



Survey

Algebraic objects via homotopy

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Abstract: Graphs associated to algebraic objects have been widely used to encode algebraic information through pairwise relations. Many algebraic relations, however, naturally involve finite collections rather than pairs. Simplicial complexes therefore provide a higher-dimensional framework for studying algebraic objects: after encoding algebraic data as a simplicial complex, one investigates the homology and homotopy type of its geometric realization. In this paper, we present several such constructions arising from commutative rings, finite groups, and cyclotomic polynomials. These examples illustrate how simplicial complexes can retain higher-order algebraic information and how the resulting topological invariants can reflect structural properties of the underlying algebraic object.

Keywords: graphs; simplicial complexes; homotopy type; commutative rings; finite groups; cyclotomic polynomial

1. Introduction

A common topic in algebra is to associate a combinatorial object to an algebraic structure and then study how the properties of this object reflect the properties of the original structure. One of the most developed forms of this idea is the association of graphs to algebraic objects. In such constructions, vertices usually represent algebraic objects while edges record a chosen pairwise algebraic relation.

In particular, graphs associated to commutative rings have been studied extensively in different ways. The zero-divisor graph of a commutative ring, studied by Anderson and Livingston [1], has vertices corresponding to non-zero zero-divisors, with vertices being adjacent if their product is zero. Related ideal-based versions were considered by Akhtar and Lee [2], where zero-divisors are replaced by ideals in order to better reflect the ideal structure of the ring. Other important examples include the intersection graph of ideals, introduced by Chakrabarty et al. [3], and the comaximal graph, introduced by Sharma and Bhatwadekar [4]. These graphs show that combinatorial methods can successfully encode ring-theoretic information.

However, many algebraic relations are not merely pairwise. For instance, one may record whether a finite collection of ideals forms a chain under inclusion, has non-zero intersection, or has non-zero product. Also, one can consider which finite sets of ring elements fail to generate the ring. On the other hand, for groups one can consider subsets of the set of irreducible character degrees with a common divisor, or sets of prime divisors occurring simultaneously in a character degree. In the cyclotomic setting, the facets of the associated complex are indexed by residue classes.

Simplicial complexes provide a natural way for encoding this higher-order information. A graph records relations among pairs of vertices, while a simplicial complex records relations among arbitrary finite collections of vertices. The main idea of this paper is that once an algebraic object is encoded as a simplicial complex, one can study the homotopy type of its geometric realization. Here we present a unified overview of several constructions of this kind. We begin with commutative rings in Section 3. Several different simplicial complexes can be associated to a ring by using ideals or elements as vertices. These include the order complex of ideals, the intersection complex of ideals, the ideal zero-divisor complex, and independence complexes of comaximal graphs and hypergraphs.

We then turn to finite groups through the set of irreducible character degrees in Section 4. The common divisor complex and the prime divisor complex associated to a set of positive integers extend the classical common divisor and prime vertex graphs. When applied to character degrees, these complexes provide a higher-dimensional viewpoint on divisibility relations among character degrees. For solvable groups, their topology is constrained by character-theoretic properties of the group, and the fundamental group can be described and bounded in terms of the corresponding prime divisors.

Finally, in Section 5, we discuss simplicial complexes associated to cyclotomic polynomials. In this case, the construction uses residues associated to the index of the cyclotomic polynomial and the homology of the resulting complexes is related to the coefficients of the polynomial. The ternary case, where the index is a product of three distinct primes, shows that homotopy type of the associated complex can carry more information than the homological interpretation.

2. Definitions and notation

In this section we recall the basic notions from combinatorial topology that will be used throughout the paper. For more details we refer the reader to [5, 6].

An abstract simplicial complex K on a vertex set V is a collection of finite subsets of V such that, whenever $\sigma \in K$ and $\tau \subseteq \sigma$, one also has $\tau \in K$. The elements of K are called simplices. If σ has $n + 1$ vertices, then σ is called an n -simplex and we write its dimension as $\dim \sigma = n$. The dimension of K is

$$\dim K = \sup\{\dim \sigma \mid \sigma \in K\}.$$

A 0-simplex is called a vertex, a 1-simplex an edge, and a 2-simplex a triangle. A nonempty subset β of α in K is called a face of α , and a simplex which is not properly contained in any other simplex of K is called a maximal face or facet. A subcollection of a simplicial complex K that contains all faces of its elements is another abstract simplicial complex called a simplicial subcomplex. In particular, the d -skeleton of K is a subcomplex of K that contains all simplices of dimension d or less.

If $\sigma \in K$, then the star of σ in K is the subcomplex

$$\text{St}_K(\sigma) = \{\tau \in K \mid \tau \cup \sigma \in K\}.$$

When it is clear from the context, we simply write $\text{St}(\sigma)$, and if $\sigma = \{v\}$ is a vertex, we write $\text{St}(v)$.

The join of two simplicial complexes K and L , with disjoint vertex sets, is the simplicial complex

$$K * L = \{\sigma \cup \tau \mid \sigma \in K, \tau \in L\}.$$

More generally, one may form iterated joins $K_1 * \cdots * K_d$.

The geometric realization of an abstract simplicial complex K , denoted by $|K|$, is the topological space obtained by replacing every abstract n -simplex by a geometric n -simplex and gluing these simplices along their common faces. In what follows, we will often identify a simplicial complex with its geometric realization when no confusion can arise.

Throughout the paper, $\widetilde{H}_*(K)$ denotes the reduced homology of the geometric realization $|K|$. Two simplicial complexes K and L are said to be homotopy equivalent, and we write

$$K \simeq L,$$

if their geometric realizations $|K|$ and $|L|$ are homotopy equivalent. A complex is contractible if it is homotopy equivalent to a point. We denote by $\mathbb{S}^n$ the n -sphere and by $\mathbb{B}^n$ the n -ball. A wedge of spheres means a one-point union of finitely many spheres.

The homotopy type of a simplicial complex can often be studied by simplifying the complex without changing its homotopy type. One such simplification is given by elementary collapses. A simplex τ is called a free face of a simplex σ if τ is a proper face of σ and σ is the unique maximal simplex of K containing τ . Removing σ together with all of its faces that contain τ is called an elementary collapse. If a complex K can be transformed into a subcomplex L by a sequence of elementary collapses, then K and L are homotopy equivalent.

