha=1}^{p-1} \bar{\lambda}(\alpha) \mathbf{e}_p(\alpha(\delta-1)) \\
&= 1 + \frac{\tau^2(\bar{\lambda})}{\tau(\lambda)} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\delta) \lambda(\delta-1) = 1 - \frac{\tau^3(\bar{\lambda})}{p}. \tag{2.21}
\end{aligned}$$

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \bar{\lambda}(\epsilon) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \bar{\lambda}(\delta) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(1-\bar{\delta}) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\epsilon-\bar{\delta}) = \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta-\epsilon) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=0}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta-\epsilon) - \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon(\epsilon-1)) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta+1-\epsilon) - \frac{\tau^3(\bar{\lambda})}{p} \\
&= \frac{1}{\tau(\lambda)} \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon-1) \sum_{\alpha=1}^{p-1} \lambda(\alpha) \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta) \mathbf{e}_p(\alpha(\delta+1-\epsilon)) - \frac{\tau^3(\bar{\lambda})}{p} \\
&= \frac{\tau(\bar{\lambda})}{\tau(\lambda)} \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon-1) \sum_{\alpha=1}^{p-1} \bar{\lambda}(\alpha) \mathbf{e}_p(\alpha(1-\epsilon)) - \frac{\tau^3(\bar{\lambda})}{p} \\
&= \frac{\tau^2(\bar{\lambda})}{\tau(\lambda)} \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon-1) \lambda(\epsilon-1) - \frac{\tau^3(\bar{\lambda})}{p} = \frac{p-3}{p} \cdot \tau^3(\bar{\lambda}). \tag{2.22}
\end{aligned}$$

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \bar{\lambda}(\delta) \lambda(\epsilon) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \bar{\lambda}(\epsilon) \lambda(\delta) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(1-\bar{\epsilon}) \bar{\lambda}(\delta-\bar{\epsilon}) \lambda(\delta) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta-\epsilon) \lambda(\delta) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon) \bar{\lambda}(\delta-\epsilon-1) \lambda(\delta) + 1 \\
&= 1 + \frac{1}{\tau(\lambda)} \cdot \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \lambda(\delta) \sum_{\alpha=1}^{p-1} \lambda(\alpha) \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon) \mathbf{e}_p(\alpha(\delta-\epsilon-1)) \\
&= 1 + \frac{\tau(\bar{\lambda})}{\tau(\lambda)} \cdot \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \lambda(\delta) \sum_{\alpha=1}^{p-1} \bar{\lambda}(\alpha) \mathbf{e}_p(\alpha(\delta-1)) \\
&= 1 + \frac{\tau^2(\bar{\lambda})}{\tau(\lambda)} \cdot \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \lambda(\delta) \lambda(\delta-1) = 1 - \frac{\tau^3(\bar{\lambda})}{p}. \tag{2.23}
\end{aligned}$$

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon-1) \bar{\lambda}(\delta\epsilon-1) \bar{\lambda}(\delta\epsilon) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(1-\bar{\epsilon}) \bar{\lambda}(\delta-\bar{\epsilon}) \bar{\lambda}(\delta) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\epsilon) \bar{\lambda}(\delta-1-\epsilon) \bar{\lambda}(\delta) + 1 \\
&= 1 + \frac{1}{\tau(\lambda)} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\delta) \sum_{\alpha=1}^{p-1} \lambda(\alpha) \sum_{\epsilon=1}^{p-1} \bar{\lambda}(\epsilon) \mathbf{e}_p(\alpha(\delta-1-\epsilon)) \\
&= 1 + \frac{\tau(\bar{\lambda})}{\tau(\lambda)} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\delta) \sum_{\alpha=1}^{p-1} \bar{\lambda}(\alpha) \mathbf{e}_p(\alpha(\delta-1)) \\
&= 1 + \frac{\tau^2(\bar{\lambda})}{\tau(\lambda)} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta-1) \bar{\lambda}(\delta) \lambda(\delta-1) = 1 - \frac{\tau^3(\bar{\lambda})}{p}. \tag{2.24}
\end{aligned}$$

