



Research article

Individual impacts of memory-mediated recovery and periodic intervention on epidemic population systems: Dynamic modeling and qualitative analysis

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Abstract: This work developed a modified Gompertz-type pest-natural enemy population model that accounts for infected natural enemies and state-dependent searching behavior. The model was then extended by incorporating two further mechanisms, namely, memory-mediated disease recovery and periodic intervention. Systematic theoretical analyses were conducted on solution boundedness, the existence and local asymptotic stability of equilibria, and transcritical bifurcation behaviors. These analyses clarify the suppression mechanism of natural-enemy predation on pest proliferation and reveal the regulatory effects of the variable searching rate and memory parameters on system dynamics. Considering that natural-enemy predation alone achieves sustainable biological control but responds slowly and cannot rapidly suppress pest outbreaks, the present work adopted an integrated strategy that combines periodic natural-enemy release with pesticide spraying to enhance pest control efficacy. The global asymptotic stability of pest-free periodic solutions was established, and parametric thresholds ensuring system persistence within a specific impulsive-period range were derived. Furthermore, sufficient criteria were provided for the global asymptotic stability of the system under pest-extinction and infected-predator-extinction scenarios, together with conditions for population persistence and the coexistence of periodic solutions. All theoretical results were validated via MATLAB numerical simulations. This study extends the theoretical framework of population dynamics and provides reliable references for integrated pest prevention and ecological regulation.

Keywords: memory-mediated recovery; periodic intervention; epidemic population model; variable searching rate

1. Introduction

In the global trend toward sustainable agriculture, the ecological management of pests has become a critical issue concerning the survival of industries and ecological balance. From an economic perspective, the burden imposed by crop diseases and pests is particularly severe. According to estimates by the United Nations Food and Agriculture Organization, global annual crop losses due to pests and diseases amount to approximately 20% to 40% of total production [1]. In China, agricultural losses caused by insect pests reach a scale of hundreds of billions of yuan. Agricultural intensification and global climate change further exacerbate the frequency, spread, and resistance of pests, posing a continuous threat to agricultural production. Although traditional chemical control yields short-term results, it tends to accelerate the evolution of pest resistance, elevate control costs, and contribute to agricultural non-point source pollution, thereby disrupting farmland ecological balance and biodiversity-revealing significant systemic flaws. Against this backdrop, green control technologies centered on ecological regulation, and physical and biological control, have emerged. Integrated pest management (IPM) is a comprehensive strategy aimed at minimizing pesticide use, whose core lies in coordinating multiple control measures to effectively suppress pest populations while minimizing adverse effects on ecosystems [2].

In ecological studies, predator-prey interactions serve as a core determinant of species behavioral traits, community structure, and systematic dynamics. Predator-prey frameworks integrated with diverse ecological mechanisms have attracted extensive research attention in existing literature [3–6]. The Gompertz model [7], as a classic sigmoidal model, is often used to fit growth data, with applications spanning studies on the growth of plants, birds, fish, and other animals [8–10]. Furthermore, the Holling-II functional response [11] is one of the most widely used functional response models in ecological dynamics, characterizing the nonlinear relationship where the predators consumption rate initially increases rapidly with prey density and then gradually saturates, while treating the searching rate as constant. However, a predator's search efficiency can vary with changes in prey density. Mondal and Samanta [12] considered a modified Holling-II system that accounts for both fear responses and density-dependent foraging effort. A related predator-prey formulation was analyzed by Yu et al. [13], also featuring fear dynamics alongside adaptive search behavior. Guo et al. [14] introduced fuzzy biological parameters into a fishery harvest setting, while Tian et al. [15] formulated a threshold-based pest-enemy model with non-smooth switching and variable hunting rates. Here, we study a class of Holling type-II pest-predator models with Gompertz growth, incorporating disease recovery memory effects and a variable searching rate. In addition, as a critical extension of integer-order calculus, fractional differential equations have emerged as powerful tools for ecological modeling. Owing to their distinctive memory and hereditary properties, such equations offer a rigorous mathematical framework for characterizing the history-dependent dynamical behaviors of biological populations [16–20].

In natural ecosystems, population growth and interspecific interactions are inherently coupled with disease transmission rather than occurring independently. Accordingly, the integration of population and epidemic dynamics enables more accurate characterization and prediction of complex ecological system behaviors [21–24]. Periodic impulsive control employs dynamic, intermittent interventions by applying pesticides impulsively during critical growth stages of crops, i.e., precise pesticide application at specific time points. Song et al. [25] examined the dynamics of a Holling-II two-prey

one-predator system subject to impulsive interventions, encompassing both biological and chemical control measures. Qian et al. [26] focused on a predator-predator framework featuring interspecific competition under periodic impulsive regulation. A three-species impulsive food chain with periodic coefficients and a prey-dependent functional response was treated by Li et al. [27]. Yan et al. [28] addressed a Leslie-Gower pest-enemy formulation that accounts for generalist natural enemies and pest clustering, putting forward a threshold-based strategy that alternates pesticide application with enemy augmentation. This work implements a periodic impulsive control strategy to curtail pest propagation.

The remainder of this paper is structured as follows. Section 2 formulates two ecological models: a fractional-order pest-natural enemy model incorporating a variable searching rate and disease recovery memory effects, and a corresponding periodic impulsive control model. Section 3 conducts theoretical analysis on the continuous model, including solution boundedness, the existence and local asymptotic stability of equilibria, and transcritical bifurcation characteristics. The global asymptotic stability of pest-free periodic solutions and system persistence are investigated. Furthermore, sufficient criteria for the global asymptotic stability and persistence of two periodic impulsive models, namely, the pest-predator and predator-infected predator systems, are derived, and the existence of coexisting periodic solutions is verified. Section 4 presents MATLAB-based numerical simulations, which validate the theoretical results and provide insightful ecological implications.

2. Model formulation and preliminaries

2.1. Model formulation

The classical Gompertz growth of the prey is governed by

$$\frac{dX}{dt} = rX \ln \frac{K}{X},$$

whose per-capita growth rate $r \ln(K/X)$ tends to $+\infty$ as $X \rightarrow 0^+$. Biologically, however, the per-capita growth rate should remain finite even when the population density is extremely low. To eliminate this singularity, we introduce a parameter c with $0 < c < K$ and modify the per-capita growth rate to

$$r \ln \frac{K}{X+c},$$

which yields

$$r \ln \frac{K}{X+c} \xrightarrow{X \rightarrow 0^+} r \ln \frac{K}{c} < \infty,$$

so that c keeps the per-capita growth rate finite at low population densities. Based on this assumption, we consider a modified Gompertz-type model for pest and infected natural enemy populations, with a variable searching rate and a Holling-II functional response, formulated as follows:

$$\begin{cases} \frac{dX}{dt} = rX(\ln K - \ln(X+c)) - \frac{b_{\max} S X^2}{(X+g)(1+hX)}, \\ \frac{dS}{dt} = \frac{vb_{\max} S X^2}{(X+g)(1+hX)} - dS - \beta S I + \frac{1}{\tau} I, \\ \frac{dI}{dt} = \beta S I - \frac{1}{\tau} I - dI, \end{cases} \quad (2.1)$$

where X denotes the pest population, S the susceptible natural enemies, and I the infected natural enemies. Since many natural enemy individuals exhibit impaired behavior after becoming infected with diseases, resulting in a significant decrease or even near-complete loss of their predation efficiency, it is assumed here that infected natural enemies do not participate in predation at all [22]. The parameter r represents the intrinsic growth rate, $K' \triangleq K - c$ represents the environmental carrying capacity of the pest, and $b(X) = \frac{b_{\max}X}{X + g}$ captures a searching rate dependent on prey density [12–15], with $b_{\max}$ being the upper bound and g the half-saturation level. The constant h denotes the handling time of a predator per prey, v the conversion efficiency, and d the natural mortality rate of the natural enemies. All parameters are positive, with $b_{\max}$, v , d all less than 1. The condition $vb_{\max} - dh > 0$ is required to ensure the survival of the natural enemy species even when the pest population is large. The term $\frac{1}{\tau}$ is the recovery rate of infected natural enemies, and τ represents the mean infection period.

Traditional ecological models adopt a constant disease recovery rate, which simplifies individual recovery behavior as instantaneous memory-free responses. Building on the foundational studies of Angstmann et al. [17] and Monteiro and Mazorche [18], this paper constructs a dynamical model via the continuous-time random walk framework. First, the concept of “waiting time” is introduced, where $\phi(t)$ represents the waiting time survival function, meaning the probability that an infected individual remains “surviving” in the infected state without being recovered, with $\psi(t) = -\frac{d\phi(t)}{dt}$, and $\psi(t) \sim t^{-1-\alpha}$ exhibiting a power-law decay distribution. This indicates that the recovery process of infected individuals has historical dependence: the longer the infection duration, the more significant the impact of persistent infection, making rapid recovery difficult. To handle the complex convolution operations in the equations, Laplace transforms are also employed in the derivation process. In this process, the memory kernel function $K(t)$ is defined, and its Laplace transform is calculated to yield the form $\frac{s^{1-\alpha}}{\tau^\alpha}$. Utilizing the Laplace transform property of the Riemann-Liouville fractional derivative $\mathcal{L}_t\{ {}_0D_t^{1-\alpha} f(t) | s \} = s^{1-\alpha} \mathcal{L}_t\{ f(t) | s \}$, the convolution inverse transform of the memory kernel is transformed into a fractional differential operator [19, 20]. This approach has been mostly used in human epidemic models. In this paper, we apply it to the recovery process of infected natural enemies in a three-dimensional predator-prey system. Ultimately, the model takes the form:

$$\begin{cases} \frac{dX(t)}{dt} = rX(t)(\ln K - \ln(X(t) + c)) - \frac{b_{\max}S(t)X(t)^2}{(X(t) + g)(1 + hX(t))}, \\ \frac{dS(t)}{dt} = \frac{vb_{\max}S(t)X(t)^2}{(X(t) + g)(1 + hX(t))} - dS(t) - \beta S(t)I(t) + \frac{1}{e^{dt}\tau^\alpha} {}_0D_t^{1-\alpha} (e^{dt}I(t)), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - \frac{1}{e^{dt}\tau^\alpha} {}_0D_t^{1-\alpha} (e^{dt}I(t)) - dI(t), \end{cases} \quad (2.2)$$

where $\frac{1}{e^{dt}\tau^\alpha}$ represents the memory kernel convolution term derived from the continuous-time random walk framework.

In agricultural pest management, pest infestations not only cause direct crop damage but also disturb ecological equilibrium, thereby undermining agricultural production stability and environmental safety. To mitigate economic losses and reduce the difficulty of long-term ecological management, proactive regulatory strategies should be executed before pest populations reach

economically damaging thresholds. Accordingly, this study proposes an integrated threshold-dependent control framework. Specifically, a predefined pest-density warning threshold is established. Once monitored pest populations exceed this critical value, targeted interventions, including quantitative pesticide spraying and artificial natural enemy release, are activated to efficiently restrain pest proliferation and spread. Considering the periodic outbreak characteristics of pest populations, a periodic impulsive control strategy is further incorporated into the ecological model (2.1):

$$\left\{ \begin{array}{l} \frac{dX(t)}{dt} = rX(t)(\ln K - \ln(X(t) + c)) - \frac{b_{max}S(t)X(t)^2}{(X(t) + g)(1 + hX(t))} \\ \frac{dS(t)}{dt} = \frac{vb_{max}S(t)X(t)^2}{(X(t) + g)(1 + hX(t))} - dS(t) - \beta S(t)I(t) + \frac{I(t)}{\tau} \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - \frac{I(t)}{\tau} - dI(t) \end{array} \right\} \quad t \neq jT_p, \quad (2.3)$$

$$\left\{ \begin{array}{l} \Delta X = -p_1X(t) \\ \Delta S = -q_2S(t) + s_T \\ \Delta I = -q_2I(t) \end{array} \right\} \quad t = jT_p.$$

Note that the recovery term in model (2.3) reduces to the classical form I/τ , rather than adopting the fractional-order memory kernel used in model (2.2). This is because the nonlocal nature of the fractional-order operator, which couples the dynamics to the entire state history over $[0, t]$, is inherently incompatible with the instantaneous jumps introduced by the impulses.

At the prescribed times $t = jT_p$ ($j \in N_+$), integrated management measures are implemented: a quantity of susceptible natural enemies (s_T) is released, while a proportion (p_1) of the pest population and a proportion (q_2) of both susceptible and infected natural enemies are simultaneously eliminated. Consequently, the densities of the pest X , susceptible natural enemies S , and infected natural enemies I instantly change to $(1 - p_1)X(t)$, $(1 - q_2)S(t) + s_T$, and $(1 - q_2)I(t)$, respectively.

3. Main work

Qualitative analysis of models (2.2) and (2.3) is conducted below.

3.1. Dynamic analysis of (2.2)

3.1.1. Boundedness of solutions

Theorem 1. All solutions $(X(t), S(t), I(t))$ of (2.2) with initial values defined on $\mathbb{R}_+^3$ are uniformly bounded.

Proof. Model (2.2) yields $\frac{dX}{dt} \leq rX(\ln K - \ln X)$. Let $A(X) = rX(\ln K - \ln X)$. It can be calculated that the maximum point of $A(X)$ occurs at $X = \frac{K}{e}$. $A\left(\frac{K}{e}\right) = \frac{rK}{e}$. That is, $\frac{dX}{dt} \leq \frac{rK}{e}$.

Consider an auxiliary system. Let $\frac{dy}{dt} = ry(\ln K - \ln y)$, $y(0) = X(0)$. By the comparison theorem, $X(t) \leq y(t)$ for all $t \geq 0$. Thus,

$$y(t) = \exp[\ln K + (\ln X(0) - \ln K)e^{-rt}] = K \left(\frac{X(0)}{K} \right)^{e^{-rt}}.$$

Therefore, $y(t) \leq \max\{X(0), K\}$, $\forall t \geq 0$. Hence, $X(t) \leq X_m = \max\{X(0), K\}$, $\forall t \geq 0$.

Since $y(t) \rightarrow K$ as $t \rightarrow \infty$, and $X(t) \leq y(t)$, it follows that: $\limsup_{t \rightarrow \infty} X(t) \leq K$. Thus, the system is ultimately uniformly bounded with respect to X .