Discrete Morse theory, introduced by Forman [7, 8], provides a systematic way of organizing such collapses. We recall only the basic consequence that will be used later. A discrete vector field V on a finite simplicial complex K is a collection of pairs $\{\alpha, \beta\}$ where α is a codimension-one face of β , meaning that $\alpha \subset \beta$ and $\dim \beta = \dim \alpha + 1$. Each simplex of K is allowed to occur in at most one such pair. Simplices which do not occur in any pair are called critical simplices.

The discrete vector field V is called acyclic if the pairings do not create any closed paths $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \beta_r, \alpha_{r+1}$ where for each $i = 0, \dots, r$, $r > 0$, $\{\alpha_i, \beta_i\}$ are pairs in V , $\beta_i \supset \alpha_{i+1} \neq \alpha_i$ and $\alpha_0 = \alpha_{r+1}$. In that case, the complex K is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical p -simplex.

Thus, if one constructs an acyclic discrete vector field with very few critical simplices, then the homotopy type of the original complex can often be read off from these critical simplices. This is the main role of discrete Morse theory in some of the examples discussed below.

3. Complexes associated to commutative rings

In this section we recall several constructions which associate simplicial complexes to a commutative ring with identity. The common feature of these constructions is that algebraic relations among ideals or elements of the ring are encoded as simplices. The homotopy type of the resulting complex then reflects structural properties of the ring.

We shall not reproduce the full proofs of the results stated below. Instead, we emphasize the construction, the resulting homotopy type, and the algebraic information captured by the topology, and refer the reader to detailed proofs in relevant papers.

Throughout this section, R denotes a commutative ring with identity. We write $\text{Max}(R)$ for the set of maximal ideals of R , $J(R)$ for the Jacobson radical of R , and $I^*(R)$ for the set of all non-zero proper ideals of R . The first three complexes described in this section consider the set of all proper non-zero ideals of R as the vertex set, so we assume R is not a field.

3.1. Order complex of ideals

Order complexes are classical objects in poset topology.

For a partially ordered set P , the order complex of P , denoted by $\Delta(P)$, is the simplicial complex whose vertices are the elements of P , and whose simplices are finite chains in P . Thus, $p_0, p_1, \dots, p_n \in \Delta(P)$ if, after reindexing if necessary, $p_0 < p_1 < \dots < p_n$.

Since many algebraic objects naturally give rise to partially ordered sets, order complexes provide a natural way of associating a simplicial complex to algebraic data. In the context of commutative rings, in [9] Vassiliev suggested studying the order complex associated to the poset of non-trivial ideals of a commutative ring. Following this suggestion, the homotopy type of the order complex of ideals was determined in [10].

Let R be a commutative ring with identity, and let $I^*(R)$ denote the set of all proper non-zero ideals of R . The order complex of ideals of R , denoted by $\Delta_{\text{ord}}(R)$, is the simplicial complex whose vertex set is $I^*(R)$, and whose simplices are finite chains of ideals. Thus,

$$\{I_0, I_1, \dots, I_n\} \in \Delta_{\text{ord}}(R)$$

if and only if, after reindexing if necessary,

$$I_0 \subset I_1 \subset \dots \subset I_n.$$

For example, Figure 1 illustrates both the poset $I^*(\mathbb{Z}_{60})$ and $|\Delta_{\text{ord}}(\mathbb{Z}_{60})|$.

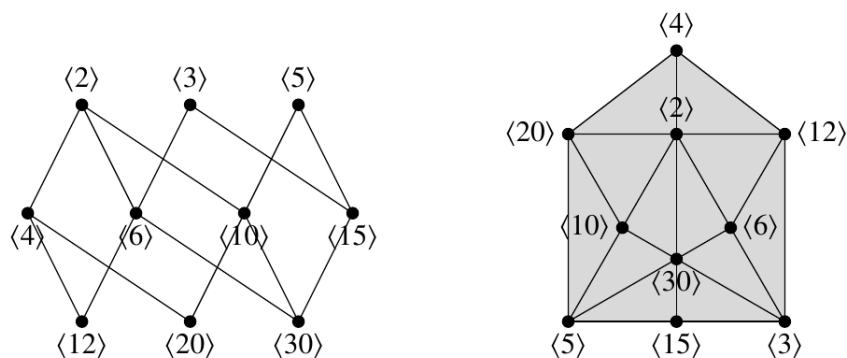


Figure 1. Poset $I^*(\mathbb{Z}_{60})$ and the complex $|\Delta_{\text{ord}}(\mathbb{Z}_{60})|$.

The homotopy type of $\Delta_{\text{ord}}(R)$ is completely determined in [10]. The result has a particularly simple form: the complex is contractible except in the case of semilocal rings with trivial Jacobson radical.

More precisely, if R is local with maximal ideal M , then every proper ideal is contained in M . Hence, every chain may be extended by adding M and $\Delta_{\text{ord}}(R)$ is a cone with apex M . Therefore,

$$|\Delta_{\text{ord}}(R)| \simeq *.$$

If R is semilocal, $|\text{Max}(R)| = m > 1$, and $J(R) = 0$, then by the Chinese remainder theorem

$$R \cong F_1 \times \cdots \times F_m,$$

where $F_1, \dots, F_m$ are fields. In this case, the non-zero proper ideals of R correspond to the proper non-empty subsets of $1, \dots, m$. Consequently, $\Delta_{\text{ord}}(R)$ is the barycentric subdivision of the boundary of an $(m-1)$ -simplex. Hence,

$$|\Delta_{\text{ord}}(R)| \simeq \partial\Delta^{m-1} \simeq S^{m-2}.$$

In all remaining cases, including semilocal rings with $J(R) \neq 0$ and rings with infinitely many maximal ideals, the complex is contractible. Therefore, the order complex of ideals is either contractible or homotopy equivalent to a sphere, with the spherical case occurring precisely for semilocal rings with trivial Jacobson radical.

3.2. Intersection complex of ideals

Another natural construction with the same vertex set is obtained by using intersections of ideals. This construction, introduced in [11], is motivated by the intersection graph of ideals, whose vertices are non-zero proper ideals and where two ideals are adjacent whenever their intersection is non-zero [3].