Combining (2.19)–(2.24) and Lemma 6, we have the identity

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta - 1) \bar{\lambda}(\delta^3 - 1) \bar{\lambda}(\epsilon^3 - 1) \lambda(\beta) \\
&= 1 - \frac{\tau^3(\bar{\lambda})}{p} + 2 \left(1 - \frac{\tau^3(\bar{\lambda})}{p} \right) + \frac{2p-6}{p} \cdot \tau^3(\bar{\lambda}) + 2 \left(1 - \frac{\tau^3(\bar{\lambda})}{p} \right) \\
&\quad + 1 - \frac{\tau^3(\bar{\lambda})}{p} + \frac{\tau^6(\lambda)}{p^2} - \frac{\tau^3(\bar{\lambda})}{p} \\
&= 6 + \frac{2p-13}{p} \cdot \tau^3(\bar{\lambda}) + \frac{\tau^6(\lambda)}{p^2}. \tag{2.25}
\end{aligned}$$

$$\begin{aligned}
& \sum_{\epsilon=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \lambda(\delta^3 \epsilon^3 - \beta - 1) \bar{\lambda}(\delta^3 - 1) \lambda(\epsilon^3 - 1) \bar{\lambda}(\beta) \\
&= \frac{1}{\tau(\bar{\lambda})} \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta^3 - 1) \lambda(\epsilon^3 - 1) \sum_{k=1}^{p-1} \bar{\lambda}(k) \sum_{\beta=1}^{p-1} \bar{\lambda}(\beta) \mathbf{e}_p(k(\delta^3 \epsilon^3 - \beta - 1)) \\
&= \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta^3 - 1) \lambda(\epsilon^3 - 1) \sum_{k=1}^{p-1} \mathbf{e}_p(k(\delta^3 \epsilon^3 - 1)) \\
&= p \cdot \sum_{\substack{\epsilon=1 \\ \delta^3 \epsilon^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta^3 - 1) \lambda(\epsilon^3 - 1) - \sum_{\epsilon=1}^{p-1} \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta^3 - 1) \lambda(\epsilon^3 - 1) \\
&= p \cdot \sum_{\delta=1}^{p-1} \bar{\lambda}(\delta^3 - 1) \lambda(\bar{\delta}^3 - 1) - \left| \sum_{\epsilon=1}^{p-1} \lambda(\epsilon - 1) (1 + \lambda(\epsilon) + \bar{\lambda}(\epsilon)) \right|^2 \\
&= p(p-4) + p + 4 - 2D = p^2 - 3p + 4 - 2D. \tag{2.26}
\end{aligned}$$

From (2.17), (2.18), (2.25), and (2.26), we have

$$\begin{aligned}
& \sum_{\substack{\delta=1 \\ p \nmid (\delta^3-1)}}^{p-1} \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \lambda(\alpha) \\
&\quad p \nmid (\delta^3-1) \delta^3 \epsilon^3 \equiv \alpha(\delta^3-1) + \beta^3(\epsilon^3-1) + 1 \pmod{p} \\
&= -(p+4-2D) + 3p^2 - 12p + 6 + \frac{2p-13}{p} \cdot \tau^3(\bar{\lambda}) \\
&\quad + \frac{\tau^6(\lambda)}{p^2} + p^2 - 3p + 4 - 2D \\
&= 4p^2 - 16p + 6 + \frac{2p-13}{p} \cdot \tau^3(\bar{\lambda}) + \frac{\tau^6(\lambda)}{p^2}. \tag{2.27}
\end{aligned}$$

Then from (2.15), (2.16), (2.27) and Lemma 1, we obtain the identity

$$\begin{aligned}
 & \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 = \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
 & \quad \alpha^3 + \beta^3 + \gamma^3 \equiv \delta^3 + \epsilon^3 + 1 \pmod{p} \quad \gamma^3 \equiv \alpha^3 (\delta^3 - 1) + \beta^3 (\epsilon^3 - 1) + 1 \pmod{p} \\
 & \quad \alpha\beta\gamma \equiv \delta\epsilon \pmod{p} \quad \gamma \equiv \delta\epsilon \pmod{p} \\
 & = \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \left(1 + \lambda(\alpha) + \bar{\lambda}(\alpha) \right) \\
 & \quad \delta^3 \epsilon^3 \equiv \alpha (\delta^3 - 1) + \beta^3 (\epsilon^3 - 1) + 1 \pmod{p} \\
 & = p^3 - 7p^2 + 33p + 4d - 19 + 4p^2 - 16p + 6 + \frac{2p-13}{p} \cdot \tau^3(\bar{\lambda}) + \frac{\tau^6(\lambda)}{p^2} \\
 & \quad + 4p^2 - 16p + 6 + \frac{2p-13}{p} \cdot \tau^3(\lambda) + \frac{\tau^6(\bar{\lambda})}{p^2} \\
 & = p^3 + p^2 - p + 2pD + D^2 - 9D - 7.
 \end{aligned}$$