Define the function $T = X + \frac{1}{\nu}(S + I)$. Then

$$\frac{dT}{dt} = rX(\ln K - \ln(X + c)) - \frac{d}{\nu}(S + I) \leq \frac{rK}{\nu} - d(T - X) \leq \frac{rK}{\nu} - dT + dX_m.$$

Let $B = \frac{rK}{\nu} + dX_m$. Then $\frac{dT}{dt} \leq B - dT$. Consequently,

$$T(t) \leq T(0)e^{-dt} + \frac{B}{d}(1 - e^{-dt}) \leq T(0)e^{-dt} + \frac{B}{d}.$$

As $t \rightarrow \infty$, we have $0 \leq T \leq \frac{B}{d}$, i.e., $0 \leq X + \frac{1}{\nu}(S + I) \leq \frac{B}{d}$. Therefore, X , S , and I are ultimately uniformly bounded. $\square$

3.1.2. Existence and stability of equilibria

Denote

$$\tilde{X} = \frac{d(1 + gh) + \sqrt{d^2(1 - gh)^2 + 4d\nu b_{\max}g}}{2(\nu b_{\max} - dh)}, \tilde{S} = \frac{\nu r \tilde{X}}{d} \ln\left(\frac{K}{\tilde{X} + c}\right)$$

and define

$$F(X) = r \ln\left(\frac{X + c}{K}\right) + \frac{b_{\max}dX}{\beta(X + g)(1 + hX)} \left(1 + \frac{1}{\tau^\alpha d^\alpha}\right).$$

Theorem 2. (i) The equilibria $E_0 = (0, 0, 0)$, $E_1 = (K - c, 0, 0)$, and $E_2 = (\tilde{X}, \tilde{S}, 0)$ exist unconditionally.

(ii) A positive equilibrium $E_3(X^*, S^*, I^*)$ exists if the condition $\beta > \beta_T \triangleq (d + \frac{1}{\tau^\alpha} d^{1-\alpha})\tilde{S}^{-1}$ is satisfied, where

$$S^* = \frac{d}{\beta} \left(1 + \frac{1}{\tau^\alpha d^\alpha}\right), \quad I^* = S^* \left(\frac{\nu b_{\max}(X^*)^2}{d(X^* + g)(1 + hX^*)} - 1\right).$$

Proof. $E_0 = (0, 0, 0)$: Directly substituting into model (2.2), the right-hand sides of all three equations are zero, so E_0 is always an equilibrium point.

$E_1 = (K - c, 0, 0)$: Let $S = I = 0$, and the first equation of model (2.2) becomes $\frac{dX}{dt} = rX(t)(\ln K - \ln(X(t) + c)) = 0$. $X = 0$ corresponds to E_0 . $X \neq 0$, so we need $\ln K = \ln(X(t) + c)$, and then $X = K - c$. From the assumption $K > c$, we know $K - c > 0$, so E_1 always exists.

$E_2 = (\tilde{X}, \tilde{S}, 0)$: Let $I = 0$. $\frac{dS}{dt} = 0$ ($S \neq 0$) gives $\frac{\nu b_{\max} X^2}{(X+g)(1+hX)} = d$, and rearranging gives $(\nu b_{\max} - dh)X^2 - d(1 + gh)X - dg = 0$. From the assumption $\nu b_{\max} - dh > 0$, the product of the two roots is $-\frac{dg}{\nu b_{\max} - dh} < 0$. That is, there is one positive root and one negative root, so the positive root is unique, namely,

$$\tilde{X} = \frac{d(1 + gh) + \sqrt{d^2(1 - gh)^2 + 4d\nu b_{\max}g}}{2(\nu b_{\max} - dh)}.$$

Then from $\frac{dX}{dt} = 0$, we have $rX(t)(\ln K - \ln(X(t) + c)) = \frac{b_{\max}SX^2}{(X+g)(1+hX)} = \frac{d}{\nu}S$, and solving gives

$$\tilde{S} = \frac{\nu r \tilde{X}}{d} \ln\left(\frac{K}{\tilde{X} + c}\right).$$

We next establish the existence of the positive equilibrium E_3 . Denote

$$\lim_{t \rightarrow \infty} X(t) = X^*, \quad \lim_{t \rightarrow \infty} S(t) = S^*, \quad \lim_{t \rightarrow \infty} I(t) = I^*.$$

Let $\frac{1}{\tau} = \delta$. Multiplying both sides of the above equation by e^{dt} gives

$$\begin{cases} \frac{d}{dt}(e^{dt}X) = e^{dt} \left[(d + r \ln K)X - rX \ln(X + c) - \frac{b_{\max} S X^2}{(X + g)(1 + hX)} \right], \\ \frac{d}{dt}(e^{dt}S) = e^{dt} \left[\frac{v b_{\max} S X^2}{(X + g)(1 + hX)} - \beta S I + \delta I \right], \\ \frac{d}{dt}(e^{dt}I) = e^{dt} [\beta S I - \delta I], \end{cases}$$

and integrating from 0 to t gives

$$\begin{cases} X(t) = X(0)e^{-dt} \\ \quad + \int_0^t e^{-d(t-\tau)} \left[(d + r \ln K)X(\tau) - rX(\tau) \ln(X(\tau) + c) - \frac{b_{\max} S(\tau) X(\tau)^2}{(X(\tau) + g)(1 + hX(\tau))} \right] d\tau, \\ S(t) = S(0)e^{-dt} + \int_0^t e^{-d(t-\tau)} \frac{v b_{\max} S(\tau) X(\tau)^2}{(X(\tau) + g)(1 + hX(\tau))} d\tau - \beta \int_0^t e^{-d(t-\tau)} S(\tau) I(\tau) d\tau \\ \quad + \frac{1}{\tau^\alpha \Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} e^{-d(t-\tau)} I(\tau) d\tau, \\ I(t) = I(0)e^{-dt} + \beta \int_0^t e^{-d(t-\tau)} S(\tau) I(\tau) d\tau - \frac{1}{\tau^\alpha \Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} e^{-d(t-\tau)} I(\tau) d\tau. \end{cases}$$

Let $g(X, S) = (d + r \ln K)X - rX \ln(X + c) - \frac{b_{\max} S X^2}{(X+g)(1+hX)}$. Taking the first equation as an example, we make the substitution $u = t - \tau$ to obtain $\int_0^t e^{-d(t-\tau)} g(X(\tau), S(\tau)) d\tau = \int_0^t e^{-du} g(X(t-u), S(t-u)) du$. By assumption, as $t \rightarrow \infty$, we have $X(t-u) \rightarrow X^*$ and $S(t-u) \rightarrow S^*$. Since g is continuous, as $t \rightarrow \infty$,

$$g(X(t-u), S(t-u)) \rightarrow g(X^*, S^*) \triangleq g^*.$$

Moreover, by Theorem 1, $g(X(t-u), S(t-u))$ is uniformly bounded with respect to u . Therefore, by the dominated convergence theorem,

$$\lim_{t \rightarrow \infty} \int_0^t e^{-du} g(X(t-u), S(t-u)) du = \int_0^\infty e^{-du} g^* du = g^* \int_0^\infty e^{-du} du = \frac{g^*}{d}.$$

Also, $X(0)e^{-dt} \rightarrow 0$ as $t \rightarrow \infty$. Substituting into the first equation yields:

$$X^* = \frac{g(X^*, S^*)}{d} = \frac{(d + r \ln K)X^* - rX^* \ln(X^* + c)}{d} - \frac{b_{\max} S^* (X^*)^2}{d(X^* + g)(1 + hX^*)}.$$

The second and third equations can be handled in the same manner. Finally, when $t \rightarrow \infty$, the equation

can be simplified to:

$$\begin{cases} X^* = \frac{(d+r\ln K)X^*}{d} - \frac{rX^*\ln(X^*+c)}{d} - \frac{b_{\max}S^*(X^*)^2}{d(X^*+g)(1+hX^*)}, \\ S^* = \frac{vb_{\max}S^*(X^*)^2}{d(X^*+g)(1+hX^*)} - \frac{\beta I^*S^*}{d} + \frac{I^*}{\tau^\alpha\Gamma(\alpha)} \int_0^{+\infty} (t-\tau)^{\alpha-1} e^{-d(t-\tau)} d\tau, \\ I^* = \frac{\beta I^*S^*}{d} - \frac{I^*}{\tau^\alpha\Gamma(\alpha)} \int_0^{+\infty} (t-\tau)^{\alpha-1} e^{-d(t-\tau)} d\tau. \end{cases}$$

Since $\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-dt} dt = d^{-\alpha}$, then

$$\begin{cases} X^* = \frac{(d+r\ln K)X^*}{d} - \frac{rX^*\ln(X^*+c)}{d} - \frac{b_{\max}S^*(X^*)^2}{d(X^*+g)(1+hX^*)}, \\ S^* = \frac{vb_{\max}S^*(X^*)^2}{d(X^*+g)(1+hX^*)} - \frac{\beta I^*S^*}{d} + \frac{I^*}{\tau^\alpha d^\alpha}, \\ I^* = \frac{\beta I^*S^*}{d} - \frac{I^*}{\tau^\alpha d^\alpha}. \end{cases} \quad (3.1)$$

From the third equation in (3.1), $I^* = \frac{\beta I^*S^*}{d} - \frac{I^*}{\tau^\alpha d^\alpha}$, and dividing both sides by I^* ($I^* \neq 0$) yields $1 = \frac{\beta S^*}{d} - \frac{1}{\tau^\alpha d^\alpha}$. We readily obtain $S^* = \frac{d}{\beta} \left(1 + \frac{1}{\tau^\alpha d^\alpha}\right) > 0$, which is valid for all parameter values. Consider

$$X^* = \frac{(d+r\ln K)X^*}{d} - \frac{rX^*\ln(X^*+c)}{d} - \frac{b_{\max}S^*(X^*)^2}{d(X^*+g)(1+hX^*)}.$$

Rearranging gives

$$r \ln \left(\frac{X^*+c}{K} \right) + \frac{b_{\max}S^*X^*}{(X^*+g)(1+hX^*)} = 0.$$

Dividing both sides by $X^* > 0$ and substituting the expression for S^* , $S^* = \frac{d}{\beta} \left(1 + \frac{1}{\tau^\alpha d^\alpha}\right)$, yields $F(X^*) = 0$, where X^* is a root of $F(X) = 0$. We have

$$F(X) = r \ln \left(\frac{X+c}{K} \right) + \frac{b_{\max}dX}{\beta(X+g)(1+hX)} \left(1 + \frac{1}{\tau^\alpha d^\alpha} \right).$$

As $X \rightarrow 0^+$, the second term tends to 0, and the first term tends to $r \ln \left(\frac{c}{K} \right)$. Since $K > c$ by assumption, $\ln \left(\frac{c}{K} \right) < 0$, hence $\lim_{X \rightarrow 0^+} F(X) < 0$. At $X = K - c$, the first term is $r \ln \left(\frac{K}{K} \right) = 0$, and the second term is positive, so $F(K-c) > 0$. By the intermediate value theorem, there exists an equilibrium $X^* \in (0, K)$ satisfying $F(X^*) = 0$.

Furthermore, Eq (3.1) gives $I^* = S^* \left(\frac{vb_{\max}(X^*)^2}{d(X^*+g)(1+hX^*)} - 1 \right)$. Define $G(X) = \frac{vb_{\max}X^2}{d(X+g)(1+hX)}$. Since $G'(X) > 0$, $G(X)$ is strictly increasing and continuous over $(0, +\infty)$. As G is strictly increasing with $\lim_{X \rightarrow 0^+} G(X) = 0$ and $\lim_{X \rightarrow \infty} G(X) = \frac{vb_{\max}}{dh} > 1$, the equation $G(X) = 1$ has a unique positive root, which is precisely $\tilde{X}$. It remains to ensure $I^* > 0$, which is equivalent to $G(X^*) > 1$, and hence to $X^* > \tilde{X}$.

Let $\Phi(X) = -F(X) = r \ln \frac{K}{X+c} - b_{\max}S^*H(X)$, which has the same zeros as F . From the defining relation of E_2 , $r \ln \frac{K}{X+c} = b_{\max}\tilde{S}H(\tilde{X})$, so

$$\Phi(\tilde{X}) = b_{\max}\tilde{S}H(\tilde{X}) - b_{\max}S^*H(\tilde{X}) = b_{\max}H(\tilde{X})(\tilde{S} - S^*).$$

Moreover $\Phi(K - c) = -b_{\max} S^* H(K - c) < 0$. Hence, if $\tilde{S} > S^*$, i.e., $\beta > \beta_T$, then $\Phi(\tilde{X}) > 0$, and by the intermediate value theorem there exists $X^* \in (\tilde{X}, K - c)$ with $\Phi(X^*) = 0$. Thus $X^* > \tilde{X}$ and $I^* = S^*(G(X^*) - 1) > 0$.

Together with $X^* > 0$ and $S^* > 0$, $E_3 = (X^*, S^*, I^*)$ is a positive equilibrium.

Remark 1. The number of positive equilibrium points cannot be determined. There may be one, two, or three. The reasons are as follows: Let $F(X) = r \ln(\frac{X+c}{K}) + \frac{lX}{(X+g)(1+hX)}$, where $l = \frac{b_{\max}d}{\beta}(1 + \frac{1}{\tau^\alpha d^\alpha})$. Denote $A(X) = \ln(\frac{K}{X+c})$, $B(X) = \frac{lX}{r(X+g)(1+hX)}$. Below, we determine the number of intersection points of $A(X)$ and $B(X)$. $A'(X) < 0$ holds identically, so $A(X)$ is monotonically decreasing. $A(K - c) = 0$. $B'(X) = \frac{l}{r} \frac{g-hX^2}{[(X+g)(1+hX)]^2}$. When $X < \sqrt{\frac{g}{h}}$, $B'(X) > 0$. When $\sqrt{\frac{g}{h}} < X < K - c$, $B'(X) < 0$. Therefore, $B(X)$ is a unimodal function that first increases and then decreases. From the above, the number of intersection points of $A(X)$ and $B(X)$ may result in the following three cases, which are one, two, and three, respectively:

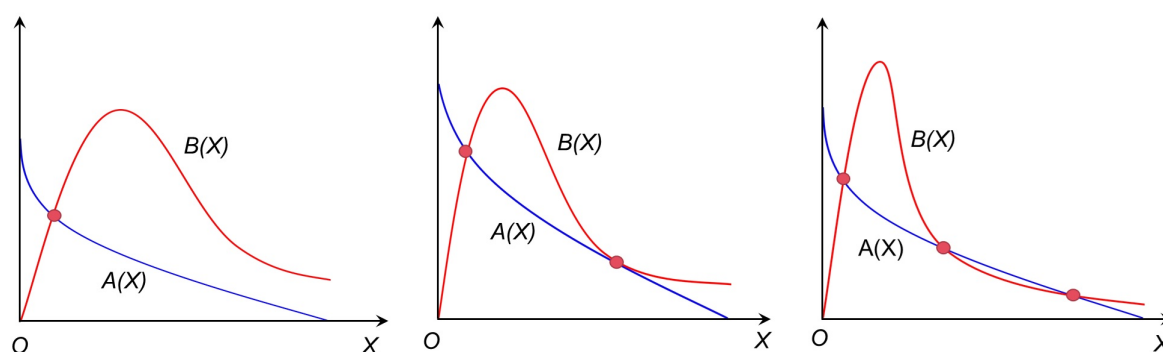


Figure 1. The number of intersection points of $A(X)$ and $B(X)$.

Therefore, the number of X^* , i.e., the number of positive equilibrium points, cannot be determined, and there may be three possible cases.

□

Theorem 3. (i) The equilibrium E_0 is unstable.

