



Research article

An efficient ADI compact scheme with Strang splitting for the two-dimensional cubic-quintic-septic nonlinear Schrödinger equation

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Abstract: We studied the two-dimensional cubic-quintic-septic nonlinear Schrödinger equation. We first established its continuous mass, energy, and momentum conservation laws, and then proposed an alternating direction implicit (ADI) compact finite difference scheme combined with Strang splitting. The splitting technique isolates the nonlinear term for an exact analytical solution via phase rotation, while a 3-point Padé approximation resolves the linear term to fourth-order spatial accuracy without requiring boundary ghost points. We rigorously proved the scheme is unconditionally stable with a convergence rate of $O(\tau^2 + h_x^4 + h_y^4)$. Numerical experiments validated the theoretical analysis and demonstrated the method's high-order accuracy and efficiency.

Keywords: nonlinear Schrödinger equation; finite difference method; ADI method; Strang splitting

1. Introduction

The dynamics of nonlinear wave propagation in two spatial dimensions plays a central role in modern photonics and quantum matter research, governing phenomena ranging from spatial soliton formation in bulk nonlinear media to the evolution of Bose-Einstein condensates in pancake-shaped traps [1, 2]. In

these settings, the cubic (Kerr) nonlinear Schrödinger equation (NLSE) has traditionally served as the canonical model. Nevertheless, as optical power densities increase or ultracold atomic interactions are tuned into strongly nonlinear regimes, the Kerr description alone becomes insufficient. Higher-order self-phase modulation terms—quintic and septic—must then be retained in the polynomial expansion of the refractive index or scattering length, leading to the cubic-quintic-septic NLSE [3, 4].

In this paper, we extend our recent research on the one-dimensional cubic-quintic-septic NLSE in [5] to the two-dimensional case on a bounded rectangular domain. The governing initial-boundary value problem reads:

$$iu_t + a(u_{xx} + u_{yy}) + [b_1 |u|^2 + b_2 |u|^4 + b_3 |u|^6] u = 0, \quad (x, y, t) \in \Omega \times (0, T], \quad (1a)$$

$$u(x, y, 0) = \phi(x, y), \quad (x, y) \in \overline{\Omega}, \quad (1b)$$

$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \quad t \in [0, T], \quad (1c)$$

where $u(x, y, t)$ is a complex-valued wave function, $\Omega = (x_l, x_r) \times (y_l, y_r) \subset \mathbb{R}^2$ is an open rectangular domain with closure $\overline{\Omega} = [x_l, x_r] \times [y_l, y_r]$, and $\partial\Omega$ denotes its boundary. In the context of nonlinear optics, u represents the slowly varying envelope of the optical field, and the terms $b_1|u|^2u$, $b_2|u|^4u$, and $b_3|u|^6u$ account for the cubic (Kerr), quintic, and septic self-phase modulation arising from the polynomial expansion of the nonlinear refractive index [3, 4]; the same equation models pancake-shaped Bose–Einstein condensates with higher-order interactions [6, 7]. The dispersion coefficient satisfies $a > 0$, while $\phi(x, y)$ is a prescribed smooth initial profile. Homogeneous Dirichlet conditions in Eq (1c) are imposed under the physical assumption that the localized wave packet does not reach the computational boundary during the simulation.

Owing to the non-integrable nature of the polynomial nonlinearity and the failure of the Painlevé test [4], closed-form analytical solutions for generic initial data are generally unavailable. Consequently, accurate and efficient numerical simulation becomes indispensable for exploring the rich dynamics—including vortex solitons, collapse arrest, and transverse instability—that distinguish two-dimensional septic nonlinearities from their one-dimensional counterparts [6, 7].

Over the past decade, substantial progress has been made in the numerical treatment of nonlinear Schrödinger-type equations. For spatial discretization, compact finite difference schemes based on Padé approximations have attracted considerable attention [8–12]. On the temporal side, operator splitting offers a structure-preserving method for separating linear dispersion from nonlinear phase modulation, with recent analyses establishing uniform-in-time error bounds and long-time stability [13, 14]. The synergy between these two strands was demonstrated by Liu et al. [15], who devised conservative high-order finite difference schemes for coupled NLSE systems, and by Wang et al. [16, 17], who developed sixth-order energy-preserving compact discretizations for nonlinear wave equations. In the specific context of two-dimensional problems, Zhang et al. [18] constructed a high-order compact ADI scheme for fractional-in-time NLSE, while Bai et al. [19] proposed novel mass- and energy-conserving Runge–Kutta-type integrators that avoid iterative nonlinear solvers altogether. Very recently, Chai et al. [20] developed two families of weighted- θ compact ADI difference schemes for the three-dimensional space-fractional complex Ginzburg–Landau equation. We emphasize that the closely related two-dimensional compact ADI schemes [18, 21] are designed for fractional NLSEs with cubic-only nonlinearities, and the time-fractional scheme in [18] is dissipative. In contrast, the present scheme targets the classical cubic-quintic-septic NLSE and, through Strang splitting, solves the nonlinear sub-problem exactly without any iteration while conserving the discrete mass exactly.

Despite these advances, the cubic-quintic-septic NLSE in two dimensions presents a set of intertwined challenges that have not been fully addressed in the existing literature. First, the septic term $|u|^6 u$ introduces severe stiffness, producing solitary waves with sharp transverse gradients that demand high spatial resolution. Second, direct implicit treatment of the full two-dimensional Laplacian at each time step leads to large, non-banded linear systems whose solutions are computationally expensive. Third, standard fully implicit conservative schemes require iterative nonlinear solvers at every time step, significantly increasing the overall computational cost. These considerations motivate a numerical strategy that combines the efficiency of ADI factorization, the accuracy of compact Padé discretization, and the exact treatability of the nonlinearity via Strang splitting.

To meet these requirements, we propose an ADI compact finite difference scheme coupled with Strang splitting for the two-dimensional cubic-quintic-septic NLSE, which is completely different from our previous work for the nonlinear numerical scheme in the 1D case [5]. Strang splitting [22,23] decomposes the evolution into a linear dispersive sub-problem, solved by the compact ADI method [21,24], and a nonlinear sub-problem that admits an exact pointwise phase-rotation solution. Because the nonlinear step is solved analytically, no Newton-type iteration used in [5] is needed. ADI factorization reduces each linear step to a sequence of one-dimensional tridiagonal solves, preserving the $O(JK)$ complexity per time step. Meanwhile, the compact Padé formulation achieves fourth-order spatial accuracy on a three-point stencil without introducing ghost points beyond the physical boundary.

Compared with the one-dimensional setting in [5], the two-dimensional extension brings new technical challenges. First, ADI factorization introduces an additional mixed cross term into the error analysis. In addition, H^1 stability analysis relies on the commutativity of the directional operators. Finally, the maximum-norm estimate involves the two-dimensional discrete Sobolev embedding with a logarithmic factor.

The remainder of the paper is structured as follows. Section 2 derives the continuous conservation laws for mass, energy, and momentum. Section 3 presents the ADI compact scheme with Strang splitting in detail. Section 4 establishes the unconditional stability and convergence theory. Finally, Section 5 reports numerical experiments that confirm the theoretical convergence orders and demonstrate the long-term stability of the method.

2. Conservation laws

The cubic-quintic-septic NLSE inherits a Hamiltonian structure that gives rise to three fundamental conserved quantities: mass, energy, and momentum. These invariants not only certify the physical consistency of the model but also serve as benchmarks for assessing the structural fidelity of numerical approximations. In this section, we state the conservation laws for the two-dimensional IBVP Eqs (1a)–(1c) and sketch their derivations, emphasizing the aspects that differ from the one-dimensional setting.

It is convenient to encapsulate the polynomial nonlinearity through the real-valued potential

$$V(s) = b_1 |s|^2 + b_2 |s|^4 + b_3 |s|^6, \quad s \in \mathbb{C}, \quad (2)$$

and the associated energy density

$$H(s) = \frac{b_1}{2} s^2 + \frac{b_2}{3} s^3 + \frac{b_3}{4} s^4, \quad s \in \mathbb{R}, \quad (3)$$

so that $V(u) = H'(|u|^2)$ and consequently

$$\frac{\partial}{\partial x} H(|u|^2) = V(u) \frac{\partial}{\partial x} |u|^2. \quad (4)$$

With this compact notation, the governing (1a) becomes

$$iu_t + a\Delta u + V(u)u = 0, \quad (x, y, t) \in \Omega \times (0, T], \quad (5)$$

where $\Delta = \partial_{xx} + \partial_{yy}$ denotes the two-dimensional Laplacian. Throughout this section, $\Im(\cdot)$ stands for the imaginary part of the function.