Let R be a commutative ring with identity, and let $I^*(R)$ denote the set of all non-zero proper ideals of R . The intersection complex of ideals of R , denoted by $\Delta_{\text{int}}(R)$, is the simplicial complex with vertex set $I^*(R)$ in which

$$\{I_0, I_1, \dots, I_n\} \in \Delta_{\text{int}}(R) \iff I_0 \cap I_1 \cap \cdots \cap I_n \neq 0.$$

This is indeed a simplicial complex, since if the intersection of a family of ideals is non-zero, then the intersection of any subfamily is also non-zero.

The 1-skeleton of $\Delta_{\text{int}}(R)$ is the intersection graph of ideals. The homotopy type of $\Delta_{\text{int}}(R)$ was determined in [11]. The result has the same general pattern as the order complex of ideals. If R is local, if R is semilocal with $J(R) \neq 0$, or if R has infinitely many maximal ideals, then $\Delta_{\text{int}}(R)$ is contractible. The only non-contractible case occurs when

$$R \cong F_1 \times \cdots \times F_m$$

is a finite product of fields, with $m > 1$. In that case,

$$|\Delta_{\text{int}}(R)| \simeq S^{m-2}.$$

Thus, as with the order complex of ideals, the intersection complex detects the number of maximal ideals when R is semilocal with $J(R) = 0$.

3.3. The ideal zero-divisor complex

A second construction uses the multiplicative structure of the ideals of R . The ideal zero-divisor complex, denoted by $\Delta_{\text{zd}}(R)$ is the simplicial complex with vertex set $I^*(R)$, where

$$\{I_0, I_1, \dots, I_n\} \in \Delta_{\text{zd}}(R) \text{ if and only if } I_0 I_1 \cdots I_n \neq 0.$$

This construction is motivated by zero-divisor graphs, but it records more than pairwise annihilation. Instead of considering only whether two ideals have zero product, the complex records which finite collections of ideals have non-zero product. An example of such complex is given for $R = \mathbb{Z}/p^9\mathbb{Z}$. We will abuse notation and by p^i denote the ideal $I = \langle p^i \rangle$. Geometric realization of this complex is represented in Figure 2.

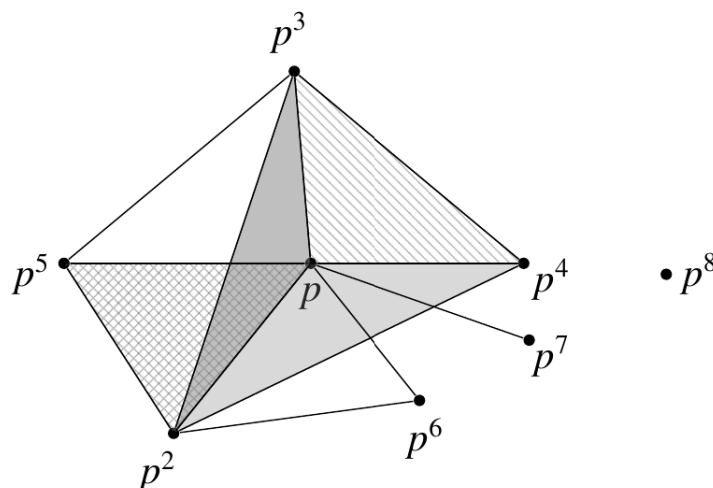


Figure 2. $|\Delta_{\text{zd}}(\mathbb{Z}/p^9\mathbb{Z})|$.

The homotopy type of $\Delta_{\text{zd}}(R)$ was determined in [12] for finite rings and for rings with infinitely many maximal ideals.

If $\text{Max}(R)$ is infinite, then $|\Delta_{\text{zd}}(R)| \simeq *$. The proof follows a pattern that appears frequently in these constructions: every finite subcomplex is contained in a larger subcomplex which is a cone. This implies, by a compactness argument and Whitehead's theorem, that the entire complex is contractible.

For finite rings, the situation is more subtle. Discrete Morse theory is used to construct an acyclic discrete vector field on $\Delta_{\text{zd}}(R)$. The unmatched, or critical, simplices determine a CW complex homotopy equivalent to $\Delta_{\text{zd}}(R)$. In this way, the complex may be reduced to a wedge of spheres, together with some exceptional disconnected cases.

For finite local rings with maximal ideal M , the homotopy type depends on the annihilator of the maximal ideal $\text{Ann}(M)$:

$$\begin{aligned} \text{Ann}(M) = 0 &\implies \Delta_{\text{zd}}(R) \text{ is contractible,} \\ \text{Ann}(M) = M &\implies \Delta_{\text{zd}}(R) \text{ is homotopy equivalent to a finite disjoint union of} \\ &\text{points,} \\ \text{Ann}(M) \neq 0, &\implies \Delta_{\text{zd}}(R) \text{ is a disjoint union of finitely many points and a} \\ \text{Ann}(M) \neq M &\text{complex homotopy equivalent to a wedge of spheres.} \end{aligned}$$

For finite non-local rings

$$|\Delta_{\text{zd}}(R)| \simeq \bigvee_{i=1}^k S^{d_i},$$

where k is the number of critical simplices of the discrete vector field and the dimensions d_i are determined by those critical simplices.

In particular, when R is a product of two fields, the complex is homotopy equivalent to two isolated points, that is S^0 , and when R is a product of $m > 2$ fields, it is a connected complex with two critical simplices (of dimensions 0 and $m - 2$) and therefore homotopy equivalent to S^{m-2} .

This construction illustrates a recurring theme: the initial simplicial complex may be combinatorially large, but its homotopy type can often be obtained by reducing it to a small collection of essential cells. The algebraic information is then reflected in the critical simplices and in the dimensions of the resulting spheres.

3.4. Independence complexes of comaximal graphs and hypergraphs

Since the fourth construction is based on independence complexes of certain graphs and hypergraphs, we first give a definition of an independence complex.

Let Γ be a simple graph with vertex set $V(\Gamma)$. A subset $\sigma \subseteq V(\Gamma)$ is independent if no two distinct vertices of σ are joined by an edge. The independence complex of Γ , denoted by $\text{Ind}(\Gamma)$, is the simplicial complex whose simplices are the independent sets of Γ .

