



Research article

Control and bifurcation analysis of a discrete predator-prey model with Holling type-IV and Leslie-Gower dynamics

Zhenxin Zhong and Wenlong Wang*

College of Science, Northeast Forestry University, Harbin 150040, China

* **Correspondence:** Email: wwl@nefu.edu.cn.

Abstract: This paper investigates the dynamical behavior of a Euler-generated discrete predator-prey map incorporating a Holling type-IV functional response and Leslie-Gower predator dynamics. Since the Leslie-Gower term is singular at zero prey density, the biologically feasible state space is restricted to $x > 0$ and $y \geq 0$. Within this domain, the existence and local stability of the boundary and positive equilibria are analyzed, and sufficient conditions for the possible configurations of positive equilibria are derived from the discriminant of the associated cubic equation. The positivity of the uncontrolled and controlled maps is also examined through analytical one-step conditions and direct numerical feasibility checks. By employing center manifold reduction and normal form theory, analytical conditions for Flip and Neimark-Sacker bifurcations at the positive equilibrium are obtained. Two stabilization approaches are then investigated. The state-feedback scheme yields an explicit triangular stability region in the gain-parameter plane, whereas the weighted hybrid scheme is equivalent to replacing the Euler step size h by an effective step size $h_{\text{eff}} = \omega h$. Numerical experiments, including convergence tests, bifurcation diagrams, maximum Lyapunov exponent (MLE) spectra, time series, and phase portraits, support the theoretical analysis and illustrate the transition among stable, periodic, quasi-periodic, and chaotic dynamics of the Euler-generated map.

Keywords: discrete predator-prey model; Holling type-IV; Leslie-Gower; bifurcation analysis; chaos control

1. Introduction

Crop losses caused by arthropod pests and plant diseases remain a major constraint on food production. Food and Agriculture Organization (FAO) assessments indicate that these biological pressures affect a substantial fraction of global crop output each year [1]. Plant pests and pathogens impose substantial economic burdens on agricultural production, particularly for major cereal crops [1, 2]. The two-spotted spider mite, *Tetranychus urticae*, represents a persistent challenge in

crop management due to its extensive host range, rapid population increase, and capacity to develop acaricide resistance, which limits the long-term effectiveness of chemical control strategies [3]. These constraints have promoted the application of phytoseiid predatory mites in orchard systems and protected cultivation environments [4]. Recent studies have further investigated alternative mite-predator associations for specific crops [5], accompanied by the expansion of commercial production of biological control agents [6].

A mathematical description of pest–natural-enemy regulation must represent two features relevant to mite systems. First, predation efficiency may decline after prey density becomes sufficiently high; this nonmonotone effect is represented by the Holling type-IV response. Second, the number of predators that can be supported changes with prey availability, which motivates the Leslie-Gower term. After numerical discretization, variations of the Euler step size may alter the stability of the resulting discrete map and generate periodic, quasi-periodic, or irregular dynamics.

Control-oriented predator-prey studies have considered several intervention mechanisms. The model proposed in [7] combined a nonmonotone response with feedback regulation, the work in [8] investigated impulsive intervention in a Leslie-Gower framework, and the study in [9] incorporated pesticide-induced effects on both interacting populations. Together, these investigations demonstrate that management-related terms can significantly reshape the stability boundaries of pest–natural-enemy models. However, the comparison of two control schemes developed for the discrete map considered here has not been addressed in these studies.

Functional responses distinguish predator-prey models by describing the relationship between prey density and the consumption rate of predators [10–12]. Conventional monotone responses are insufficient for situations where predation efficiency increases initially but decreases when prey density becomes excessively high. To describe this inhibitory mechanism, we adopt the Holling type-IV functional response, which is expressed as

$$p(x) = \frac{mx}{b + x^2}, \quad (1.1)$$

whose maximum occurs at an intermediate prey density. A broad class of predator-prey models can be constructed by introducing different forms of growth, loss, and interaction functions to describe species relationships. Consider the general system (1.2),

$$\begin{cases} \frac{dx}{dt} = H(x) - F(x, y), \\ \frac{dy}{dt} = \sigma F(x, y) - G(y), \end{cases} \quad (1.2)$$

where $H(x)$ describes the prey growth process in the absence of predators, $F(x, y)$ characterizes the reduction of prey caused by predator-prey interactions, and $G(y)$ represents the loss of the predator population. By selecting different forms of these functions, the system can reproduce classical predator-prey models and incorporate additional nonlinear ecological mechanisms [13–18].

One important extension is the Leslie-Gower predator-prey model, which assumes that the predator carrying capacity is proportional to the prey population. The Leslie-Gower model can be written as

$$\begin{cases} \frac{dx}{dt} = H(x) - F(x, y), \\ \frac{dy}{dt} = \gamma y \left(1 - \frac{y}{\delta x}\right), \end{cases} \quad (1.3)$$

where γ denotes the intrinsic growth rate of predators and δ represents the conversion efficiency parameter. Because of its biological relevance, the Leslie-Gower model (1.3) has been widely studied in the literature [19–22], and several ecological mechanisms such as prey refuge, ratio dependence, predator interference, and generalized functional responses have been incorporated into predator-prey models to provide more realistic descriptions of ecological interactions [23–25]. Independently, the Leslie-Gower formulation, which ties the predator carrying capacity to prey abundance, provides a biologically realistic density-dependent mechanism and has been extended to incorporate prey refuges, ratio dependence, and predator interference [26, 27]. Despite this progress, most continuous models treat either a Holling type-IV response or Leslie-Gower dynamics in isolation. A rigorous stability and bifurcation analysis of the combined continuous system is available only in a limited number of studies, while its discrete-time counterpart has not been systematically investigated.

Another important feature of predator-prey systems is that simple ecological interactions can generate complex nonlinear dynamics. Early investigations revealed that deterministic population models may produce irregular behaviors even when their mathematical structures are relatively simple. May showed that nonlinear ecological models can exhibit complicated dynamical responses [28], and Hastings and Powell later demonstrated chaotic behavior in ecological food-chain systems [29]. In nonlinear dynamics, the classical theorem of Li and Yorke showed that the existence of period-three cycles implies chaotic dynamics [30]. Subsequent studies explored chaos generation and bifurcation mechanisms in predator-prey models and related ecological systems [31–33]. Meanwhile, various approaches for regulating chaotic states in nonlinear systems have been developed, including the classical Ott–Grebogi–Yorke (OGY) method [34]. Beyond ecological applications, similar complex dynamics have also been reported in other discrete nonlinear systems. Wei et al. [35] investigated a two-dimensional discrete Cournot map and established the occurrence of Flip and Neimark-Sacker bifurcations together with chaotic dynamics, further illustrating the broad relevance of bifurcation and chaos analysis in nonlinear discrete maps.

Compared with continuous-time formulations, discrete-time predator-prey models provide a natural description for populations with seasonal reproduction, separated generations, or periodically sampled observations [36–38]. Owing to the iterative nature of the mapping process, discrete models can display a variety of dynamical transitions, including period-doubling (Flip) bifurcations, Neimark-Sacker bifurcations, quasi-periodic motion, and chaotic attractors [39–43]. Recent discrete ecological models have incorporated additional biological factors, such as random disturbances, fear effects, and environmental fluctuations, to improve ecological realism [44–47]. These developments indicate that introducing realistic mechanisms may significantly enrich the underlying nonlinear behaviors.

From the perspective of nonlinear dynamics, discrete ecological systems can also be viewed as multidimensional discrete maps, in which nonlinear interactions among state variables may generate complex iterative dynamics. Bucolo et al. [48] investigated multidimensional discrete chaotic maps and highlighted the rich dynamical structures that can arise in nonlinear multidimensional mappings. In this context, system (1.5) constitutes a two-dimensional nonlinear discrete map with biologically

motivated interactions between prey and predator populations. The present analysis may therefore also provide a basis for extending this framework to higher-dimensional ecological models involving multiple interacting populations or additional state variables.

Despite these advances, the discrete Leslie-Gower predator-prey model combined with a Holling type-IV functional response remains insufficiently explored. Existing studies have rarely provided a complete analytical framework covering the existence and stability properties of all equilibria, explicit bifurcation boundaries, and corresponding bifurcation criteria for this model class. Moreover, although Holling-type functional responses and Leslie-Gower predator dynamics have been extensively studied, most investigations have focused on continuous-time systems. Therefore, a comprehensive analysis that integrates discrete modeling, bifurcation theory, and control strategies for the Leslie-Gower predator-prey system with a Holling type-IV functional response is still needed.

Controlling oscillatory and chaotic dynamics arising from Flip and Neimark-Sacker bifurcations are an important topic in nonlinear discrete systems. Stability analysis and chaos suppression have also been investigated in a variety of nonlinear dynamical models. For example, Wei et al. [49] studied the Jacobi stability of a five-dimensional self-exciting homopolar disc dynamo and demonstrated the suppression of complex hidden-attractor dynamics by impulsive control. A range of control approaches has been developed, including the classical OGY method [34], state feedback control, weighted control schemes, and impulsive control. In ecological models, Huang [7] analyzed feedback control under anti-predator behavior, Li et al. [8] studied impulsive control for a Leslie-Gower system, and Dai et al. [9] examined pesticide sublethal effects within a predator-prey framework. Nevertheless, comparatively few studies have examined state feedback and weighted hybrid stabilization within the same discrete predator-prey framework. This motivates the comparative control analysis developed in the present study.

Inspired by these previous investigations, we introduce a continuous predator-prey model incorporating the Holling type-IV functional response and Leslie-Gower predator regulation. The model is formulated as

$$\begin{cases} \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{mx}{b + x^2}y, \\ \frac{dy}{dt} = sy\left(1 - \frac{y}{\theta x}\right), \end{cases} \quad (1.4)$$

where $x(t)$ and $y(t)$ describe the prey and predator densities at time t , respectively. The parameters r and s correspond to the growth coefficients of prey and predator populations, k represents the maximum environmental support level of prey, and θ controls the prey-dependent carrying capacity term in the Leslie-Gower formulation.

To investigate the discrete dynamics associated with system (1.4), we apply the forward Euler scheme with step size $h > 0$. Here, h is treated as the discretization step and subsequently as a bifurcation parameter of the resulting discrete map. It should not be interpreted directly as a biological generation time, sampling interval, or management cycle. For sufficiently small h , the Euler map provides a first-order approximation to the continuous system, whereas increasing h may alter the stability properties of the discretized system and generate dynamics that are specific to the discrete map. The resulting discrete system is given by

$$\begin{cases} x_{n+1} = x_n + h \left[rx_n \left(1 - \frac{x_n}{k} \right) - \frac{mx_n}{b + x_n^2} y_n \right], \\ y_{n+1} = y_n + h \left[sy_n \left(1 - \frac{y_n}{\theta x_n} \right) \right], \end{cases} \quad (1.5)$$

where $h > 0$ denotes the forward Euler discretization step. The local truncation error of the forward Euler scheme is of order $O(h^2)$, while the accumulated global error is of order $O(h)$. In addition, if μ_i denotes an eigenvalue of the Jacobian matrix of the continuous system at an equilibrium, the corresponding multiplier of the Euler map is $\lambda_i = 1 + h\mu_i$. Therefore, the local stability and bifurcation boundaries of the discrete system depend explicitly on h .

Positivity of the discrete map. Since the Leslie-Gower term is singular at $x = 0$, the biological feasibility of the Euler-generated map requires explicit consideration. Define the biologically feasible domain as

$$\Omega = \{(x, y) : x > 0, y \geq 0\}.$$

For $(x_n, y_n) \in \Omega$, system (1.5) can be rewritten in the multiplicative form

$$x_{n+1} = x_n \Phi_h(x_n, y_n), \quad y_{n+1} = y_n \Psi_h(x_n, y_n),$$

where

$$\Phi_h(x, y) = 1 + h \left[r \left(1 - \frac{x}{k} \right) - \frac{my}{b + x^2} \right],$$

and

$$\Psi_h(x, y) = 1 + hs \left(1 - \frac{y}{\theta x} \right).$$

Therefore, for a state $(x_n, y_n) \in \Omega$, a sufficient condition for the next iterate to remain biologically feasible is

$$\Phi_h(x_n, y_n) > 0, \quad \Psi_h(x_n, y_n) \geq 0.$$

These inequalities are equivalent to

$$\frac{my_n}{b + x_n^2} < \frac{1}{h} + r \left(1 - \frac{x_n}{k} \right),$$

and

$$y_n \leq \theta x_n \left(1 + \frac{1}{hs} \right).$$

Since the left-hand side of the first inequality is nonnegative, its right-hand side must be positive, which further requires

$$0 < x_n < k \left(1 + \frac{1}{hr} \right).$$

Consequently, a sufficient one-step feasible set can be defined as

$$\mathcal{D}_h = \left\{ (x, y) \in \Omega : 0 < x < k \left(1 + \frac{1}{hr} \right), \quad 0 \leq y \leq \theta x \left(1 + \frac{1}{hs} \right), \quad y < \frac{b + x^2}{m} \left[\frac{1}{h} + r \left(1 - \frac{x}{k} \right) \right] \right\}.$$

For every $(x_n, y_n) \in \mathcal{D}_h$, the next iterate satisfies

$$x_{n+1} > 0, \quad y_{n+1} \geq 0.$$

Hence, if the above feasibility conditions are satisfied at every iteration, the corresponding trajectory remains in Ω .

It should be emphasized that the whole positive quadrant Ω is not assumed to be forward invariant for arbitrary values of the Euler step size h . In particular, sufficiently large values of h may violate the above inequalities. Therefore, in addition to the analytical one-step conditions, the numerical simulations reported in Section 6 are checked directly for $x_n > 0$ and $y_n \geq 0$ without applying any clipping, projection, or artificial replacement of negative state values.

This study investigates a discrete-time Leslie-Gower predator-prey model incorporating a Holling type-IV functional response system (1.5), which is derived from the continuous system (1.4) by applying the forward Euler scheme. The forward Euler discretization of the continuous predator-prey model provides a two-dimensional nonlinear map in which discretization-induced changes in equilibrium stability, bifurcation, and control can be investigated explicitly. Accordingly, system (1.5) is treated in this study as a Euler-generated discrete dynamical map rather than as an independently derived generational ecological model. Explicit conditions are derived for the existence of positive equilibria, while the local dynamics of the boundary equilibrium E_1 and the positive equilibrium E^* are analytically characterized. Furthermore, the critical conditions for the occurrence of Flip bifurcation and Neimark-Sacker bifurcation at E^* are obtained, together with closed-form expressions for the bifurcation discriminants θ_1 and θ_2 for Flip and L for Neimark-Sacker, which determine the bifurcation direction and amplitude. These analytical findings provide quantitative criteria for identifying transitions associated with pest population fluctuations, offering additional insights into the dynamics of discrete ecological models. To regulate the complex dynamics induced by bifurcations, two stabilization approaches are investigated: state feedback control and weighted hybrid control. For the state-feedback scheme, a triangular stability region is obtained analytically in the (ω_1, ω_2) gain plane. For the hybrid scheme, the weighting parameter $\omega \in (0, 1)$ modifies the effective Euler step size according to $h_{\text{eff}} = \omega h$. Numerical bifurcation diagrams and MLE spectra are used to compare their stabilization characteristics within the Euler-generated discrete map.

The remaining sections are arranged as follows. Section 2 presents the model derivation and investigates the existence and local stability of the boundary and positive equilibria of the discrete model. Section 3 derives the existence conditions and the normal form of the Flip bifurcation at the positive equilibrium, and Section 4 carries out the corresponding analysis for the Neimark-Sacker bifurcation. Section 5 designs two control strategies-state feedback control and hybrid control-to suppress the chaotic and oscillatory dynamics induced by these bifurcations and characterizes the resulting stability regions. Section 6 provides numerical simulations, including bifurcation diagrams, MLE spectra, and phase portraits, to validate the theoretical results. Finally, Section 7 summarizes the main conclusions of the paper.

2. System formulation and stability analysis of equilibrium points

First, we determine the equilibrium points of system (1.5) and classify their corresponding types. The Jacobian matrix of system (1.5) evaluated at an arbitrary equilibrium point $E(x, y)$ is given by

$$J(x, y) = \begin{pmatrix} 1 + h \left(r - \frac{2xr}{k} - \frac{my(b-x^2)}{(b+x^2)^2} \right) & -h \frac{mx}{b+x^2} \\ h \frac{sy^2}{\theta x^2} & 1 + h \left(s - \frac{2sy}{\theta x} \right) \end{pmatrix}. \quad (2.1)$$

Solving the equilibrium equations of system (1.5) gives the boundary equilibrium $E_1(k, 0)$ and the positive equilibrium $E^*(x^*, y^*)$.

2.1. Existence and stability of the boundary equilibrium

Let λ_1 and λ_2 be the two eigenvalues of the characteristic equation associated with the Jacobian matrix (2.1). Substituting each equilibrium point into the Jacobian matrix yields the following theorem.

Theorem 1. For the boundary equilibrium $E_1(k, 0)$, the eigenvalues are $\lambda_1 = 1 - hr$ and $\lambda_2 = 1 + hs$, and the equilibrium is unstable.

2.2. Positive equilibria and their types

2.2.1. The existence analysis of the positive equilibrium point

The positive equilibrium $E^*(x^*, y^*)$ of model (1.5) must satisfy the following relations:

$$y^* = \theta x^*, \quad x^3 - kx^2 + \left(\frac{km\theta}{r} + b \right) x - kb = 0.$$

Denote:

$$A = k, \quad B = \frac{km\theta}{r} + b, \quad C = kb.$$

With these notations, the equilibrium condition can be rewritten as the following cubic equation:

$$f(x) = x^3 - Ax^2 + Bx - C = 0. \quad (2.2)$$

For the cubic Eq (2.2), define its discriminant by

$$\Delta = A^2B^2 - 4B^3 - 4A^3C - 27C^2 + 18ABC.$$

The sign of Δ determines the structure of the real roots: $\Delta > 0$ corresponds to three distinct real roots, $\Delta = 0$ to multiple real roots, and $\Delta < 0$ to one real root and a pair of complex conjugate roots.