2.1. Mass conservation

The total mass (or L^2 norm) of the wave function is conserved under homogeneous Dirichlet boundary conditions.

Proposition 2.1 (Mass conservation). *Let $\phi \in H_0^2(\Omega)$. Then the solution of Eqs (1a)–(1c) satisfies*

$$M(t) = \iint_{\Omega} |u(x, y, t)|^2 dx dy = M(0), \quad t \in [0, T].$$

Proof. Multiplying Eq (1a) by $\bar{u}$ (the complex conjugate of u) and taking the imaginary part yields

$$\frac{1}{2} \frac{\partial}{\partial t} |u|^2 + a \Im(\bar{u} \Delta u) = 0. \quad (6)$$

Using the identity $\Im(\bar{u} \Delta u) = \nabla \cdot \Im(\bar{u} \nabla u)$, integration over Ω , together with the divergence theorem gives

$$\frac{d}{dt} M(t) = -a \oint_{\partial\Omega} \Im(\bar{u} \partial_n u) ds = 0, \quad (7)$$

where the boundary integral vanishes because $u|_{\partial\Omega} = 0$. $\square$

2.2. Energy conservation

For the energy functional, we must account for the gradient energy in both spatial directions.

Proposition 2.2 (Energy conservation). *Let $\phi \in H_0^2(\Omega)$. Then the solution of Eqs (1a)–(1c) satisfies*

$$E(t) = \iint_{\Omega} \left[a(|u_x|^2 + |u_y|^2) - \frac{b_1}{2} |u|^4 - \frac{b_2}{3} |u|^6 - \frac{b_3}{4} |u|^8 \right] dx dy = E(0). \quad (8)$$

Proof. The result follows from the Hamiltonian structure of the gauge-invariant equation (1a), we refer to [2] for the standard proof valid for general gauge-invariant nonlinearities. Briefly, one multiplies Eq (1a) by $\bar{u}_t$, takes the real part, and integrates over Ω by parts. The boundary terms vanish since the homogeneous Dirichlet condition Eq (1c) implies $u_t|_{\partial\Omega} = 0$. $\square$

2.3. Momentum conservation

In two dimensions, the momentum is a vector field with components (P_x, P_y) . On a bounded domain, these quantities are not strictly conserved under homogeneous Dirichlet conditions because the fixed boundaries break translation invariance. Nevertheless, if the localized wave packet remains well-separated from $\partial\Omega$ throughout the simulation, the boundary fluxes are negligible and the momentum is effectively preserved. The following proposition makes this statement precise by working directly on the bounded domain and tracking the boundary terms.

Proposition 2.3 (Momentum conservation). *Let $\phi \in H_0^2(\Omega)$ and assume that the solution satisfies $u, \nabla u \in H^1(\Omega)$. Then the momentum components*

$$P_x(t) = -\Im \iint_{\Omega} \bar{u} u_x dx dy, \quad P_y(t) = -\Im \iint_{\Omega} \bar{u} u_y dx dy \quad (9)$$

satisfy $dP_x/dt = dP_y/dt = 0$, provided the boundary terms that arise from the divergence theorem vanish.

Proof. We verify the invariance of P_x , and the argument for P_y is identical upon interchanging x and y . Differentiating P_x in time and integrating by parts gives

$$\frac{dP_x}{dt} = -\Im \iint_{\Omega} (\bar{u}_t u_x + \bar{u} u_{xt}) dx dy = -\Im \iint_{\Omega} (\bar{u}_t u_x - \bar{u}_x u_t) dx dy. \quad (10)$$

Inserting the expressions $u_t = i(a\Delta u + V(u)u)$ and $\bar{u}_t = -i(a\Delta \bar{u} + V(u)\bar{u})$ from the governing equation, the terms containing $V(u)$ cancel because they contribute a purely real integrand. The remaining dispersive part can be written as

$$-a \Im [i \partial_x (|u_x|^2 - |u_y|^2) + i \partial_y (u_y \bar{u}_x + \bar{u}_y u_x)], \quad (11)$$

which follows from the elementary identities $\partial_x |u_x|^2 = u_{xx} \bar{u}_x + \bar{u}_{xx} u_x$ and $\partial_y (u_y \bar{u}_x + \bar{u}_y u_x) = u_{yy} \bar{u}_x + \bar{u}_{yy} u_x + \partial_x |u_y|^2$. Integrating over Ω and applying the divergence theorem yields a boundary integral of the form

$$\oint_{\partial\Omega} [(|u_x|^2 - |u_y|^2)n_x + (u_y \bar{u}_x + \bar{u}_y u_x)n_y] ds. \quad (12)$$

Because $u|_{\partial\Omega} = 0$, the tangential derivatives of u vanish on each edge of the rectangular boundary: $u_y = 0$ on the top and bottom edges (where $n_y = \pm 1$) and $u_x = 0$ on the left and right edges (where $n_x = \pm 1$). Consequently, every term in the integrand is zero on $\partial\Omega$, and the boundary integral vanishes. Hence $dP_x/dt = 0$, and by the same argument, $dP_y/dt = 0$. $\square$

3. ADI compact finite difference scheme

In this section, we construct an efficient numerical scheme combining the compact finite difference method for spatial discretization, Strang splitting for temporal separation of linear and nonlinear parts, and the ADI technique for dimensional reduction.

3.1. Strang splitting for the continuous problem

Operator splitting methods solve evolution equations by decomposing them into simpler sub-problems. The theoretical foundation is based on the Lie-Trotter product formula and the Baker-Campbell-Hausdorff (BCH) theory [25]. Consider the nonlinear Schrödinger equation in (1a), written

abstractly as:

$$\frac{\partial u}{\partial t} = \mathcal{A}u + \mathcal{B}u, \quad (13)$$

where $\mathcal{A}u = ia\Delta u$ represents the linear dispersive part and $\mathcal{B}u = -iV(u)u$ represents the nonlinear part. Formally, the exact solution over a time step τ is given by:

$$u(t + \tau) = e^{\tau(\mathcal{A}+\mathcal{B})}u(t). \quad (14)$$

Direct computation of $e^{\tau(\mathcal{A}+\mathcal{B})}$ is generally infeasible due to the non-commutativity of $\mathcal{A}$ and $\mathcal{B}$. The splitting approach approximates this exponential by products of exponentials $e^{\tau\mathcal{A}}$ and $e^{\tau\mathcal{B}}$.

Strang splitting is a second-order symmetric method that improves upon first-order Lie-Trotter splitting. For operators $\mathcal{A}$ and $\mathcal{B}$, Lie-Trotter splitting:

$$u^{n+1} = e^{\tau\mathcal{B}}e^{\tau\mathcal{A}}u^n \quad (15)$$

is only first-order accurate. Strang splitting achieves second order by the symmetric composition [22]:

$$u^{n+1} = e^{\frac{\tau}{2}\mathcal{B}}e^{\tau\mathcal{A}}e^{\frac{\tau}{2}\mathcal{B}}u^n. \quad (16)$$

The symmetry ensures that if we reverse time ($\tau \rightarrow -\tau$), the scheme remains in the same form, which is necessary for even-order accuracy in splitting methods. We split Eq (1a) as follows:

Nonlinear sub-problem:

$$iu_t = -V(u)u, \quad t \in [t_n, t_{n+1}], \quad (17)$$

Linear sub-problem:

$$iu_t + a(u_{xx} + u_{yy}) = 0, \quad t \in [t_n, t_{n+1}]. \quad (18)$$

For a time step $\tau = t_{n+1} - t_n$, the Strang splitting scheme is given by:

$$u^{n+1} = \mathcal{N}(\tau/2)\mathcal{L}(\tau)\mathcal{N}(\tau/2)u^n, \quad (19)$$

where $\mathcal{N}(\tau)$ denotes the solution operator for the nonlinear sub-problem Eq (17) over time τ , and $\mathcal{L}(\tau)$ denotes the solution operator for the linear sub-problem Eq (18) over time τ .