More generally, let H be a hypergraph with vertex set $V(H)$ and hyperedge set $E(H)$. A subset $\sigma \subseteq V(H)$ is independent if it contains no hyperedge of H . The independence complex of H is the simplicial complex whose simplices are independent sets of H .

Now, recall that the comaximal graph $\Gamma(R)$, introduced by Sharma and Bhatwadekar [4], has vertex set R , with two distinct vertices $x, y \in R$ adjacent whenever

$$Rx + Ry = R.$$

Similarly, one may define the comaximal hypergraph $H(R)$, whose hyperedges are finite subsets $\{x_0, \dots, x_n\} \subseteq R$ satisfying

$$Rx_0 + \dots + Rx_n = R.$$

The corresponding independence complexes encode non-generation. In order to avoid trivial cone points, one usually restricts the vertex set by removing units and elements of the Jacobson radical. Thus, one considers $R \setminus (U(R) \cup J(R))$ as the relevant vertex set and denotes the corresponding graph and hypergraph by $\Gamma'_2(R)$ and $H'(R)$, respectively.

For the comaximal hypergraph, the independence complex $\text{Ind } H'(R)$ consists of finite sets which do not generate the entire ring:

$$\{x_0, \dots, x_n\} \in \text{Ind } H'(R) \iff Rx_0 + \dots + Rx_n \neq R.$$

Equivalently, the elements $x_0, \dots, x_n$ are contained in a common maximal ideal.

For the comaximal graph, the independence complex $\text{Ind } \Gamma'_2(R)$ is defined by the pairwise condition

$$\{x_0, \dots, x_n\} \in \text{Ind } \Gamma'_2(R) \iff Rx_i + Rx_j \neq R$$

for all $i \neq j$.

The homotopy types of these complexes were determined in [13]. If R is local, then both complexes are empty since every non-unit lies in the maximal ideal and $J(R)$ is the maximal ideal. If $\text{Max}(R)$ is infinite, then both complexes are contractible:

$$|\text{Ind } H'(R)| \simeq * \quad \text{and} \quad |\text{Ind } \Gamma'_2(R)| \simeq *.$$

For semilocal rings, the hypergraph and graph cases differ. If R is semilocal and $|\text{Max}(R)| = m > 1$, then $|\text{Ind}_{H'(R)}| \simeq \partial\Delta^{m-1} \simeq S^{m-2}$. Thus, the independence complex of the comaximal hypergraph again detects the number of maximal ideals through the dimension of a sphere.

On the other hand, for the comaximal graph one obtains

$$|\text{Ind}_{\Gamma'_2(R)}| \simeq \begin{cases} \text{two discrete points,} & m = 2, \\ *, & m > 2. \end{cases}$$

This comparison shows that the choice of algebraic relation used to define simplices is essential. The comaximal graph records only pairwise generation of the unit ideal, while the comaximal hypergraph records generation by arbitrary finite subsets. As a result, the hypergraph independence complex retains higher-dimensional information which is not visible in the graph case and its homotopy type reflects the number of maximal ideals of the ring.

3.5. Remarks on the ring-theoretic constructions

The constructions discussed in this section show that a single algebraic object, a commutative ring R , may give rise to several different simplicial complexes depending on which algebraic relation is chosen to define the simplices. In the order complex of ideals, simplices are chains under inclusion. In the intersection complex, simplices are families of ideals with non-zero total intersection. In the ideal zero-divisor complex, simplices are families of ideals whose product is non-zero. Finally, in the independence complexes of comaximal graphs and hypergraphs, simplices are determined by failure to generate the unit ideal.

Although these constructions use different algebraic relations, their homotopy types often reflect similar structural features of the ring. At the same time, the different constructions do not encode exactly the same information. The order complex records the poset structure of ideals, the intersection and zero-divisor complexes record different multiplicative or intersection-theoretic relations among ideals, while the comaximal constructions record how elements generate the whole ring. Thus, the choice of simplicial complex determines which part of the ring structure becomes visible topologically.

4. Complexes associated to finite groups

In this section we discuss simplicial complexes associated to finite groups through their irreducible character degrees. Graphs associated to character degrees have been extensively studied as a way of extracting information about the structure of the underlying group; see, for example, the overview [14]. A natural higher-dimensional analogue is obtained by replacing these graphs with simplicial complexes.

This approach was proposed by Isaacs (see [14] Section 12), who suggested studying simplicial complexes associated to character degrees. In particular, one considers the common divisor simplicial complex and the prime divisor simplicial complex. These complexes were later studied by Jensen [15] and Wheeler [16]. Jensen investigated the character degree simplicial complex of a finite group, and Wheeler studied fundamental groups of these complexes. The results discussed below continue this line of work by comparing the two complexes up to homotopy and by studying the fundamental group in the solvable case.

Let G be a finite group and let $\text{Irr}(G)$ denote the set of irreducible complex characters of G . The set of irreducible character degrees is

$$\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}.$$

It is well known that $\text{cd}(G)$ contains important information about the structure of G . A classical way to study this set is through associated graphs, such as the common divisor graph and the prime vertex graph. The simplicial complexes considered below extend these graph constructions by retaining higher-dimensional divisibility information.

4.1. Common divisor and prime divisor complexes

We first define the relevant complexes for an arbitrary set X of positive integers. Let $X^* = X \setminus \{1\}$. The common divisor complex associated to X , denoted by $\mathcal{G}(X)$, is the simplicial complex whose vertex set is X^* , and whose simplices are finite subsets of X^* with non-trivial common divisor. Thus,

$$\sigma \in \mathcal{G}(X) \iff \sigma \subseteq X^*, \sigma \text{ is finite, and } \gcd(\sigma) \neq 1.$$

In particular, the 1-skeleton of $\mathcal{G}(X)$ is the common divisor graph of X .

Let $\pi(x)$ denote the set of prime divisors of a positive integer x , and let $\rho(X) = \bigcup_{x \in X} \pi(x)$ be the set of all primes dividing elements of X . The prime divisor complex associated to X , denoted by $\mathcal{D}(X)$, is the simplicial complex whose vertex set is $\rho(X)$, and whose simplices are finite sets of primes whose product divides some element of X . Equivalently,

$$\tau \in \mathcal{D}(X) \iff \tau \subseteq \rho(X), \tau \text{ is finite, and there exists } x \in X \text{ such that } \prod_{p \in \tau} p \mid x.$$

The 1-skeleton of $\mathcal{D}(X)$ is the prime vertex graph associated to X .