Through calculations, we can obtain the following conclusions.

Lemma 1. Let

$$f(x) = x^3 - Ax^2 + Bx - C,$$

where

$$A = k, \quad B = \frac{km\theta}{r} + b, \quad C = kb.$$

Then the positive equilibria of system (1.5) can be classified according to the discriminant Δ of $f(x)$ as follows:

(i) If $\Delta < 0$, Eq (2.2) has exactly one real root x^* , which satisfies

$$0 < x^* < k.$$

Hence, system (1.5) has a unique positive equilibrium

$$E^*(x^*, \theta x^*).$$

(ii) If

$$\Delta = 0, \quad B = \frac{A^2}{3}, \quad C = \frac{A^3}{27},$$

then Eq (2.2) has a triple positive root

$$x_0 = \frac{A}{3} = \frac{k}{3}.$$

Hence, system (1.5) has one distinct positive equilibrium

$$E^*(x_0, \theta x_0),$$

with algebraic multiplicity three.

(iii) If

$$\Delta = 0, \quad B < \frac{A^2}{3},$$

then Eq (2.2) has two distinct positive roots, one of which is a double root. Therefore, system (1.5) has two distinct positive equilibria

$$E_1^* = (x_1^*, \theta x_1^*), \quad E_2^* = (x_2^*, \theta x_2^*),$$

where

$$0 < x_1^* < x_2^* < k.$$

(iv) If $\Delta > 0$, Eq (2.2) has three distinct positive roots

$$0 < x_1^* < x_2^* < x_3^* < k.$$

Hence, system (1.5) has three distinct positive equilibria

$$E_i^* = (x_i^*, \theta x_i^*), \quad i = 1, 2, 3.$$

Remark 1. From the ecological viewpoint, the dynamical behavior near the unique positive equilibrium is of particular interest. Since $f(0) = -C = -kb < 0$ and $f(k) = k^3 - Ak^2 + Bk - C = \frac{k^2 m \theta}{r} > 0$, the equation has at least one positive root in $(0, k)$. To guarantee the uniqueness of the positive equilibrium, the parameters are assumed to satisfy the conditions in case (i). In the following analysis, these conditions are adopted and the unique positive root is denoted by x^* .

For convenience, denote the unique positive root as x^* . Then the positive equilibrium can be written as:

$$E^*(x^*, y^*) = (x^*, \theta x^*).$$

2.2.2. The stability analysis of the unique positive equilibrium $E^*(x^*, y^*)$

The Jacobian matrix evaluated at the positive equilibrium E^* is given by:

$$J(E^*) = \begin{pmatrix} 1 + h \left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) & -h \frac{mx^*}{b + (x^*)^2} \\ h \frac{s(y^*)^2}{\theta(x^*)^2} = hs\theta & 1 + h \left(s - \frac{2sy^*}{\theta x^*} \right) = 1 - hs \end{pmatrix}.$$

Using the relation $y^* = \theta x^*$, the above matrix can be rewritten as:

$$J(E^*) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where:

$$\begin{aligned} a_{11} &= 1 + h \left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) \\ a_{12} &= -\frac{hmx^*}{b + (x^*)^2} \\ a_{21} &= hs\theta \\ a_{22} &= 1 - hs \end{aligned}$$

The corresponding characteristic polynomial of $J(E^*)$ is expressed as:

$$F(\lambda) = \lambda^2 - P\lambda + Q = 0 \quad (2.3)$$

where:

$$\begin{aligned} P &= \text{tr}(J(E^*)) = a_{11} + a_{22} \\ Q &= \det(J(E^*)) = a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

Let λ_1, λ_2 denote the eigenvalues of $J(E^*)$. Then:

$$\lambda_1 + \lambda_2 = P$$

$$\lambda_1 \lambda_2 = Q$$

Moreover, we have $F(1) = 1 - P + Q$, $F(-1) = 1 + P + Q$.

The following standard stability criterion for a second-order discrete system is recalled from [50].

Lemma 2. Let

$$F(\lambda) = \lambda^2 - P\lambda + Q$$

be the characteristic polynomial with roots λ_1 and λ_2 . According to the Jury stability criterion [50], the following conclusions hold:

(A₁) Both eigenvalues lie strictly inside the unit circle, that is, $|\lambda_1| < 1$ and $|\lambda_2| < 1$, if and only if

$$1 - P + Q > 0, \quad 1 + P + Q > 0, \quad 1 - Q > 0.$$

(A₂) Both eigenvalues lie strictly outside the unit circle, that is, $|\lambda_1| > 1$ and $|\lambda_2| > 1$, if and only if

$$|Q| > 1, \quad Q(1 - P + Q) > 0, \quad Q(1 + P + Q) > 0.$$

(A₃) One eigenvalue lies inside and the other lies outside the unit circle if and only if

$$(1 - P + Q)(1 + P + Q) < 0.$$

(A₄) The eigenvalues form a pair of complex conjugate roots on the unit circle if and only if

$$Q = 1, \quad -2 < P < 2.$$

Based on the above Jury criterion, the local dynamical properties of the unique positive equilibrium $E^*(x^*, y^*)$ are summarized as follows.

Theorem 2. Assume that the parameters satisfy the conditions in case (i) of Lemma 1. Let λ_1 and λ_2 denote the eigenvalues of the characteristic equation $F(\lambda) = 0$. Then the following properties hold:

(A₁) The positive equilibrium $E^*(x^*, y^*)$ is a sink, and hence locally asymptotically stable, if and only if

$$1 - P + Q > 0, \quad 1 + P + Q > 0, \quad 1 - Q > 0.$$

Equivalently,

$$-(1 + Q) < P < 1 + Q, \quad Q < 1.$$

(A₂) The positive equilibrium $E^*(x^*, y^*)$ is a source if and only if

$$|Q| > 1, \quad Q(1 - P + Q) > 0, \quad Q(1 + P + Q) > 0.$$

(A₃) The positive equilibrium $E^*(x^*, y^*)$ is a saddle if and only if

$$(1 - P + Q)(1 + P + Q) < 0.$$

The equality cases correspond to nonhyperbolic stability boundaries. In particular,

$$F(1) = 1 - P + Q = 0$$

implies that one eigenvalue is $\lambda = 1$, while

$$F(-1) = 1 + P + Q = 0$$

implies that one eigenvalue is $\lambda = -1$. Moreover, when

$$Q = 1, \quad |P| < 2,$$

the eigenvalues form a complex conjugate pair on the unit circle.

3. Flip bifurcation analysis of equilibrium points

3.1. Existence conditions for flip bifurcation

To investigate the Flip bifurcation near the positive equilibrium $E^*(x^*, y^*)$, we introduce the translated variables $\bar{x}_{n+1} = x_{n+1} - x^*$ and $\bar{y}_{n+1} = y_{n+1} - y^*$, where h is regarded as the bifurcation parameter.

Under this transformation, system (1.5) becomes:

$$\begin{cases} \bar{x}_{n+1} = \bar{x}_n + h \left[r(\bar{x}_n + x^*) \left(1 - \frac{\bar{x}_n + x^*}{k} \right) - (\bar{y}_n + y^*) \frac{m(\bar{x}_n + x^*)}{b + (\bar{x}_n + x^*)^2} \right], \\ \bar{y}_{n+1} = \bar{y}_n + h \left[(\bar{y}_n + y^*) \left(s \left(1 - \frac{\bar{y}_n + y^*}{\theta(\bar{x}_n + x^*)} \right) \right) \right]. \end{cases} \quad (3.1)$$

Applying the Taylor expansion around the equilibrium $E^*(x^*, y^*)$, the system can be locally expressed as:

$$\begin{pmatrix} \bar{x}_{n+1} \\ \bar{y}_{n+1} \end{pmatrix} = J \Big|_{E^*} \begin{pmatrix} \bar{x}_n \\ \bar{y}_n \end{pmatrix} + h \begin{pmatrix} \psi_1(\bar{x}_n, \bar{y}_n) \\ \psi_2(\bar{x}_n, \bar{y}_n) \end{pmatrix}, \quad (3.2)$$

in which

$$\begin{aligned} \psi_1(\bar{x}_n, \bar{y}_n) &= j_1 \bar{x}_n \bar{y}_n + j_2 \bar{x}_n^2 + j_3 \bar{x}_n^3 + j_4 \bar{x}_n^2 \bar{y}_n + j_5 \bar{x}_n^3 \bar{y}_n \\ &\quad + O(|\bar{x}_n + \bar{y}_n|^5), \end{aligned}$$

and

$$\begin{aligned} \psi_2(\bar{x}_n, \bar{y}_n) &= k_1 \bar{x}_n \bar{y}_n + k_2 \bar{x}_n^2 + k_3 \bar{y}_n^2 + k_4 \bar{x}_n^3 + k_5 \bar{x}_n^2 \bar{y}_n \\ &\quad + k_6 \bar{x}_n \bar{y}_n^2 + k_7 \bar{x}_n^3 \bar{y}_n + O(|\bar{x}_n + \bar{y}_n|^5), \end{aligned}$$

where

$$\begin{aligned} j_1 &= - \left(\frac{m}{b + (x^*)^2} - \frac{2m(x^*)^2}{(b + (x^*)^2)^2} \right), \\ j_2 &= - \frac{r}{k} - \theta x^* \left(\frac{4m(x^*)^3}{(b + (x^*)^2)^3} - \frac{3mx^*}{(b + (x^*)^2)^2} \right), \\ j_3 &= -\theta x^* \left(-\frac{8m(x^*)^4}{(b + (x^*)^2)^4} + \frac{8m(x^*)^2}{(b + (x^*)^2)^3} - \frac{m}{(b + (x^*)^2)^3} \right), \\ j_4 &= - \left(\frac{4m(x^*)^3}{(b + (x^*)^2)^3} - \frac{3mx^*}{(b + (x^*)^2)^2} \right), \\ j_5 &= - \left(-\frac{8m(x^*)^4}{(b + (x^*)^2)^4} + \frac{8m(x^*)^2}{(b + (x^*)^2)^3} - \frac{m}{(b + (x^*)^2)^3} \right), \end{aligned}$$

and

$$\begin{aligned} k_1 &= \frac{2s}{x^*}, & k_2 &= -\frac{s\theta}{x^*}, & k_3 &= -\frac{s}{\theta x^*}, \\ k_4 &= \frac{s\theta}{(x^*)^2}, & k_5 &= \frac{2s}{(x^*)^2}, & k_6 &= \frac{s}{\theta(x^*)^2}, & k_7 &= \frac{2s}{(x^*)^3}. \end{aligned}$$

To determine the critical step size at which one eigenvalue crosses -1 , we impose the Flip bifurcation condition $F(-1) = 0$. Solving this condition for h gives the critical value

$$h^* = \frac{-(A_1 + D) - \sqrt{(A_1 - D)^2 + 4B_1C_1}}{A_1D - B_1C_1}. \quad (3.3)$$

where

$$A_1 = r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2},$$

$$B_1 = -\frac{mx^*}{b + (x^*)^2}, \quad C_1 = s\theta, \quad D = -s.$$

We choose the solution corresponding to the negative sign because the positive branch gives $h^* = 29.04$ with $|Q| \approx 24.3 > 1$, which does not satisfy the stability requirement $|Q| < 1$. For the negative branch, we obtain $h^* \approx 2.2959$ and $|Q| \approx 0.842 < 1$, which agrees with the numerical results presented in Section 6.

Since Eq (3.3) satisfies $F(-1) = 0$, one eigenvalue of the linearized system (3.2) is equal to -1 . Assuming that the modulus of the remaining eigenvalue is smaller than one, it follows that $|Q| < 1$. Let δ be a sufficiently small perturbation of the parameter h around the critical value h^* , namely $h = h^* + \delta$, where $|\delta| \ll 1$. Therefore, system (3.2) can be rewritten in the perturbed form:

$$\begin{pmatrix} \bar{x}_{n+1} \\ \bar{y}_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + (h^* + \delta) \left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) & (h^* + \delta) \left(-\frac{mx^*}{b + (x^*)^2} \right) \\ (h^* + \delta)s\theta & 1 + (h^* + \delta)(-s) \end{pmatrix} \begin{pmatrix} \bar{x}_n \\ \bar{y}_n \end{pmatrix} + (h^* + \delta) \begin{pmatrix} \psi_1(\bar{x}_n, \bar{y}_n) \\ \psi_2(\bar{x}_n, \bar{y}_n) \end{pmatrix}. \quad (3.4)$$

The characteristic equation associated with the perturbed Jacobian matrix is given by:

$$F(\lambda) = \lambda^2 - [2 + (h^* + \delta)(A_1 + D)]\lambda + 1 + (h^* + \delta)(A_1 + D) + (h^* + \delta)^2(A_1D - B_1C_1).$$

The two eigenvalues can be expressed as:

$$\lambda_{1,2} = \frac{2 + (h^* + \delta)(A_1 + D)}{2} \pm \frac{\sqrt{[2 + (h^* + \delta)(A_1 + D)]^2 - 4[1 + (h^* + \delta)(A_1 + D) + (h^* + \delta)^2(A_1D - B_1C_1)]}}{2}.$$

The transversality condition is obtained as follows:

$$\left. \frac{d\lambda_{1,2}(\delta)}{d\delta} \right|_{\delta=0} = \frac{A_1 + D \pm \sqrt{(A_1 - D)^2 + 4B_1C_1}}{2}.$$

Provided that

$$\left. \frac{d\lambda_{1,2}(\delta)}{d\delta} \right|_{\delta=0} \neq 0,$$

the Flip bifurcation takes place at the equilibrium E^* .

3.2. Direction of the flip bifurcation at $E^*(x^*, y^*)$

Introduce the matrix

$$Z = \begin{pmatrix} 1 + h^* \left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) & h^* \left(-\frac{mx^*}{b + (x^*)^2} \right) \\ h^* s\theta & 1 + h^*(-s) \end{pmatrix} = \begin{pmatrix} 1 + h^*A_1 & h^*B_1 \\ h^*C_1 & 1 + h^*D \end{pmatrix}. \quad (3.5)$$

When one eigenvalue of matrix Z equals -1 , namely $\lambda_1 = -1$, the associated eigenvector can be written as

$$\eta_1 = \begin{pmatrix} -h^* \left(-\frac{mx^*}{b + (x^*)^2} \right) \\ 2 + h^*r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \end{pmatrix}.$$

For the remaining eigenvalue λ_2 , the corresponding eigenvector is expressed as

$$\eta_2 = \begin{pmatrix} -h^* \left(-\frac{mx^*}{b + (x^*)^2} \right) \\ 1 + h^*r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} - \lambda_2 \end{pmatrix}.$$

Construct the transformation matrix $T = (\eta_1, \eta_2)$, which takes the form

$$T = \begin{pmatrix} T_1 & T_1 \\ 1 + T_2 & -\lambda_2 + T_2 \end{pmatrix}. \quad (3.6)$$

The inverse of the transformation matrix is obtained as

$$T^{-1} = -\frac{1}{T_1(1 + \lambda_2)} \begin{pmatrix} -\lambda_2 + T_2 & -T_1 \\ -(1 + T_2) & T_1 \end{pmatrix}. \quad (3.7)$$

where

$$T_1 = -h^* \left(-\frac{mx^*}{b + (x^*)^2} \right),$$

$$T_2 = 1 + h^*r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2}.$$

Using the transformation

$$\begin{pmatrix} S_n \\ I_n \end{pmatrix} = T^{-1} \begin{pmatrix} \bar{x}_n \\ \bar{y}_n \end{pmatrix}$$

system (3.4) can be transformed into

$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} S_n \\ I_n \end{pmatrix} - \frac{1}{T_1(1 + \lambda_2)} \begin{pmatrix} g_1(\bar{x}_n, \bar{y}_n, \delta) \\ g_2(\bar{x}_n, \bar{y}_n, \delta) \end{pmatrix}. \quad (3.8)$$

in which

$$\begin{aligned} g_1(\bar{x}_n, \bar{y}_n, \delta) &= m_1 \bar{x}_n \delta + m_2 \bar{y}_n \delta + m_3 \bar{x}_n^2 + m_4 \bar{x}_n \bar{y}_n \\ &\quad + m_5 \bar{x}_n \bar{y}_n \delta + m_6 \bar{x}_n^2 \delta + m_7 \bar{x}_n^3 + m_8 \bar{x}_n^2 \bar{y}_n \\ &\quad + m_9 \bar{x}_n^3 \delta + m_{10} \bar{x}_n^2 \bar{y}_n \delta + O(|\bar{x}_n| + |\bar{y}_n| + |\delta|)^4, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned}
 g_2(\bar{x}_n, \bar{y}_n, \delta) = & n_1 \bar{x}_n \delta + n_2 \bar{y}_n \delta + n_3 \bar{x}_n^2 + n_4 \bar{x}_n \bar{y}_n \\
 & + n_5 \bar{x}_n \bar{y}_n \delta + n_6 \bar{x}_n^2 \delta + n_7 \bar{x}_n^3 + n_8 \bar{x}_n^2 \bar{y}_n \\
 & + n_9 \bar{x}_n^3 \delta + n_{10} \bar{x}_n^2 \bar{y}_n \delta + O(|\bar{x}_n| + |\bar{y}_n| + |\delta|)^4.
 \end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
 m_1 &= (-\lambda_2 + T_2)A_1 - T_1 C_1 \\
 &= (-\lambda_2 + T_2) \left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) - T_1 s\theta, \\
 m_2 &= (-\lambda_2 + T_2)B_1 - T_1 D = (-\lambda_2 + T_2) \left(-\frac{mx^*}{b + (x^*)^2} \right) - T_1(-s), \\
 m_3 &= h^* m_5, \quad m_4 = h^* m_9, \\
 m_5 &= (-\lambda_2 + T_2)j_3 - T_1 k_3, \quad m_6 = (-\lambda_2 + T_2)j_1 - T_1 k_1, \\
 m_7 &= h^* m_6, \quad m_8 = h^* m_{10}, \\
 m_9 &= (-\lambda_2 + T_2)j_2 - T_1 k_2, \quad m_{10} = (-\lambda_2 + T_2)j_4 - T_1 k_4, \\
 n_1 &= (-1 - T_2)(-A_1) + T_1 C_1 \\
 &= (-1 - T_2) \left(-\left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) \right) + T_1 s\theta, \\
 n_2 &= (-1 - T_2)B_1 + T_1 D = (-1 - T_2) \left(-\frac{mx^*}{b + (x^*)^2} \right) + T_1(-s), \\
 n_3 &= h^* n_5, \quad n_4 = h^* n_9, \\
 n_5 &= (-1 - T_2)j_3 + T_1 k_3, \quad n_6 = (-1 - T_2)j_1 + T_1 k_1, \\
 n_7 &= h^* n_6, \quad n_8 = h^* n_{10}, \\
 n_9 &= (-1 - T_2)j_2 + T_1 k_2, \quad n_{10} = (-1 - T_2)j_4 + T_1 k_4.
 \end{aligned} \tag{3.11}$$