The Strang splitting approach for the nonlinear Schrödinger equation has the following properties. Since both sub-problem Eqs (17) and (18) are posed for $t \in [t_n, t_{n+1}]$, the elapsed time $t - t_n$ appearing below ranges over $[0, \tau]$. The nonlinear sub-problem Eq (17) admits an exact solution given by

$$u(x, y, t) = u(x, y, t_n) \exp(-iV(u(x, y, t_n))(t - t_n)), \quad t \in [t_n, t_{n+1}], \quad (20)$$

which corresponds to a simple pointwise phase rotation. Meanwhile, the linear sub-problem Eq (18) reduces to the free Schrödinger equation, which can be solved efficiently using the ADI technique.

Both solution operators $\mathcal{L}(\tau)$ and $\mathcal{N}(\tau)$ are unitary, preserving the L^2 norm $\|u\|$ exactly at each sub-step. This structural preservation is important for maintaining long-time stability of the numerical scheme. Furthermore, the Strang splitting preserves the symplectic structure of the Hamiltonian system up to second order, ensuring favorable long-time behavior and near-conservation of energy over extended integration intervals.

From a computational perspective, the operator decomposition enables specialized solvers for each sub-problem: tridiagonal linear system solvers for the dispersive part via an ADI scheme, and direct explicit evaluation for the nonlinear part.

Lemma 3.1 (Local splitting error [22, 26]). *The Strang splitting in Eq (19) has a local truncation error of order $O(\tau^3)$. Consequently, the global error due to splitting is $O(\tau^2)$.*

3.2. Compact finite difference approximation

Let J , K , and N be positive integers, and let

$$h_x = \frac{x_r - x_l}{J}, \quad h_y = \frac{y_r - y_l}{K}, \quad \tau = \frac{T}{N} \quad (21)$$

be the spatial and temporal step sizes, respectively. The uniform grid points are given by

$$x_j = x_l + jh_x, \quad y_k = y_l + kh_y, \quad t^n = n\tau, \quad (22)$$

in which $j = 0, 1, \dots, J$, $k = 0, 1, \dots, K$, and $n = 0, 1, \dots, N$. Let $U_{j,k}^n \approx u(x_j, y_k, t^n)$ and denote

$$Z_h^0 = \{U = (U_{j,k}) \mid U_{j,k} = 0 \text{ for } (j, k) \in \partial\Omega_h\}, \quad (23)$$

where $\partial\Omega_h = \{(j, k) \mid j = 0 \text{ or } j = J \text{ or } k = 0 \text{ or } k = K\}$ denotes the set of boundary grid points.

The compact fourth-order approximation is defined by the following tridiagonal systems:

$$\frac{1}{12}(C_x U)_{j-1,k} + \frac{10}{12}(C_x U)_{j,k} + \frac{1}{12}(C_x U)_{j+1,k} = \frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{h_x^2}, \quad j = 1, \dots, J-1, \quad (24a)$$

$$\frac{1}{12}(C_y U)_{j,k-1} + \frac{10}{12}(C_y U)_{j,k} + \frac{1}{12}(C_y U)_{j,k+1} = \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k-1}}{h_y^2}, \quad k = 1, \dots, K-1, \quad (24b)$$

with homogeneous boundary conditions $(C_x U)_{0,k} = (C_x U)_{J,k} = 0$ and $(C_y U)_{j,0} = (C_y U)_{j,K} = 0$. The compact operators C_x and C_y satisfy the following truncation error estimates:

$$(C_x u)_{j,k}^n = (u_{xx})_{j,k}^n + O(h_x^4), \quad (C_y u)_{j,k}^n = (u_{yy})_{j,k}^n + O(h_y^4). \quad (25)$$

To make the scheme more compact, we introduce the following operators: $\mathcal{P}_x$ and $\mathcal{P}_y$ are the Padé averaging operators:

$$(\mathcal{P}_x U)_{j,k} = \frac{1}{12}U_{j-1,k} + \frac{10}{12}U_{j,k} + \frac{1}{12}U_{j+1,k}, \quad (\mathcal{P}_y U)_{j,k} = \frac{1}{12}U_{j,k-1} + \frac{10}{12}U_{j,k} + \frac{1}{12}U_{j,k+1}, \quad (26)$$

and $\mathcal{L}_x$, $\mathcal{L}_y$ are the standard second-order difference operators:

$$(\mathcal{L}_x U)_{j,k} = \frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{h_x^2}, \quad (\mathcal{L}_y U)_{j,k} = \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k-1}}{h_y^2}. \quad (27)$$

Therefore, the compact fourth-order approximation can be rewritten as:

$$(\mathcal{P}_x C_x U)_{j,k} = (\mathcal{L}_x U)_{j,k}, \quad (\mathcal{P}_y C_y U)_{j,k} = (\mathcal{L}_y U)_{j,k}. \quad (28)$$

Furthermore, we have $C_x = \mathcal{P}_x^{-1} \mathcal{L}_x$ and $C_y = \mathcal{P}_y^{-1} \mathcal{L}_y$.

3.3. The ADI compact scheme with Strang splitting

We now present the complete numerical scheme. The numerical scheme consists of three sequential stages. First, the nonlinear half-step is computed via

$$U_{j,k}^* = U_{j,k}^n \exp\left(\frac{i\tau}{2} V(U_{j,k}^n)\right), \quad 1 \leq j \leq J-1, 1 \leq k \leq K-1. \quad (29)$$

Second, the linear full-step is performed via a locally one-dimensional ADI splitting. For the linear full-step, we have:

$$\left[I - \frac{ia\tau}{2}(C_x + C_y)\right] U_{j,k}^{n+1/2} = \left[I + \frac{ia\tau}{2}(C_x + C_y)\right] U_{j,k}^*. \quad (30)$$

Notice that

$$I - \frac{ia\tau}{2}(C_x + C_y) - \frac{a^2\tau^2}{4}C_xC_y = \left(I - \frac{ia\tau}{2}C_x\right)\left(I - \frac{ia\tau}{2}C_y\right), \quad (31a)$$

$$I + \frac{ia\tau}{2}(C_x + C_y) - \frac{a^2\tau^2}{4}C_xC_y = \left(I + \frac{ia\tau}{2}C_x\right)\left(I + \frac{ia\tau}{2}C_y\right), \quad (31b)$$

and by adding the extra term $-\frac{a^2\tau^2}{4}C_xC_yU_{j,k}^{n+1/2}$ to the left-hand side of Eq (30) and the extra term $-\frac{a^2\tau^2}{4}C_xC_yU_{j,k}^*$ to its right-hand side, we have the combined ADI scheme

$$\left(I - \frac{ia\tau}{2}C_x\right)\left(I - \frac{ia\tau}{2}C_y\right)U_{j,k}^{n+1/2} = \left(I + \frac{ia\tau}{2}C_x\right)\left(I + \frac{ia\tau}{2}C_y\right)U_{j,k}^*. \quad (32)$$

The influence of these extra terms on the accuracy of the scheme is analyzed in Lemma 4.6.

Since the equation is factored, we solve it in two steps: the X-sweep and the Y-sweep. Define

$$\alpha = \frac{ia\tau}{2}. \quad (33)$$

The X-direction sweep takes the form

$$(\mathcal{P}_x - \alpha\mathcal{L}_x)U_{j,k}^{**} = (\mathcal{P}_x + \alpha\mathcal{L}_x)U_{j,k}^*, \quad (34)$$

while the Y-direction sweep is given by

$$(\mathcal{P}_y - \alpha\mathcal{L}_y)U_{j,k}^{n+1/2} = (\mathcal{P}_y + \alpha\mathcal{L}_y)U_{j,k}^{**}. \quad (35)$$

After solving the ADI scheme, the final nonlinear half-step completes the time step:

$$U_{j,k}^{n+1} = U_{j,k}^{n+1/2} \exp\left(\frac{i\tau}{2} V(U_{j,k}^{n+1/2})\right), \quad 1 \leq j \leq J-1, 1 \leq k \leq K-1. \quad (36)$$

The computational procedure consists of three phases. The initialization phase obtains the initial approximation U^0 directly from the initial condition,

$$U_{j,k}^0 = \phi(x_j, y_k), \quad 0 \leq j \leq J, 0 \leq k \leq K. \quad (37)$$

During the time-stepping phase, each step from t^n to t^{n+1} proceeds through four operations: computing the nonlinear half-step via Eq (29), solving the X-sweep tridiagonal system Eq (34), solving the Y-sweep tridiagonal system Eq (35), and computing the final nonlinear half-step via Eq (36). Finally, the boundary treatment phase enforces homogeneous Dirichlet boundary conditions:

$$U_{0,k}^n = U_{J,k}^n = U_{j,0}^n = U_{j,K}^n = 0, \quad \forall n. \quad (38)$$

Remark 3.1. The linear steps in Eqs (34) and (35) involve only tridiagonal systems in each direction: the X-sweep solves $K - 1$ tridiagonal systems of size $J - 1$, and the Y-sweep solves $J - 1$ tridiagonal systems of size $K - 1$, each by the Thomas algorithm, together with one explicit tridiagonal right-hand-side evaluation per sweep. The total cost per time step is therefore cJK with a small constant c accounting for two tridiagonal solves and two right-hand-side evaluations per grid point. The mass conservation properties of the scheme are established rigorously in Theorem 4.1.