Although the complexes $\mathcal{G}(X)$ and $\mathcal{D}(X)$ have different vertex sets, they encode the same divisibility data from complementary points of view. The common divisor complex records which elements of X share a common prime divisor, while the prime divisor complex records which sets of primes occur simultaneously in some element of X .

A relationship between these two complexes was already observed at the level of fundamental groups by Wheeler [16]. The stronger statement is that the complexes are in fact homotopy equivalent:

$$|\mathcal{G}(X)| \simeq |\mathcal{D}(X)|.$$

This was proved in [17] by considering both complexes as posets and defining order-reversing maps between them. A simplex in $\mathcal{G}(X)$ is sent to the set of prime divisors of the greatest common divisor of its vertices, while a simplex in $\mathcal{D}(X)$ is sent to the set of elements of X divisible by the product of the primes in that simplex. By using Quillen's homotopy property 1.3 (see [18]), these maps induce homotopy equivalences on the corresponding order complexes.

There is also a useful reduction on the common divisor side. An element $x \in X$ is called prime-divisor-maximal if $\pi(x)$ is maximal, with respect to inclusion, among the sets $\pi(y)$ for $y \in X$. If $\Omega \subseteq X$ is a set of representatives of the prime-divisor-maximal classes, then

$$|\mathcal{G}(X)| \simeq |\mathcal{G}(\Omega)|.$$

Thus, one may often replace X by a smaller set without changing the homotopy type of the associated complex.

Example 1. A simple example illustrates the relation between the two complexes. Let $X = \{6, 10, 15\}$. Then, the common divisor complex $\mathcal{G}(X)$ has vertices 6, 10, 15. Any two of these integers have a non-trivial common divisor but $\gcd(6, 10, 15) = 1$. Therefore, $\mathcal{G}(X)$ is the boundary of a 2-simplex, and hence, $|\mathcal{G}(X)| \simeq \mathbb{S}^1$.

On the other hand, $\rho(X) = \{2, 3, 5\}$. The prime divisor complex $\mathcal{D}(X)$ has edges $\{2, 3\}$, $\{2, 5\}$, and $\{3, 5\}$, corresponding respectively to 6, 10, and 15, but it has no 2-simplex because no element of X is divisible by $2 \cdot 3 \cdot 5$. Hence, $\mathcal{D}(X)$ is also the boundary of a 2-simplex, so $|\mathcal{D}(X)| \simeq \mathbb{S}^1$.

This example shows explicitly how the same homotopy type may arise from two different but related descriptions of divisibility data.

4.2. Character degree complexes

We now apply these constructions to the set of irreducible character degrees of a finite group G . We define

$$\mathcal{G}(G) := \mathcal{G}(\text{cd}(G)) \quad \text{and} \quad \mathcal{D}(G) := \mathcal{D}(\text{cd}(G)).$$

The complex $\mathcal{G}(G)$ is the common divisor complex of the character degrees of G , while $\mathcal{D}(G)$ is the corresponding prime divisor complex. The 1-skeleton of $\mathcal{D}(G)$ is the usual character degree graph.

Since $\mathcal{G}(X)$ and $\mathcal{D}(X)$ are homotopy equivalent for every set X of positive integers, we immediately obtain

$$|\mathcal{G}(G)| \simeq |\mathcal{D}(G)|.$$

Therefore, from the point of view of homotopy type, one may work either with character degrees as vertices or with primes as vertices.

This equivalence is useful because the two complexes emphasize different aspects of the same algebraic information. The complex $\mathcal{G}(G)$ keeps the character degrees themselves as vertices and records their common divisors. The complex $\mathcal{D}(G)$ replaces character degrees by their prime divisors and records which primes occur together in a character degree, and passing between these two viewpoints preserves the homotopy type.

4.3. Character degree complexes for solvable groups

The topology of these complexes becomes meaningful when G is solvable. Classical results about character degree graphs of solvable groups impose strong restrictions on the possible shape of $\mathcal{D}(G)$, and hence also on $\mathcal{G}(G)$.

If G is solvable and $\mathcal{D}(G)$ is disconnected, then it has at most two connected components, and these components are simplices. Thus, in the disconnected case, the homotopy type is comparatively simple.

The connected case is more subtle. When $|\rho(\text{cd}(G))| = 1$ the common divisor complex is a cone and the prime divisor complex is a point and hence contractible. When $|\rho(\text{cd}(G))| \geq 2$, the fundamental group of the common divisor complex is free. More precisely, if G is a finite solvable group and $\mathcal{G}(G)$ is connected, then $\pi_1(\mathcal{G}(G))$ is a free group. Since $\mathcal{G}(G)$ and $\mathcal{D}(G)$ are homotopy equivalent, the same conclusion holds for $\mathcal{D}(G)$.

Moreover, the rank of this free group can be bounded in terms of the number of primes which divide irreducible character degrees. So, assuming $|\rho(\text{cd}(G))| \geq 2$,

$$\text{rk } \pi_1(\mathcal{D}(G)) \leq |\rho(\text{cd}(G))| - 2.$$

This shows that the fundamental group of the associated complex is constrained by purely character-theoretic data.

The proof, given in [17], uses a standard presentation of the fundamental group of a connected simplicial complex. One chooses a maximal tree in the 1-skeleton, assigns generators to the remaining edges, and imposes relations coming from 2-simplices. In the solvable case, character-theoretic restrictions imply that the resulting group is free and that many of the potential generators are redundant.

4.4. Remarks on the group-theoretic construction

The constructions in this section differ from the ring-theoretic constructions of the previous section in an important way. In the ring case, the simplices are defined directly from algebraic relations among ideals or elements of the ring. In the group-theoretic case, the group is first represented by an arithmetic invariant, the set $\text{cd}(G)$, and the simplicial complexes are then built from divisibility relations inside this set.