(ii) The equilibrium E_1 is locally asymptotically stable if $d > d_{\max} \triangleq \frac{vb_{\max}K^2}{(K+g)(1+hK)}$.

(iii) The equilibrium E_2 achieves local asymptotic stability when the following two conditions hold simultaneously:

$$\frac{\beta v r \tilde{X}}{d} \ln\left(\frac{K}{\tilde{X}}\right) < \frac{1}{\tau^\alpha} d^{1-\alpha} + d, \quad \frac{b_{\max} \tilde{X} \tilde{S} (h \tilde{X}^2 - g)}{[(\tilde{X} + g)(1 + h \tilde{X})]^2} < \frac{r \tilde{X}}{\tilde{X} + c}.$$

(iv) The equilibrium E_3 is locally asymptotically stable provided that

$$r \ln\left(\frac{K}{X^* + c}\right) - \frac{rX^*}{X^* + c} - \frac{b_{\max} X^* S^* (X^* + hgX^* + 2g)}{[(X^* + g)(1 + hX^*)]^2} < 0.$$

Proof. Consider the Laplace transform of the fractional-order term $e^{-dt} {}_0D_t^{1-\alpha}(e^{dt}I(t))$ as follows:

$$\mathcal{L}\left\{e^{-dt} {}_0D_t^{1-\alpha}(e^{dt}I(t))\right\} = \hat{I}(s)(s+d)^{1-\alpha} = \hat{I}(s)\left(d^{1-\alpha} + (1-\alpha)d^{-\alpha}s + O(s^2)\right),$$

where $\hat{I}(s) = \mathcal{L}\{I(t)\}$. Since the Laplace transform is a linear operator, the inverse can be taken term by term [17], yielding

$$\begin{aligned} e^{-dt} {}_0D_t^{1-\alpha}(e^{dt}I(t)) &= \mathcal{L}^{-1}\left\{\hat{I}(s)\left(d^{1-\alpha} + (1-\alpha)d^{-\alpha}s + O(s^2)\right)\right\} \\ &= d^{1-\alpha}I(t) + (1-\alpha)d^{-\alpha}\frac{dI(t)}{dt} + \mathcal{L}^{-1}\{O(s^2)\}. \end{aligned}$$

By the final value theorem of the Laplace transform, if $\hat{f}(t)$ has a limit as $t \rightarrow \infty$, and $\hat{f}(s)$ is analytic in $\text{Re}(s) \geq 0$ except possibly at $s = 0$, then $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0^+} s\hat{f}(s)$. Denote $\mathcal{L}^{-1}\{\hat{I}(s) \cdot O(s^2)\}(t) \triangleq r(t)$. Then its Laplace transform is $\hat{I}(s) \cdot O(s^2) = \hat{r}(s)$. Here $\mathcal{L}^{-1}\{\hat{I}(s) \cdot O(s^2)\}$ is precisely the aforementioned $\mathcal{L}^{-1}\{O(s^2)\}$, and in the text, the preceding factor $\hat{I}(s)$ is omitted for brevity, but strictly speaking, the notation $O(s^2)$ itself should be understood as a compound symbol that already includes the factor $\hat{I}(s)$. By Theorem 1, $I(t)$ is uniformly bounded, hence $\lim_{s \rightarrow 0^+} s\hat{I}(s) = \lim_{t \rightarrow \infty} I(t) = I^*$. Therefore, $\lim_{t \rightarrow \infty} r(t) = \lim_{s \rightarrow 0^+} s\hat{r}(s) = \lim_{s \rightarrow 0^+} s\hat{I}(s) \cdot O(s^2) = I^* \cdot 0 = 0$. Thus we obtain $\mathcal{L}^{-1}\{O(s^2)\} = 0$. Then we obtain

$$\lim_{t \rightarrow \infty} \left\{e^{-dt} {}_0D_t^{1-\alpha}(e^{dt}I(t))\right\} = I^*d^{1-\alpha}.$$

Consequently,

$$\lim_{t \rightarrow \infty} \frac{\partial}{\partial I} \left\{e^{-dt} {}_0D_t^{1-\alpha}(e^{dt}I(t))\right\} = d^{1-\alpha}.$$

Therefore, the Jacobian matrix of model (2.2) with respect to (X, S, I) is computed as follows:

$$J = \begin{pmatrix} r(\ln K - \ln(X+c)) - \frac{rX}{X+c} - \frac{b_{\max}XS(X+gX+2g)}{[(X+g)(1+hX)]^2} & -\frac{b_{\max}X^2}{(X+g)(1+hX)} & 0 \\ \frac{vb_{\max}XS(X+gX+2g)}{[(X+g)(1+hX)]^2} & \frac{vb_{\max}X^2}{(X+g)(1+hX)} - d - \beta I & -\beta S + \frac{1}{\tau^\alpha}d^{1-\alpha} \\ 0 & \beta I & \beta S - \frac{1}{\tau^\alpha}d^{1-\alpha} - d \end{pmatrix}.$$

(i) Evaluated at $E_0(0, 0, 0)$, we have

$$J|_{E_0} = \begin{pmatrix} r \ln \frac{K}{c} & 0 & 0 \\ 0 & -d & \frac{1}{\tau^\alpha}d^{1-\alpha} \\ 0 & 0 & -\frac{1}{\tau^\alpha}d^{1-\alpha} - d \end{pmatrix}.$$

The resulting eigenvalues are $\lambda_1 = r \ln \frac{K}{c} > 0$, $\lambda_2 = -d < 0$, and $\lambda_3 = -\frac{1}{\tau^\alpha}d^{1-\alpha} - d < 0$. Consequently, E_0 is an unstable saddle point.

(ii) Evaluated at $E_1(K - c, 0, 0)$, we have

$$J|_{E_1} = \begin{pmatrix} -\frac{r(K-c)}{K} & -\frac{b_{\max}K^2}{(K+g)(1+hK)} & 0 \\ 0 & \frac{vb_{\max}K^2}{(K+g)(1+hK)} - d & \frac{1}{\tau^\alpha}d^{1-\alpha} \\ 0 & 0 & -\frac{1}{\tau^\alpha}d^{1-\alpha} - d \end{pmatrix}.$$

The corresponding characteristic equation reads

$$\left(\lambda + \frac{r(K-c)}{K}\right)\left(\lambda - \frac{vb_{\max}K^2}{(K+g)(1+hK)} + d\right)\left(\lambda + \frac{1}{\tau^\alpha}d^{1-\alpha} + d\right) = 0.$$

According to Floquet theory [29], all eigenvalues are negative provided that $d > d_{\max} \triangleq \frac{vb_{\max}K^2}{(K+g)(1+hK)}$, which guarantees the local asymptotic stability of E_1 . By contrast, E_1 becomes unstable when $d < d_{\max}$.

(iii) Evaluated at $E_2(\tilde{X}, \tilde{S}, 0)$, we have

$$J|_{E_2} = \begin{pmatrix} r \ln\left(\frac{K}{\tilde{X}+c}\right) - r\frac{\tilde{X}}{\tilde{X}+c} - \frac{b_{\max}\tilde{X}\tilde{S}(\tilde{X}+hg\tilde{X}+2g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2} & -\frac{b_{\max}\tilde{X}^2}{(\tilde{X}+g)(1+h\tilde{X})} & 0 \\ \frac{vb_{\max}\tilde{X}\tilde{S}(\tilde{X}+hg\tilde{X}+2g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2} & \frac{vb_{\max}\tilde{X}^2}{(\tilde{X}+g)(1+h\tilde{X})} - d & -\beta\tilde{S} + \frac{1}{\tau^\alpha}d^{1-\alpha} \\ 0 & 0 & \beta\tilde{S} - \frac{1}{\tau^\alpha}d^{1-\alpha} - d \end{pmatrix}.$$

One eigenvalue is explicitly derived as $\lambda_4 = \beta\tilde{S} - \frac{1}{\tau^\alpha}d^{1-\alpha} - d$. Accordingly, the local stability of E_2 necessitates the condition

$$\frac{\beta vr\tilde{X}}{d} \ln\left(\frac{K}{\tilde{X}+c}\right) < \frac{1}{\tau^\alpha}d^{1-\alpha} + d.$$

Consider the 2×2 leading principal submatrix

$$J|_{E_{22}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = r \ln\left(\frac{K}{\tilde{X}+c}\right) - r\frac{\tilde{X}}{\tilde{X}+c} - \frac{b_{\max}\tilde{X}\tilde{S}(\tilde{X}+hg\tilde{X}+2g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2}, \quad a_{12} = -\frac{b_{\max}(\tilde{X})^2}{(\tilde{X}+g)(1+h\tilde{X})},$$

$$a_{21} = \frac{vb_{\max}\tilde{X}\tilde{S}(\tilde{X}+hg\tilde{X}+2g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2}, \quad a_{22} = \frac{vb_{\max}(\tilde{X})^2}{(\tilde{X}+g)(1+h\tilde{X})} - d.$$

From Eq (2.2), $a_{22} = 0$ under the condition $I = 0$. In addition, it can be verified that $a_{12} < 0$ and $a_{21} > 0$. The local stability of E_2 requires the trace condition $\text{tr}(J|_{E_{22}}) = a_{11} + a_{22} < 0$ and the determinant condition $\det(J|_{E_{22}}) = a_{11}a_{22} - a_{12}a_{21} > 0$. Substituting $a_{22} = 0$ simplifies the stability criteria to $a_{11} < 0$, that is, $r \ln\left(\frac{K}{\tilde{X}+c}\right) - \frac{r\tilde{X}}{\tilde{X}+c} - \frac{b_{\max}\tilde{X}\tilde{S}(\tilde{X}+hg\tilde{X}+2g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2} < 0$. From the first equation of model (2.2),

we have $r \ln\left(\frac{K}{\tilde{X}+c}\right) = \frac{b_{\max}\tilde{S}\tilde{X}}{(\tilde{X}+g)(1+h\tilde{X})}$. Substituting this into the above inequality yields $\frac{b_{\max}\tilde{S}\tilde{X}}{(\tilde{X}+g)(1+h\tilde{X})} - \frac{r\tilde{X}}{\tilde{X}+c} - \frac{b_{\max}\tilde{S}\tilde{X}(\tilde{X}+hg\tilde{X}+2g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2} < 0$. After simplification, we obtain

$$\frac{b_{\max}\tilde{S}\tilde{X}(h\tilde{X}^2 - g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2} < \frac{r\tilde{X}}{\tilde{X}+c}.$$

Overall, E_2 is locally asymptotically stable if

$$\frac{\beta v r \tilde{X}}{d} \ln\left(\frac{K}{\tilde{X}+c}\right) < \frac{1}{\tau^\alpha} d^{1-\alpha} + d, \quad \frac{b_{\max}\tilde{S}\tilde{X}(h\tilde{X}^2 - g)}{[(\tilde{X}+g)(1+h\tilde{X})]^2} < \frac{r\tilde{X}}{\tilde{X}+c}$$

are satisfied.

(iv) Evaluated at $E_3 = (X^*, S^*, I^*)$, we have

$$J|_{E_3} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

where

$$\begin{aligned} b_{11} &= r \ln\left(\frac{K}{X^*+c}\right) - \frac{rX^*}{X^*+c} - \frac{b_{\max}X^*S^*(X^*+hgX^*+2g)}{[(X^*+g)(1+hX^*)]^2}, & b_{12} &= -\frac{b_{\max}(X^*)^2}{(X^*+g)(1+hX^*)}, & b_{13} &= 0, \\ b_{21} &= \frac{vb_{\max}X^*S^*(X^*+hgX^*+2g)}{[(X^*+g)(1+hX^*)]^2}, & b_{22} &= \frac{vb_{\max}(X^*)^2}{(X^*+g)(1+hX^*)} - \beta I^* - d, & b_{23} &= -\beta S^* + \frac{1}{\tau^\alpha} d^{1-\alpha}, \\ b_{31} &= 0, & b_{32} &= \beta I^*, & b_{33} &= \beta S^* - \frac{1}{\tau^\alpha} d^{1-\alpha} - d. \end{aligned}$$

From Eq (3.1), we obtain $b_{33} = 0$, $b_{23} = -d$, and $b_{22} = -\frac{I^* d^{1-\alpha}}{S^* \tau^\alpha} < 0$. The corresponding characteristic equation is formulated as $\lambda^3 - (b_{22} + b_{11})\lambda^2 + (b_{11}b_{22} - b_{23}b_{32} - b_{12}b_{21})\lambda + b_{11}b_{23}b_{32} = 0$, where the coefficients are defined as $A = -b_{22} - b_{11}$, $B = b_{11}b_{22} - b_{23}b_{32} - b_{12}b_{21}$, and $C = b_{11}b_{23}b_{32}$. Simplification yields $AB - C = -b_{11}^2 b_{22} + b_{11}b_{12}b_{21} - b_{11}b_{22}^2 + b_{22}b_{23}b_{32} + b_{12}b_{21}b_{22}$. By virtue of the Hurwitz stability criterion, the inequalities $A > 0$, $C > 0$, and $AB - C > 0$ are satisfied whenever $b_{11} < 0$, thereby ensuring local asymptotic stability of the equilibrium point. $\square$

Remark 2. The coefficient b_{11} is given by

$$b_{11} = r \ln\left(\frac{K}{X^*+c}\right) - \frac{rX^*}{X^*+c} - \frac{b_{\max}X^*S^*(X^*+hgX^*+2g)}{[(X^*+g)(1+hX^*)]^2}.$$

Define $G(X) = \ln\left(\frac{K}{X+c}\right) - \frac{X}{X+c}$. If $X^* > \bar{X} \triangleq G^{-1}(0)$, then $b_{11} < 0$. Thus, $X^* > \bar{X}$ provides a sufficient condition for E_3 to be locally asymptotically stable.

3.1.3. Transcritical bifurcation

When $\beta = \beta_T \triangleq (d + \frac{1}{\tau^\alpha} d^{1-\alpha}) \tilde{S}^{-1}$, the equilibria E_2 and E_3 coincide. To analyze the transcritical bifurcation occurring at E_2 , we choose the transmission rate β as the bifurcation parameter.

Theorem 4. When $\beta = \beta_T$, a transcritical bifurcation occurs in model (2.2).