Using the inverse relation

$$\begin{pmatrix} \bar{x}_n \\ \bar{y}_n \end{pmatrix} = T \begin{pmatrix} S_n \\ I_n \end{pmatrix},$$

and substituting it into the nonlinear terms $g_1(\bar{x}_n, \bar{y}_n, \delta)$ and $g_2(\bar{x}_n, \bar{y}_n, \delta)$ in system (3.8), we obtain

$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} -S_n \\ \lambda_2 I_n \end{pmatrix} + \begin{pmatrix} g_3(S_n, I_n, \delta) \\ g_4(S_n, I_n, \delta) \end{pmatrix}. \tag{3.12}$$

with

$$\begin{aligned}
 g_3(S_n, I_n, \delta) = & p_1 S_n \delta + p_2 I_n \delta + p_3 S_n^2 + p_4 S_n I_n + p_5 I_n^2 \\
 & + p_6 S_n^2 \delta + p_7 S_n I_n \delta + p_8 I_n^2 \delta \\
 & + p_9 S_n^3 + p_{10} S_n^2 I_n + p_{11} S_n I_n^2 + p_{12} I_n^3 \\
 & + O(|S_n| + |I_n| + |\delta|)^4.
 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
 g_4(S_n, I_n, \delta) = & q_1 S_n \delta + q_2 I_n \delta + q_3 S_n^2 + q_4 S_n I_n + q_5 I_n^2 \\
 & + q_6 S_n^2 \delta + q_7 S_n I_n \delta + q_8 I_n^2 \delta \\
 & + q_9 S_n^3 + q_{10} S_n^2 I_n + q_{11} S_n I_n^2 + q_{12} I_n^3 \\
 & + O(|S_n| + |I_n| + |\delta|)^4.
 \end{aligned} \tag{3.14}$$

in which,

$$\begin{aligned}
 p_1 &= m_1 T_1 + m_2(1 + T_2), \\
 p_2 &= m_1 T_1 + m_2(-\lambda_2 + T_2), \\
 p_3 &= m_3(1 + T_2)^2 + m_4 T_1(1 + T_2), \\
 p_4 &= (1 - \lambda_2 + 2T_2)(m_3 + m_4 T_1), \\
 p_5 &= m_3(-\lambda_2 + T_2)^2 + m_4 T_1(-\lambda_2 + T_2), \\
 p_6 &= m_5(1 + T_2)^2 + m_9 T_1(1 + T_2), \\
 p_7 &= (1 - \lambda_2 + 2T_2)(m_5 + m_9 T_1), \\
 p_8 &= m_5(-\lambda_2 + T_2)^2 + m_9 T_1(-\lambda_2 + T_2), \\
 p_9 &= m_7(1 + T_2)^3 + m_8 T_1(1 + T_2)^2, \\
 p_{10} &= 3m_7(1 + T_2)^2(-\lambda_2 + T_2) + m_8 T_1(1 + T_2)^2 \\
 &\quad + m_8 T_1(1 - \lambda_2 + 2T_2), \\
 p_{11} &= 3m_7(1 + T_2)(-\lambda_2 + T_2)^2 + m_8 T_1(-\lambda_2 + T_2)^2 \\
 &\quad + m_8 T_1(1 - \lambda_2 + 2T_2), \\
 p_{12} &= m_7(-\lambda_2 + T_2)^3 + m_8 T_1(-\lambda_2 + T_2)^2, \\
 p_{13} &= m_9(1 + T_2)^3 + m_{10} T_1(1 + T_2)^2, \\
 p_{14} &= 3m_9(1 + T_2)^2(-\lambda_2 + T_2) + m_{10} T_1(1 + T_2)^2 \\
 &\quad + m_{10} T_1(1 - \lambda_2 + 2T_2), \\
 q_1 &= n_1 T_1 + n_2(1 + T_2), \\
 q_2 &= n_1 T_1 + n_2(-\lambda_2 + T_2), \\
 q_3 &= n_3(1 + T_2)^2 + n_4 T_1(1 + T_2), \\
 q_4 &= (1 - \lambda_2 + 2T_2)(n_3 + n_4 T_1), \\
 q_5 &= n_3(-\lambda_2 + T_2)^2 + n_4 T_1(-\lambda_2 + T_2), \\
 q_6 &= n_5(1 + T_2)^2 + n_9 T_1(1 + T_2), \\
 q_7 &= (1 - \lambda_2 + 2T_2)(n_5 + n_9 T_1), \\
 q_8 &= n_5(-\lambda_2 + T_2)^2 + n_9 T_1(-\lambda_2 + T_2), \\
 q_9 &= n_7(1 + T_2)^3 + n_8 T_1(1 + T_2)^2, \\
 q_{10} &= 3n_7(1 + T_2)^2(-\lambda_2 + T_2) + n_8 T_1(1 + T_2)^2 \\
 &\quad + n_8 T_1(1 - \lambda_2 + 2T_2), \\
 q_{11} &= 3n_7(1 + T_2)(-\lambda_2 + T_2)^2 + n_8 T_1(-\lambda_2 + T_2)^2 \\
 &\quad + n_8 T_1(1 - \lambda_2 + 2T_2), \\
 q_{12} &= n_7(-\lambda_2 + T_2)^3 + n_8 T_1(-\lambda_2 + T_2)^2.
 \end{aligned} \tag{3.15}$$

The above coordinate transformation separates the critical eigendirection associated with $\lambda_1 = -1$ from the remaining eigendirection. The coefficients p_i and q_i therefore provide the nonlinear terms required for the subsequent center manifold reduction and normal form analysis.

Theorem 3. Let

$$h^* = \frac{-(A_1 + D) - \sqrt{(A_1 - D)^2 + 4B_1C_1}}{A_1D - B_1C_1},$$

satisfy $|Q| < 1$. Then the equilibrium E^* experiences a Flip bifurcation when the parameter h varies in a sufficiently small neighborhood of h^* and θ_1, θ_2 are both nonzero.

Proof: According to the center manifold and normal form theory for discrete dynamical systems [51], we consider a sufficiently small neighborhood of the origin. When $\delta = 0$, the system possesses a central manifold passing through $(0, 0)$, which can be represented as

$$W^c(0, 0) = \{(x_n, y_n) : y_n = g(x_n, \delta) = g_{20}x_n^2 + g_{11}x_n\delta + O(|x_n + \delta|^6)\}. \quad (3.16)$$

Substituting Eq (3.16) into Eq (3.12), we obtain

$$g_{20} = \frac{q_3}{1 - \lambda_2}, \quad g_{11} = -\frac{q_1}{1 + \lambda_2}. \quad (3.17)$$

Substituting these center manifold coefficients into the transformed system reduces the local two-dimensional dynamics to a one-dimensional map along the critical eigendirection. Accordingly, the restriction of the system to the center manifold can be written as

$$\begin{aligned} x_{n+1} &= f(x_n, \delta) = -x_n + g_3(x_n, g(x_n, \delta), \delta) \\ &= -x_n + r_1x_n\delta + r_2x_n^2 + r_3x_n^2\delta + r_4x_n\delta^2 + r_5x_n^3 + r_6x_n^3\delta + r_7x_n^2\delta^2 + r_8x_n^3\delta^2. \end{aligned} \quad (3.18)$$

with

$$\begin{aligned} r_1 &= p_1, \quad r_2 = p_3, \quad r_3 = p_2g_{20} + p_6 + p_4g_{11}, \quad r_4 = p_2g_{11}, \\ r_5 &= p_4g_{20} + p_9, \quad r_6 = p_7g_{20} + p_{13} + 2p_5g_{20}g_{11} + p_{10}g_{11}, \\ r_7 &= g_{11}(p_7 + p_5g_{11}), \quad r_8 = 2p_8g_{20}g_{11} + p_{14}g_{11} + p_{11}g_{11}^2. \end{aligned} \quad (3.19)$$

For the occurrence of a Flip bifurcation in system (3.1), the following nondegeneracy conditions are required:

$$\theta_1 = \left(\frac{\partial^2 f}{\partial x_n \partial \delta} + \frac{1}{2} \frac{\partial f}{\partial \delta} \frac{\partial^2 f}{\partial x_n^2} \right) \Big|_{(0,0)} = r_1 \neq 0, \quad (3.20)$$

$$\theta_2 = \left(\frac{1}{6} \frac{\partial^3 f}{\partial x_n^3} + \left(\frac{1}{2} \frac{\partial^2 f}{\partial x_n^2} \right)^2 \right) \Big|_{(0,0)} = \frac{1}{6} r_5 + \frac{1}{4} r_2^2 \neq 0. \quad (3.21)$$

Since $\theta_1 \neq 0$ and $\theta_2 \neq 0$, the Flip bifurcation occurs at the equilibrium point E^* .

Theorem 4. Suppose that the assumptions of Theorem 3 are satisfied, namely, $\theta_1 \neq 0$ and $\theta_2 \neq 0$ at $h = h^*$. Then system (1.5) undergoes a Flip bifurcation at the positive equilibrium E^* as h crosses the critical value h^* . Moreover,

- (i) If $\theta_2 > 0$, the bifurcation is subcritical, and the period-2 orbit bifurcating from E^* is unstable;
- (ii) If $\theta_2 < 0$, the bifurcation is supercritical, and the period-2 orbit bifurcating from E^* is stable.

4. Neimark-sacker bifurcation analysis of equilibrium points

4.1. Existence conditions for Neimark-Sacker bifurcation

After obtaining the Flip bifurcation conditions at E^* , we next investigate the Neimark-Sacker bifurcation. Taking h as the bifurcation parameter again, the critical value h^* is determined from the condition $Q = 1$, namely,

$$h = h^* = \frac{A_1 + D}{B_1 C_1 - A_1 D} \quad (4.1)$$

which satisfies $Q = 1$. According to the standard Neimark-Sacker bifurcation theory for discrete dynamical systems [51], the conditions $Q = 1$ and $|P| < 2$ imply that λ_1 and λ_2 form a pair of complex conjugate eigenvalues on the unit circle. Together with the transversality and nonresonance conditions verified below, these spectral conditions provide the basis for the occurrence of a Neimark-Sacker bifurcation at E^* .

To analyze the local behavior near h^* , we introduce a small parameter perturbation δ by setting $h = h^* + \delta$ with $|\delta| \ll 1$. The corresponding characteristic equation is expressed as:

$$F(\lambda) = \lambda^2 - P(\delta)\lambda + Q(\delta) = 0 \quad (4.2)$$

where

$$\begin{aligned} P(\delta) &= 2 + (h^* + \delta) \left[r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} - s \right] \\ Q(\delta) &= 1 + (h^* + \delta) \left[r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} - s \right] \\ &\quad + (h^* + \delta)^2 \left[\left(r - \frac{2rx^*}{k} - \frac{m\theta x^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right) (-s) - \left(-\frac{mx^*}{b + (x^*)^2} \right) s\theta \right]. \end{aligned}$$

The eigenvalues of the perturbed system are given by:

$$\lambda_{1,2}(\delta) = \frac{P(\delta) \pm i\sqrt{4Q(\delta) - P^2(\delta)}}{2} \quad (4.3)$$

Moreover,

$$\lambda_{1,2}(0) = \frac{J_{11} + J_{22} \pm i\sqrt{4(J_{11}J_{22} - J_{12}J_{21}) - (J_{11} + J_{22})^2}}{2}. \quad (4.4)$$

Since

$$|\lambda_{1,2}(\delta)| = \sqrt{Q(\delta)},$$

the transversality condition for the Neimark-Sacker bifurcation is

$$\left. \frac{d}{d\delta} |\lambda_{1,2}(\delta)| \right|_{\delta=0} = \frac{1}{2} Q'(0) \neq 0,$$

where

$$Q'(0) = A_1 + D + 2h^*(A_1 D - B_1 C_1).$$

This condition guarantees that the pair of complex conjugate multipliers crosses the unit circle transversally as the bifurcation parameter varies.

In addition, the standard nonresonance condition requires

$$\lambda_{1,2}^j(0) \neq 1, \quad j = 1, 2, 3, 4.$$

Since $Q(0) = 1$ and $|P(0)| < 2$ already exclude the cases $\lambda = 1$ and $\lambda = -1$, the remaining strong resonances are excluded by imposing

$$P(0) \neq -1, \quad P(0) \neq 0.$$

Under these spectral, transversality, and nonresonance conditions, the local system satisfies the generic assumptions required for a Neimark-Sacker bifurcation.

The corresponding complex eigenvectors can be chosen consistently with the real coordinate transformation used below. Since

$$2v = \sqrt{4(J_{11}J_{22} - J_{12}J_{21}) - (J_{11} + J_{22})^2},$$

the eigenvectors associated with $\lambda_{1,2} = u \pm iv$ may be written as

$$\alpha_{1,2} = \begin{pmatrix} 2J_{12} \\ J_{22} - J_{11} \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 2v \end{pmatrix}. \quad (4.5)$$

4.2. Direction of the Neimark-Sacker bifurcation at $E^*(x^*, y^*)$

Let

$$u = \frac{1}{2}(J_{11} + J_{22}), \quad v = \frac{1}{2} \sqrt{4(J_{11}J_{22} - J_{12}J_{21}) - (J_{11} + J_{22})^2}.$$

Taking the real and imaginary parts of the eigenvector in (4.5) as the columns of the transformation matrix gives

$$T = \begin{pmatrix} 2J_{12} & 0 \\ J_{22} - J_{11} & 2v \end{pmatrix}. \quad (4.6)$$

With this choice,

$$T^{-1}JT = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}.$$

The inverse transformation matrix is

$$T^{-1} = \begin{pmatrix} \frac{1}{2J_{12}} & 0 \\ -\frac{J_{22} - J_{11}}{4vJ_{12}} & \frac{1}{2v} \end{pmatrix}. \quad (4.7)$$

Introduce the new variables

$$\begin{pmatrix} S_n \\ I_n \end{pmatrix} = T^{-1} \begin{pmatrix} \bar{x}_n \\ \bar{y}_n \end{pmatrix}.$$

At the critical value $h = h^*$, system (3.2) is therefore transformed into

$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} S_n \\ I_n \end{pmatrix} + \begin{pmatrix} f(S_n, I_n) \\ g(S_n, I_n) \end{pmatrix}. \quad (4.8)$$

For convenience, define

$$A_2 = 2J_{12}, \quad B_2 = J_{22} - J_{11}, \quad C_2 = 2v.$$

Then the inverse relation is

$$\bar{x}_n = A_2 S_n, \quad \bar{y}_n = B_2 S_n + C_2 I_n.$$

Using the nonlinear functions ψ_1 and ψ_2 introduced in (3.2), the nonlinear terms in (4.8) are

$$f(S_n, I_n) = \frac{h^*}{A_2} \psi_1(A_2 S_n, B_2 S_n + C_2 I_n),$$

and

$$g(S_n, I_n) = h^* \left[-\frac{B_2}{A_2 C_2} \psi_1(A_2 S_n, B_2 S_n + C_2 I_n) + \frac{1}{C_2} \psi_2(A_2 S_n, B_2 S_n + C_2 I_n) \right].$$

Retaining terms up to third order gives

$$f(S_n, I_n) = s_1 S_n^2 + s_2 S_n I_n + s_3 S_n^3 + s_4 S_n^2 I_n + O((|S_n| + |I_n|)^4),$$

and

$$\begin{aligned} g(S_n, I_n) = & t_1 S_n^2 + t_2 S_n I_n + t_3 I_n^2 \\ & + t_4 S_n^3 + t_5 S_n^2 I_n + t_6 S_n I_n^2 + O((|S_n| + |I_n|)^4). \end{aligned}$$

The corresponding coefficients are

$$s_1 = h^* (j_1 B_2 + j_2 A_2),$$

$$s_2 = h^* j_1 C_2,$$

$$s_3 = h^* (j_3 A_2^2 + j_4 A_2 B_2),$$

$$s_4 = h^* j_4 A_2 C_2,$$

$$t_1 = \frac{h^*}{C_2} \left[-B_2^2 j_1 - A_2 B_2 j_2 + A_2 B_2 k_1 + A_2^2 k_2 + B_2^2 k_3 \right],$$

$$t_2 = h^* [-B_2 j_1 + A_2 k_1 + 2B_2 k_3],$$

$$t_3 = h^* C_2 k_3,$$

$$t_4 = \frac{h^*}{C_2} \left[-A_2^2 B_2 j_3 - A_2 B_2^2 j_4 + A_2^3 k_4 + A_2^2 B_2 k_5 + A_2 B_2^2 k_6 \right],$$

$$t_5 = h^* \left[-A_2 B_2 j_4 + A_2^2 k_5 + 2A_2 B_2 k_6 \right],$$

$$t_6 = h^* A_2 C_2 k_6.$$

Remark 2. The absence of several monomial terms in f and g can be explained by the structure of the coordinate transformation. Under this transformation, $\bar{x}_n = A_2 S_n$ depends only on S_n , whereas $\bar{y}_n = B_2 S_n + C_2 I_n$ has a linear dependence on I_n . The nonlinear function ψ_1 defined in (3.2) contains $\bar{x}_n \bar{y}_n$ as its highest-order term involving $\bar{y}_n$, and therefore no quadratic or cubic terms solely composed of $\bar{y}_n$ are generated. Since $f = h^* \psi_1 / A_2$, the coefficients corresponding to I_n^2 , $S_n I_n^2$, and I_n^3 in f are identically zero.