This proves Lemma 7. $\square$

3. Proof of the theorem

The proofs of our theorems are now straightforward, thanks to the seven basic lemmas from the preceding section. In particular, when $3 \mid (p-1)$, we invoke Lemma 2 to obtain the trigonometric identity

$$\sum_{\alpha=0}^{p-1} \mathbf{e}_p(m\alpha) = \begin{cases} p, & \text{if } p \mid m; \\ 0, & \text{if } p \nmid m. \end{cases}$$

From the orthogonality of characters modulo p , we obtain

$$\begin{aligned}
 & \sum_{\chi \pmod{p}} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 \\
 & = p(p-1) \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \sum_{\phi=1}^{p-1} e\left(\frac{\alpha + \beta + \gamma - \delta - \epsilon - \phi}{p}\right) \\
 & \quad \alpha^3 + \beta^3 + \gamma^3 \equiv \delta^3 + \epsilon^3 + \phi^3 \pmod{p} \\
 & \quad \alpha\beta\gamma \equiv \delta\epsilon\phi \pmod{p} \\
 & = p(p-1) \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} \sum_{\phi=1}^{p-1} \mathbf{e}_p(f(\alpha + \beta + \gamma - \delta - \epsilon - 1)) \\
 & \quad \alpha^3 + \beta^3 + \gamma^3 \equiv \delta^3 + \epsilon^3 + 1 \pmod{p} \\
 & \quad \alpha\beta\gamma \equiv \delta\epsilon \pmod{p} \\
 & = p^2(p-1) \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 - p(p-1) \cdot \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\
 & \quad \alpha^3 + \beta^3 + \gamma^3 \equiv \delta^3 + \epsilon^3 + 1 \pmod{p} \quad \alpha\beta\gamma \equiv \delta\epsilon \pmod{p} \\
 & \quad \alpha + \beta + \gamma \equiv \delta + \epsilon + 1 \pmod{p}
 \end{aligned}$$

$$= p^2(p-1)(7p^2 - 29p + 44) - \frac{1}{p-1} \cdot \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3) \right|^6.$$

This proves Theorem 1.

From Lemma 7, we have

$$\begin{aligned} & \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 \\ &= p^2(p-1) \sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3\equiv\delta^3+\epsilon^3+1 \bmod p}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma\equiv\delta\epsilon \bmod p}}^{p-1} \sum_{\substack{\gamma=1 \\ \alpha+\beta+\gamma\equiv\delta+\epsilon+1 \bmod p}}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 - p(p-1) \cdot \sum_{\substack{\alpha=1 \\ \alpha^3+\beta^3+\gamma^3\equiv\delta^3+\epsilon^3+1 \bmod p}}^{p-1} \sum_{\substack{\beta=1 \\ \alpha\beta\gamma\equiv\delta\epsilon \bmod p}}^{p-1} \sum_{\gamma=1}^{p-1} \sum_{\delta=1}^{p-1} \sum_{\epsilon=1}^{p-1} 1 \\ &= p^2(p-1)(7p^2 - 29p + 44) \\ &\quad - p(p-1)(p^3 + p^2 - p + 2pD + D^2 - 9D - 7) \\ &= p(p-1)(6p^3 - 30p^2 + 45p - 2Dp - D^2 + 9D + 7). \end{aligned}$$

This completes the proof of Theorem 2.

As a consistency check, take the smallest prime $p = 7$ satisfying $p \equiv 1 \pmod{3}$. Recall that D is uniquely determined by $4p = D^2 + 27B^2$ with $D \equiv 1 \pmod{3}$. For $p = 7$, we have $4 \cdot 7 = 28 = 1^2 + 27 \cdot 1^2$, so $D = 1$ and $B = 1$. Substituting into Theorem 2 yields an integer divisible by $p(p-1)$, which provides a numerical sanity check for our formula.

4. Conclusions

The main results of this paper are to prove the following two identities:

Let p be an odd prime with $3 \mid (p-1)$, then we have

$$\begin{aligned} & \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3 + \alpha) \right|^6 \\ &= p^2(p-1)(7p^2 - 29p + 44) - \frac{1}{p-1} \cdot \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3) \right|^6 \end{aligned}$$

and

$$\frac{1}{p \cdot (p-1)^2} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{\alpha=1}^{p-1} \chi(\alpha) \mathbf{e}_p(m\alpha^3) \right|^6 = p^3 + p^2 - p + 2pD + D^2 - 9D - 7.$$

In fact, we have solved an open problem proposed by Liu and Zhang [2].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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