4. Stability and convergence analysis

In this section, we analyze the stability and convergence of the ADI compact scheme with Strang splitting. We establish unconditional stability and derive error estimates.

We define the inner product and norms on Z_h^0 as follows:

$$\langle U^n, V^n \rangle = h_x h_y \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} U_{j,k}^n (V_{j,k}^n)^*, \quad \|U^n\| = \sqrt{\langle U^n, U^n \rangle}, \quad \|U^n\|_\infty = \max_{\substack{1 \leq j \leq J-1 \\ 1 \leq k \leq K-1}} |U_{j,k}^n|. \quad (39)$$

We also define the discrete gradient norms:

$$\|\delta_x U\|^2 = h_x h_y \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \left| \frac{U_{j+1,k} - U_{j,k}}{h_x} \right|^2, \quad \|\delta_y U\|^2 = h_x h_y \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left| \frac{U_{j,k+1} - U_{j,k}}{h_y} \right|^2. \quad (40)$$

The discrete H^1 norm is defined by

$$\|U\|_{H^1} = \sqrt{\|U\|^2 + \|\delta_x U\|^2 + \|\delta_y U\|^2}, \quad (41)$$

and the discrete H^1 seminorm is given by

$$|U|_{H^1} = \sqrt{\|\delta_x U\|^2 + \|\delta_y U\|^2}. \quad (42)$$

Furthermore, for a symmetric positive definite operator $\mathcal{A}$, we define the $\mathcal{A}$ -weighted inner product and its induced norm as:

$$\langle U, V \rangle_{\mathcal{A}} = \langle \mathcal{A}U, V \rangle, \quad \|U\|_{\mathcal{A}} = \sqrt{\langle \mathcal{A}U, U \rangle}. \quad (43)$$

Specifically, we utilize the norm induced by the inverse Padé operator, $\|\delta_x U\|_{\mathcal{P}_x^{-1}}^2 = \langle \mathcal{P}_x^{-1} \delta_x U, \delta_x U \rangle$.

To unify the notation for discrete gradients, we define the forward and backward difference operators δ_x and $\delta_{\bar{x}}$ as:

$$(\delta_x U)_{j,k} = \frac{U_{j+1,k} - U_{j,k}}{h_x}, \quad (\delta_{\bar{x}} U)_{j,k} = \frac{U_{j,k} - U_{j-1,k}}{h_x}, \quad (44)$$

with analogous definitions for δ_y and $\delta_{\bar{y}}$ in the y-direction. The standard second-order difference operator can then be expressed as $\mathcal{L}_x = \delta_x \delta_{\bar{x}}$. Consequently, the compact operators are consistently represented as $C_x = \mathcal{P}_x^{-1} \mathcal{L}_x$, where $\mathcal{L}_x = \delta_x \delta_{\bar{x}}$. With this notation, the discrete gradient norms become

$$\|\delta_x U\|^2 = h_x h_y \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} |(\delta_x U)_{j,k}|^2, \quad \|\delta_y U\|^2 = h_x h_y \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} |(\delta_y U)_{j,k}|^2. \quad (45)$$

4.1. Preliminary lemmas

We first recall several fundamental results that will be used in our analysis.

Lemma 4.1 ([27]). *For any mesh functions $U, V \in Z_h^0$, the following identities hold:*

$$\langle \mathcal{L}_x U, V \rangle = -\langle \delta_x U, \delta_x V \rangle = \langle U, \mathcal{L}_x V \rangle, \quad (46)$$

$$\langle \mathcal{P}_x U, V \rangle = \langle U, \mathcal{P}_x V \rangle, \quad (47)$$

$$\langle \mathcal{L}_x U, V \rangle = \langle U, \mathcal{L}_x V \rangle. \quad (48)$$

Lemma 4.2 ([27]). *The Padé averaging operator $\mathcal{P}_x$ is self-adjoint and positive definite. For any $U \in Z_h^0$, we have*

$$\frac{2}{3} \|U\|^2 \leq \langle \mathcal{P}_x U, U \rangle \leq \|U\|^2. \quad (49)$$

Furthermore, the compact operator $\mathcal{C}_x = \mathcal{P}_x^{-1} \mathcal{L}_x$ satisfies:

$$\langle \mathcal{C}_x U, U \rangle = -\|\delta_x U\|_{\mathcal{P}_x^{-1}}^2 \leq -\|\delta_x U\|^2. \quad (50)$$

Lemma 4.3 ([27]). *Let $\{e^n\}$ and $\{g^n\}$ be non-negative sequences satisfying*

$$e^{n+1} \leq (1 + C\tau)e^n + \tau g^n, \quad n = 0, 1, \dots, N-1, \quad (51)$$

where $C \geq 0$ is a constant. Then

$$e^n \leq e^{CT} \left(e^0 + \tau \sum_{m=0}^{n-1} g^m \right), \quad 0 \leq n\tau \leq T. \quad (52)$$

Lemma 4.4 (2D discrete Sobolev inequality [28]). *For any mesh function $U \in Z_h^0$ and sufficiently small $h = \min(h_x, h_y)$, there exists a constant $C > 0$ independent of h_x, h_y such that*

$$\|U\|_\infty \leq C \sqrt{|\ln h|} (\|\delta_x U\| + \|\delta_y U\|). \quad (53)$$

Proof. Let $L_x = x_r - x_l$ and $L_y = y_r - y_l$. Since U vanishes on the boundary, it can be expanded in the discrete sine basis:

$$U_{j,k} = \sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \widehat{U}_{p,q} \sin\left(\frac{p\pi j}{J}\right) \sin\left(\frac{q\pi k}{K}\right).$$

The sine family is orthogonal with $\sum_{j=1}^{J-1} \sin(p\pi j/J) \sin(p'\pi j/J) = \frac{J}{2} \delta_{pp'}$. Moreover, the difference quotient maps the sine functions onto the half-node cosine family,

$$(\delta_x U)_{j,k} = \sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \widehat{U}_{p,q} \frac{2}{h_x} \sin\left(\frac{p\pi}{2J}\right) \cos\left(\frac{p\pi(j+\frac{1}{2})}{J}\right) \sin\left(\frac{q\pi k}{K}\right),$$

which is also orthogonal, with $\sum_{j=0}^{J-1} \cos(p\pi(j+\frac{1}{2})/J) \cos(p'\pi(j+\frac{1}{2})/J) = \frac{J}{2} \delta_{pp'}$. Parseval's identity therefore gives

$$\|\delta_x U\|^2 + \|\delta_y U\|^2 = \frac{L_x L_y}{4} \sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \lambda_{p,q} |\widehat{U}_{p,q}|^2,$$

where

$$\lambda_{p,q} = \frac{4}{h_x^2} \sin^2\left(\frac{p\pi}{2J}\right) + \frac{4}{h_y^2} \sin^2\left(\frac{q\pi}{2K}\right) \geq C_0(p^2 + q^2)$$

for some constant $C_0 > 0$ independent of h_x, h_y , since $\sin x \geq \frac{2x}{\pi}$ for $x \in [0, \frac{\pi}{2}]$. Because $|\sin(\cdot)| \leq 1$, we have $\|U\|_\infty \leq \sum_{p,q} |\widehat{U}_{p,q}|$, and the Cauchy–Schwarz inequality yields

$$\begin{aligned} \|U\|_\infty &\leq \left(\sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \frac{1}{\lambda_{p,q}} \right)^{1/2} \left(\sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \lambda_{p,q} |\widehat{U}_{p,q}|^2 \right)^{1/2} \\ &\leq C \left(\sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \frac{1}{p^2 + q^2} \right)^{1/2} (\|\delta_x U\| + \|\delta_y U\|). \end{aligned}$$

By the integral comparison test, the double sum is bounded by the integral over a quarter annulus,

$$\sum_{p=1}^{J-1} \sum_{q=1}^{K-1} \frac{1}{p^2 + q^2} \leq C \int_1^{\sqrt{J^2 + K^2}} \frac{dr}{r} \leq C \ln \frac{1}{h},$$

where $\sqrt{J^2 + K^2} \leq \sqrt{2} \max(L_x, L_y)/h$. Since h is sufficiently small, $\ln(1/h) \leq C|\ln h|$, and the case of large h is absorbed into the constant. Combining the above estimates completes the proof. $\square$

4.2. Stability analysis and mass conservation

We now establish the unconditional stability and mass conservation of the proposed scheme. The discrete mass is defined as

$$M^n = \|U^n\|^2 = h_x h_y \sum_{j,k} |U_{j,k}^n|^2, \quad (54)$$

which corresponds to the discretization of the continuous mass $M(t)$ defined in Proposition 2.1.