For

$$g = h^* \left(-\frac{B_2}{A_2 C_2} \psi_1 + \frac{1}{C_2} \psi_2 \right),$$

the terms $k_3 \bar{y}_n^2$ and $k_6 \bar{x}_n \bar{y}_n^2$ contained in ψ_2 defined in (3.2) contribute to the terms $t_3 I_n^2$ and $t_6 S_n I_n^2$, which have already been included above. Meanwhile, ψ_2 does not contain any cubic term of $\bar{y}_n$, resulting in a zero coefficient of I_n^3 in g . Hence, the obtained expressions of f and g include all nonzero terms up to the third order.

To determine the direction and stability of the Neimark-Sacker bifurcation, we introduce the complex variable

$$z_n = S_n + i I_n.$$

For the linear part of system (4.8), we choose

$$\lambda_1 = u - iv, \quad \lambda_2 = u + iv = \bar{\lambda}_1.$$

The complex form of the transformed map can then be written locally as

$$\begin{aligned} z_{n+1} = & \lambda_1 z_n + \tau_{20} z_n^2 + \tau_{11} z_n \bar{z}_n + \tau_{02} \bar{z}_n^2 \\ & + \tau_{21} z_n^2 \bar{z}_n + O(|z_n|^4), \end{aligned}$$

where the coefficients required for the first Lyapunov coefficient are obtained from the quadratic and cubic coefficients of f and g as follows:

$$\tau_{20} = \frac{1}{4} [(s_1 + t_2) + i(t_1 - t_3 - s_2)],$$

$$\tau_{11} = \frac{1}{2} [s_1 + i(t_1 + t_3)],$$

$$\tau_{02} = \frac{1}{4} [(s_1 - t_2) + i(t_1 - t_3 + s_2)],$$

and

$$\tau_{21} = \frac{1}{8} [(3s_3 + t_5) + i(3t_4 + t_6 - s_4)].$$

According to the normal form theory for Neimark-Sacker bifurcation [51], the corresponding first Lyapunov coefficient is

$$L = -\operatorname{Re}\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\tau_{20}\tau_{11}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2 + \operatorname{Re}(\lambda_2\tau_{21}). \quad (4.9)$$

The condition $L \neq 0$ is the corresponding nondegeneracy condition determining the type and stability of the bifurcating invariant closed curve.

Theorem 5. Suppose that

$$h = h^*, \quad Q(0) = 1, \quad |P(0)| < 2,$$

and that the nonresonance conditions

$$P(0) \neq -1, \quad P(0) \neq 0$$

hold. If, in addition,

$$Q'(0) \neq 0$$

and the first Lyapunov coefficient defined in (4.9) satisfies

$$L \neq 0,$$

then system (1.5) undergoes a Neimark-Sacker bifurcation at the positive equilibrium E^* as h passes through h^* .

If $L < 0$, the bifurcating invariant closed curve is attracting, whereas if $L > 0$, the bifurcating invariant closed curve is repelling. The side of the bifurcation point on which the invariant curve exists is determined jointly by the signs of L and the transversality coefficient $Q'(0)$.

5. Chaos control

Chaos control aims to regulate unstable periodic orbits (UPOs) contained in chaotic attractors by introducing appropriately designed perturbations. The basic idea is to utilize the intrinsic characteristics of chaotic systems and modify the evolution of trajectories, so that irregular motions can be transformed into stable and predictable behaviors. For discrete-time dynamical systems, various control methods have been developed to improve system stability. In this work, two control schemes are considered, namely the state feedback control method and the hybrid control method.

5.1. State feedback control

The state feedback method stabilizes chaotic dynamics by introducing a feedback term into the original system. The control parameters are determined according to the local properties of the system near the target equilibrium, so that the unstable equilibrium can be transformed into a stable state. By introducing the feedback term into the prey equation, the original system (1.5) is transformed into the following controlled system:

$$\begin{cases} x_{n+1} = x_n + h \left[rx_n \left(1 - \frac{x_n}{k} \right) - y_n \frac{mx_n}{b + x_n^2} \right] - u_n, \\ y_{n+1} = y_n + h \left[y_n \left(s \left(1 - \frac{y_n}{\theta x_n} \right) \right) \right], \end{cases} \quad (5.1)$$

where $u_n = \omega_1(x_n - x^*) + \omega_2(y_n - y^*)$ denotes the feedback term introduced near the equilibrium point E^* , and ω_1, ω_2 represent the corresponding control gains.

Positivity under state feedback control. Using the functions Φ_h and Ψ_h defined for the uncontrolled map, system (5.1) can be written as

$$x_{n+1} = x_n \Phi_h(x_n, y_n) - u_n, \quad y_{n+1} = y_n \Psi_h(x_n, y_n).$$

Therefore, for $x_n > 0$ and $y_n \geq 0$, a sufficient one-step condition for biological feasibility of the state-feedback-controlled map is

$$u_n < x_n \Phi_h(x_n, y_n), \quad \Psi_h(x_n, y_n) \geq 0.$$

Equivalently, the prey positivity condition can be written explicitly as

$$\omega_1(x_n - x^*) + \omega_2(y_n - y^*) < x_n \left\{ 1 + h \left[r \left(1 - \frac{x_n}{k} \right) - \frac{m y_n}{b + x_n^2} \right] \right\},$$

while the predator positivity condition is

$$1 + h s \left(1 - \frac{y_n}{\theta x_n} \right) \geq 0.$$

Unlike the uncontrolled system, global positivity of the state-feedback-controlled map cannot be guaranteed for arbitrary gain values because the additive control term u_n may alter the sign of the prey update. The controller considered here is therefore interpreted locally around the positive equilibrium E^* . Since $u(E^*) = 0$ and E^* lies in the interior of the biologically feasible domain, continuity of the controlled map guarantees that sufficiently small perturbations of E^* have positive one-step images. For the numerical simulations presented in Section 6, biological feasibility is additionally verified at every iteration without clipping, projection, or artificial replacement of negative state values.

The Jacobian matrix at E^* becomes

$$J_n(E^*) = \begin{pmatrix} J_{11} - \omega_1 & J_{12} - \omega_2 \\ J_{21} & J_{22} \end{pmatrix}. \quad (5.2)$$

The characteristic equation is obtained as:

$$\lambda^2 - (J_{11} - \omega_1 + J_{22}) \lambda + (J_{11} - \omega_1) J_{22} - (J_{12} - \omega_2) J_{21} = 0. \quad (5.3)$$

For the eigenvalues λ_1, λ_2 of (5.2), the trace and determinant relations can be written as

$$\lambda_1 + \lambda_2 = J_{11} - \omega_1 + J_{22}, \quad (5.4)$$

$$\lambda_1 \lambda_2 = (J_{11} - \omega_1) J_{22} - (J_{12} - \omega_2) J_{21}. \quad (5.5)$$

The stability boundaries in the (ω_1, ω_2) plane are determined by the cases in which the eigenvalues reach the unit circle. Specifically, the conditions $\lambda = 1$, $\lambda = -1$, and $\lambda_1 \lambda_2 = 1$ generate the three marginal stability boundaries considered below.

Theorem 6. The triangular region surrounded by L_1 , L_2 , and L_3 in Eqs (5.6)–(5.8) accommodates stable eigenvalues.

Proof: The stability region can be obtained from the following conditions:

$$\lambda = \pm 1, \quad \lambda_1 \lambda_2 = 1.$$

With $\lambda_1 \lambda_2 = 1$, Eq (5.5) gives

$$L_1 : (J_{11} - \omega_1) J_{22} - (J_{12} - \omega_2) J_{21} = 1. \quad (5.6)$$

Equation (5.6) is the first marginal stability condition.

When $\lambda_1 = 1$, we obtain

$$L_2 : \omega_1(1 - J_{22}) + \omega_2 J_{21} = J_{11} + J_{22} - 1 - J_{11} J_{22} + J_{21} J_{12}. \quad (5.7)$$

When $\lambda_1 = -1$, we obtain

$$L_3 : \omega_1(1 + J_{22}) - \omega_2 J_{21} = J_{11} + J_{22} + 1 + J_{11} J_{22} - J_{21} J_{12}. \quad (5.8)$$

The lines L_1 , L_2 , and L_3 represent the marginal stability boundaries corresponding to eigenvalues on the unit circle. The triangular region enclosed by these boundaries is the local stability region in which both eigenvalues lie inside the unit circle.

5.2. Hybrid control

The second stabilization approach introduces a weighting parameter $\omega \in (0, 1)$ by forming a weighted combination of the Euler update and the identity map. Unlike the state-feedback scheme, this formulation does not introduce an additional additive feedback term. The hybrid-controlled map is formulated as follows:

$$\begin{cases} x_{n+1} = \omega x_n + \omega h \left[r x_n \left(1 - \frac{x_n}{k} \right) - y_n \frac{m x_n}{b + x_n^2} \right] + (1 - \omega) x_n, \\ y_{n+1} = \omega y_n + \omega h \left[y_n \left(s \left(1 - \frac{y_n}{\theta x_n} \right) \right) \right] + (1 - \omega) y_n. \end{cases} \quad (5.9)$$

Here, $\omega \in (0, 1)$ is the weighting parameter. As shown below, system (5.9) is algebraically equivalent to the original Euler map with the effective step size $h_{\text{eff}} = \omega h$. Hence, the role of ω is to rescale the discrete evolution rather than to represent a direct state-feedback intervention.

Positivity under hybrid control. The positivity condition of the hybrid-controlled system can be obtained directly from its update structure. Since

$$\omega x_n + (1 - \omega) x_n = x_n, \quad \omega y_n + (1 - \omega) y_n = y_n,$$

system (5.9) can be equivalently rewritten as

$$x_{n+1} = x_n \left\{ 1 + \omega h \left[r \left(1 - \frac{x_n}{k} \right) - \frac{m y_n}{b + x_n^2} \right] \right\},$$

and

$$y_{n+1} = y_n \left\{ 1 + \omega h s \left(1 - \frac{y_n}{\theta x_n} \right) \right\}.$$

Hence, the hybrid-controlled map has the same multiplicative positivity structure as the uncontrolled Euler map, with the effective step size

$$h_{\text{eff}} = \omega h.$$

Because $\omega \in (0, 1)$, we have $h_{\text{eff}} > 0$. Therefore, for $x_n > 0$ and $y_n \geq 0$, a sufficient one-step condition for the next iterate to remain biologically feasible is

$$1 + \omega h \left[r \left(1 - \frac{x_n}{k} \right) - \frac{m y_n}{b + x_n^2} \right] > 0,$$

and

$$1 + \omega h s \left(1 - \frac{y_n}{\theta x_n} \right) \geq 0.$$

Equivalently, these conditions can be written as

$$\frac{m y_n}{b + x_n^2} < \frac{1}{\omega h} + r \left(1 - \frac{x_n}{k} \right),$$

and

$$y_n \leq \theta x_n \left(1 + \frac{1}{\omega h s} \right).$$

The first inequality further requires

$$0 < x_n < k \left(1 + \frac{1}{\omega h r} \right).$$

Thus, the sufficient feasibility conditions for the hybrid-controlled system are obtained directly from those of the uncontrolled Euler map by replacing h with ωh . As in the uncontrolled case, these one-step inequalities are checked along the numerical trajectories, and the simulations reported in Section 6 are performed without clipping, projection, or artificial replacement of negative state values.

The Jacobian matrix of system (5.9) at E^* is given by:

$$J_{E^*} = \begin{pmatrix} 1 + \omega h \left[r - \frac{2x^*r}{k} - \frac{my^*(b - (x^*)^2)}{(b + (x^*)^2)^2} \right] & \omega h \left(-\frac{mx^*}{b + (x^*)^2} \right) \\ \omega h \left(\frac{s(y^*)^2}{\theta(x^*)^2} \right) & 1 + \omega h \left(s - \frac{2sy^*}{\theta x^*} \right) \end{pmatrix} = \begin{pmatrix} J_{11}^\omega & J_{12}^\omega \\ J_{21}^\omega & J_{22}^\omega \end{pmatrix}. \quad (5.10)$$

To determine the range of the control weight ω that stabilizes the positive equilibrium, we apply the unit-circle stability criterion to the characteristic polynomial associated with the Jacobian matrix (5.10). This converts the local stability requirement into explicit inequalities involving ω .

Theorem 7. System (5.9) is locally asymptotically stable if:

$$1 - (J_{11}^\omega + J_{22}^\omega) + J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega > 0, \quad (5.11)$$

$$1 + (J_{11}^\omega + J_{22}^\omega) + J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega > 0, \quad (5.12)$$

$$J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega < 1. \quad (5.13)$$

Proof. The characteristic polynomial corresponding to the Jacobian matrix (5.10) is

$$F(\lambda) = \lambda^2 - (J_{11}^\omega + J_{22}^\omega)\lambda + J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega = 0. \quad (5.14)$$

According to the standard stability criterion for discrete systems [50], the equilibrium point is locally asymptotically stable when all characteristic roots of (5.14) are located within the unit circle. This requirement is equivalent to the conditions $F(1) > 0$, $F(-1) > 0$, and $F(0) < 1$. Specifically,

$$F(1) = 1 - (J_{11}^\omega + J_{22}^\omega) + J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega,$$

$$F(-1) = 1 + (J_{11}^\omega + J_{22}^\omega) + J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega,$$

$$F(0) = J_{11}^\omega J_{22}^\omega - J_{12}^\omega J_{21}^\omega.$$

Therefore, the local asymptotic stability of system (5.9) follows directly from conditions (5.11)–(5.13). $\square$

6. Numerical simulations

To distinguish the small-step approximation property from the large-step dynamics of the Euler-generated map, we first examine the convergence of system (1.5) toward the continuous system (1.4). We take $r = 1.0$, $s = 0.5$, $k = 3.0$, $m = 1.5$, $b = 1.0$, $\theta = 0.5$, and $(x_0, y_0) = (1.0, 0.5)$ over the interval $0 \leq t \leq 5$. A high-accuracy numerical solution of system (1.4) is used as the reference solution. The maximum error is defined by

$$E_h = \max_n \|(x_n, y_n) - (x_{\text{ref}}(t_n), y_{\text{ref}}(t_n))\|_\infty.$$

The resulting errors are listed in Table 1. As the step size is halved, the error is also approximately halved, confirming the expected first-order convergence of the forward Euler discretization. This result verifies the consistency of the Euler map with the continuous model for sufficiently small h . In contrast, the bifurcation and chaotic regimes obtained for substantially larger values of h are interpreted in this study as discretization-induced dynamics of the resulting discrete map rather than as quantitatively accurate approximations of the continuous-time trajectories.

Table 1. Convergence of the forward Euler map toward the continuous-time reference solution.

Step size h	Maximum error E_h
0.200	2.4818×10^{-2}
0.100	1.2187×10^{-2}
0.050	6.0404×10^{-3}
0.025	3.0067×10^{-3}

Numerical verification of biological feasibility. In addition to the analytical positivity conditions derived above, the biological feasibility of the numerical simulations was verified directly. No clipping, projection, or artificial replacement of negative state values was used. For the complete parameter intervals employed in the uncontrolled Flip and Neimark-Sacker simulations, the state-feedback-controlled simulations, and the hybrid-controlled simulations, the conditions

$$x_n > 0, \quad y_n \geq 0$$

were checked at every iteration. All trajectories retained in the revised bifurcation diagrams, MLE calculations, and phase portraits remained in the biologically feasible domain throughout the corresponding parameter intervals. In particular, the hybrid-controlled Neimark-Sacker scan was restricted to $\delta \in [-0.30, 1.10]$ so that only biologically feasible trajectories are reported.

The previous analytical results establish the local properties of the system around its equilibria. To further investigate the global dynamical characteristics, including bifurcation transitions, chaotic behaviors, and the long-term evolution under different initial states, numerical experiments are carried out in this section. Specifically, the influences of the step size parameter h , prey and predator growth rates r and s , carrying capacity k , predation parameter m , half-saturation parameter b , and Leslie-Gower coefficient θ on the dynamics of system (1.5) are investigated.