Proof. It follows that $\det J(E_2) = 0$. Let V' and W' denote the eigenvectors corresponding to the zero eigenvalue of $J(E_2)$ and its transpose $J^T(E_2)$, respectively. Straightforward calculation yields

$$V' = \begin{pmatrix} V'_1 \\ V'_2 \\ V'_3 \end{pmatrix} = \begin{pmatrix} -\frac{c_{12}c_{23}}{c_{12}c_{21} - c_{11}c_{22}} \\ \frac{c_{12}c_{21} - c_{11}c_{22}}{c_{12}c_{21} - c_{11}c_{22}} \\ 1 \end{pmatrix}, \quad W' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= r \ln \left(\frac{K}{X^* + c} \right) - \frac{rX^*}{X^* + c} - \frac{b_{\max} X^* S^* (X^* + hgX^* + 2g)}{[(X^* + g)(1 + hX^*)]^2}, \quad c_{12} = -\frac{b_{\max} (X^*)^2}{(X^* + g)(1 + hX^*)}, \\ c_{21} &= \frac{\nu b_{\max} X^* S^* (X^* + hgX^* + 2g)}{[(X^* + g)(1 + hX^*)]^2}, \quad c_{22} = \frac{\nu b_{\max} (X^*)^2}{(X^* + g)(1 + hX^*)} - d, \\ c_{23} &= -\beta S^* + \frac{1}{\tau^\alpha} d^{1-\alpha} = -d. \end{aligned}$$

In addition,

$$F_\beta(E_2; \beta) = \begin{pmatrix} 0 \\ -SI \\ SI \end{pmatrix}_{(E_2; \beta)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$DF_\beta(E_2; \beta)V' = \begin{pmatrix} 0 \\ -S^* \\ S^* \end{pmatrix}_{(E_2; \beta)} = S^* \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{d\tau^\alpha + d^{1-\alpha}}{\beta\tau^\alpha} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Regarding the transversality condition, we have

$$W'^T F_\beta(E_2; \beta) = 0, \quad W'^T [DF_\beta(E_2; \beta)V'] = \frac{d\tau^\alpha + d^{1-\alpha}}{\beta\tau^\alpha} \neq 0,$$

$$W'^T [D^2 F_\beta(E_2; \beta)(V', V')] \neq 0.$$

Obviously,

$$\begin{aligned}\frac{\partial^2 F_1}{\partial X^2} &= -r \cdot \frac{X+2c}{(X+c)^2} + \frac{-2b_{\max}S(X+hgX+g)}{[(X+g)(1+hX)]^2} \\ &\quad + \frac{b_{\max}XS(X+hgX+2g)[2X(1+h^2g^2+4hg)+4h^2X^3+(6hX^2+2g)(gh+1)]}{[(X+g)(1+hX)]^4}, \\ \frac{\partial^2 F_1}{\partial S^2} &= \frac{\partial^2 F_1}{\partial I^2} = \frac{\partial^2 F_1}{\partial S \partial I} = \frac{\partial^2 F_1}{\partial X \partial I} = 0, \quad \frac{\partial^2 F_1}{\partial X \partial S} = -\frac{b_{\max}X[2g+(1+gh)X]}{[(X+g)(1+hX)]^2}, \\ \frac{\partial^2 F_2}{\partial X^2} &= \nu b_{\max}S \cdot \frac{-2h(1+gh)X^3-6ghX^2+2g^2}{[(X+g)(1+hX)]^3}, \\ \frac{\partial^2 F_2}{\partial S^2} &= 0, \quad \frac{\partial^2 F_2}{\partial I^2} = 0, \quad \frac{\partial^2 F_2}{\partial S \partial I} = -\beta, \quad \frac{\partial^2 F_2}{\partial X \partial I} = 0, \quad \frac{\partial^2 F_2}{\partial X \partial S} = \frac{\nu b_{\max}X[2g+(1+gh)X]}{[(X+g)(1+hX)]^2}, \\ \frac{\partial^2 F_3}{\partial X^2} &= \frac{\partial^2 F_3}{\partial S^2} = \frac{\partial^2 F_3}{\partial I^2} = \frac{\partial^2 F_2}{\partial X \partial I} = \frac{\partial^2 F_2}{\partial X \partial S} = 0, \quad \frac{\partial^2 F_3}{\partial I \partial S} = \beta.\end{aligned}$$

Thus,

$$\begin{aligned}D^2F_{\beta}(E_2; \beta)(V', V') &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial X^2} V_1'^2 + \frac{\partial^2 F_1}{\partial S^2} V_2'^2 + \frac{\partial^2 F_1}{\partial I^2} V_3'^2 + 2 \frac{\partial^2 F_1}{\partial X \partial S} V_1' V_2' + 2 \frac{\partial^2 F_1}{\partial X \partial I} V_1' V_3' + 2 \frac{\partial^2 F_1}{\partial S \partial I} V_2' V_3' \\ \frac{\partial^2 F_2}{\partial X^2} V_1'^2 + \frac{\partial^2 F_2}{\partial S^2} V_2'^2 + \frac{\partial^2 F_2}{\partial I^2} V_3'^2 + 2 \frac{\partial^2 F_2}{\partial X \partial S} V_1' V_2' + 2 \frac{\partial^2 F_2}{\partial X \partial I} V_1' V_3' + 2 \frac{\partial^2 F_2}{\partial S \partial I} V_2' V_3' \\ \frac{\partial^2 F_3}{\partial X^2} V_1'^2 + \frac{\partial^2 F_3}{\partial S^2} V_2'^2 + \frac{\partial^2 F_3}{\partial I^2} V_3'^2 + 2 \frac{\partial^2 F_3}{\partial X \partial S} V_1' V_2' + 2 \frac{\partial^2 F_3}{\partial X \partial I} V_1' V_3' + 2 \frac{\partial^2 F_3}{\partial S \partial I} V_2' V_3' \end{pmatrix} \\ &\neq 0.\end{aligned}$$

Therefore, according to Sotomayor's theorem [30], a transcritical bifurcation occurs when $\beta = \beta_T$. $\square$

3.2. Dynamic analysis of a periodically pulsed control model

3.2.1. Pest-eradication periodic solution

When considering only the absence of pests and disregarding the presence of disease-carrying predators, the following simplified model was obtained:

$$\left\{ \begin{array}{l} \frac{dX(t)}{dt} = 0 \\ \frac{dS(t)}{dt} = -dS(t) \\ \frac{dI(t)}{dt} = 0 \end{array} \right\} \quad t \neq jT_p, \quad (3.2)$$

$$\left\{ \begin{array}{l} \Delta X = 0 \\ \Delta S = -q_2 S(t) + s_T \\ \Delta I = 0 \end{array} \right\} \quad t = jT_p.$$

Lemma 1. *Model (3.2) possesses a positive periodic solution $\bar{S}(t)$, and all solutions $S(t)$ initiated at $S(0^+) = S_0 \geq 0$ approach $\bar{S}(t)$ as $t \rightarrow \infty$. Furthermore, if $S_0 \geq \frac{s_T}{1-(1-q_2)e^{-dT_p}}$, then $S(t) \geq \bar{S}(t)$; if $S_0 < \frac{s_T}{1-(1-q_2)e^{-dT_p}}$, then $S(t) < \bar{S}(t)$.*

Here, $\bar{S}(t) = \frac{s_T e^{-d(t-jT_p)}}{1-(1-q_2)e^{-dT_p}}$, $t \in (jT_p, (j+1)T_p]$, $j \in \mathbb{Z}_+$, $\bar{S}(0^+) = \frac{s_T}{1-(1-q_2)e^{-dT_p}}$. Therefore, model (3.2) has a pest-extinction periodic solution $\bar{z}(t) = (0, \bar{S}(t), 0) = \left(0, \frac{s_T e^{-d(t-jT_p)}}{1-(1-q_2)e^{-dT_p}}, 0\right)$ without considering infected natural enemies.

Theorem 5. Model (3.2) exhibits a disease-free pest-eradication periodic solution $\bar{z}(t) = (0, \bar{S}(t), 0)$ for $(j-1)T_p \leq t < jT_p$. Additionally, for any trajectory $S(t)$ governed by (3.2), $S(t)$ approaches $\bar{S}(t)$ asymptotically.

Proof. Obviously,

$$\bar{z}(t) = (0, \bar{S}(t), 0) = \left(0, \frac{s_T e^{-d(t-jT_p)}}{1-(1-q_2)e^{-dT_p}}, 0\right),$$

and $\bar{z}(0^+) = (0, \bar{S}(0^+), 0)$ is the disease-free pest-eradication periodic solution of model (3.2).

For $0 < t \leq T_p$, we have $S(t) = S(0^+)e^{-dt}$ and $S(T_p^+) = (1-q_2)S(T_p) + s_T$.

For $T_p < t \leq 2T_p$, we have $S(t) = S(T_p^+)e^{-d(t-T_p)}$ and $S(2T_p^+) = (1-q_2)S(2T_p) + s_T = (1-q_2)S(T_p^+)e^{-dT_p} + s_T$.

Similarly, for $(j-1)T_p < t \leq jT_p$, we have $S(t) = S((j-1)T_p^+)e^{-d(t-(j-1)T_p)}$ and $S((j-1)T_p^+) = (1-q_2)S(jT_p) + s_T$.

Then, by induction, we obtain $S(jT_p^+) = \left[(1-q_2)e^{-dT_p}\right]^j S(0^+) + \frac{1-[1-(1-q_2)e^{-dT_p}]^j}{1-(1-q_2)e^{-dT_p}} s_T$. Therefore, for $jT_p < t \leq (j+1)T_p$,

$$S(t) = S(jT_p^+)e^{-d(t-jT_p)} = (1-q_2)^j \left[S(0^+) - \frac{s_T}{1-(1-q_2)e^{-dT_p}} \right] e^{-dt} + \frac{s_T e^{-d(t-jT_p)}}{1-(1-q_2)e^{-dT_p}}.$$

As $t \rightarrow \infty$, since $d > 0$, we get $S(t) \rightarrow \bar{S}(t)$. □

Theorem 6. If $T_p < -\frac{\ln(1-p_1)}{r \ln \frac{K}{c}}$ and $\frac{\beta s_T}{d[1-(1-q_2)e^{-dT_p}]}(1-e^{-dT_p}) < \ln \frac{1}{1-q_2} + \left(\frac{1}{\tau} + d\right)T_p$, then $\bar{z}(t) = (0, \bar{S}(t), 0)$ possesses local asymptotic stability and global attractiveness.

Proof. First, the local asymptotic stability of $(0, \bar{S}(t), 0)$ is proved using Floquet theory [29].

Let $\sigma_1(t) \triangleq X(t)$, $\sigma_2(t) \triangleq S(t) - \bar{S}(t)$, $\sigma_3(t) \triangleq I(t)$. Then $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$ satisfy

$$\begin{pmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} \sigma_1(0) \\ \sigma_2(0) \\ \sigma_3(0) \end{pmatrix}, \quad 0 \leq t < T_p,$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} r \ln \frac{K}{c} & 0 & 0 \\ 0 & -d & -\beta \bar{S}(t) + \frac{1}{\tau} \\ 0 & 0 & \beta \bar{S}(t) - \frac{1}{\tau} - d \end{pmatrix} \Phi(t),$$

and $\Phi(0) = I$.

Since,

$$\begin{pmatrix} \sigma_1(jT_p^+) \\ \sigma_2(jT_p^+) \\ \sigma_3(jT_p^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-q_2 & 0 \\ 0 & 0 & 1-q_2 \end{pmatrix} \begin{pmatrix} \sigma_1(jT_p) \\ \sigma_2(jT_p) \\ \sigma_3(jT_p) \end{pmatrix},$$

then

$$\begin{aligned}
 M = \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-q_2 & 0 \\ 0 & 0 & 1-q_2 \end{pmatrix} \Phi(T_p) &= \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-q_2 & 0 \\ 0 & 0 & 1-q_2 \end{pmatrix} \begin{pmatrix} e^{r \ln \frac{K}{c} T_p} & 0 & 0 \\ 0 & e^{-dT_p} & e^{\int_0^{T_p} [-\beta \bar{S}(s) + \frac{1}{\tau}] ds} \\ 0 & 0 & e^{\int_0^{T_p} [\beta \bar{S}(t) - \frac{1}{\tau} - d] ds} \end{pmatrix} \\
 &= \begin{pmatrix} (1-p_1)e^{r \ln \frac{K}{c} T_p} & 0 & 0 \\ 0 & (1-q_2)e^{-dT_p} & (1-q_2)e^{\int_0^{T_p} [-\beta \bar{S}(s) + \frac{1}{\tau}] ds} \\ 0 & 0 & (1-q_2)e^{\int_0^{T_p} [\beta \bar{S}(t) - \frac{1}{\tau} - d] ds} \end{pmatrix}.
 \end{aligned}$$

The eigenvalues can be obtained as follows:

$$\mu_1 = (1-p_1)e^{r \ln \frac{K}{c} T_p}, \mu_2 = (1-q_2)e^{-dT_p}, \mu_3 = (1-q_2) \exp\left[\frac{\beta s_T}{d[1-(1-q_2)e^{-dT_p}]} (1 - e^{-dT_p}) - \left(\frac{1}{\tau} + d\right) T_p\right].$$

According to Floquet theory [29], if $T_p < -\frac{\ln(1-p_1)}{r \ln \frac{K}{c}}$ and $\frac{\beta s_T}{d[1-(1-q_2)e^{-dT_p}]} (1 - e^{-dT_p}) < \ln \frac{1}{1-q_2} + \left(\frac{1}{\tau} + d\right) T_p$, then $\bar{z}(t) = (0, \bar{S}(t), 0)$ is locally asymptotically stable.

Let $\bar{\sigma} > 0$ be a sufficiently small constant satisfying

$$(1-p_1) \exp\left\{\int_0^{T_p} \left(r(\ln K - \ln(X+c)) - \frac{b_{\max} X(\bar{S}(t) - \bar{\sigma})}{(X+g)(1+hX)}\right) dt\right\} < (1-p_1)e^{r \ln \frac{K}{c} T_p} < 1.$$

Define

$$\tilde{\rho} \triangleq (1-p_1) \exp\left\{\int_0^{T_p} \left(r(\ln K - \ln(X+c)) - \frac{b_{\max} X(\bar{S}(t) - \bar{\sigma})}{(X+g)(1+hX)}\right) dt\right\}.$$

From model (2.3), we obtain

$$\begin{cases} \left\{ \begin{array}{l} \frac{dS}{dt} \geq -dS(t) - \beta S(t)I(t) + \frac{I(t)}{\tau} \\ \frac{dI}{dt} = \beta S(t)I(t) - \frac{I(t)}{\tau} - dI(t) \end{array} \right\} & t \neq jT_p, \\ \left\{ \begin{array}{l} S(jT_p^+) = (1-q_2)S(jT_p) + s_T \\ I(jT_p^+) = (1-q_2)I(jT_p) \end{array} \right\} & t = jT_p. \end{cases} \quad (3.3)$$

For the model

$$\begin{cases} \left\{ \begin{array}{l} \frac{dU}{dt} = -dU(t) - \beta U(t)I(t) + \frac{I(t)}{\tau} \\ \frac{dI}{dt} = \beta U(t)I(t) - \frac{I(t)}{\tau} - dI(t) \end{array} \right\} & t \neq jT_p, \\ \left\{ \begin{array}{l} U(jT_p^+) = (1-q_2)U(jT_p) + s_T \\ I(jT_p^+) = (1-q_2)I(jT_p) \end{array} \right\} & t = jT_p, \end{cases} \quad (3.4)$$

with $U(0) = S(0)$, by the comparison principle for impulsive differential systems, $S(t) \geq U(t) \rightarrow \bar{S}(t)$. Therefore, we have

$$S(t) \geq U(t) > \bar{S}(t) - \bar{\sigma}$$

for sufficiently large t . Without loss of generality, it is supposed that the above relation holds for all time t . Since

$$\begin{cases} \frac{dX}{dt} \leq rX(\ln K - \ln(X + c)) - \frac{b_{\max}X^2(\bar{S} - \bar{\sigma})}{(X + g)(1 + hX)}, & t \neq jT_p, \\ X(jT_p^+) = (1 - p_1)X(jT_p), & t = jT_p, \end{cases} \quad (3.5)$$

then $X((j + 1)T_p) \leq \tilde{\rho}X(jT_p)$, so $X(jT_p) \leq X(0^+)\tilde{\rho}^j$ and as $j \rightarrow \infty$, since $\tilde{\rho} < 1$, we have $X(jT_p) \rightarrow 0$. Also, since for $jT_p < t < (j + 1)T_p$, $0 < X(t) \leq X(jT_p)(1 - p_1)e^{r \ln \frac{K}{c} T_p}$, we have $X(t) \rightarrow 0$ as $j \rightarrow \infty$. Similarly, it can be rigorously demonstrated that $I(t) \rightarrow 0$ as $j \rightarrow \infty$.