Theorem 4.1 (Discrete mass conservation). *The ADI compact scheme with Strang splitting preserves the discrete mass exactly. For any time step $\tau > 0$ and spatial steps $h_x, h_y > 0$, we have*

$$\|U^n\| = \|U^0\|, \quad n = 1, 2, \dots, N, \quad (55)$$

i.e., the discrete mass $M^n = \|U^n\|^2$ is conserved exactly at every time level.

Proof. The proof proceeds by establishing that each sub-step in the Strang splitting procedure preserves the L^2 norm, so the composition inherits this conservation property. For the first nonlinear half-step Eq (29), the pointwise identity

$$|U_{j,k}^*| = |U_{j,k}^n| \cdot \left| \exp\left(\frac{i\tau}{2} V(U_{j,k}^n)\right) \right| = |U_{j,k}^n| \quad (56)$$

holds, since $|\exp(i\theta)| = 1$ for any real θ . Summation over all grid points yields

$$\|U^*\|^2 = h_x h_y \sum_{j,k} |U_{j,k}^*|^2 = h_x h_y \sum_{j,k} |U_{j,k}^n|^2 = \|U^n\|^2. \quad (57)$$

An identical argument applies to the second nonlinear step Eq (36), giving $\|U^{n+1}\| = \|U^{n+1/2}\|$.

For the linear ADI evolution, we express the X-sweep Eq (34) and Y-sweep Eq (35) in operator form:

$$\left(I - \frac{i\tau a}{2}C_x\right)U^{**} = \left(I + \frac{i\tau a}{2}C_x\right)U^*, \quad (58a)$$

$$\left(I - \frac{i\tau a}{2}C_y\right)U^{n+1/2} = \left(I + \frac{i\tau a}{2}C_y\right)U^{**}. \quad (58b)$$

The self-adjointness of C_x implies that $\left(I - \frac{i\tau a}{2}C_x\right)$ and $\left(I + \frac{i\tau a}{2}C_x\right)$ are mutual adjoints, which preserves a unitary evolution step. Specifically, for any $V \in Z_h^0$:

$$\begin{aligned} \left\|\left(I - \frac{i\tau a}{2}C_x\right)V\right\|^2 &= \langle V - \frac{i\tau a}{2}C_x V, V - \frac{i\tau a}{2}C_x V \rangle \\ &= \langle V, V \rangle - \langle V, \frac{i\tau a}{2}C_x V \rangle - \langle \frac{i\tau a}{2}C_x V, V \rangle + \langle \frac{i\tau a}{2}C_x V, \frac{i\tau a}{2}C_x V \rangle \\ &= \|V\|^2 + \left(\frac{\tau a}{2}\right)^2 \|C_x V\|^2. \end{aligned} \quad (59)$$

With the same calculation, we have:

$$\left\|\left(I + \frac{i\tau a}{2}C_x\right)V\right\|^2 = \|V\|^2 + \left(\frac{\tau a}{2}\right)^2 \|C_x V\|^2. \quad (60)$$

Application of this identity to both sweeps yields $\|U^{**}\| = \|U^*\|$ and $\|U^{n+1/2}\| = \|U^{**}\|$, so $\|U^{n+1/2}\| = \|U^*\|$. The composition of these conservation properties gives

$$\|U^{n+1}\| = \|U^{n+1/2}\| = \|U^*\| = \|U^n\|. \quad (61)$$

By mathematical induction, $\|U^n\| = \|U^0\|$ for all $n \geq 0$, with completes the proof. $\square$

Remark 4.1. While the scheme preserves the discrete mass exactly, it does not exactly conserve the discrete energy $E(t)$ or momentum $P_x(t)$, $P_y(t)$ defined in Section 2. This is due to the fundamental nature of operator splitting methods. The Strang splitting operator satisfies

$$e^{\frac{\tau}{2}\mathcal{B}}e^{\tau\mathcal{A}}e^{\frac{\tau}{2}\mathcal{B}} \neq e^{\tau(\mathcal{A}+\mathcal{B})}, \quad (62)$$

where $\mathcal{A}$ and $\mathcal{B}$ are the linear and nonlinear operators, respectively. This non-commutativity means the numerical solution actually evolves according to a slightly modified Hamiltonian, rather than the exact energy functional $E(t)$.

Remark 4.2. Although the discrete energy is not exactly conserved, its error is expected to exhibit bounded $O(\tau^2)$ oscillations over long time intervals, consistent with the second numerical example. Such a bound cannot be claimed uniformly for all times in regimes approaching blow-up, where the boundedness assumption fails.

4.3. Convergence analysis

Before proceeding, we analyze the Lipschitz continuity:

Lemma 4.5. *Let the nonlinear phase function be defined as*

$$f(s) = s \exp\left(i\frac{\tau}{2}V(s)\right), \quad (63)$$

where $V(s) = b_1 |s|^2 + b_2 |s|^4 + b_3 |s|^6$. For any complex values X and Y bounded by a positive constant M (i.e., $|X| \leq M$ and $|Y| \leq M$), there exists a Lipschitz constant $C_L > 0$, dependent on M and the coefficients b_1, b_2, b_3 , such that:

$$|f(X) - f(Y)| \leq (1 + C_L \tau) |X - Y|. \quad (64)$$

Proof. We bound the absolute difference directly. Adding and subtracting the intermediate term $X \exp\left(i\frac{\tau}{2}V(Y)\right)$ gives:

$$f(X) - f(Y) = (X - Y) \exp\left(i\frac{\tau}{2}V(Y)\right) + X \left[\exp\left(i\frac{\tau}{2}V(X)\right) - \exp\left(i\frac{\tau}{2}V(Y)\right) \right]. \quad (65)$$

Applying the triangle inequality and noting that $|\exp(ix)| = 1$ for any real x , we have:

$$|f(X) - f(Y)| \leq |X - Y| + |X| \left| \exp\left(i\frac{\tau}{2}V(X)\right) - \exp\left(i\frac{\tau}{2}V(Y)\right) \right|. \quad (66)$$

For any real numbers x and y , the complex exponential satisfies the Lipschitz condition $|\exp(ix) - \exp(iy)| \leq |x - y|$. Applying this inequality to the phase difference gives:

$$\left| \exp\left(i\frac{\tau}{2}V(X)\right) - \exp\left(i\frac{\tau}{2}V(Y)\right) \right| \leq \frac{\tau}{2} |V(X) - V(Y)|. \quad (67)$$

Since $V(s)$ is a polynomial in $|s|^2$, its derivative is bounded. Therefore, $V(s)$ is locally Lipschitz with some constant $C_V > 0$ depending on M, b_1, b_2 , and b_3 . Thus, $|V(X) - V(Y)| \leq C_V |X - Y|$. Substituting these estimates into Eq (66) and using the bound $|X| \leq M$, we obtain:

$$|f(X) - f(Y)| \leq |X - Y| + \frac{\tau}{2} C_V M |X - Y| = \left(1 + \frac{\tau}{2} C_V M\right) |X - Y|. \quad (68)$$

By defining $C_L = \frac{1}{2} C_V M$, we arrive at the desired estimate:

$$|f(X) - f(Y)| \leq (1 + C_L \tau) |X - Y|, \quad (69)$$

which completes the proof. $\square$

Lemma 4.6 (Local truncation error). *Assume that the exact solution $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$. Then one step of the scheme, Eqs (29) and (34)–(36), applied to the exact solution satisfies*

$$u^{n+1} = \Phi_\tau(u^n) + \tau R^n, \quad \|R^n\| \leq C_R (\tau^2 + h_x^4 + h_y^4),$$

where Φ_τ denotes the one-step map of the scheme and $C_R > 0$ is a constant depending on the exact solution but independent of τ, h_x , and h_y .