Nevertheless, the same general principle remains: algebraic information is encoded combinatorially, and the homotopy type of the resulting complex reveals structural constraints. In the case of finite solvable groups, the topology of the character degree complexes is not arbitrary; the number of connected components, the freeness of the fundamental group, and bounds on its rank are all controlled by character-theoretic properties of the group.

5. Complexes associated to cyclotomic polynomials

In this section, we discuss simplicial complexes associated to cyclotomic polynomials. The idea for these simplicial complexes originally appeared in the work of Bolker [19] on the transportation problem of higher dimensional rectangular arrays and later in Kalai [20] and Adin [21] on higher-dimensional matrix-tree theorems.

This construction, studied by Musiker and Reiner in [22], gives a topological interpretation of the coefficients of the cyclotomic polynomial. It is especially interesting because, in most cases, the homotopy type of the associated complex reflects the coefficients completely, while in the ternary case additional homotopical phenomena appear.

Let n be a positive integer and let $\Phi_n(x)$ denote the n -th cyclotomic polynomial. Thus, $\Phi_n(x)$ is the minimal polynomial over $\mathbb{Q}$ of a primitive n -th root of unity. It is monic and irreducible, with degree $\varphi(n)$, and can be written as

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j),$$

where ζ is a primitive n -th root of unity.

The coefficients of cyclotomic polynomials are integers and have been studied from many different points of view. The construction described below associates to $\Phi_n(x)$ a family of simplicial complexes whose homology, and in favorable cases homotopy type, encodes these coefficients.

Since the coefficients of $\Phi_n(x)$ for a general n may be reduced to the squarefree case, it is enough to consider $n = p_1 p_2 \cdots p_d$ where $p_1, \dots, p_d$ are distinct primes.

5.1. The Musiker-Reiner construction

For each prime p_i , let K_{p_i} be the 0-dimensional simplicial complex with p_i vertices labeled by the residues

$$0 \bmod p_i, 1 \bmod p_i, \dots, (p_i - 1) \bmod p_i.$$

Consider the simplicial join

$$K_{p_1, \dots, p_d} := K_{p_1} * \dots * K_{p_d}.$$

This is a pure $(d - 1)$ -dimensional simplicial complex. Its facets are labeled by residues $\{j_1 \bmod p_1, \dots, j_d \bmod p_d\}$. By the Chinese remainder theorem, each corresponds to a unique residue modulo n . We denote the corresponding facet by $F_{j \bmod n}$.

For a subset $A \subseteq \{0, 1, \dots, \varphi(n)\}$, let K_A be the subcomplex of $K_{p_1, \dots, p_d}$ generated by the facets $F_{j \bmod n}$, where

$$j \in A \cup \{\varphi(n) + 1, \varphi(n) + 2, \dots, n - 1\}.$$

In particular, $K_\emptyset$ is generated by the facets indexed by residues strictly larger than $\varphi(n)$, while the complexes $K_{\{j\}}$ are obtained from $K_\emptyset$ by adding one additional facet $K_{\{j\}} = K_\emptyset \cup F_{j \bmod n}$. Thus, the transition from $K_\emptyset$ to $K_{\{j\}}$ records the effect of attaching the facet corresponding to the coefficient c_j of $\Phi_n(x)$.

The main point of the construction is that the topology of $K_\emptyset$ and of the complexes $K_{\{j\}}$ reflects the coefficients of the cyclotomic polynomial.

5.2. Homology and coefficients

Write

$$\Phi_n(x) = \sum_{j=0}^{\varphi(n)} c_j x^j,$$

and let $[z_{j \bmod n}]$ denote the oriented boundary of the facet $F_{j \bmod n}$. In [22], Musiker and Reiner proved that

$$\widetilde{H}_*(K_\emptyset) \cong \widetilde{H}_*(\mathbb{S}^{d-2}),$$

and that $\widetilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$ is generated by the cycle $[z_{\varphi(n) \bmod n}]$. Moreover, for every $j = 0, 1, \dots, \varphi(n)$, one has

$$[z_{j \bmod n}] = c_j [z_{\varphi(n) \bmod n}] \quad \text{in} \quad \widetilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}.$$

Thus, the coefficient c_j appears as the degree with which the boundary of the facet $F_{j \bmod n}$ represents the fundamental homology class of $K_\emptyset$.

Equivalently, the homology of $K_{\{j\}}$ is the same as the homology of a space obtained by attaching a $(d - 1)$ -ball to $\mathbb{S}^{d-2}$ by a map of degree c_j :

$$\widetilde{H}_*(K_{\{j\}}) \cong \widetilde{H}_*(\mathbb{B}^{d-1} \cup_{f_j} \mathbb{S}^{d-2}), \quad \deg(f_j) = c_j.$$

Therefore, the coefficients of $\Phi_n(x)$ are recovered from the homology of these associated simplicial complexes.

5.3. From homology to homotopy

The homological statement above has a stronger homotopical form in almost all dimensions. Musiker and Reiner proved in [22] that for $d \geq 4$ and every $A \subseteq \{0, 1, \dots, \varphi(n)\}$ the complex K_A is simply connected. Consequently, for $d \neq 3$, one has

$$K_\emptyset \simeq \mathbb{S}^{d-2} \text{ and } K_{\{j\}} \simeq \mathbb{S}^{d-2} \cup_{f_j} \mathbb{B}^{d-1}, \text{ where } \deg(f_j) = c_j.$$

In these cases, the homotopy type of $K_{\{j\}}$ recovers the magnitude of the coefficient c_j through the degree of the attaching map. Thus, for $d \neq 3$, homology and homotopy are aligned: the homology calculation not only detects the coefficient c_j , but also describes the homotopy type of the corresponding complex.

The exceptional case is $d = 3$, that is, when $n = pqr$ is a product of three distinct primes. In this case, the complexes $K_\emptyset$ and $K_{\{j\}}$ are two-dimensional and the homology computation alone does not determine the homotopy type. Musiker and Reiner asked whether the same homotopical conclusion remains true in this ternary case, is $K_\emptyset \simeq \mathbb{S}^1$, and $K_{\{j\}} \simeq \mathbb{S}^1 \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$?

It turns out that the answer is not uniform: in some ternary cases the expected homotopy type holds, while in others it fails.