For $0 < \chi < \tilde{\chi}$, $\tilde{\chi} = \frac{(d+r \ln K)\tilde{\chi}}{d} - \frac{r\tilde{\chi} \ln(\tilde{\chi}+c)}{d} - \frac{b_{\max}S^*(\tilde{\chi})^2}{d(\tilde{\chi}+g)(1+h\tilde{\chi})}$, then $\exists T_0 > 0$ ensures that $0 < X(t) < \chi$ ($t \geq T_0$). Assume $0 < X(t) < \chi$ ($t \geq 0$). Since

$$-dS(t) \leq \frac{dS(t)}{dt} \leq \left(\frac{vb_{\max}\chi^2}{(\chi + g)(1 + h\chi)} - d \right) S(t),$$

then we have $\bar{S}(t) \leq S(t) \leq v_1(t) \rightarrow \bar{v}_1(t)$ ($t \rightarrow \infty$), where $v_1(t)$ is the solution of model (3.6):

$$\begin{cases} \frac{dv_1(t)}{dt} = \left(\frac{vb_{\max}\chi^2}{(\chi + g)(1 + h\chi)} - d \right) v_1(t), & t \neq jT_p, \\ \Delta v_1(t) = -q_2 v_1(t) + s_T, & t = jT_p, \\ v_1(0^+) = S(0^+), \end{cases} \quad (3.6)$$

and then

$$\bar{v}_1(t) = \frac{s_T \exp \left\{ \left(\frac{vb_{\max}\chi^2}{(\chi + g)(1 + h\chi)} - d \right) (t - jT_p) \right\}}{1 - (1 - q_2) \exp \left\{ \left(\frac{vb_{\max}\chi^2}{(\chi + g)(1 + h\chi)} - d \right) T_p \right\}}, \quad jT_p < t \leq (j + 1)T_p.$$

Therefore, $\forall \epsilon_1 > 0, \exists T_1 > 0$, such that $\bar{S}(t) - \epsilon_1 < S(t) < \bar{v}_1(t) + \epsilon_1$ ($t > T_1$). Consequently, letting $\chi \rightarrow 0$ yields the bounds $\bar{S}(t) - \epsilon_1 < S(t) < \bar{S}(t) + \epsilon_1$ for $t > T_1$. This implies the convergence $S(t) \rightarrow \bar{S}(t)$ as $t \rightarrow \infty$. Clearly, $I(t) \rightarrow \bar{I}(t)$ as $t \rightarrow \infty$. Hence the conclusion is proved. $\square$

3.2.2. Persistence

Lemma 2. *The solution of model (2.3) is bounded, that is, when t is sufficiently large, there must exist a constant $M > 0$ such that $\max\{X(t), S(t), I(t)\} \leq M$.*

Proof. Let $L(t) = S(t) + I(t) + vX(t)$. Then, for $t \neq jT_p$, we have

$$D^+L(t) = v \frac{dX}{dt} + \frac{dS}{dt} + \frac{dI}{dt} = vrX(\ln K - \ln(X + c)) - d(S + I).$$

Therefore,

$$D^+L(t) + dL(t) = vrX(\ln K - \ln(X + c)) + vdX.$$

Let $G(X) = vrX(\ln K - \ln(X + c)) + vdX$, and $G''(X) < 0$ for all $X \geq 0$. From $G'(X) = 0$, we obtain $\ln K + \frac{d}{r} - \ln(\check{X} + c) - \frac{\check{X}}{\check{X} + c} = 0$, and also $G(\check{X}) = vr\left[\left(\ln K + \frac{d}{r}\right)\check{X} - \check{X}\ln(\check{X} + c)\right]$. From the extremum condition, we get $G(\check{X}) = vr\frac{(\check{X})^2}{\check{X} + c} = M_0$. Then,

$$D^+L(t) + dL(t) \leq M_0.$$

For the model

$$\begin{cases} D^+L(t) \leq -dL(t) + M_0, \\ L(jT_p^+) \leq L(jT_p) + s_T, \end{cases} \quad (3.7)$$

we have

$$\begin{aligned} L(t) &\leq L(0)e^{-dt} + \int_0^t M_0 e^{-d(t-\bar{s})} d\bar{s} + \sum_{t > jT_p} s_T e^{-d(t-jT_p)} \\ &< L(0)e^{-dt} + \frac{M_0}{d}(1 - e^{-dt}) + s_T \frac{e^{dT_p}}{e^{dT_p} - 1} \\ &\rightarrow \frac{M_0}{d} + s_T \frac{e^{dT_p}}{e^{dT_p} - 1} \quad (t \rightarrow \infty). \end{aligned}$$

Therefore, $L(t)$ is bounded, i.e., $\exists M = \frac{M_0}{d} + s_T \frac{e^{dT_p}}{e^{dT_p} - 1} > 0$, such that $\max\{X(t), S(t), I(t)\} < L(t) \leq M$. The proof is complete.

For model (2.3), if $\exists M_1, M_2, M_3 > 0$ and $\bar{T} > 0$, ensuring that for every $t \geq \bar{T}$, we have $M_1 \leq X(t) \leq M_2, M_1 \leq S(t) \leq M_2, M_1 \leq I(t) \leq M_2$, then the model is said to be persistent. $\square$

Choose $X_M > 0, \varepsilon > 0$,

$$r \ln \frac{K}{X_M + c} - \frac{b_{\max} X_M}{(X_M + g)(1 + hX_M)} > 0,$$

and

$$\rho_0 \triangleq (1 - p_1) \exp \left[r \ln \left(\frac{K}{X_M + c} \right) T_p - \frac{b_{\max} X_M \left(\int_{j_1 T_p}^{(\bar{j}_1 + 1) T_p} \bar{u}(t) dt + \varepsilon T_p \right)}{(X_M + g)(1 + hX_M)} \right] > 1.$$

Define $A \triangleq \frac{vb_{\max} X_M^2}{(X_M + g)(1 + hX_M)} - d$.

Theorem 7. When $A > 0$, model (2.3) is persistent if

$$\max \left\{ \frac{-\ln(1 - p_1) + \frac{b_{\max} X_M \int_{j_1 T_p}^{(\bar{j}_1 + 1) T_p} \bar{u}(t) dt}{(X_M + g)(1 + hX_M)}}{r \ln \frac{K}{X_M + c} - \frac{b_{\max} \varepsilon X_M}{(X_M + g)(1 + hX_M)}}, \frac{-\ln(1 - q_2)}{\beta m_1 - \delta_2} \right\} < T_p < \frac{-\ln(1 - q_2)}{A}.$$

When $A \leq 0$, model (2.3) is persistent provided that

$$\max \left\{ \frac{-\ln(1 - p_1) + \frac{b_{\max} X_M \int_{j_1 T_p}^{(\bar{j}_1 + 1) T_p} \bar{u}(t) dt}{(X_M + g)(1 + hX_M)}}{r \ln \frac{K}{X_M + c} - \frac{b_{\max} \varepsilon X_M}{(X_M + g)(1 + hX_M)}}, \frac{-\ln(1 - q_2)}{\beta m_1 - \delta_2} \right\} < T_p.$$

Proof. According to Lemma 2, without loss of generality, assume that for $t \geq 0$, we already have $\max\{X(t), S(t), I(t)\} < M$.

(i) Since $\frac{dS}{dt} \geq -dS - \beta SI > -(d + \beta M)S$, let $\delta_1 = d + \beta M > 0$. Denote $\underline{S}(t)$ as the solution of the system

$$\begin{cases} \frac{d\underline{S}}{dt} = -\delta_1 \underline{S}, & t \neq jT_P, \\ \underline{S}(jT_P^+) = (1 - q_2)\underline{S}(jT_P) + s_T, \end{cases} \quad (3.8)$$

satisfying $\underline{S}(0) = S(0)$. We obtain

$$\underline{S}(t) = \frac{s_T e^{-\delta_1(t-jT_P)}}{1 - (1 - q_2)e^{-\delta_1 T_P}}, t \in (jT_P, (j+1)T_P].$$

Choose $\varepsilon_0 > 0$ and let $m_1 = \frac{s_T e^{-\delta_1 T_P}}{1 - (1 - q_2)e^{-\delta_1 T_P}} - \varepsilon_0$. By the comparison theorem, $\exists T_{P_0} > 0$ such that $S(t) > m_1$ as $t > T_{P_0}$.

(ii) Prove that there exists a positive constant m_2 such that $I(t) > m_2$ for sufficiently large t .

From (i), there exists $T_{P_0} > 0$ such that for $t > T_{P_0}$, we have $\frac{dI}{dt} \geq \beta m_1 I - \left(\frac{1}{\tau} + d\right)I$. Let $\delta_2 = \frac{1}{\tau} + d > 0$. Denote $\underline{I}(t)$ as the solution of the model:

$$\begin{cases} \frac{d\underline{I}}{dt} = \underline{I}(\beta m_1 - \delta_2), & t \neq jT_P, \\ \underline{I}(jT_P^+) = (1 - q_2)\underline{I}(jT_P). \end{cases}$$

From the impulsive condition, it is easy to obtain $\underline{I}((j+1)T_P^+) = (1 - q_2)e^{(\beta m_1 - \delta_2)T_P} \underline{I}(jT_P^+)$. Assume

$\beta m_1 > \delta_2$. When $T_P > \frac{\ln \frac{1}{1-q_2}}{\beta m_1 - \delta_2}$, the impulsive multiplier satisfies $(1 - q_2)e^{(\beta m_1 - \delta_2)T_P} > 1$, so the sequence $\{\underline{I}(jT_P^+)\}$ is increasing and the lower solution $\underline{I}(t)$ converges to a positive T_P -periodic solution $\underline{I}^*(t)$. Choose $\varepsilon_0 > 0$ and let $m_2 = \min_{t \in [0, T_P]} \underline{I}^*(t) - \varepsilon_0 > 0$. By the comparison theorem, there exists $T_{P_1} > 0$ such that $I(t) \geq \underline{I}(t) > m_2$ for $t > T_{P_1}$.

(iii) Finally, demonstrate the presence of a positive constant m_3 satisfying $X(t) > m_3$ for all large enough t .

Step 1: Next, we show that $X(t) < X_M$ cannot persist for all $t > 0$. Otherwise, since

$$\frac{dS}{dt} = \frac{vb_{\max} S X^2}{(X + g)(1 + hX)} - dS + \frac{I}{\tau} \leq \frac{vb_{\max} S X_M^2}{(X_M + g)(1 + hX_M)} - dS + \frac{I}{\tau},$$

we have

$$\begin{cases} \frac{dS}{dt} \leq \frac{vb_{\max} S X_M^2}{(X_M + g)(1 + hX_M)} - dS + \frac{1}{\tau} I \\ \frac{dI}{dt} = \beta S I - \frac{I}{\tau} - dI \\ S(jT_P^+) = (1 - q_2)S(jT_P) + s_T \\ I(jT_P^+) = (1 - q_2)I(jT_P) \end{cases} \quad \begin{matrix} t \neq jT_P, \\ t = jT_P. \end{matrix}$$

By virtue of the comparison theorem, as $t \rightarrow \infty$, it follows that $S(t) \leq u(t) \rightarrow \bar{u}(t)$, where $u(0^+) = S(0^+)$, and for $t \in [jT_p, (j+1)T_p)$,

$$\left\{ \begin{array}{l} \frac{du}{dt} = \frac{vb_{\max}uX_M^2}{(X_M + g)(1 + hX_M)} - du + \frac{1}{\tau}I \\ \frac{dI}{dt} = \beta uI - \frac{I}{\tau} - dI \\ u(jT_p^+) = (1 - q_2)u(jT_p) + s_T \\ I(jT_p^+) = (1 - q_2)I(jT_p) \end{array} \right\} \quad \begin{array}{l} t \neq jT_p, \\ t = jT_p. \end{array} \quad (3.9)$$

We now prove the existence of $\bar{u}(t)$.

Solving model (3.9) on the interval $(jT_p^+, (j+1)T_p)$, the general solution is given by

$$u = e^{\left(\frac{vb_{\max}X_M^2}{(X_M + g)(1 + hX_M)} - d\right)(t - jT_p)} \left[u(jT_p^+) + \int_{jT_p}^t e^{-\left(\frac{vb_{\max}X_M^2}{(X_M + g)(1 + hX_M)} - d\right)(\xi - jT_p)} \frac{1}{\tau} I(\xi) d\xi \right].$$

Let $U_j = u(jT_p^+)$, $A = \frac{vb_{\max}X_M^2}{(X_M + g)(1 + hX_M)} - d$, $B = \int_0^{T_p} e^{-A\xi} \frac{1}{\tau} I(\xi) d\xi$. From the impulsive condition, we obtain

$$U_{j+1} = u((j+1)T_p^+) = (1 - q_2)u((j+1)T_p^-) + s_T = (1 - q_2)e^{AT_p}(U_j + B) + s_T.$$

Let $K = (1 - q_2)e^{AT_p}$, $C = (1 - q_2)e^{AT_p}B + s_T$, which yields the difference equation $U_{j+1} = KU_j + C$. Construct the function $f(U) = KU + C$, so that the difference equation becomes $f(U_j) = U_{j+1}$. Solving this difference equation reveals that it has a unique positive fixed point $U^+ = \frac{C}{1-K}$, where $K < 1$. Hence, the trajectory of (3.9) starting from U^+ constitutes a positive periodic orbit, whose stability reduces to that of the fixed point U^+ for the associated map $U_{j+1} = f(U_j)$.

We now prove the global asymptotic stability of U^+ . Since $f'(U) = K > 0$, $f(U)$ is monotonically increasing with respect to U .