For this simulation, we take $r = 1.0$, $s = 0.5$, $k = 3.0$, and $h = 0.3$. Since $y_0 = 0$ places the initial condition exactly on the invariant line $y = 0$, the predator remains identically zero for all $n \geq 0$ and the prey converges monotonically to the carrying capacity $k = 3.0$. For the present parameter set, the eigenvalues are $\lambda_1 = 0.7$ and $\lambda_2 = 1.15$, so E_1 is a saddle, consistently with the instability stated in Theorem 1.

Figure 1 illustrates representative numerical trajectories associated with the four positive-equilibrium configurations summarized in Lemma 1. Panels (a,b) correspond to Case (i), where $\Delta < 0$ and a unique positive equilibrium $E^* \approx (2.1360, 1.0680)$ exists. Under the selected initial condition, both x_n and y_n approach the corresponding equilibrium levels. Panels (c,d) correspond to Case (ii), where $\Delta = 0$ and Eq (2.2) possesses the triple positive root $x^* = 1.5$, giving the positive equilibrium $E^* = (1.5, 0.75)$. The initial condition is chosen at this degenerate equilibrium so that the corresponding equilibrium levels are displayed directly. Panels (e,f) correspond to Case (iii), where $\Delta = 0$ and two distinct positive equilibria coexist, namely $E_1^* = (1.0, 0.5)$ and $E_2^* = (2.5, 1.25)$, with E_1^* associated with a double root. Both equilibrium levels are indicated simultaneously, while the selected trajectory approaches E_2^* . Panels (g,h) correspond to Case (iv), where $\Delta > 0$ and three distinct positive equilibria coexist. The three equilibrium levels E_1^* , E_2^* , and E_3^* are all indicated under the same parameter values, while the selected trajectory approaches E_3^* . The dashed horizontal lines therefore provide a direct visual correspondence between the equilibrium configurations predicted by Lemma 1 and the numerical trajectories without changing the parameter values within each case.

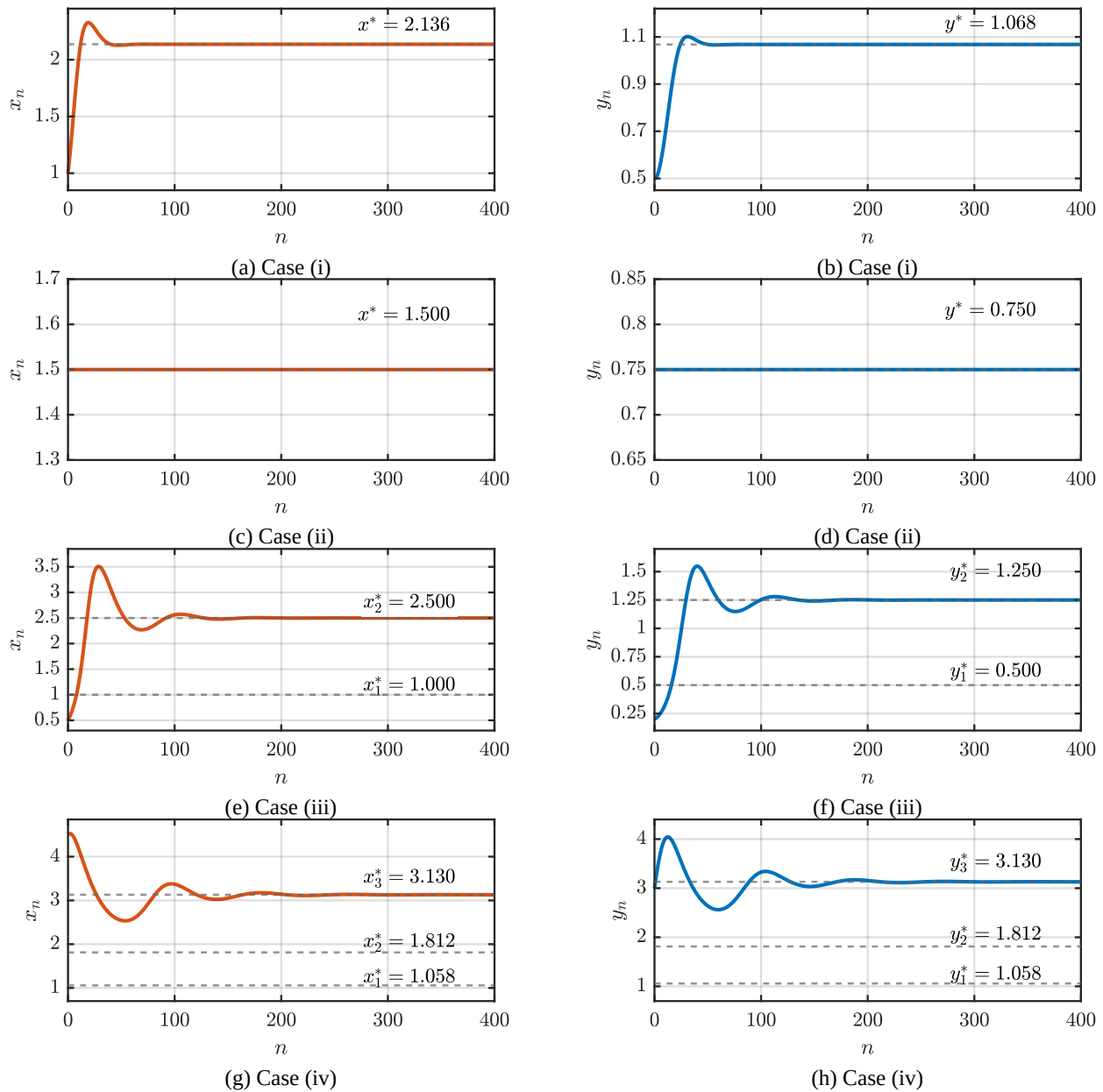


Figure 1. Time series of the prey and predator populations and the corresponding positive equilibrium levels for the four equilibrium configurations in Lemma 1. Panels (a,b), (c,d), (e,f), and (g,h) correspond to Cases (i)–(iv), respectively. The left and right panels in each row show x_n and y_n , respectively. The dashed horizontal lines indicate all distinct positive equilibrium levels existing under the same parameter set.

Panels (a,b), Case (i): $r = 1.0$, $k = 3.0$, $m = 1.5$, $b = 1.0$, $\theta = 0.5$, $s = 0.5$, and $h = 0.3$, with initial value $(x_0, y_0) = (1.0, 0.5)$. The unique positive equilibrium is $E^* \approx (2.1360, 1.0680)$.

Panels (c,d), Case (ii): $r = 1.0$, $k = 4.5$, $m = 8k/(27\theta) \approx 2.6667$, $b = k^2/27 = 0.75$, $\theta = 0.5$, $s = 0.5$, and $h = 0.3$. Equation (2.2) has the triple positive root $x^* = 1.5$, corresponding to the positive equilibrium $E^* = (1.5, 0.75)$. The initial value is chosen as $(x_0, y_0) = (1.5, 0.75)$ to display this degenerate equilibrium directly.

Panels (e,f), Case (iii): $r = 1.0$, $k = 4.5$, $m = 196/81 \approx 2.4198$, $b = 5/9 \approx 0.5556$, $\theta = 0.5$, $s = 0.5$, and $h = 0.3$, with initial value $(x_0, y_0) = (0.5, 0.2)$. The two distinct positive equilibria are $E_1^* = (1.0, 0.5)$ and $E_2^* = (2.5, 1.25)$, where E_1^* corresponds to a double root. Both equilibrium levels are displayed under the same parameter set.

Panels (g,h), Case (iv): $r = 1.0$, $k = 6.0$, $m = 1.65$, $b = 1.0$, $\theta = 1.0$, $s = 0.5$, and $h = 0.3$, with initial value $(x_0, y_0) = (4.5, 3.0)$. The three distinct positive equilibria are approximately $E_1^* = (1.0578, 1.0578)$, $E_2^* = (1.8122, 1.8122)$, and $E_3^* = (3.1300, 3.1300)$. All three equilibrium levels are displayed simultaneously under the same parameter set.

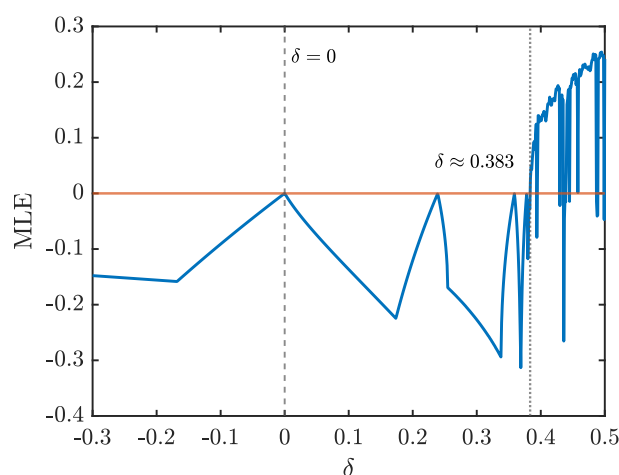


Figure 2. MLE of system (1.5) with respect to the perturbation parameter δ , where $h = h^* + \delta$ and $h^* \approx 2.2959$. The parameters are $r = 1.0$, $k = 5.0$, $m = 1.5$, $b = 1.0$, $s = 1.5$, $\theta = 1.0$, with initial value $(x_0, y_0) = (2.002, 1.998)$. The MLE remains negative at $\delta = 0.18$ and first becomes positive near $\delta \approx 0.38$.

As illustrated in Figures 2 and 3, we analyze the Flip bifurcation diagrams of the prey population x_n and the predator population y_n of system (1.5), together with the corresponding MLE, all plotted with respect to the perturbation parameter δ , where the discrete step size is taken as $h = h^* + \delta$ and $h^* \approx 2.2959$ denotes the critical step size at which the Flip bifurcation occurs. The numerical results show that for $\delta < 0$, the MLE remains strictly negative and the positive equilibrium $E^*(x^*, y^*) = (2, 2)$ is asymptotically stable. As δ crosses zero, the system becomes unstable via a Flip (period-doubling) bifurcation and enters a stable period-2 orbit, which is consistent with the theoretical condition $F(-1) = 0$ and $|Q| \approx 0.842 < 1$. With a further increase of δ , successive period-doubling processes lead to period-4, period-8, and higher-order periodic orbits. A recalculation of the MLE shows that it remains negative at $\delta = 0.18$ and first becomes positive near $\delta \approx 0.38$, indicating the onset of a chaotic regime. Further variations of the MLE reflect the presence of periodic windows embedded within the chaotic

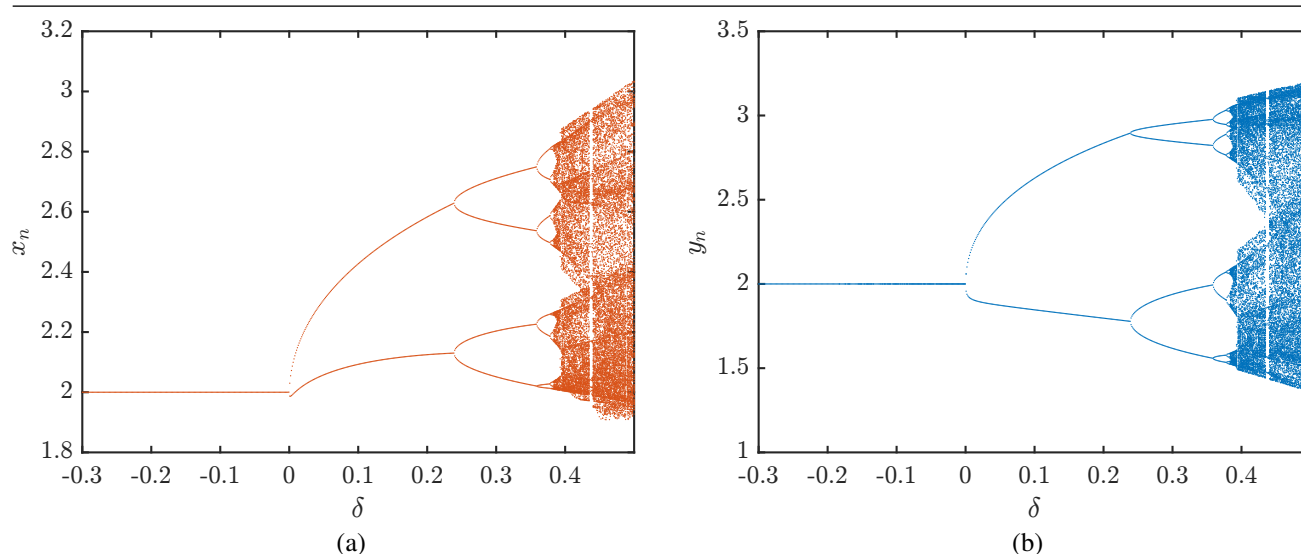


Figure 3. Flip bifurcation diagram of x_n and y_n . Take the values of the parameters $r = 1.0$, $k = 5.0$, $m = 1.5$, $b = 1.0$, $s = 1.5$, $\theta = 1.0$, and the initial value is $(x_0, y_0) = (2.002, 1.998)$.

parameter region. The high consistency between the bifurcation structures of x_n and y_n and the sign-change behavior of the MLE supports the theoretical Flip-bifurcation results.

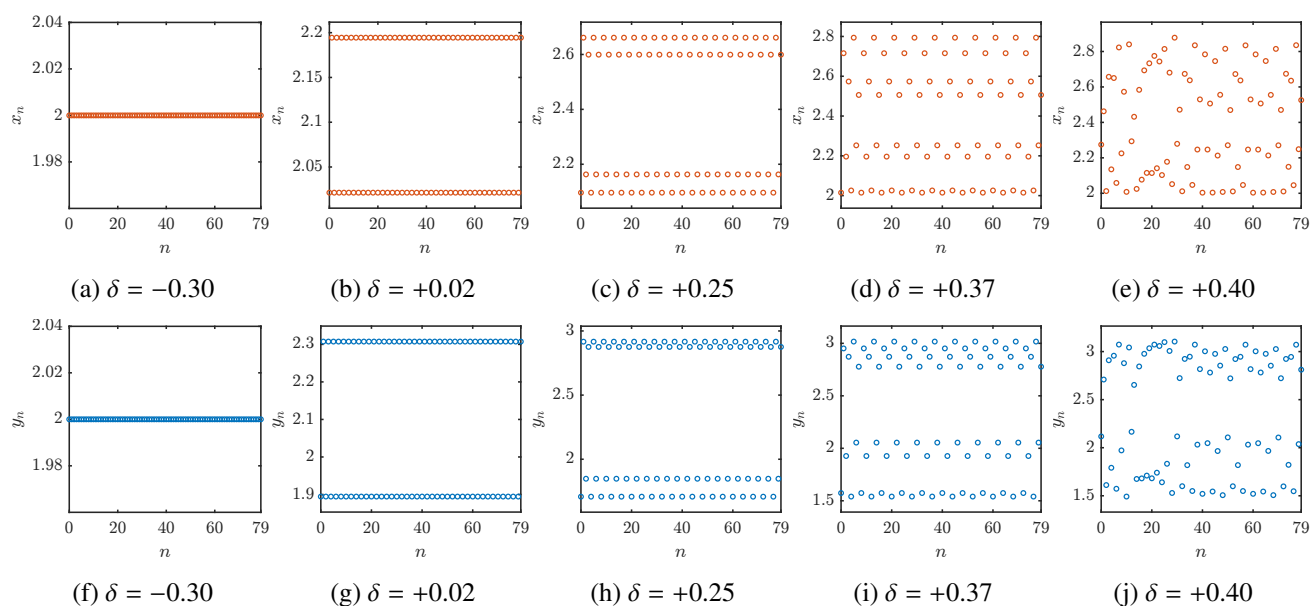


Figure 4. Representative prey and predator time series illustrating the dynamical evolution of system (1.5) as the additive perturbation parameter δ varies, where $h = h^* + \delta$ and $h^* \approx 2.2959$. The parameters are $r = 1.0$, $k = 5.0$, $m = 1.5$, $b = 1.0$, $s = 1.5$, $\theta = 1.0$, with initial value $(x_0, y_0) = (2.002, 1.998)$.

Figure 4 illustrates representative prey and predator trajectories as the additive perturbation

parameter δ varies, where $h = h^* + \delta$ and $h^* \approx 2.2959$. For $\delta = -0.30$ [panels (a) and (f)], both state variables converge to the positive equilibrium $E^* = (2, 2)$. At $\delta = +0.02$ [panels (b) and (g)], the equilibrium has lost stability through a Flip bifurcation and a period-2 orbit is observed. When δ increases to $+0.25$ [panels (c) and (h)], the trajectories exhibit a higher-order period-doubled state. At $\delta = +0.37$ [panels (d) and (i)], a higher-period orbit appears close to the onset of chaotic dynamics. Finally, at $\delta = +0.40$ [panels (e) and (j)], the trajectories become irregular, consistent with the positive MLE obtained after the first clear MLE zero crossing near $\delta \approx 0.383$. These numerical results illustrate the period-doubling route toward chaos in the Euler-generated discrete map.