5.4. The first counterexample: $n = 105$

The smallest ternary case in which the expected homotopy type fails is $n = 105 = 3 \cdot 5 \cdot 7$. In [23], it was shown that $K_\emptyset$ is not homotopy equivalent to $\mathbb{S}^1$. Although

$$\widetilde{H}_*(K_\emptyset) \cong \widetilde{H}_*(\mathbb{S}^1),$$

the fundamental group of $K_\emptyset$ is not isomorphic to $\mathbb{Z}$.

More precisely, by choosing a maximal tree in the 1-skeleton and using the standard presentation of the fundamental group of a two-dimensional simplicial complex, one obtains

$$\pi_1(K_\emptyset) \cong \langle a, b \mid ab^2a^{-1}b^{-2}a^{-1}bab^{-1}a^{-1}b^{-1} = 1 \rangle.$$

This group has abelianization $\mathbb{Z}$, as expected from the homology computation. However, the group itself is non-abelian. For example, one may construct an epimorphism from this group onto the non-abelian group S_3 . Hence,

$$\pi_1(K_\emptyset) \not\cong \mathbb{Z}, \text{ and therefore } K_\emptyset \not\simeq \mathbb{S}^1.$$

The same phenomenon occurs for a complex obtained by adding one facet. For $j = 7$, the complex $K_{\{7\}}$ has the homology predicted by the coefficient $c_7 = -2$, but its fundamental group is not isomorphic to $\mathbb{Z}_2$. Therefore, $K_{\{7\}}$ is not homotopy equivalent to the expected space $\mathbb{S}^1 \cup_{f_7} \mathbb{B}^2$, where $\deg(f_7) = -2$.

This example is important because it shows that the homological interpretation of the coefficients remains valid, but the homotopy type can carry more information.

In addition to $n = 105$, the reference [23] also considers the case $n = 385 = 5 \cdot 7 \cdot 11$ which exhibits nontrivial behavior. There, the value $|c_{119}|$ of $\Phi_{385}(x)$ equals 3, so the expected fundamental group of $K_{\{119\}}$ would be $\mathbb{Z}_3$. Nevertheless, a more delicate analysis using Fox calculus and elementary ideals shows that the corresponding fundamental group is not isomorphic to $\mathbb{Z}_3$. Similarly, the fundamental group of $K_\emptyset$ is not isomorphic to $\mathbb{Z}$.

5.5. The family $n = 3 \cdot 5 \cdot p$

A more systematic analysis of the ternary case was carried out in [24] for $n = 3 \cdot 5 \cdot p$, where $p \geq 7$ is a prime. In this family, the homotopy type depends on the residue class of p modulo 15.

The positive cases are

$$p \equiv 1, 2, 13, 14 \pmod{15}.$$

For these congruence classes, one has $K_\emptyset \simeq \mathbb{S}^1$. Moreover, for every $j = 0, 1, \dots, \varphi(n)$, we have $K_{\{j\}} \simeq \mathbb{S}^1 \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$.

The negative cases are

$$p \equiv 4, 7, 8, 11 \pmod{15}.$$

For these congruence classes, $K_\emptyset \neq \mathbb{S}^1$.

The case $p = 7$, corresponding to $n = 105$, was the first counterexample obtained in [23], and the remaining congruence classes in this family are treated in [24] where the proof uses discrete Morse theory. The complex $K_\emptyset$ is decomposed into a union of star subcomplexes. For $n = 3 \cdot 5 \cdot p$, one can write

$$K_\emptyset = \bigcup_{i=8}^{p+7} \text{St}(i),$$

after labeling the vertices in a convenient way. The stars $\text{St}(8), \dots, \text{St}(p)$ admit acyclic discrete vector fields with no critical 2-simplices. Therefore, the main difficulty is transferred to the smaller subcomplex

$$\bigcup_{i=p+1}^{p+7} \text{St}(i).$$

For primes in the positive congruence classes, one constructs acyclic discrete vector fields whose only critical cells are one 0-simplex and one 1-simplex. By Forman's theorem (see [7, 8]), the complex is then homotopy equivalent to a CW complex with one 0-cell and one 1-cell. Since the homology is already known to be that of $\mathbb{S}^1$, this gives $K_\emptyset \simeq \mathbb{S}^1$.

For primes in the negative congruence classes, the complex $K_\emptyset$ is shown to have a non-cyclic or non-abelian fundamental group. In several cases, the resulting fundamental groups are related to Baumslag-Solitar groups or to the group obtained in the case $n = 105$. Hence, the complex cannot be homotopy equivalent to $\mathbb{S}^1$.

Thus, the family $n = 3 \cdot 5 \cdot p$ shows that the homotopy type of the cyclotomic complex is sensitive to arithmetic congruence conditions on the prime p .

5.6. Further positive results for $n = pqr$

The previous results show that the expected homotopy type does not always hold in the ternary case. Nevertheless, there are broad classes of ternary cyclotomic polynomials for which the expected homotopy type does hold.

Let $n = pqr$, where $p < q < r$ are primes. If

$$r \equiv \pm 1 \pmod{pq},$$

then $K_\emptyset \simeq \mathbb{S}^1$. Moreover, for every $j = 0, 1, \dots, \varphi(n)$, $K_{\{j\}} \simeq \mathbb{S}^1 \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$.

This was proved in [25] and the proof is based on simplifying the complex without changing its homotopy type. When r is sufficiently large compared to p and q , and in particular when $r \equiv \pm 1 \pmod{pq}$, many facets of K_0 can be removed by elementary collapses. After these collapses, the remaining complex decomposes into star subcomplexes whose structure can be controlled explicitly. The congruence condition determines the pattern in which these stars intersect and allows one to reduce the complex to a one-dimensional complex with the homotopy type of $\mathbb{S}^1$.

This result gives a positive answer to the Musiker-Reiner question for a large class of ternary cyclotomic polynomials.

5.7. Remarks on the polynomial construction

The cyclotomic construction differs from the ring and group constructions discussed in the previous sections. In the ring case, simplices are defined directly by relations among ideals or elements of the ring, while in the group case, simplices are defined by divisibility relations among character degrees. Here, the simplicial complex is built from residue data associated to the index n of the cyclotomic polynomial, and the topology of the complex reflects the coefficients of $\Phi_n(x)$.