If the initial value $U(0) = U_0$ satisfies $0 < U_0 < U^+$, then we have $0 < f(U_0) < f(U^+) = U^+$, i.e., $U_1 < U^+$. By induction, for all $j \in \mathbb{N}_+$, we have $0 < U_j < U^+$. Moreover, since $U_{j+1} - U_j > 0$, it follows that when $0 < U_0 < U^+$, we have $0 < U_j < U^+$ (for $j \in \mathbb{N}^+$) and $\{U_j\}$ increases monotonically to U^+ .

Similarly, it can be proved that when $U_0 > U^+$, we have $U_j > U^+$ (for $j \in \mathbb{N}^+$), and $\{U_j\}$ decreases monotonically to U^+ . Therefore, U^+ is globally asymptotically stable. Consequently, the periodic solution $\bar{u}(t)$ of model (3.9) with U^+ is globally asymptotically stable. When $A > 0$, the existence of a unique positive periodic solution to model (3.9) requires that $T_p < -\frac{\ln(1-q_2)}{A}$. If $A \leq 0$, no such constraint on T_p is needed, and the unique positive periodic solution exists unconditionally.

Therefore, $\exists T_2 > 0$, $\exists n_1 \in \mathbb{N}_+$, ensuring that for every $t > n_1 T_p > T_2$, we have

$$S(t) \leq \bar{S}(t) + \varepsilon \leq \bar{u}(t) + \varepsilon. \quad (3.10)$$

This also implies

$$\begin{aligned} \frac{dX}{dt} &= rX(\ln K - \ln(X + c)) - \frac{b_{\max}SX^2}{(X + g)(1 + hX)} \\ &\geq X \left(r \ln K - r \ln(X_M + c) - \frac{b_{\max}(\bar{u}(t) + \varepsilon)X_M}{(X_M + g)(1 + hX_M)} \right) \quad (t > T_2, X > \sqrt{\frac{g}{h}}). \end{aligned}$$

Choose $\bar{j}_1 \in \mathbb{N}_+$ such that $\bar{j}_1 T_p \geq T_2$. Then integrating gives

$$X((\bar{j}_1 + 1)T_p) \geq X(\bar{j}_1 T_p^+) \exp \left\{ \int_{\bar{j}_1 T_p}^{(\bar{j}_1 + 1)T_p} \left[r \ln K - r \ln(X_M + c) - \frac{b_{\max}(\bar{u}(t) + \varepsilon)X_M}{(X_M + g)(1 + hX_M)} \right] dt \right\} = \rho_0 X(\bar{j}_1 T_p),$$

which leads to

$$X((\bar{j}_1 + k)T_p) \geq X(\bar{j}_1 T_p) \rho_0^k \rightarrow \infty \quad (k \rightarrow \infty), \quad (3.11)$$

which contradicts Lemma 2. Therefore, $\exists t_1 > 0$ satisfying $X(t_1) \geq X_M$.

Step 2: We discuss two cases.

Case 1: A time $T_3 > 0$ can be found beyond which $X(t) \geq X_M$ holds for all t . In this case, we can simply take $m_3 = X_M$.

Case 2: Suppose there is a sequence of intervals $\{[\alpha_k, \gamma_k]\}$ for which $X(t) < X_M$ holds on the union $\bigcup_{k=1}^{\infty} [\alpha_k, \gamma_k] := B$, while $X(t) \geq X_M$ elsewhere.

Let $[\alpha, \gamma]$ be any fixed interval $[\alpha_k, \gamma_k]$ from this sequence. It suffices to establish the existence of $m_3 > 0$ satisfying $X(t) > m_3$ for $t \in [\alpha, \gamma]$.

Let $\alpha \in [lT_p, (l+1)T_p)$, where l is some positive integer. Clearly, we have $X(\alpha^+) \geq (1 - p_1)X_M$. Denote $\bar{T} = (n_1 + l + 1)$, where n_1 is determined by Eq (3.10).

(a) First, assume $\bar{T} < \gamma$. From the proofs of (i) and (ii), without loss of generality, assume that for $t \geq 0$, we already have $\max\{X(t), S(t), I(t)\} < M$. Consider the model

$$\begin{cases} \frac{d\underline{X}_1(t)}{dt} = \underline{X}_1(t) \left[r(\ln K - \ln(M + c)) - \frac{b_{\max}M^2}{(M + g)(1 + hM)} \right], & t \neq jT_p, \\ \underline{X}_1(jT_p^+) = (1 - p_1)\underline{X}_1(jT_p). \end{cases} \quad (3.12)$$

Denote the solution of model (3.12) satisfying the initial condition $\underline{X}_1(lT_p^+) = (1 - p_1)X_M$ as $\underline{X}_1(t)$, and let $\underline{X}_1(\bar{T}) = \ddot{X}$. Applying the comparison principle of impulsive differential equations, for $t \geq \alpha$, we have $\underline{X}_1(t) \leq X(t)$.

Clearly, we can take M large enough that $r(\ln K - \ln(M + c)) - \frac{b_{\max}M^2}{(M+g)(1+hM)} < 0$, so $\underline{X}_1(t)$ is monotonically decreasing. Consider the model

$$\begin{cases} \frac{d\underline{X}_2(t)}{dt} = \underline{X}_2(t) \left[r \ln \frac{K}{X_M + c} - \frac{b_{\max}(\bar{u}(t) + \varepsilon)X_M}{(X_M + g)(1 + hX_M)} \right], & t \neq jT_p, \\ \underline{X}_2(jT_p^+) = (1 - p_1)\underline{X}_2(jT_p). \end{cases} \quad (3.13)$$

Denote $\underline{X}_2(t)$ as the solution of (3.13) satisfying $\underline{X}_2(\bar{T}) = \ddot{X}$. By the argument used for inequality (3.11), there is some $n_2 \in \mathbb{N}_+$ for which $\underline{X}_2(\bar{T} + n_2 T_p) \geq X_M$. By the comparison principle, on $[\bar{T}, \bar{T} + n_2 T_p]$, we have $X(t) \geq \underline{X}_2(t)$. Therefore, $X(\bar{T} + n_2 T_p) \geq X_M$. Since on $[\bar{T}, \gamma]$, $X(t) < X_M$, it follows that $\bar{T} + n_2 T_p > \gamma$.

In summary: $\min_{t \in [\alpha, \gamma]} X(t) \geq \min_{t \in [\alpha, \gamma]} \underline{X}_1(t) \geq \min_{t \in [lT_p^+, \bar{T} + n_2 T_p]} \underline{X}_1(t) = \underline{X}_1((\bar{T} + n_2 T_p)^-)$.

Since (3.12) is a pulse system with constant coefficients, the value of the trajectory issuing from the fixed initial value $\underline{X}_1(0^+) = (1 - p_1)X_M$ after k pulse periods depends only on the number of periods k , and is independent of the starting pulse instant lT_p (that is, of the shift l). Hence, although $\bar{T} = (n_1 + l + 1)T_p$ contains l , which depends on α , the value $\underline{X}_1(\bar{T} + n_2 T_p)$ coincides exactly with the

value of that trajectory, issuing from $(1 - p_1)X_M$, after $(n_1 + n_2 + 1)$ periods, and is therefore independent of l (and consequently of α). Taking n_1, n_2 to be the smallest positive integers satisfying (3.10) and $\underline{X}_2(\bar{T} + n_2 T_p) \geq X_M$, respectively, both are independent of α, γ , and thus $\underline{X}_1(\bar{T} + n_2 T_p)$ is independent of α, γ .

Moreover, since $\underline{X}_1$ is monotonically decreasing on $[lT_p, \bar{T} + n_2 T_p]$ and $X(t) \geq \underline{X}_1(t)$, we have

$$\min_{t \in [\alpha, \gamma]} X(t) \geq \underline{X}_1((\bar{T} + n_2 T_p)^-).$$

Let $m'_3 = \underline{X}_1((\bar{T} + n_2 T_p)^-) > 0$, which is independent of α, γ . Then we have $X(t) > m'_3$ ($t \in [\alpha, \gamma]$), hence $X(t) > m'_3$ ($t \in B$).

(b) Suppose $\bar{T} \geq \gamma$. In this case, $[\alpha, \gamma] \subseteq [lT_p, \bar{T}]$. Since $\underline{X}_1$ is monotonically decreasing on $[lT_p, \bar{T}]$ and $X(t) \geq \underline{X}_1(t)$, we have

$$\min_{t \in [\alpha, \gamma]} X(t) \geq \min_{t \in [lT_p, \bar{T}]} \underline{X}_1(t) = \underline{X}_1(\bar{T}^-).$$

By the shift-invariance of the constant-coefficient pulse system (3.12), $\underline{X}_1(\bar{T})$ equals the value of the trajectory issuing from $(1 - p_1)X_M$ after $(n_1 + 1)$ periods, and is therefore independent of l (and consequently of α). Taking n_1 to be the smallest positive integer satisfying (3.10), the value $\underline{X}_1(\bar{T}^-)$ is independent of α, γ . Let $m''_3 = \underline{X}_1(\bar{T}^-) > 0$ and then $X(t) \geq m''_3$ for $t \in [\alpha, \gamma]$, hence $X(t) \geq m''_3$ for all $t \in B$.

Finally, let $m_3 = \min\{m'_3, m''_3\}$. Consequently, $X(t) > m_3$ for $t \in B$.

In summary, let $m = \min\{m_1, m_2, m_3\}$. Then for sufficiently large t , we have $X(t) > m$, $S(t) > m$, $I(t) > m$. Therefore, the system is persistent. $\square$

Corollary 1. Let $(X(t), S(t), I(t))$ denote an arbitrary solution to model (2.3). If the condition

$$-\frac{\ln(1 - q_2)}{\beta m - \delta_2} < T_p < -\frac{\ln(1 - p_1)}{r \ln \frac{K}{c}}$$

holds, then $X(t) \rightarrow 0$ as $t \rightarrow \infty$, while both S and I are persistent. Furthermore, by a similar argument to that in the proof of Theorem 3, it can be shown that if

$$T_p < -\frac{\ln(1 - p_1)}{r \ln \frac{K}{c}}$$

and

$$\left[e^{\int_0^{T_p} [-\beta I_1(s) - d] ds} + e^{\int_0^{T_p} [\beta S_1(s) - \frac{1}{\tau} - d] ds} + \sqrt{\Delta_2} \right] < \frac{2}{1 - q_2},$$

where

$$\Delta_2 = \left[e^{\int_0^{T_p} [-\beta I_1(s) - d] ds} - e^{\int_0^{T_p} [\beta S_1(s) - \frac{1}{\tau} - d] ds} \right]^2 + 4e^{\int_0^{T_p} [-\beta S_1(s) + \beta I_1(s) + \frac{1}{\tau}] ds},$$

then the solution $z_1(t) = (0, S_1(t), I_1(t))$ is globally asymptotically stable.

Corollary 2. Let $(X(t), S(t), I(t))$ denote an arbitrary solution to model (2.3). If the following conditions hold:

$$\max \left\{ \frac{-\ln(1 - p_1) + \frac{b_{\max} X_M \int_{j_1 T_p}^{(\bar{j}_1 + 1) T_p} \bar{u}(t) dt}{(X_M + g)(1 + h X_M)}}{r \ln \left(\frac{K}{X_M + c} \right) - \frac{b_{\max} e X_M}{(X_M + g)(1 + h X_M)}}, 0 \right\} < T_p < -\frac{\ln(1 - q_2)}{\frac{v b_{\max} X_M^2}{(X_M + g)(1 + h X_M)} - d},$$

$$\frac{\beta s_T}{d[1 - (1 - q_2)e^{-dT_P}]}(1 - e^{-dT_P}) < \ln \frac{1}{1 - q_2} + \left(\frac{1}{\tau} + d\right)T_P,$$

then, as $t \rightarrow \infty$, we have $I(t) \rightarrow 0$, while both S and X are persistent. Additionally, following an argument analogous to that in the proof of Theorem 3, it can be shown that if

$$\beta \int_0^{T_P} S_2(s) \, ds < \left(\frac{1}{\tau} + d\right)T_P - \ln(1 - q_2)$$

and

$$(1 - p_1)e^{R_1 T_P} + (1 - q_2)e^{R_4 T_P} \pm \sqrt{\Delta_3} < 2,$$

where

$$\begin{aligned} R_1 &= r \ln \frac{K}{X_2 + c} - \frac{rX_2}{X_2 + c} - \frac{b_{\max}XS_2(X_2 + hgX_2 + 2g)}{[X_2 + g](1 + hX_2)}, \\ R_2 &= -\frac{b_{\max}X_2^2}{(X_2 + g)(1 + hX_2)}, \quad R_3 = \frac{vb_{\max}X_2S_2(X_2 + hgX_2 + 2g)}{[(X_2 + g)(1 + hX_2)]^2}, \quad R_4 = \frac{vb_{\max}X_2^2}{(X_2 + g)(1 + hX_2)} - d, \\ \Delta_3 &= \left[(1 - p_1)e^{R_1 T_P} - (1 - q_2)e^{R_4 T_P}\right]^2 + 4(1 - p_1)(1 - q_2)e^{(R_2 + R_3)T_P}, \end{aligned}$$

then the solution $z_2(t) = (X_2(t), S_2(t), 0)$ is globally asymptotically stable.

3.2.3. Coexistence periodic solution

When the infected natural enemies are not considered, the following model is obtained:

$$\left\{ \begin{array}{l} \frac{dX(t)}{dt} = rX(t)(\ln K - \ln(X(t) + c)) - \frac{b_{\max}S(t)X(t)^2}{(X(t) + g)(1 + hX(t))} \\ \frac{dS(t)}{dt} = \frac{vb_{\max}S(t)X(t)^2}{(X(t) + g)(1 + hX(t))} - dS(t) \\ \Delta X = -p_1X(t) \\ \Delta S = -q_2S(t) + s_T \end{array} \right\} \quad \begin{array}{l} t \neq jT_p, \\ t = jT_p. \end{array} \quad (3.14)$$

Theorem 8. When $T_p > \overline{T_p} \triangleq -\frac{\ln(1 - p_1)}{r \ln \frac{K}{c}}$, a supercritical coexistence periodic solution can bifurcate from model (3.14).