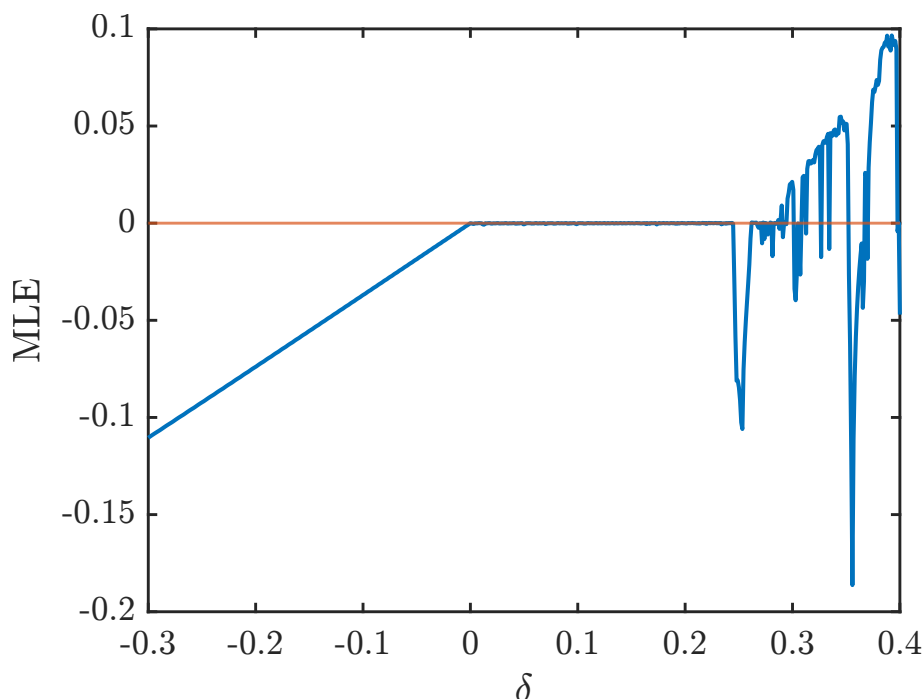


Figure 5. MLE corresponding to Figure 6. Take the values of the parameters $r = 1.0$, $s = 0.5$, $k = 3.0$, $m = 1.5$, $b = 1.0$, $\theta = 0.5$, $h^* \approx 2.8037$, and the initial value is approximately $(2.137, 1.067)$.

As shown in Figures 5–7, we present the Neimark-Sacker (NS) bifurcation diagrams of the prey population x_n and the predator population y_n of system (1.5), together with the corresponding MLE diagram and phase portraits in the (x_n, y_n) plane. All results are plotted with respect to the perturbation parameter δ , where the discrete step size is taken as $h = h^* + \delta$ and $h^* \approx 2.8037$ denotes the critical step size at which the NS bifurcation occurs. The numerical results show that when $\delta < 0$, the MLE remains negative and the positive equilibrium E^* maintains local asymptotic stability. As δ passes through zero, the equilibrium becomes unstable through an NS bifurcation, and a stable invariant closed curve appears around it, in agreement with the theoretical conditions $|P(0)| < 2$ and $Q(0) = 1$. With a further increase of δ , the invariant curve expands and undergoes further complex dynamical transitions, eventually leading to the destruction of quasi-periodic motion and the onset of chaos. Once δ exceeds approximately 0.2, the MLE crosses the zero axis and becomes positive, manifesting itself as dense chaotic bands in the bifurcation diagrams. The high consistency among the bifurcation structures of

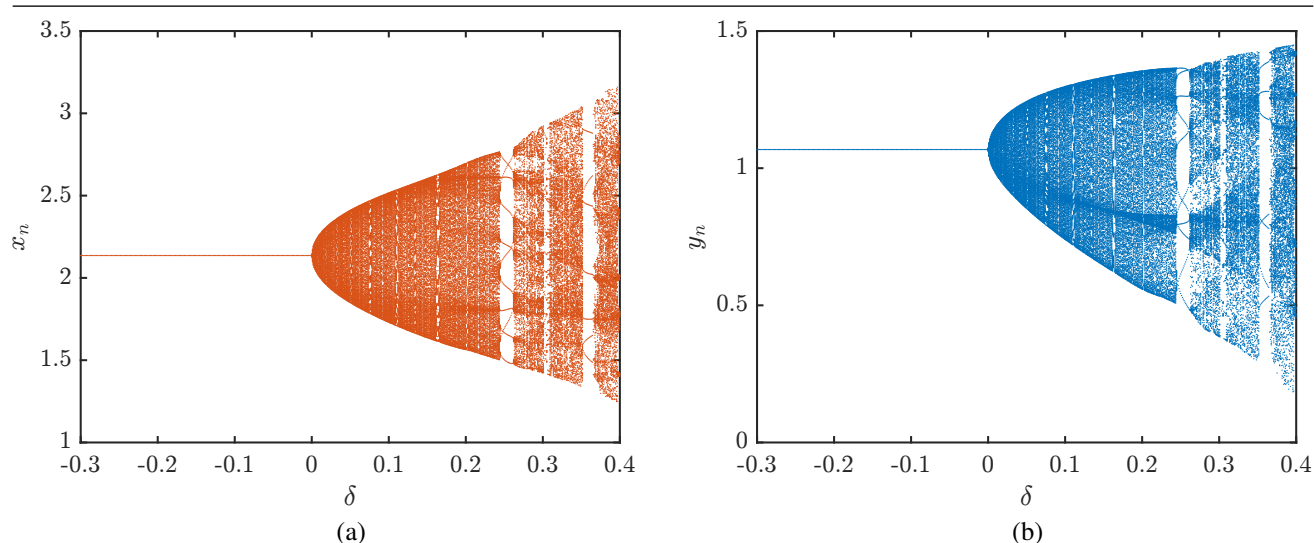


Figure 6. Bifurcation diagrams of the prey population x_n and predator population y_n with respect to the perturbation parameter δ . Take the values of the parameters $r = 1.0$, $s = 0.5$, $k = 3.0$, $m = 1.5$, $b = 1.0$, $\theta = 0.5$, $h^* \approx 2.8037$, and the initial value is approximately $(2.137, 1.067)$.

x_n and y_n , the sign-change behavior of the MLE, and the phase portraits confirms the validity of the theoretical NS bifurcation analysis.

Figures 5–7 present the numerical results for the Neimark-Sacker bifurcation of system (1.5) under the parameter set $r = 1.0$, $s = 0.5$, $k = 3.0$, $m = 1.5$, $b = 1.0$, and $\theta = 0.5$. The discrete step size is written as $h = h^* + \delta$, where $h^* \approx 2.8037$ is the critical value predicted by the theoretical analysis. Specifically, Figure 5 shows the MLE as a function of δ , Figure 6 gives the corresponding bifurcation diagrams of the prey population x_n and predator population y_n , and Figure 7 presents phase portraits at six representative values of δ .

For $\delta < 0$, the MLE is negative and the numerical trajectories approach the positive equilibrium, indicating local asymptotic stability. As δ passes through zero, the equilibrium loses stability through a Neimark-Sacker bifurcation, consistently with the theoretical conditions $Q(0) = 1$ and $|P(0)| < 2$. A closed invariant curve then appears around the equilibrium and gradually expands as δ increases. The phase portraits in Figure 7 illustrate this transition: the trajectories converge toward the equilibrium at $\delta = -0.15$ and -0.02 , while closed invariant curves are observed after the bifurcation at $\delta = 0.02$, 0.08 , and 0.15 . At $\delta = 0.30$, the invariant structure becomes strongly deformed and the dynamics are substantially more complex. Consistently, positive MLE values appear in portions of the larger- δ parameter range, indicating the emergence of chaotic dynamics. Overall, the MLE, bifurcation diagrams, and phase portraits provide mutually consistent numerical evidence for the Neimark-Sacker bifurcation and the subsequent nonlinear evolution of the Euler-generated discrete map.

Figures 8–12 systematically illustrate the suppression of the Flip bifurcation in the discrete Leslie-Gower predator-prey model with a Holling type-IV functional response before and after the application of state feedback control. For the Flip bifurcation, the parameters considered are: $r = 3.2$, $k = 3.0$, $m = 3.0$, $b = 0.5$, $s = 0.5$, $\theta = 0.5$, with the positive equilibrium (x^*, y^*) satisfying $y^* = \theta x^*$. To clarify how the selection of feedback gains shapes the controlled dynamics, two representative gain pairs were

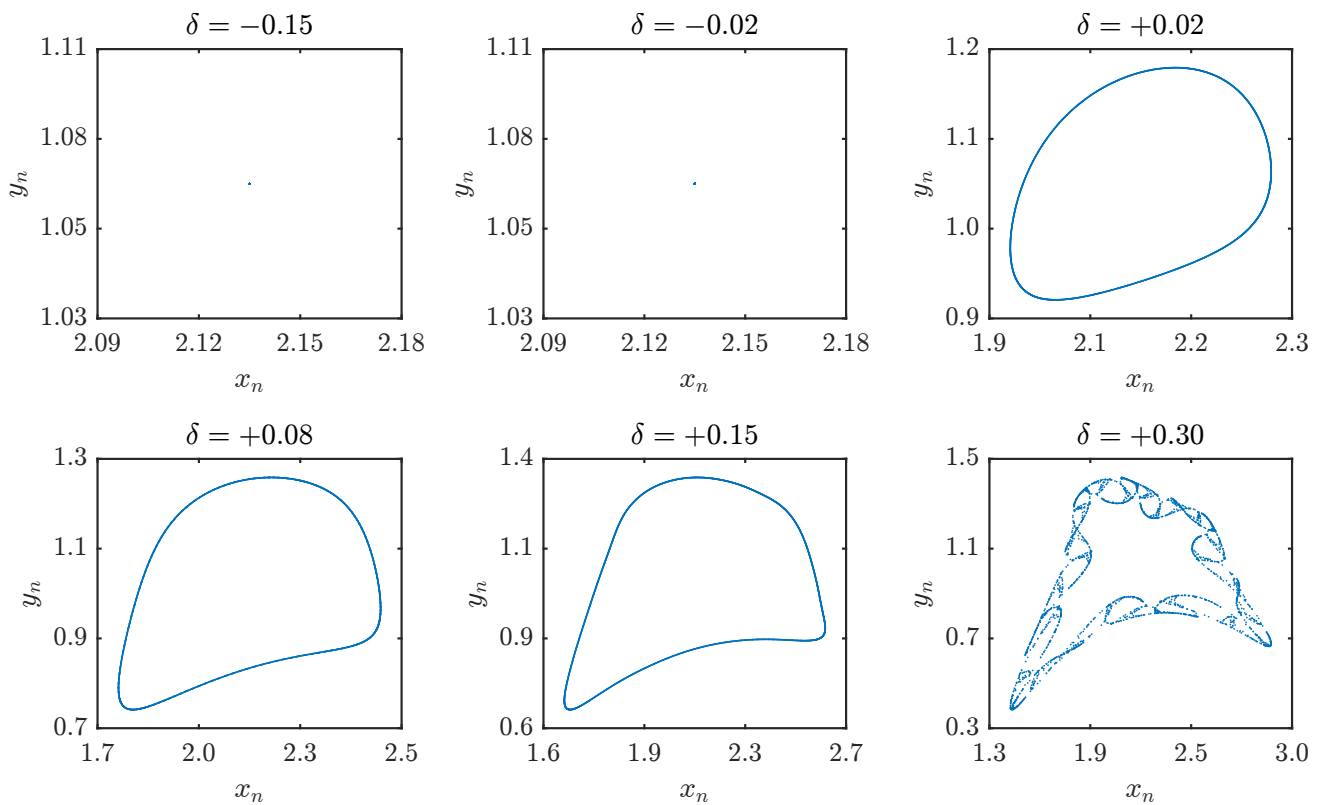


Figure 7. Phase portraits in the (x_n, y_n) plane for six representative values of δ . Take the values of the parameters $r = 1.0, s = 0.5, k = 3.0, m = 1.5, b = 1.0, \theta = 0.5, h^* \approx 2.8037$, and the initial value is approximately $(2.137, 1.067)$.

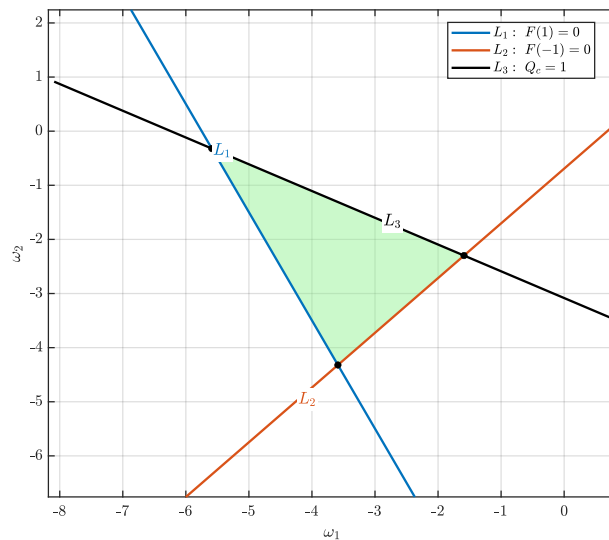


Figure 8. Stability region for the controlled system in the (ω_1, ω_2) plane when Flip bifurcation occurs (green triangular region).

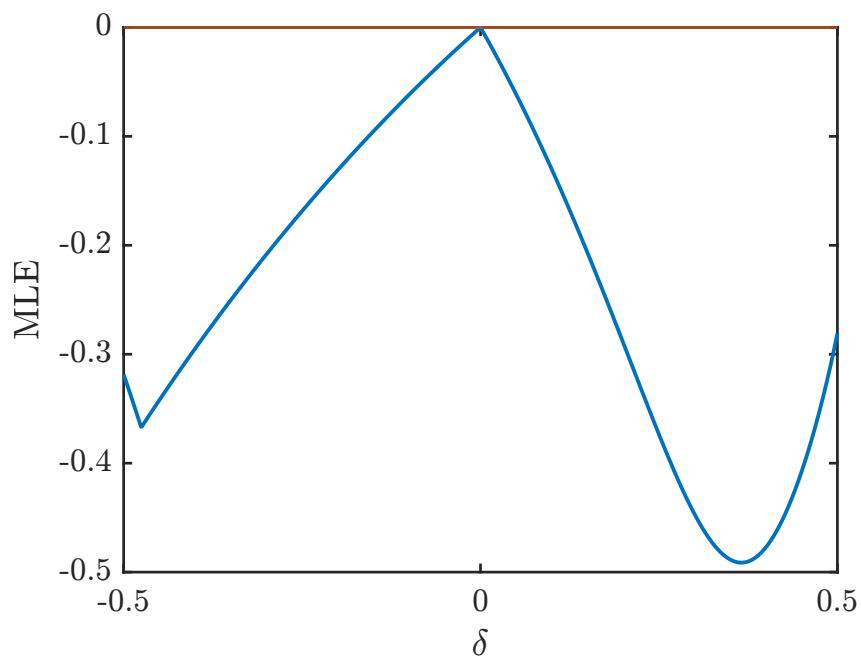


Figure 9. MLE of the controlled system as a function of the perturbation parameter δ .

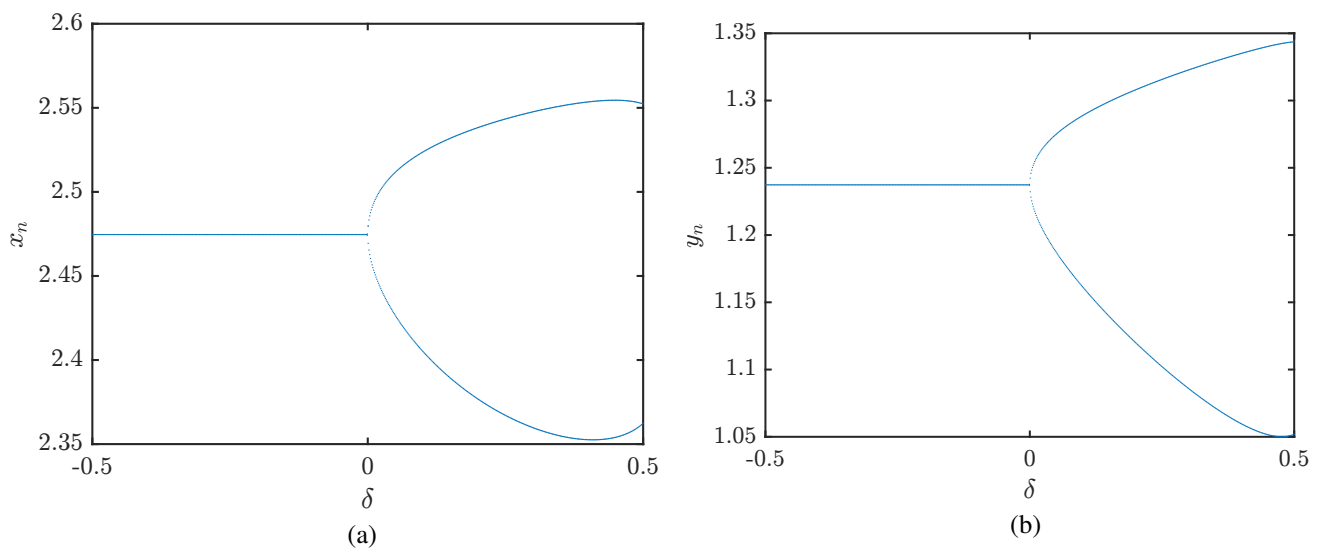


Figure 10. Bifurcation diagrams of the prey population x_n and predator population y_n versus δ for the controlled system.

taken from the stability region of Figure 8 and compared over an identical range of the perturbation parameter δ .

Under the first pair, $(\omega_1, \omega_2) = (-2.588, -3.310)$, the MLE in Figure 9 does not become positive at any point in the examined interval, so chaotic motion is suppressed entirely. The bifurcation diagram in Figure 10 shows that the coexistence equilibrium (x^*, y^*) remains stable through the tested positive values of δ and only at larger δ gives way to a bounded period-2 orbit rather than to chaos; the Flip bifurcation is thus shifted to a substantially larger δ , and the most complex attainable state is capped at small-amplitude periodic coexistence.