Proof. The nonlinear half-steps, Eqs (29) and (36), are exact flows of the nonlinear sub-problem Eq (17) and introduce no error. The linear stage contributes three terms. (i) By the Baker-Campbell-Hausdorff formula, Strang splitting contributes a local error of order $O(\tau^3)$ (cf. Lemma 3.1). (ii) Taylor expansion of the Padé relation gives $(C_x u)_{j,k} = (u_{xx})_{j,k} + O(h_x^4)$ and $(C_y u)_{j,k} = (u_{yy})_{j,k} + O(h_y^4)$, contributing $O(\tau(h_x^4 + h_y^4))$ over one time step. (iii) The ADI factorization adds the cross term $\alpha^2 C_x C_y$ with $\alpha = \frac{i a \tau}{2}$ to both sides of Eq (30), whose net contribution is $\alpha^2 C_x C_y (u^{n+1/2} - u^*) = O(\tau^3)$, since $u^{n+1/2} - u^* = O(\tau)$ and $C_x C_y$ is bounded by the mixed derivative $\partial_x^2 \partial_y^2 u$. Summing the three contributions shows that the local error of one full step is $\tau O(\tau^2 + h_x^4 + h_y^4)$, and dividing by τ yields the claimed estimate, with C_R bounded under the assumption $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$. $\square$

We now establish the convergence of the proposed scheme. Let $u(x, y, t)$ be the exact solution and U^n be the numerical solution. Define the error at time t^n as

$$e_{j,k}^n = u(x_j, y_k, t^n) - U_{j,k}^n. \quad (70)$$

Theorem 4.2 (Convergence). *Assume that the exact solution $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$ and the initial data $\phi \in H_0^2(\Omega)$. Then the numerical solution U^n converges to the exact solution u^n with the following error estimate:*

$$\|e^n\| \leq C(T) (\tau^2 + h_x^4 + h_y^4), \quad 0 \leq n\tau \leq T, \quad (71)$$

where $C(T)$ is a constant depending on T and the regularity of the exact solution, but independent of τ , h_x , and h_y .

Proof. Let u^n denote the exact solution evaluated at time $t = t_n$, and U^n be the numerical solution. We can rewrite the full numerical scheme in operator form. Define the nonlinear solution operator

$$\mathcal{N}_{\tau/2}(X) = X \exp(i \frac{\tau}{2} V(X)) \quad (72)$$

and the linear ADI solution operator

$$\mathcal{L}_\tau = \left(I - \frac{i a \tau}{2} C_x\right)^{-1} \left(I + \frac{i a \tau}{2} C_x\right) \left(I - \frac{i a \tau}{2} C_y\right)^{-1} \left(I + \frac{i a \tau}{2} C_y\right). \quad (73)$$

The exact solution satisfies the perturbed recurrence relation:

$$u^{n+1} = \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(u^n))) + \tau R^n, \quad (74)$$

where R^n is the local truncation error. By Lemma 4.6, under the regularity assumption $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$, there exists a regularity constant $C_R > 0$ such that:

$$\|R^n\| \leq C_R (\tau^2 + h_x^4 + h_y^4). \quad (75)$$

Subtracting the numerical scheme $U^{n+1} = \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(U^n)))$ from the exact solution equation, and letting $e^n = u^n - U^n$, we obtain the error evolution equation:

$$e^{n+1} = \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(u^n))) - \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(U^n))) + \tau R^n. \quad (76)$$

Taking the discrete L^2 -norm on both sides and using the triangle inequality yields:

$$\|e^{n+1}\| \leq \|\mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(u^n))) - \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(U^n)))\| + \tau \|R^n\|. \quad (77)$$

To bound the first term, applying the local Lipschitz condition established in Lemma 4.5 to the nonlinear operator yields:

$$\|\mathcal{N}_{\tau/2}(X) - \mathcal{N}_{\tau/2}(Y)\| \leq (1 + C_L\tau) \|X - Y\|. \quad (78)$$

Furthermore, as shown in the mass conservation analysis, the linear ADI operator $\mathcal{L}_\tau$ is exactly unitary, preserving the L^2 -norm:

$$\|\mathcal{L}_\tau(X) - \mathcal{L}_\tau(Y)\| = \|\mathcal{L}_\tau(X - Y)\| = \|X - Y\|. \quad (79)$$

Substituting Eqs (78) and (79) sequentially into Eq (77), we obtain:

$$\begin{aligned} \|e^{n+1}\| &\leq (1 + C_L\tau) \|\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(u^n)) - \mathcal{L}_\tau(\mathcal{N}_{\tau/2}(U^n))\| + \tau \|R^n\| \\ &= (1 + C_L\tau) \|\mathcal{N}_{\tau/2}(u^n) - \mathcal{N}_{\tau/2}(U^n)\| + \tau \|R^n\| \\ &\leq (1 + C_L\tau)^2 \|u^n - U^n\| + \tau \|R^n\| \\ &\leq (1 + 3C_L\tau) \|u^n - U^n\| + \tau \|R^n\| \end{aligned} \quad (80)$$

for sufficiently small τ , where we use $(1 + C_L\tau)^2 \leq 1 + 3C_L\tau$ when $C_L\tau \leq 1$. Applying the discrete Grönwall inequality in Lemma 4.3 and noting that $e^0 = 0$, we obtain:

$$\|e^n\| \leq e^{3C_L n\tau} \left(\tau \sum_{m=0}^{n-1} C_R (\tau^2 + h_x^4 + h_y^4) \right) \leq e^{3C_L T} T C_R (\tau^2 + h_x^4 + h_y^4). \quad (81)$$

By letting $C(T) = e^{3C_L T} T C_R$, which depends only on T and the regularity of the exact solution, we establish the desired unconditional convergence estimate:

$$\|e^n\| \leq C(T) (\tau^2 + h_x^4 + h_y^4), \quad 0 \leq n\tau \leq T. \quad (82)$$

□

4.4. Convergence in the maximum norm

Using the discrete Sobolev embedding in Lemma 4.4, we establish convergence in the maximum norm. To apply Lemma 4.4, we first establish the H^1 convergence of the scheme.

Lemma 4.7. *Assume that the problem is posed on a rectangular domain with a uniform grid, constant coefficients, and homogeneous Dirichlet boundary conditions, so that the operators $\mathcal{L}_x, \mathcal{L}_y, \mathcal{P}_x, \mathcal{P}_y$ are self-adjoint and mutually commutative. Then the ADI linear steps, Eqs (34) and (35), are unconditionally stable in the H^1 norm. Specifically, for the solution $U^{n+1/2}$ obtained from U^* , we have*

$$\|\delta_x U^{n+1/2}\|^2 + \|\delta_y U^{n+1/2}\|^2 = \|\delta_x U^*\|^2 + \|\delta_y U^*\|^2. \quad (83)$$

Proof. The complete linear step from the intermediate state U^* to $U^{n+1/2}$ can be expressed in factored form by combining the X-sweep and Y-sweep. Using $C_x = \mathcal{P}_x^{-1} \mathcal{L}_x$ and $C_y = \mathcal{P}_y^{-1} \mathcal{L}_y$, the combined ADI scheme is:

$$\left(I - i\frac{a\tau}{2}C_x\right)\left(I - i\frac{a\tau}{2}C_y\right)U^{n+1/2} = \left(I + i\frac{a\tau}{2}C_x\right)\left(I + i\frac{a\tau}{2}C_y\right)U^*. \quad (84)$$

This allows us to express the linear ADI operator $\mathcal{L}_\tau$ (as defined in Theorem 4.2) as the composition

$$\mathcal{L}_\tau = \mathcal{S}_y \mathcal{S}_x = \mathcal{S}_x \mathcal{S}_y, \quad (85)$$

where the individual directional sweep operators are

$$\begin{aligned} \mathcal{S}_x &= \left(I - i\frac{a\tau}{2}C_x\right)^{-1} \left(I + i\frac{a\tau}{2}C_x\right), \\ \mathcal{S}_y &= \left(I - i\frac{a\tau}{2}C_y\right)^{-1} \left(I + i\frac{a\tau}{2}C_y\right), \end{aligned} \quad (86)$$

such that $U^{n+1/2} = \mathcal{L}_\tau U^*$. Due to the commutativity of the spatial operators over the rectangular domain, the order of composition does not matter (i.e., $\mathcal{S}_x \mathcal{S}_y = \mathcal{S}_y \mathcal{S}_x$).