Proof. Applying the transformation $x_1(t) = S(t)$, $x_2(t) = X(t)$ yields the model:

$$\left\{ \begin{array}{l} \frac{dx_1}{dt} = \frac{vb_{\max}x_1x_2^2}{(x_2 + g)(1 + hx_2)} - dx_1 \\ \frac{dx_2}{dt} = rx_2(\ln K - \ln(x_2 + c)) - \frac{b_{\max}x_1x_2^2}{(x_2 + g)(1 + hx_2)} \\ \Delta x_1 = -q_2x_1 + s_T \\ \Delta x_2 = -p_1x_2 \end{array} \right\} \quad \begin{array}{l} t \neq jT_p, \\ t = jT_p. \end{array} \quad (3.15)$$

Hence, we obtain

$$\begin{aligned} F_1(x_1, x_2) &= \frac{vb_{\max}x_1x_2^2}{(x_2 + g)(1 + hx_2)} - dx_1, \\ F_2(x_1, x_2) &= rx_2(\ln K - \ln(x_2 + c)) - \frac{b_{\max}x_1x_2^2}{(x_2 + g)(1 + hx_2)}, \\ \theta_1(x_1, x_2) &= (1 - q_2)x_1(t) + s_T, \\ \theta_2(x_1, x_2) &= (1 - p_1)x_2(t), \\ \xi(t) &= (x_s(t), 0) = (\bar{S}(t), 0), \end{aligned}$$

where $F_1, F_2, \theta_1, \theta_2$ are sufficiently smooth functions, θ_1, θ_2 are strictly positive, and $F_2(x_1, 0) = \theta_2(x_1, 0) = 0$. When $x_2(t) = 0$, model (3.15) reduces to

$$\begin{cases} \frac{dx_1(t)}{dt} = F_1(x_1(t), 0), & t \neq jT_p, \\ x_1(jT_p^+) = \theta_1(x_1(jT_p), 0), & t = jT_p, \end{cases}$$

where the stable $\bar{T}_p$ -periodic solution is denoted by

$$x_s = \frac{s_T e^{-d(t-jT_p)}}{1 - (1 - q_2)e^{-dT_p}}.$$

Thus, $\xi(t) = (x_s(t), 0)$ is a boundary periodic solution of model (3.15). Let $x_0 = x_s(0)$ and $\xi(0) = (x_0, 0)$. It follows that

$$\begin{cases} \frac{\partial \Phi_1(t, x_0)}{\partial x_1} = e^{-dt}, & \frac{\partial \Phi_2(t, x_0)}{\partial x_2} = e^{r \ln \frac{K}{c} t} = \left(\frac{K}{c}\right)^{rt}, & \frac{\partial \Phi_1(t, x_0)}{\partial x_2} = 0, \\ \frac{\partial \theta_1(t, x_0)}{\partial x_1} = 1 - q_2, & \frac{\partial \theta_2(t, x_0)}{\partial x_2} = 1 - p_1, \\ d'_0 = 1 - (1 - p_1)e^{r \ln \frac{K}{c} \bar{T}_p}, & a'_0 = 1 - (1 - q_2)e^{-d\bar{T}_p} > 0, & b'_0 = 0, \\ \frac{\partial^2 \Phi_2(t, x_0)}{\partial x_1 \partial x_2} = 0, & \frac{\partial^2 \Phi_2(t, x_0)}{\partial x_2^2} = \left(\frac{K}{c}\right)^{rt} \int_0^t \left(-\frac{2r}{c} - \frac{2b_{\max}x_s(u)}{g}\right) du, & \frac{\partial^2 \Phi_2(t, x_0)}{\partial x_2 \partial \tau} = r \ln \frac{K}{c} \left(\frac{K}{c}\right)^{rt}, \\ \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} = 0, & \frac{\partial^2 \theta_2}{\partial x_2^2} = 0. \end{cases}$$

Hence we have:

$$B = -(1 - p_1)r \ln \frac{K}{c} \left(\frac{K}{c}\right)^{r\bar{T}_p}, \quad C = (1 - p_1) \left(\frac{K}{c}\right)^{r\bar{T}_p} \int_0^{\bar{T}_p} \left(\frac{2r}{c} + \frac{2bx_s(u)}{g}\right) du.$$

Therefore, the condition $BC < 0$ implies the bifurcation of a supercritical coexistence periodic solution off the boundary periodic solution in model (3.14). $\square$

Remark 3. When pests are not considered, the following model is obtained:

$$\left\{ \begin{array}{l} \frac{dS}{dt} = -dS(t) - \beta S(t)I(t) + \frac{I(t)}{\tau} \\ \frac{dI}{dt} = \beta S(t)I(t) - \frac{I(t)}{\tau} - dI(t) \\ \Delta S = -q_2 S(t) + s_T \\ \Delta I = -q_2 I(t) \end{array} \right\} \quad t \neq jT_P, \quad (3.16)$$

$$\left\{ \begin{array}{l} \Delta S = -q_2 S(t) + s_T \\ \Delta I = -q_2 I(t) \end{array} \right\} \quad t = jT_P.$$

Let $\overline{T}_p \triangleq \min\{T_p : g(T_p) > 1\}$, where $g(T_p) = (1 - q_2)e^{\frac{\beta s_T}{d[1-(1-q_2)e^{-dT_p}]}(1-e^{-dT_p})-(\frac{1}{\tau}+d)T_p}$. The stable periodic solution with period $\overline{T}_p$ is given by

$$s_s = \frac{s_T e^{-d(t-jT_p)}}{1 - (1 - q_2)e^{-dT_p}}.$$

Similar to the proof method of Theorem 7, define

$$B = -\left[\frac{\beta s_T e^{-d\overline{T}_p} + (1 - q_2)^2}{1 - (1 - q_2)e^{-d\overline{T}_p}} - \frac{1}{\tau} - d + \beta \overline{T}_p \right], C = -\frac{2(1 - q_2)\beta \overline{T}_p I_1}{1 - (1 - q_2)e^{-d\overline{T}_p}} - \beta I_2,$$

where $H = \frac{\beta s_T}{d[1-(1-q_2)e^{-d\overline{T}_p}]}$, $A = \frac{1}{\tau} + d$, $I_1 = e^{-d\overline{T}_p} \int_0^{\overline{T}_p} \left[-dH + \frac{1}{\tau} e^{du} \right] \exp \left[H(1 - e^{-du}) - Au \right] du$, and $I_2 = \int_0^{\overline{T}_p} \int_0^u \frac{e^{-d(u-v)} [-dHe^{-dv} + \frac{1}{\tau}] \exp [H(1 - e^{-dv}) - Av] dv}{\exp [H(1 - e^{-du}) - Au]} du$. Therefore, when $BC < 0$, model (3.16) undergoes a bifurcation from the boundary periodic solution, giving rise to a supercritical coexistence periodic solution.

4. Simulations and verifications

4.1. Numerical simulations of the continuous model

Numerical simulations were performed in MATLAB to present the primary results and rigorously validate the effectiveness of the control method. The parameters are taken as follows: $r = 0.5$, $K = 100$, $b_{max} = 0.2$, $g = 10$, $h = 0.05$, $v = 0.6$, $d = 0.15$, $\beta = 0.005$, $\tau = 1.5$.

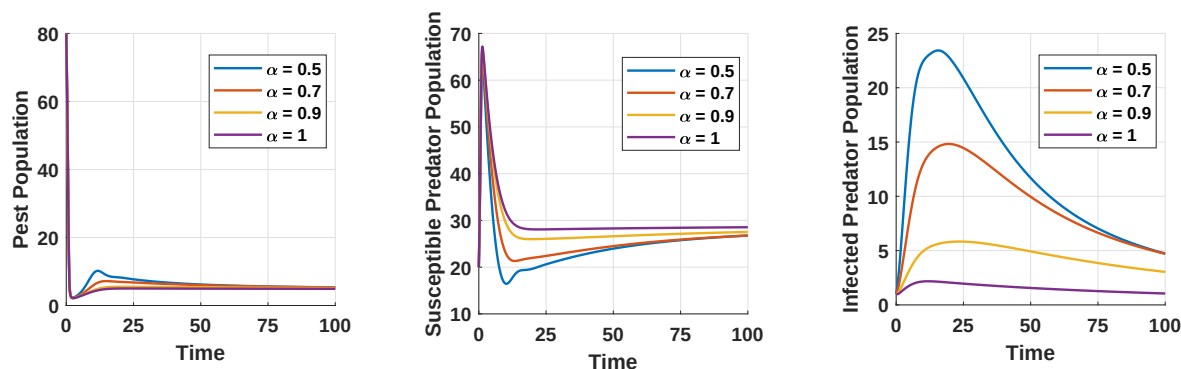


Figure 2. Variation trends of X , S , and I densities under different values of α .

From Eq (3.1), we have $S^* = \frac{d}{\beta} \left(1 + \frac{1}{\tau^\alpha d^\alpha}\right)$ and $I^* = \left(1 + \frac{1}{\tau^\alpha d^\alpha}\right) \left[\frac{vb_{max}(X^*)^2}{\beta(X^*+g)(1+hX^*)} - \frac{d}{\beta}\right]$. It is straightforward to obtain $\frac{dS^*}{d\alpha} = -\frac{d \ln(\tau d)}{\beta(\tau d)^\alpha}$.

If $0 < \tau d < 1$, then $\ln(\tau d) < 0$ and $\frac{dS^*}{d\alpha} > 0$, which means that the susceptible predator equilibrium S^* increases with the fractional order α . The variation trends of X^* and I^* with respect to α are not easily derived analytically; they are therefore observed through numerical simulations, as shown in Figure 2. It can be seen that a larger α leads to a lower pest density X^* and a lower infected predator density I^* . The parameter α reflects the memory effect in disease recovery. A smaller α corresponds to a stronger memory effect, meaning that the dynamic evolution of the populations is more influenced by the cumulative and lasting impact of historical infection states. Consequently, the recovery rate of infected predators becomes slower, and fewer individuals are converted back to the susceptible class. This implies that management strategies should account for the memory effect associated with disease progression in infected predators and the distribution of recovery times, rather than focusing solely on current infection counts.

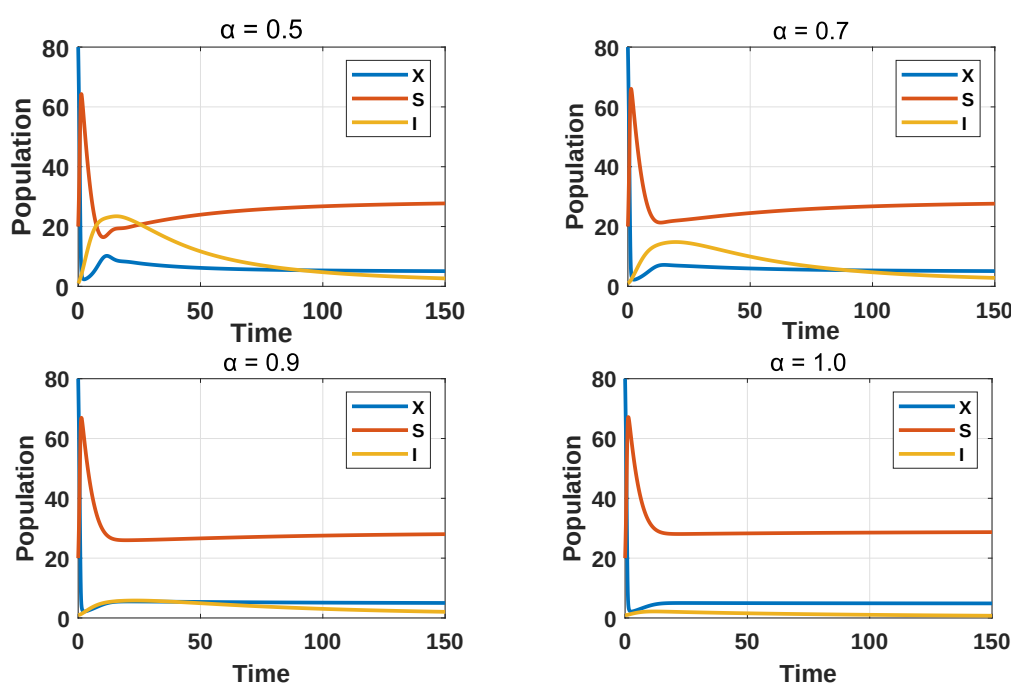


Figure 3. Temporal evolution of each population under different values of α .

Figure 3 shows how the fractional order α affects the dynamics of the three populations: pests, susceptible predators, and infected predators. It can be observed that a smaller α (indicating a stronger memory effect) reflects a longer historical dependence in the infection-recovery process of predators. Mathematically, this corresponds to a recovery-time distribution with a heavier tail (a long-tailed distribution), meaning that the time predators remain in the infected state is more uncertain. A small fraction of predators may carry the pathogen or remain infected for a prolonged period, thereby slowing down the convergence of all populations to the equilibrium. Furthermore, because the recovery of infected predators is relatively slow, their residence time in the population increases, leading to a higher proportion of infected predators. Hence, the fractional order α is not

merely a parameter in the mathematical model; it reflects the intensity of temporal memory in the infection-recovery process within the ecosystem, which has important theoretical significance for understanding disease transmission dynamics and evaluating the long-term effectiveness of control strategies.

The influence of the variable searching rate on the pest population is illustrated in Figure 4. Clearly, the equilibrium pest density X^* decreases as the variable searching rate $b(X^*)$ increases. When the searching ability of predators for pests is enhanced, the probability of pests being captured rises, which suppresses the survival and reproduction rates of the pest population, ultimately leading to a lower steady-state density. This result fully demonstrates that the variable searching rate is a key factor regulating pest population dynamics. Investigating the impact of the variable searching rate on pest density provides important theoretical references and a scientific basis for developing integrated pest-management strategies in the field.

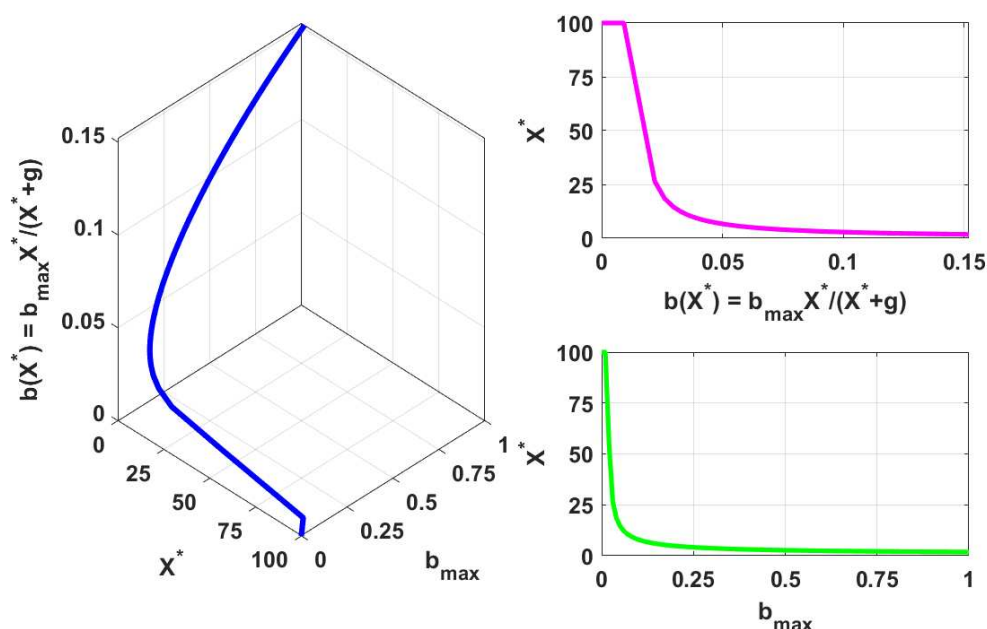


Figure 4. Three-dimensional surface of the pest equilibrium density X^* versus the state-dependent searching rate $b(X^*)$ and the maximum searching rate b_{max} .