Under the second gain pair, $(\omega_1, \omega_2) = (-3.038, -2.982)$, the MLE in Figure 11 remains strongly negative over most of the investigated interval and increases toward zero only near the upper end of the parameter range. To show the corresponding loss of stability more clearly, Figure 12 presents a magnified view of the bifurcation structure over $0.40 \leq \delta \leq 0.50$. The controlled equilibrium remains stable before the right-hand transition region is reached, after which increasingly complex dynamical behavior develops. The complete numerical scan used for the stability and feasibility analysis nevertheless covers the full interval $-0.35 \leq \delta \leq 0.50$.

The two selections therefore embody a clear design trade-off. The first pair (Figures 9 and 10) delivers comparatively soft stabilization. Although it does not maintain the equilibrium over as wide a parameter range, the system evolves into bounded, regular period-2 oscillations as δ increases, without entering a chaotic regime. The MLE remains non-positive throughout the examined interval. The second pair (Figures 11 and 12) provides stronger stabilization and keeps both populations close to the coexistence equilibrium over a wider range of δ , as indicated by the strongly negative MLE near $\delta = 0$. However, it exhibits a relatively sharp stability boundary, beyond which increasingly complex dynamics emerge as δ increases.

The two gain pairs therefore illustrate different mathematical stabilization characteristics within the controlled discrete map. The feedback gains ω_1 and ω_2 are treated here as abstract control parameters. Relating these gains quantitatively to specific ecological interventions would require an independently calibrated control model and empirical validation.

Figures 13–16 illustrate the state-feedback stabilization of the Neimark-Sacker bifurcation of system (5.1). The parameters are $r = 1.0$, $s = 0.5$, $k = 3.0$, $m = 1.5$, $b = 1.0$, and $\theta = 0.5$, yielding the unique positive equilibrium $E^*(x^*, y^*) \approx (2.1360, 1.0680)$ and the uncontrolled critical step size $h^* \approx 2.8037$. Figure 13 shows the local stability region in the (ω_1, ω_2) plane obtained from the unit-circle stability conditions. The green triangular region consists of feedback-gain combinations for which the eigenvalues of the controlled Jacobian lie inside the unit circle. The representative gain pair $(\omega_1, \omega_2) = (-0.50, -0.50)$, located inside this region, is used in the subsequent numerical simulations shown in Figures 14–16.

The bifurcation diagram and MLE in Figures 14 and 15 show that the selected feedback gains substantially extend the stable parameter interval relative to the uncontrolled Neimark-Sacker threshold at $\delta = 0$. The controlled equilibrium remains stable over a broad positive range of δ , and the loss of stability occurs only near the upper part of the investigated interval. Positive MLE values subsequently appear near the far-right end of the scan, indicating the development of chaotic dynamics. Figure 16 provides representative phase portraits at $\delta = +0.70, +0.85, +0.92, +1.00, +1.05$, and $+1.12$, showing the transition from convergence toward the equilibrium to closed invariant curves, increasingly deformed invariant structures, and finally irregular dynamics. These results demonstrate

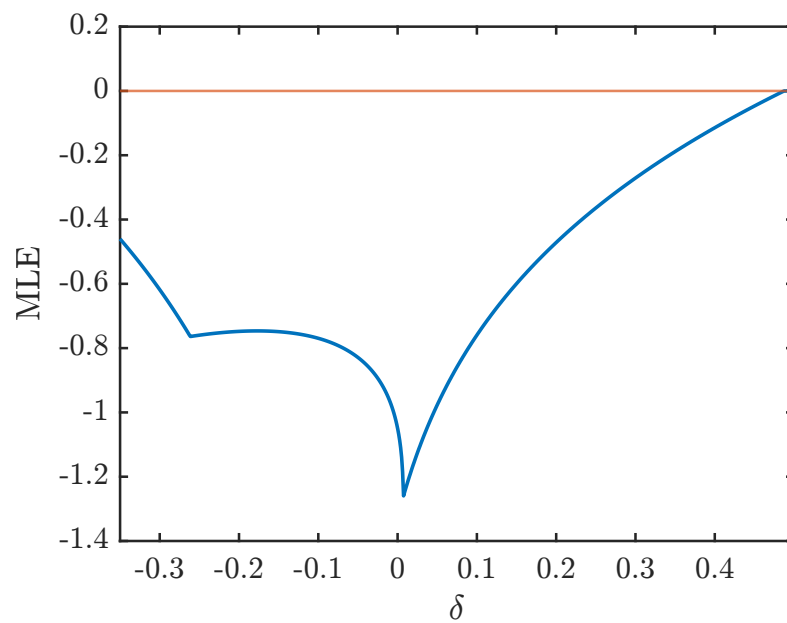


Figure 11. MLE of the controlled system as a function of the perturbation parameter δ .

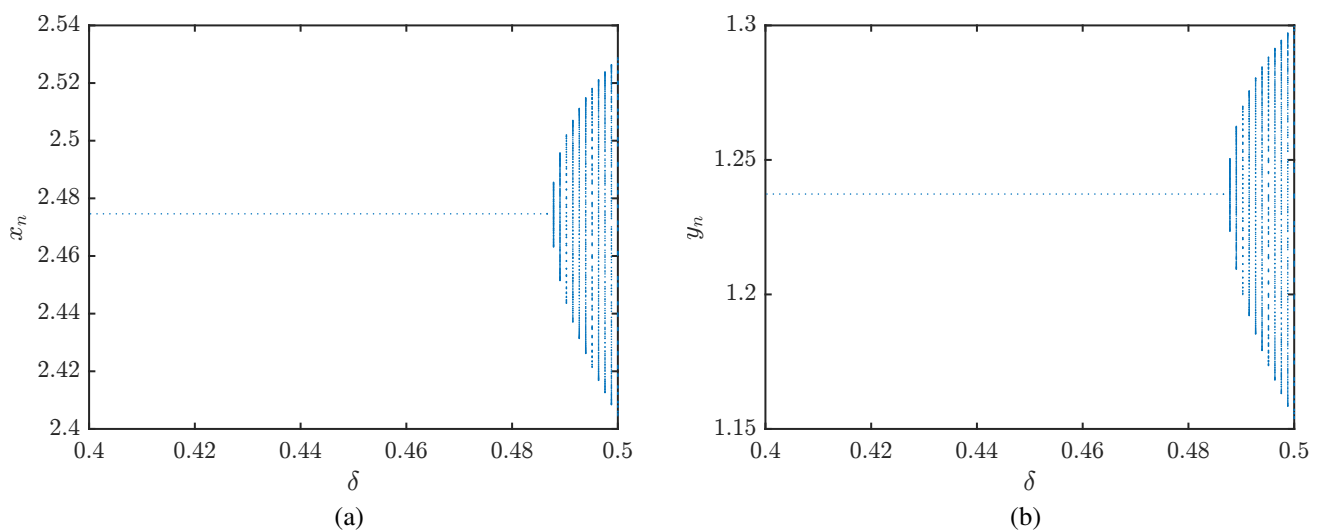


Figure 12. Magnified bifurcation diagrams of the prey population x_n and predator population y_n for the state-feedback-controlled system over $0.40 \leq \delta \leq 0.50$. The full numerical scan and feasibility verification were performed over $-0.35 \leq \delta \leq 0.50$.

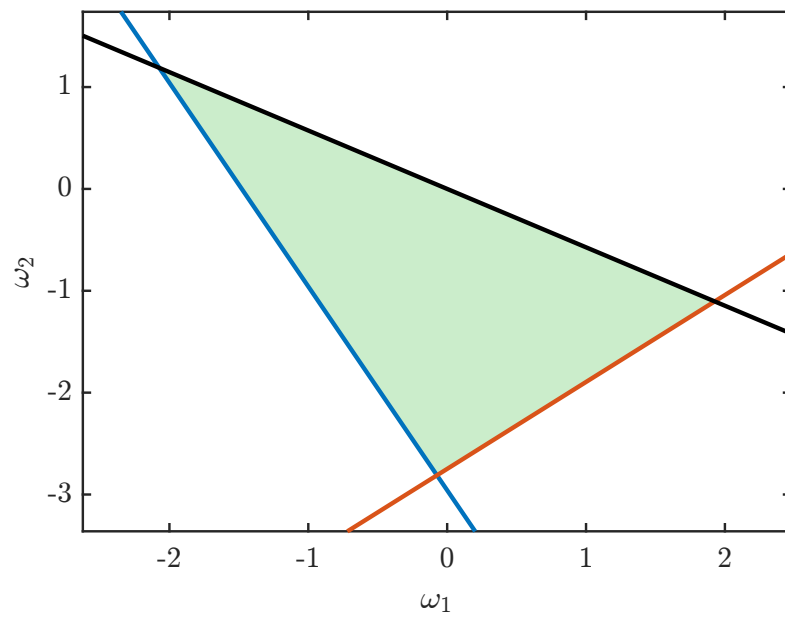


Figure 13. Stability region of the state-feedback-controlled system (5.1) in the (ω_1, ω_2) plane at the Neimark-Sacker bifurcation point. The green triangular region represents the feedback gains satisfying the local unit-circle stability conditions, and the marked point $(\omega_1, \omega_2) = (-0.50, -0.50)$ is selected for the numerical simulations in Figures 14–16.

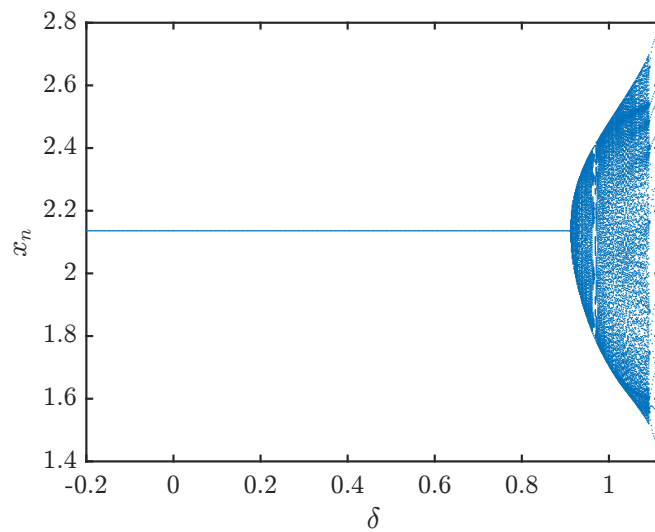


Figure 14. Bifurcation diagram of the controlled system (5.1) with respect to the perturbation parameter δ ($\omega_1 = -0.50$, $\omega_2 = -0.50$, $h^* \approx 2.8037$).

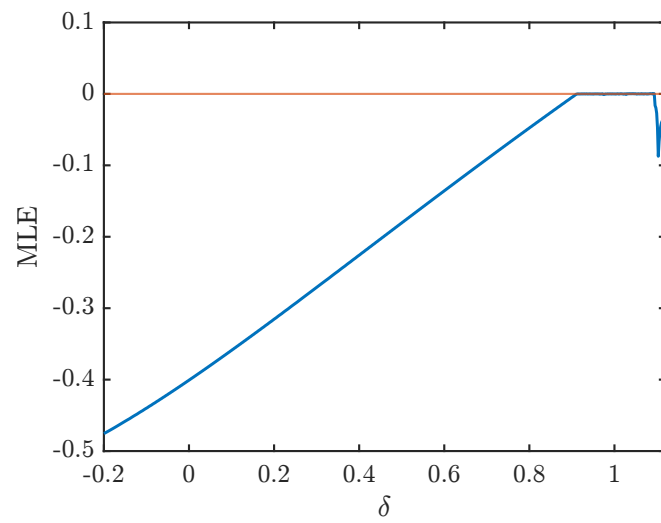


Figure 15. Variation of the MLE of the controlled system with respect to δ .

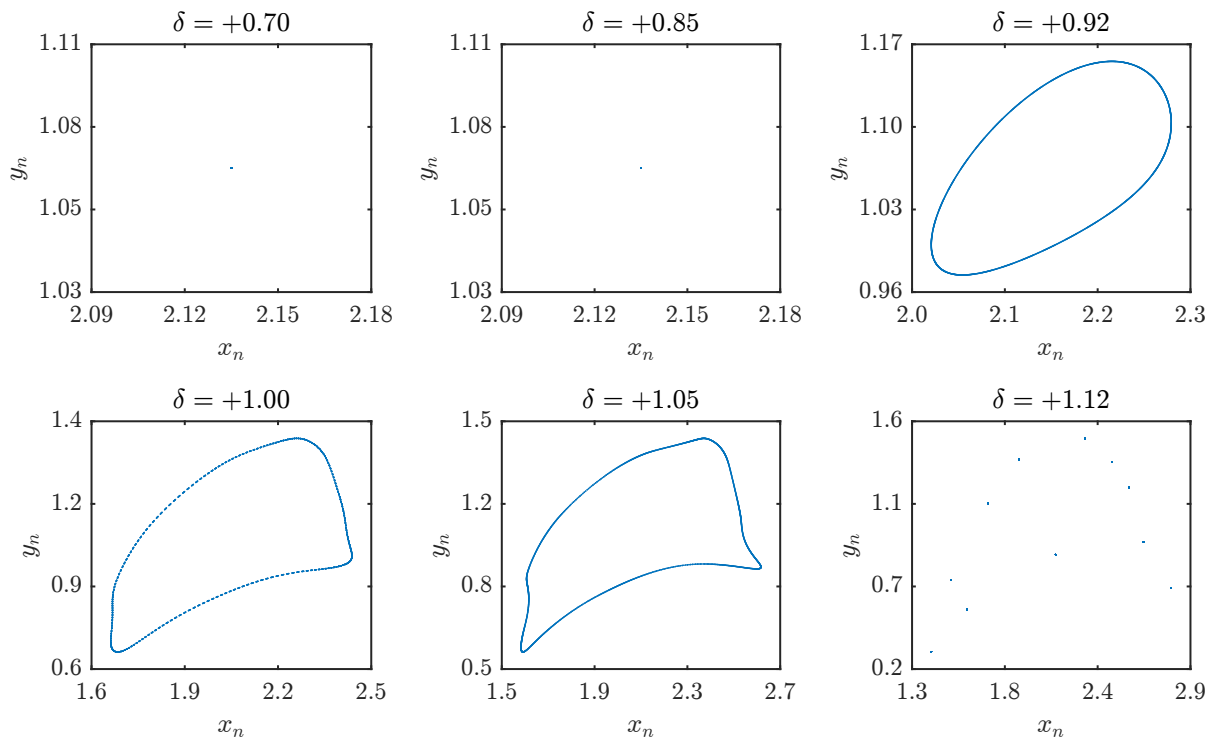


Figure 16. Phase portraits of the controlled system (5.1) at $\delta = +0.70, +0.85, +0.92, +1.00, +1.05$, and $+1.12$.

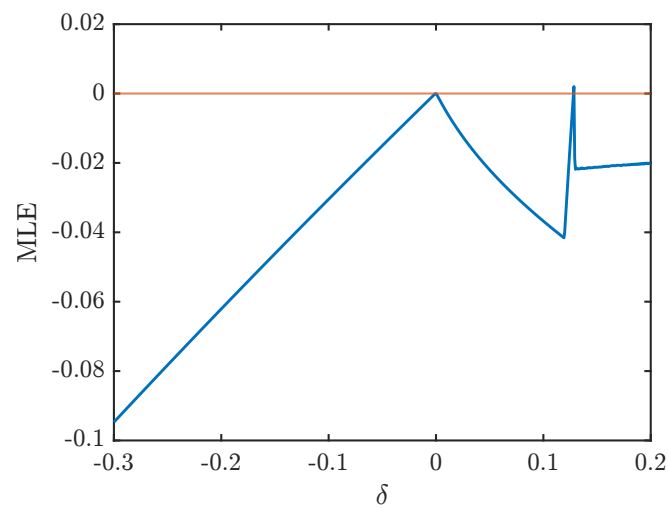


Figure 17. MLE of the controlled system as a function of the perturbation parameter δ .

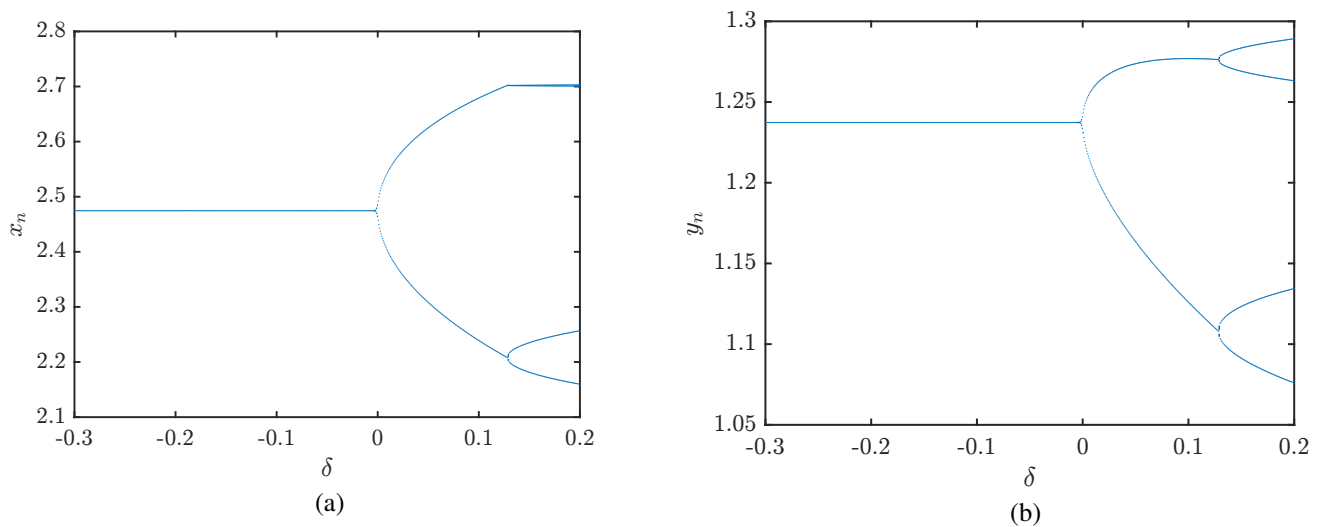


Figure 18. Bifurcation diagrams of the prey population x_n and predator population y_n versus δ for the controlled system.

the stabilization effect of the selected gain pair within the Euler-generated controlled map.