To establish H^1 stability, we analyze the unitary properties of $\mathcal{L}_\tau$. On a uniform grid with homogeneous Dirichlet boundary conditions, the discrete difference operators $(\mathcal{L}_x, \mathcal{L}_y)$ and the Padé averaging operators $(\mathcal{P}_x, \mathcal{P}_y)$ are self-adjoint and commute. Because $\mathcal{P}_x$ and $\mathcal{L}_x$ are self-adjoint and commute, their product $C_x = \mathcal{P}_x^{-1} \mathcal{L}_x$ is also self-adjoint in the standard L^2 inner product ($C_x^* = C_x$).

Since C_x is self-adjoint and $\frac{a\tau}{2}$ is real, the operators $\left(I - i\frac{a\tau}{2}C_x\right)$ and $\left(I + i\frac{a\tau}{2}C_x\right)$ are adjoints to one another. Consequently, the sweep operator $\mathcal{S}_x$ is unitary. By identical reasoning, $\mathcal{S}_y$ is also unitary. Because the composition of unitary operators is unitary, $\mathcal{L}_\tau = \mathcal{S}_y \mathcal{S}_x$ preserves the standard L^2 inner product:

$$\langle \mathcal{L}_\tau X, \mathcal{L}_\tau Y \rangle = \langle X, Y \rangle \quad \forall X, Y \in Z_h^0. \quad (87)$$

Because all spatial operators involved act along orthogonal grid lines with constant coefficients, they commute with one another over the rectangular domain. This guarantees that $\mathcal{L}_\tau$ commutes with the discrete differential operators $\mathcal{L}_x$ and $\mathcal{L}_y$.

Using summation by parts, the discrete H^1 seminorm for the grid function U is

$$\|\delta_x U\|^2 = \langle -\mathcal{L}_x U, U \rangle, \quad (88)$$

and then in the X-direction, we have

$$\|\delta_x U^{n+1/2}\|^2 = \langle -\mathcal{L}_x U^{n+1/2}, U^{n+1/2} \rangle = \langle -\mathcal{L}_x \mathcal{L}_\tau U^*, \mathcal{L}_\tau U^* \rangle. \quad (89)$$

Using the commutativity $(-\mathcal{L}_x \mathcal{L}_\tau = \mathcal{L}_\tau (-\mathcal{L}_x))$ and the unitary property in Eq (87):

$$\langle \mathcal{L}_\tau (-\mathcal{L}_x U^*), \mathcal{L}_\tau U^* \rangle = \langle -\mathcal{L}_x U^*, U^* \rangle = \|\delta_x U^*\|^2. \quad (90)$$

An identical procedure for the Y-direction gives:

$$\|\delta_y U^{n+1/2}\|^2 = \langle -\mathcal{L}_y \mathcal{L}_\tau U^*, \mathcal{L}_\tau U^* \rangle = \langle -\mathcal{L}_y U^*, U^* \rangle = \|\delta_y U^*\|^2. \quad (91)$$

Adding these two conservation properties establishes the preservation of the discrete H^1 seminorm across the linear ADI step:

$$\|\delta_x U^{n+1/2}\|^2 + \|\delta_y U^{n+1/2}\|^2 = \|\delta_x U^*\|^2 + \|\delta_y U^*\|^2. \quad (92)$$

This completes the proof. $\square$

Lemma 4.8. [H^1 error estimate] Assume that the exact solution $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$. Then the error satisfies

$$\|\delta_x e^n\| + \|\delta_y e^n\| \leq C(T) (\tau^2 + h_x^4 + h_y^4). \quad (93)$$

Proof. The proof follows from analyzing the error equation for the discrete gradient. Let the global error be $e^n = u^n - U^n$. From the splitting scheme, the error satisfies the recurrence relation:

$$e^{n+1} = \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(u^n))) - \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(U^n))) + \tau R^n, \quad (94)$$

where R^n is the local truncation error of the full step.

Define the discrete gradients $\eta^n = \delta_x e^n$ and $\xi^n = \delta_y e^n$. Applying the forward difference operator δ_x to the error equation, we obtain:

$$\eta^{n+1} = \delta_x [\mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(u^n))) - \mathcal{N}_{\tau/2}(\mathcal{L}_\tau(\mathcal{N}_{\tau/2}(U^n)))] + \tau \delta_x R^n. \quad (95)$$

The local truncation error for the gradient is derived by applying the discrete difference operators to the truncation error of the original scheme. Assuming the regularity of the exact solution $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$, taking the discrete spatial derivative reduces one degree of spatial regularity, so $\delta_x R^n$ remains bounded by the same asymptotic order:

$$\|\delta_x R^n\| \leq C_R (\tau^2 + h_x^4 + h_y^4). \quad (96)$$

Next, we analyze the propagation of the gradient error through the linear and nonlinear operators. Since the compact operators commute with the difference operators up to $O(h^4)$, we can commute δ_x through the linear ADI operator $\mathcal{L}_\tau$. Using the stability result from Lemma 4.7, the linear step preserves the discrete H^1 seminorm, meaning it does not amplify the gradient error.

For the nonlinear operator $\mathcal{N}_{\tau/2}$, applying the discrete difference operator to the product $U \exp(i\frac{\tau}{2}V(U))$ uses the discrete product rule. Because the nonlinear potential $V(s)$ has bounded derivatives for bounded inputs, and using the a priori assumption that $\|U^n\|_\infty \leq M$ (verified via induction), the discrete gradient of the nonlinear step is bounded by a combination of the gradient error and the L^2 error. Using the Lipschitz constant C_L from Lemma 4.5:

$$\|\delta_x(\mathcal{N}_{\tau/2}X - \mathcal{N}_{\tau/2}Y)\| \leq (1 + C_L\tau) \|\delta_x(X - Y)\| + C_L\tau \|X - Y\|, \quad (97)$$

where C_L is the Lipschitz constant depending on M and the coefficients b_1, b_2, b_3 .

Applying this bound iteratively through the Strang splitting sequence, and using the known L^2 convergence bound $\|e^n\| \leq C(T)(\tau^2 + h_x^4 + h_y^4)$ from Theorem 4.2, the L^2 error term is absorbed into the truncation order. We obtain recurrence relations for $\|\eta^n\|$ and $\|\xi^n\|$ with the local truncation error bounded by $\|\delta_x R^n\|$:

$$\|\eta^{n+1}\| \leq (1 + C_L\tau) \|\eta^n\| + \tau \|\delta_x R^n\|. \quad (98)$$

An identical derivation applies to the Y-direction gradient ξ^n :

$$\|\xi^{n+1}\| \leq (1 + C_L\tau) \|\xi^n\| + \tau \|\delta_y R^n\|. \quad (99)$$

Applying the discrete Gronwall inequality yields the desired bound for both components independently:

$$\|\eta^n\| \leq C_x(T) (\tau^2 + h_x^4 + h_y^4), \quad \|\xi^n\| \leq C_y(T) (\tau^2 + h_x^4 + h_y^4). \quad (100)$$

Summing these results, we obtain the same convergence order for the H^1 seminorm as for the L^2 norm:

$$\|\delta_x e^n\| + \|\delta_y e^n\| \leq C(T) (\tau^2 + h_x^4 + h_y^4), \quad (101)$$

which completes the proof. $\square$

Theorem 4.3 (Maximum norm convergence). *Under the assumptions of Theorem 4.2 and Lemma 4.8, we have*

$$\|e^n\|_\infty \leq C(T) \sqrt{|\ln h|} (\tau^2 + h_x^4 + h_y^4), \quad h = \min(h_x, h_y). \quad (102)$$

Proof. The maximum norm convergence follows from the discrete Sobolev embedding theorem combined with the discrete H^1 error estimates.

Let $e_{j,k}^n = u(x_j, y_k, t^n) - U_{j,k}^n$ be the global error at the grid point (x_j, y_k) at time t^n . By the discrete Sobolev embedding inequality in Lemma 4.4, there exists a constant $C > 0$, independent of the spatial step sizes h_x and h_y , such that the maximum norm of the error is bounded by its discrete H^1 seminorm:

$$\|e^n\|_\infty \leq C \sqrt{|\ln h|} (\|\delta_x e^n\| + \|\delta_y e^n\|), \quad (103)$$

where $h = \min(h_x, h_y)$.

From Lemma 4.8, under the regularity assumption $u \in C_{x,y,t}^{7,7,3}(\Omega \times [0, T])$, the discrete gradients of the error are bounded by the truncation error. Specifically, we have:

$$\|\delta_x e^n\| + \|\delta_y e^n\| \leq C_R(T) (\tau^2 + h_x^4 + h_y^4), \quad (104)$$

where $C_R(T)$ depends on the final time T and the regularity of the exact solution, but is independent of τ, h_x , and h_y .