This makes the cyclotomic example useful for illustrating the distinction between homological and homotopical information. The Musiker-Reiner theorem shows that the homology of the complexes K_0 and $K_{[j]}$ is closely related to the coefficients of $\Phi_n(x)$. In the ternary case, however, the homotopy type need not be determined by this homological information alone. In particular, the fundamental group may distinguish complexes which have the expected homology but are not homotopy equivalent to the expected spaces.

6. Conclusions and perspectives

The examples discussed in this paper show that simplicial complexes provide a useful and flexible way of translating algebraic information into topology. Although commutative rings, finite groups, and cyclotomic polynomials are different types of algebraic objects, the constructions considered here follow a common pattern: one begins with an algebraic relation, encodes finite collections satisfying that relation as simplices, and then studies the homotopy type of the resulting complex.

A natural direction for further research is to associate simplicial complexes to algebraic objects in other meaningful ways. Many algebraic structures come with natural families of subobjects, invariants, or relations among finite collections of elements. Whenever such relations are genuinely higher-dimensional, rather than only pairwise, studying the associated simplicial complex may be suitable. One may therefore look for new constructions associated to groups, modules, non-commutative rings, semigroups, or other algebraic objects, with the aim of identifying which algebraic properties are reflected in the homotopy type of the resulting complex.

Another possible direction is to revisit existing graph constructions in algebra and ask whether they admit natural higher-dimensional analogues as passing from such graphs to simplicial complexes may give additional information. The examples reviewed here suggest that this approach can have meaningful interpretation and provide a useful bridge between algebra, combinatorics, and topology.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999), 434–447. <https://doi.org/10.1006/jabr.1998.7840>
2. R. Akhtar, L. Lee, Homology of zero-divisors, *Rocky Mountain J. Math.*, **37** (2007), 1105–1126. <https://doi.org/10.1216/rmjm/1187453099>
3. I. Chakrabarty, S. Ghosh, T. K. Mukherjee, M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.*, **309** (2009), 5381–5392. <https://doi.org/10.1016/j.disc.2008.11.034>
4. P. K. Sharma, S. M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra*, **176** (1995), 124–127. <https://doi.org/10.1006/jabr.1995.1236>
5. D. Kozlov, *Combinatorial Algebraic Topology*, 1st edition, Springer, 2008. <https://doi.org/10.1007/978-3-540-71962-5>
6. J. R. Munkres, *Elements of Algebraic Topology*, 1st edition, CRC Press, 1984. <https://doi.org/10.1201/9780429493911>
7. R. Forman, Morse theory for cell complexes, *Adv. Math.*, **134** (1998), 90–145. <https://doi.org/10.1006/aima.1997.1650>
8. R. Forman, A user's guide to discrete Morse theory, *Semin. Lotharingien Comb.*, **48** (2002), 1–35.
9. V. A. Vassiliev, Topology of discriminants and their complements, in *Proceedings of the International Congress of Mathematicians*, (1995), 209–226. https://doi.org/10.1007/978-3-0348-9078-6_16
10. N. Milošević, Z. Z. Petrović, Order complex of ideals in a commutative ring with identity, *Czech. Math. J.*, **65** (2015), 947–952. <https://doi.org/10.1007/s10587-015-0219-9>
11. N. Milošević, Kompleks presjeka ideala, in *Symposium Mathematics and Applications*, (2015), 48–51.
12. N. Milošević, Z. Z. Petrović, Ideal zero-divisor complex, *J. Commut. Algebra*, **9** (2017), 243–261. <https://doi.org/10.1216/JCA-2017-9-2-243>
13. N. Milošević, Independence complexes of comaximal graphs of commutative rings with identity, *Publ. Inst. Math.*, **98** (2015), 109–117.
14. M. L. Lewis, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, *Rocky Mountain J. Math.*, **38** (2008), 175–211. <https://doi.org/10.1216/RMJ-2008-38-1-175>

15. S. Jensen, The character degree simplicial complex of a finite group, *J. Algebra*, **440** (2015), 33–48. <https://doi.org/10.1016/j.jalgebra.2015.06.001>
16. E. Wheeler, Fundamental groups of simplicial complexes, *J. Algebra*, **478** (2017), 283–295. <https://doi.org/10.1016/j.jalgebra.2017.01.023>
17. A. K. Matijević, N. Milošević, Z. Z. Petrović, Simplicial complexes associated to character degrees of solvable groups, *J. Algebra Appl.*, (2025), 2650231. <https://doi.org/10.1142/S0219498826502312>
18. D. Quillen, Homotopy properties of the poset of nontrivial p -subgroups of a group, *Adv. Math.*, **28** (1978), 101–128. [https://doi.org/10.1016/0001-8708\(78\)90058-0](https://doi.org/10.1016/0001-8708(78)90058-0)
19. E. D. Bolker, Simplicial geometry and transportation polytopes, *Trans. Am. Math. Soc.*, **217** (1976), 121–142. <https://doi.org/10.2307/1997562>
20. G. Kalai, Enumeration of $\mathbb{Q}$ -acyclic simplicial complexes, *Israel J. Math.*, **45** (1983), 337–351. <https://doi.org/10.1007/BF02804017>
21. R. M. Adin, Counting colorful multi-dimensional trees, *Combinatorica*, **12** (1992), 247–260. <https://doi.org/10.1007/BF01285814>
22. G. Musiker, V. Reiner, The cyclotomic polynomial topologically, *J. Reine Angew. Math.*, **687** (2014), 113–132. <https://doi.org/10.1515/crelle-2012-0051>
23. A. Kostić, N. Milošević, Z. Z. Petrović, Note on the cyclotomic polynomial topologically, *Exp. Math.*, **31** (2022), 669–675. <https://doi.org/10.1080/10586458.2019.1680464>
24. A. Kostić, On the simplicial complexes associated to the cyclotomic polynomial, *Kragujevac J. Math.*, **47** (2023), 309–329. <https://doi.org/10.46793/KgJMat2302.309K>
25. A. Kostić, N. Milošević, Z. Z. Petrović, Topological analysis of ternary cyclotomic polynomials, *Kragujevac J. Math.*, **51** (2027), 1321–1335.



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