Figures 17 and 18 systematically illustrate the delay and suppression of the Flip bifurcation in the discrete Leslie-Gower predator-prey model with a Holling type-IV functional response under hybrid control. The parameters considered are: $r = 3.2$, $k = 3.0$, $m = 3.0$, $b = 0.5$, $s = 0.5$, $\theta = 0.5$, with the positive equilibrium (x^*, y^*) satisfying $y^* = \theta x^*$. Under the first weight, $\omega \approx 0.25$ (giving a theoretical cascade stretching factor $1/\omega \approx 4$), the MLE in Figure 17 remains strictly non-positive across nearly the entire scan, crossing zero only at the far right end near $\delta \approx 0.13$. The bifurcation diagrams in Figure 18 show that the period-doubling cascade of both prey density x_n and predator density y_n is pushed substantially rightward: the first period-2 bifurcation does not appear until $\delta \approx 0.1$, with period-4 and the onset of chaos postponed accordingly.

Under the second weight, $\omega = 0.2489$ (with a marginally larger nominal stretching factor $1/\omega \approx 4.018$), the MLE in Figure 19 follows a qualitatively similar but quantitatively shifted profile: it stays negative through most of the interval yet crosses zero noticeably earlier, near $\delta \approx 0.1$. Correspondingly, the bifurcation diagrams in Figure 20 reveal that the period-2 onset of x_n has migrated leftward to $\delta \approx 0$ and that of y_n to roughly $\delta \approx -0.1$, so that the entire period-doubling cascade is compressed back toward the original instability threshold of the uncontrolled system.

The two selections therefore embody a clear design trade-off in the placement of the bifurcation cascade rather than in its qualitative outcome. The first weight (Figures 17 and 18) realizes a wide-window stabilization: the period-doubling structure is stretched across a substantial positive interval of δ , and the buffer between the equilibrium and the chaotic regime is maximal. The second weight (Figures 19 and 20), despite a slightly larger nominal stretching factor, produces a narrower realized window: the cascade is repositioned much closer to the original bifurcation point and the system loses stable coexistence sooner as δ grows. The contrast makes explicit that the theoretical factor $1/\omega$ alone does not determine effective performance—coupling between ω , the delay Δh , and the underlying nonlinear structure can amplify a small change in ω into a pronounced shift of the cascade.

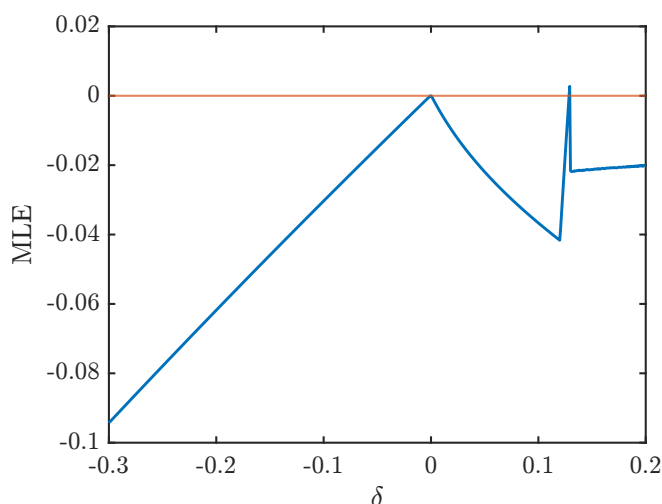


Figure 19. MLE of the controlled system as a function of the perturbation parameter δ .

These results show that the weighting parameter ω can substantially modify the location of the bifurcation cascade in the Euler-generated map. Since ω acts through the effective step size $h_{\text{eff}} = \omega h$, it

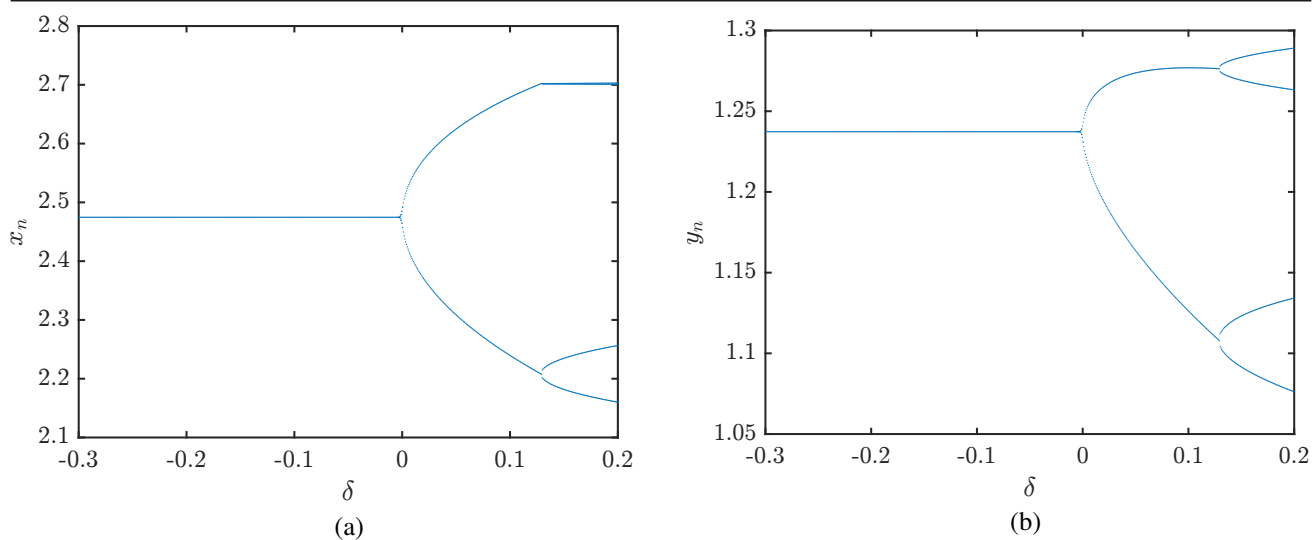


Figure 20. Bifurcation diagrams of the prey population x_n and predator population y_n versus δ for the controlled system.

is interpreted here as a mathematical weighting parameter rather than as a directly calibrated ecological intervention intensity.

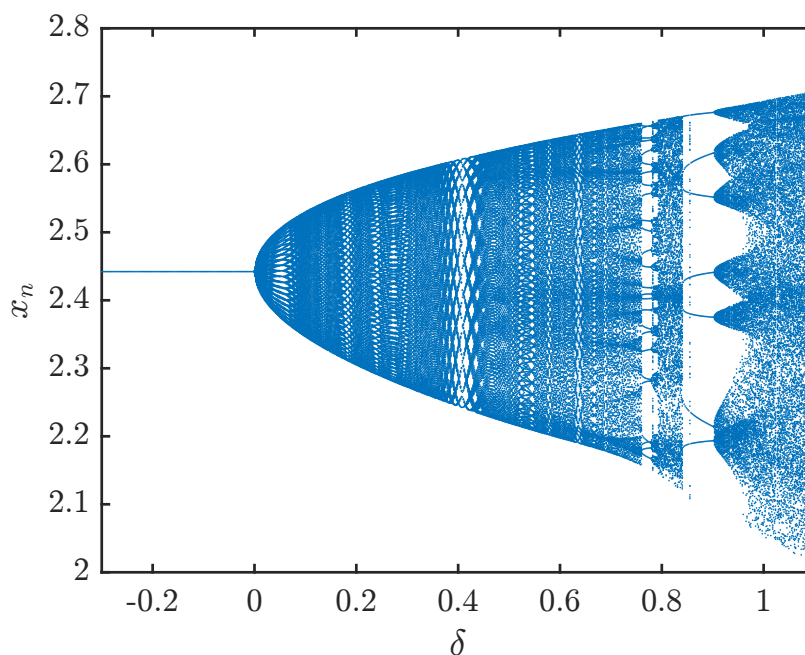


Figure 21. Bifurcation diagram of the hybrid-controlled system with respect to the perturbation parameter δ over the interval $[-0.30, 1.10]$ for $\omega = 0.5$.

Figures 21–23 illustrate the dynamics of the hybrid-controlled system near the Neimark-Sacker bifurcation. The parameters are $r = 1.1$, $k = 3.0$, $m = 0.5$, $b = 1.2$, $s = 0.2$, $\theta = 1.2$, and the

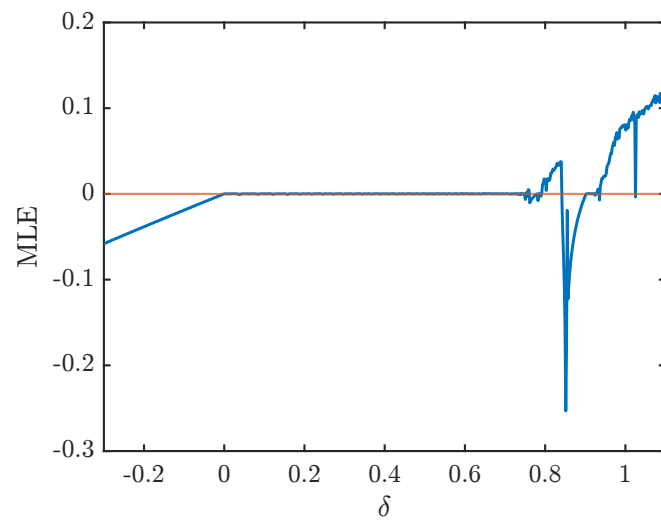


Figure 22. MLE of the hybrid-controlled system with respect to the perturbation parameter δ over the interval $[-0.30, 1.10]$ for $\omega = 0.5$.

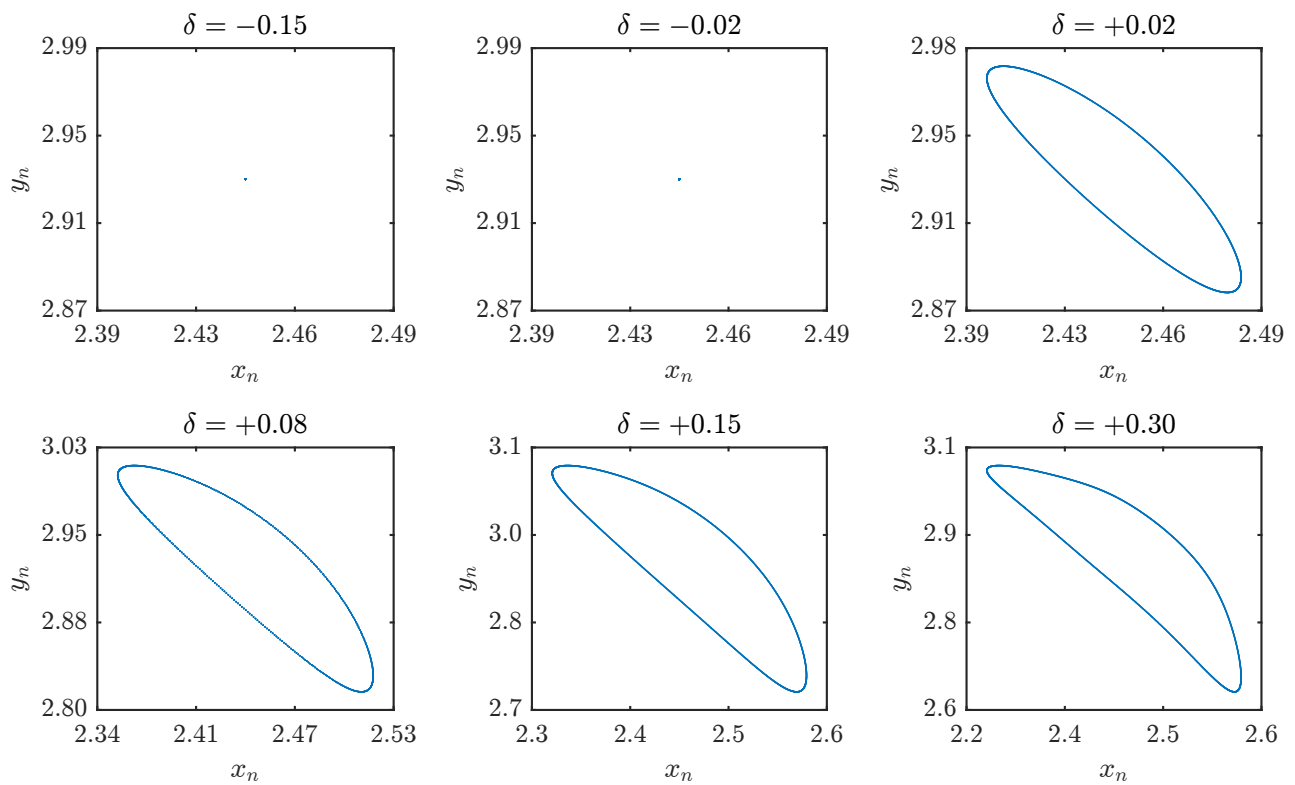


Figure 23. Phase portraits of the hybrid-controlled system in the (x_n, y_n) plane at six representative values of the perturbation parameter: $\delta = -0.15, -0.02, +0.02, +0.08, +0.15,$ and $+0.30$, with $\omega = 0.5$.

control weight is fixed at $\omega = 0.5$. The corresponding positive equilibrium is approximately $E^* = (2.4421, 2.9306)$.

Figure 21 presents the bifurcation diagram of the hybrid-controlled map over the interval $\delta \in [-0.30, 1.10]$, while Figure 22 shows the corresponding MLE. For negative values of δ , the orbit remains close to the positive equilibrium and the MLE is negative. As δ increases through the controlled Neimark-Sacker threshold, the equilibrium loses stability and quasi-periodic dynamics emerge. With further increases in δ , the bifurcation structure becomes increasingly complex and the MLE eventually becomes positive, indicating the appearance of chaotic dynamics.

Figure 23 shows representative phase portraits at $\delta = -0.15, -0.02, +0.02, +0.08, +0.15$, and $+0.30$. For the negative perturbations, the trajectory remains close to the positive equilibrium. Immediately beyond the bifurcation threshold, a small closed invariant curve appears around the equilibrium. As δ increases, the invariant curve expands and deforms, illustrating the progressive development of quasi-periodic oscillations in the hybrid-controlled Euler map. These phase portraits are consistent with the corresponding bifurcation diagram and MLE results.

7. Conclusions

In this paper, we investigated an Euler-generated discrete predator-prey map incorporating a Holling type-IV functional response and Leslie-Gower predator dynamics. Because the Leslie-Gower term is singular at zero prey density, the analysis was restricted to the biologically feasible domain $x > 0$ and $y \geq 0$. Sufficient one-step positivity conditions were derived for the uncontrolled map, and biological feasibility was also examined for the two controlled systems. The numerical trajectories used in the bifurcation diagrams, MLE calculations, and phase portraits were checked directly to ensure that they remained biologically feasible without clipping or artificial correction of negative state values. The possible configurations of positive equilibria were characterized through the discriminant of the associated cubic equation, and the local stability properties of the boundary and positive equilibria were determined using the eigenvalues of the corresponding Jacobian matrices.

The Flip and Neimark-Sacker bifurcations of the positive equilibrium were then analyzed using center manifold reduction and normal form theory. Critical values of the Euler step size were obtained from the corresponding unit-circle conditions, together with the nondegeneracy, transversality, and nonresonance conditions required for the two bifurcations. Numerical bifurcation diagrams, MLE spectra, time series, and phase portraits were consistent with the theoretical predictions and illustrated the transition from stable equilibrium behavior to periodic, quasi-periodic, and more complex dynamics. A small-step convergence test further confirmed the expected first-order convergence of the forward Euler method to the continuous system. Consequently, the large-step bifurcation and chaotic regimes discussed in this work should be interpreted as nonlinear dynamics of the Euler-generated discrete map rather than as quantitatively accurate trajectories of the original continuous-time system.

Two stabilization approaches were considered. For state feedback control, the local stability conditions lead to a triangular stability region in the (ω_1, ω_2) gain plane, from which representative feedback gains were selected for numerical investigation. For the hybrid-controlled map, the weighting parameter $\omega \in (0, 1)$ modifies the effective Euler step size according to $h_{\text{eff}} = \omega h$, and the corresponding local stability conditions can be expressed explicitly in terms of ω . The numerical results show that suitable control parameters can enlarge the parameter interval over which the

positive equilibrium remains stable and can delay the appearance of complex discrete-map dynamics. These results provide a theoretical comparison of the two stabilization mechanisms, but the control parameters should presently be regarded as mathematical control quantities rather than as directly calibrated ecological management variables.

The ecological mechanisms represented by the Holling type-IV response and the Leslie-Gower term remain relevant to predator-prey interactions, but direct ecological interpretation of the discretization-induced bifurcation thresholds and control parameters requires additional validation. Future work should therefore consider discrete models derived directly from biological life-cycle or sampling mechanisms, parameter estimation using experimental or field population data, and comparisons between such models and the Euler-generated map studied here. Further extensions may include stage structure, time delays, environmental stochasticity, spatial dispersal, seasonal forcing, and alternative food resources. Establishing an empirical relationship between mathematical control parameters and measurable management actions would also be necessary before the proposed stabilization schemes could be used for practical biological-control decision making.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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