4.2. Numerical simulations of the periodically pulsed control model

(i) Pest-eradication periodic solution

The parameters in (2.3) are taken as: $r = 0.1$, $K = 50$, $c = 0.05$, $b_{max} = 0.5$, $g = 5$, $h = 0.01$, $v = 0.4$, $d = 0.05$, $\tau = 15$, $\beta = 0.02$, $p_1 = 0.5$, $q_2 = 0.22$, $s_T = 5$, $T_p = 1.7$.

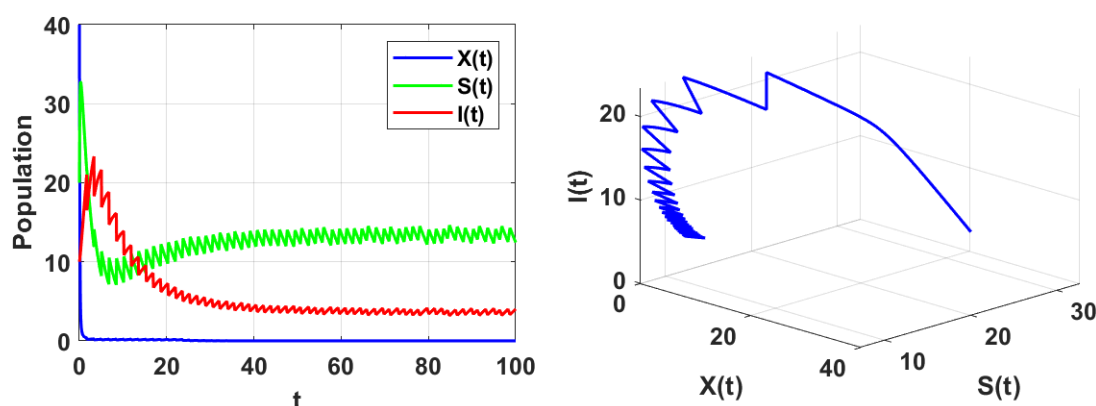


Figure 5. Time-domain responses and phase-plane trajectories of model (2.3)'s solution for $T_p = 1.7$.

When the threshold condition holds, the pest-eradication periodic solution exhibits local asymptotic stability and global attractivity, as illustrated in Figure 5. Although this condition can achieve the goal of eradicating pests, it requires frequent control interventions, which is impractical for real-world systems.

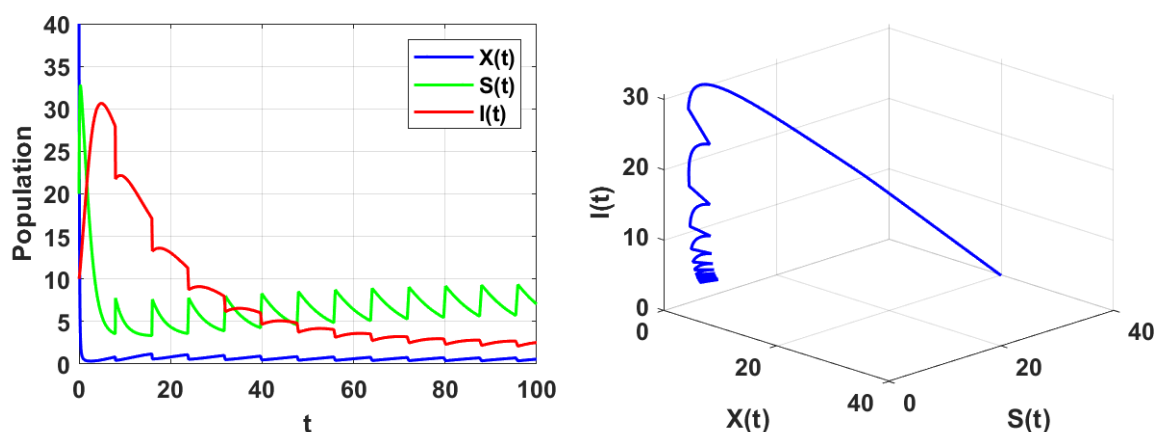


Figure 6. Time-domain responses and phase-plane trajectories of model (2.3)'s solution for $T_p = 8$.

Figure 6 illustrates the case where $T_p = 8$. Here the threshold fails, destabilizing the pest-free cycle and giving rise to a coexisting periodic orbit.

(ii) Coexistence order-1 periodic solution

When $b_{max} = 0.2$, $g = 10$, $h = 0.05$, $v = 0.6$, $d = 0.1$, $\tau = 1.5$, $\beta = 0.02$, $r = 0.5$, $K = 100$, $c = 0.05$, $p_1 = 0.2$, $q_2 = 0.1$, $T_p = 5$, and $s_T = 30$, models (3.16), (2.3), and (3.14) all have coexisting periodic solutions, as shown in Figure 7.

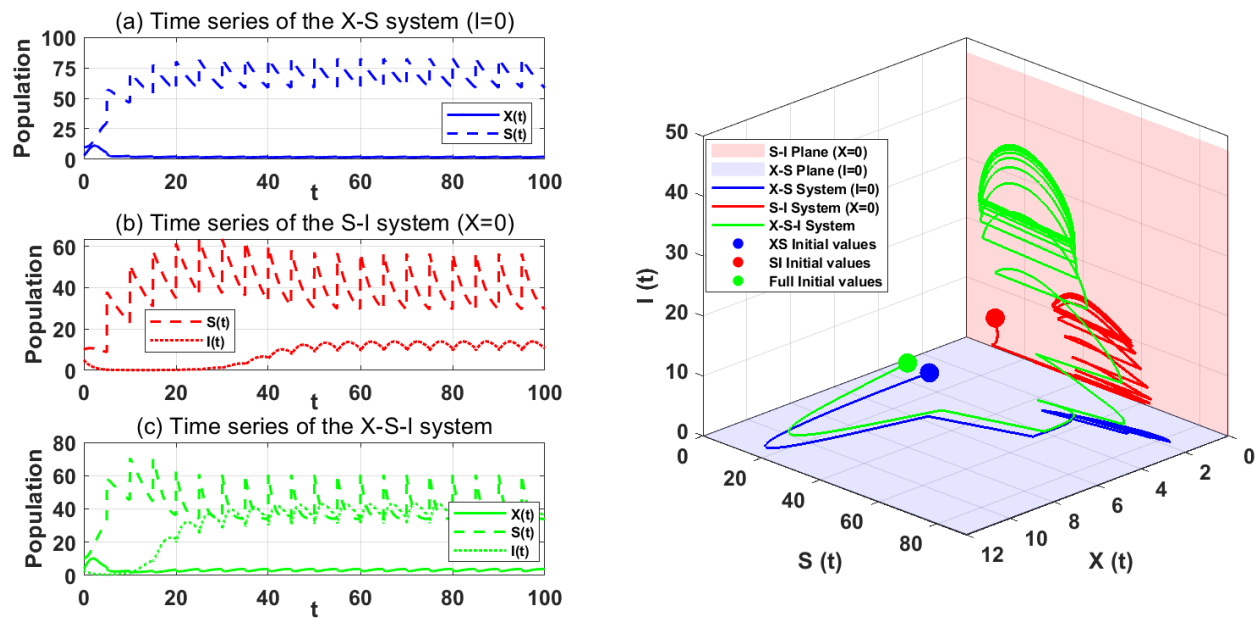


Figure 7. Time series and phase diagrams of periodic solutions for models (3.14), (3.16), and (2.3) for $T_p = 5$, $s_T = 30$.

When $b_{max} = 0.2$, $g = 10$, $h = 0.05$, $v = 0.6$, $d = 0.1$, $\tau = 1.5$, $\beta = 0.02$, $r = 0.5$, $K = 100$, $c = 0.05$, $p_1 = 0.2$, $q_2 = 0.1$, $T_p = 10$, and $s_T = 3$, the condition is not satisfied, and models (3.16) and (2.3) have no coexisting periodic solutions, as shown in Figure 8.

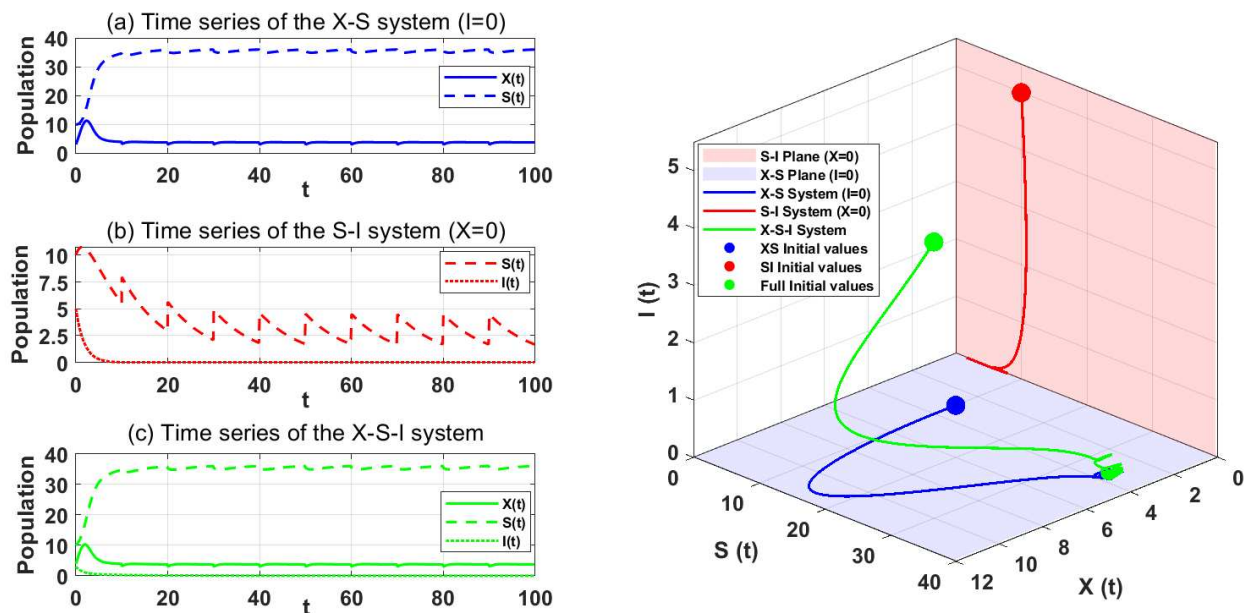


Figure 8. Time series and phase diagrams of periodic solutions for models (3.14), (3.16), and (2.3) for $T_p = 10$, $s_T = 3$.

Figure 9 illustrates the coexisting periodic solutions corresponding to different T_p values. The pest density at the control instant declines with the increase in T_p . This finding indicates that more frequent intervention does not necessarily yield superior control performance. Hence, optimal intervention strategies should be tailored to practical application scenarios rather than following a simplistic high-frequency intervention principle.

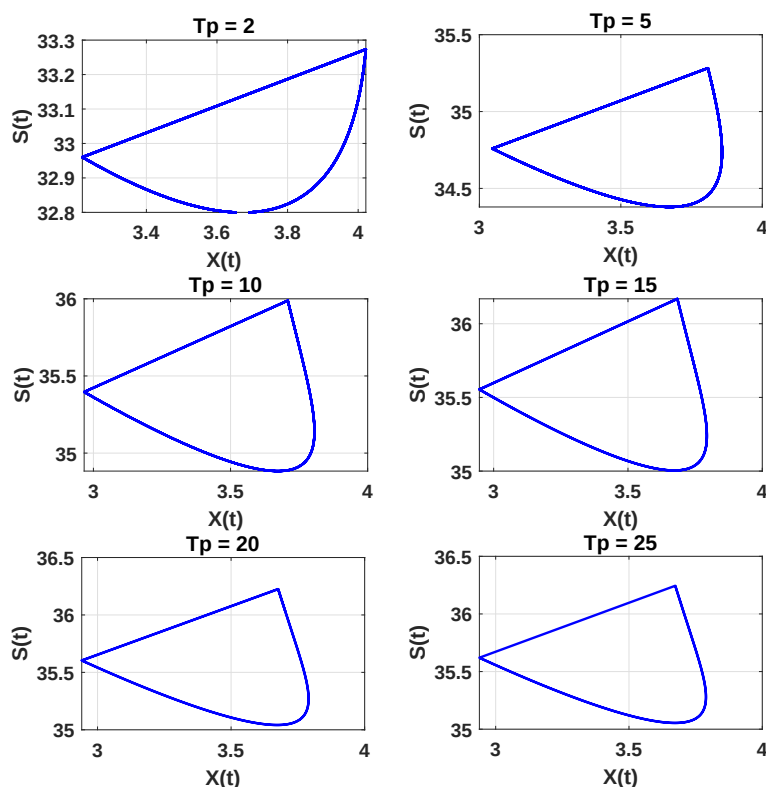


Figure 9. Phase diagrams of coexisting periodic solutions of model (3.14) for different values of T_p .

5. Conclusions

Agricultural pests pose a serious threat to the stability of agricultural production and food security. Current pest-regulation strategies fall into three main categories: chemical applications, biological interventions, and IPM. To effectively manage the spread of pests, this paper proposes a study on a fractional-order pest-infected predator model with a variable searching rate and disease recovery memory effects. It discusses solution boundedness in the continuous model, establishes whether equilibria exist and are locally asymptotically stable, and identifies transcritical bifurcations. The research results indicate that an increase in the fractional order promotes the population of susceptible predators while inhibiting the populations of pests and infected predators. Moreover, the smaller the fractional order, representing a stronger “memory effect”, the slower the convergence rate of all populations to the equilibrium point and the higher the infection proportion among predators. Predators play a certain inhibitory role in the population growth of pests, and the maximum variable searching rate has a direct impact on the positive equilibrium point.

To prevent the spread of pests, this paper adopts a control strategy involving the periodic release of predators and the application of insecticides, constructing a periodic impulsive model. Theoretical results show that a pest-eradication periodic solution exists, and its impulsive control period has a certain influence on its stability. Once the critical criterion is satisfied, global asymptotic stability is exhibited by the pest-eradication periodic solution. However, in agricultural ecosystems, completely eradicating pests is neither a necessary goal nor often sustainably achievable. In practice, the key to control lies in suppressing the pest population below economic or ecological thresholds to prevent significant damage. Therefore, this paper demonstrates the conditions for the global asymptotic stability of the pest-free periodic solution and finds that the model can maintain a persistent state when the impulsive period falls within a certain range. Additionally, conditions for the global asymptotic stability of the remaining two populations when either the pests or the infected predators become extinct, as well as sufficient conditions for persistence, are provided. Consequently, by selecting an appropriate impulsive control period, it is possible to achieve pest eradication while allowing predators to persist permanently. Finally, the existence of a coexisting periodic solution for these two second-order periodic impulsive models is also demonstrated. This provides a certain theoretical basis for pest management strategies.

The study shows that the incorporation of infected predators, a variable predator searching rate, and fractional-order dynamics further enriches the dynamical characteristics of biological systems, offering a theoretical reference for research on integrated pest management.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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