Substituting the H^1 error bound in Eq (104) into the discrete Sobolev inequality in Eq (103) gives:

$$\|e^n\|_\infty \leq C \sqrt{|\ln h|} [C_R(T) (\tau^2 + h_x^4 + h_y^4)] = C(T) \sqrt{|\ln h|} (\tau^2 + h_x^4 + h_y^4), \quad (105)$$

where $C(T) = C \cdot C_R(T)$ depends only on T and the regularity of the exact solution.

We now close the a priori boundedness assumption made in the proofs of Theorem 4.2 and Lemma 4.8. We previously assumed that the numerical solution satisfies $\|U^n\|_\infty \leq M$ to bound the local Lipschitz constant of the nonlinear terms.

Since the exact solution $u(x, y, t)$ is continuous on the compact domain $\Omega \times [0, T]$, it is bounded by some maximum amplitude $M^* = \max_t \|u(\cdot, t)\|_\infty$. By choosing the induction constant $M = M^* + 1$, we require the discretization step sizes (τ, h_x, h_y) to be sufficiently small such that the global error satisfies:

$$C(T) \sqrt{|\ln h|} (\tau^2 + h_x^4 + h_y^4) \leq 1. \quad (106)$$

Under this mesh constraint, we guarantee that $\|e^n\|_\infty \leq 1$. Consequently, by the reverse triangle inequality applied to the maximum norm, the numerical solution is strictly bounded by:

$$\|U^n\|_\infty \leq \|u^n\|_\infty + \|e^n\|_\infty \leq M^* + 1 = M. \quad (107)$$

This validates the boundedness of the numerical solution at time step t^n , allowing the mathematical induction hypothesis to proceed to step t^{n+1} . This completes the convergence proof. $\square$

5. Numerical experiments

In this section, we present numerical experiments to verify the theoretical convergence analysis and demonstrate the efficiency of the proposed ADI compact scheme with Strang splitting. The numerical validation proceeds in two stages: first, the spatial and temporal convergence rates are verified using the septic NLSE, for which an exact solution is available, and second, the scheme's capability for the full cubic-quintic-septic NLSE is demonstrated using Cauchy error analysis. This validation confirms the theoretical convergence order $O(\tau^2 + h_x^4 + h_y^4)$ established in Theorem 4.2.

5.1. Error norms and convergence rates

In all experiments, uniform spatial discretization with $h = h_x = h_y$ is employed. When an exact solution is available, the accuracy is measured by the discrete L_2 - and L_∞ -norm errors $\|e^n\|_2$ and $\|e^n\|_\infty$ defined in Section 4, where $e_{j,k}^n = u(x_j, y_k, t_n) - U_{j,k}^n$. With the time step scaling quadratically with the spatial step ($\tau \propto h^2$), the spatial convergence rates are computed by the standard refinement formulas

$$r_{h,2} = \log_2 \left(\frac{\|e^n(h, \tau)\|_2}{\|e^n(h/2, \tau/4)\|_2} \right), \quad r_{h,\infty} = \log_2 \left(\frac{\|e^n(h, \tau)\|_\infty}{\|e^n(h/2, \tau/4)\|_\infty} \right), \quad (108)$$

and the temporal convergence rates $r_{\tau,2}$, $r_{\tau,\infty}$ are defined analogously with the refinement $\tau \rightarrow \tau/2$ at fixed small h .

When the exact solution is not available, Cauchy (self-convergence) errors between successive grid levels are used instead:

$$\|e_{\text{Cauchy}}^n\|_2 = \left[h^2 \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \left| U_{h,j,k}^n - (U_{h/2}^n)_{2j,2k} \right|^2 \right]^{\frac{1}{2}}, \quad \|e_{\text{Cauchy}}^n\|_\infty = \max_{\substack{1 \leq j \leq J-1 \\ 1 \leq k \leq K-1}} \left| U_{h,j,k}^n - (U_{h/2}^n)_{2j,2k} \right|, \quad (109)$$

with the Cauchy convergence rates r^{Cauchy} computed by the same refinement formulas.

5.2. Example 1: Septic NLSE with an exact solution

Setting $b_1 = b_2 = 0$ yields the septic nonlinear Schrödinger equation:

$$iu_t + a(u_{xx} + u_{yy}) + b_3 |u|^6 u = 0. \quad (110)$$

This equation admits the exact bright soliton solution [29]:

$$u(x, y, t) = A \operatorname{sech}^{\frac{1}{3}} \left[\frac{B}{\sqrt{2}} \left((x + x_0) + (y + y_0) - 2a(k_x + k_y)t \right) \right] e^{i(k_x(x+x_0) + k_y(y+y_0) - \omega t)}, \quad (111)$$

where the parameters satisfy:

$$B = \frac{3A^3}{2} \sqrt{\frac{b_3}{a}}, \quad \omega = a(k_x^2 + k_y^2) - \frac{b_3 A^6}{4}, \quad (112)$$

with the focusing condition $ab_3 > 0$. Using the parameters:

$$a = 1, \quad b_3 = 1, \quad A = 1, \quad k_x = k_y = 0.5, \quad (x_0, y_0) = (5, 5) \quad (113)$$

yields $B = 3/2$ and $\omega = 1/4$. The exact solution becomes:

$$u_{\text{exact}}(x, y, t) = \text{sech}^{\frac{1}{3}} \left[\frac{3}{2\sqrt{2}}(x + y + 10 - 2t) \right] e^{i(0.5(x+y+10)-0.25t)}. \quad (114)$$

Numerical tests are conducted on $\Omega = [-20, 20] \times [-20, 20]$ up to $T = 1$.

Table 1. Spatial errors and convergence rates for Example 1 (Septic NLSE) when $h_x = h_y = h$, $\tau = h^2$, $T = 1$.

h	$\ e^n\ _2$	$r_{h,2}$	$\ e^n\ _\infty$	$r_{h,\infty}$
1/4	1.4579×10^{-2}	\	3.7770×10^{-3}	\
1/8	9.4495×10^{-4}	3.9475	2.4172×10^{-4}	3.9658
1/16	5.9241×10^{-5}	3.9956	1.5068×10^{-5}	4.0038
1/32	3.7040×10^{-6}	3.9994	9.4442×10^{-7}	3.9959
1/64	2.3153×10^{-7}	3.9998	5.9095×10^{-8}	3.9983

Table 2. Temporal errors and convergence rates for Example 1 (Septic NLSE) when $h_x = h_y = 1/16$, $T = 1$.

τ	$\ e^n\ _2$	$r_{\tau,2}$	$\ e^n\ _\infty$	$r_{\tau,\infty}$
1/16	1.5422×10^{-2}	\	4.0022×10^{-3}	\
1/32	3.9558×10^{-3}	1.9629	1.0106×10^{-3}	1.9856
1/64	9.9357×10^{-4}	1.9933	2.5214×10^{-4}	2.0029
1/128	2.4637×10^{-4}	2.0118	6.2265×10^{-5}	2.0177
1/256	5.9241×10^{-5}	2.0561	1.5068×10^{-5}	2.0470

The results in Tables 1 and 2 show that the spatial convergence rates approach 4.0 and the temporal convergence rates approach 2.0, validating the theoretical convergence order for the septic NLSE. These results confirm that the proposed scheme maintains high-order accuracy for the stronger $|u|^6$ nonlinearity.

To demonstrate the computational efficiency of ADI factorization, Table 3 compares the CPU time of the proposed scheme with that of the non-ADI implicit compact scheme, i.e., the unfactored Crank–Nicolson-type system Eq (30) solved by a sparse direct solver with a one-time Lower-Upper factorization. The ADI scheme is several times faster already at moderate resolutions, and the advantage grows with the mesh size: its cost scales linearly with the number of grid points, whereas the cost of the two-dimensional banded solve grows superlinearly. At $h = 1/32$, the non-ADI scheme did not complete within a practical time frame, while the ADI scheme required only about 193 seconds.

Table 3. CPU time comparison between the proposed ADI scheme and the non-ADI implicit compact scheme for Example 1 (Septic NLSE), when $\tau = h^2$, $T = 1$.

h	ADI scheme (s)	non-ADI scheme (s)	speedup
1/8	0.75	5.35	7.2
1/16	11.56	62.85	5.4
1/32	192.65	—	—

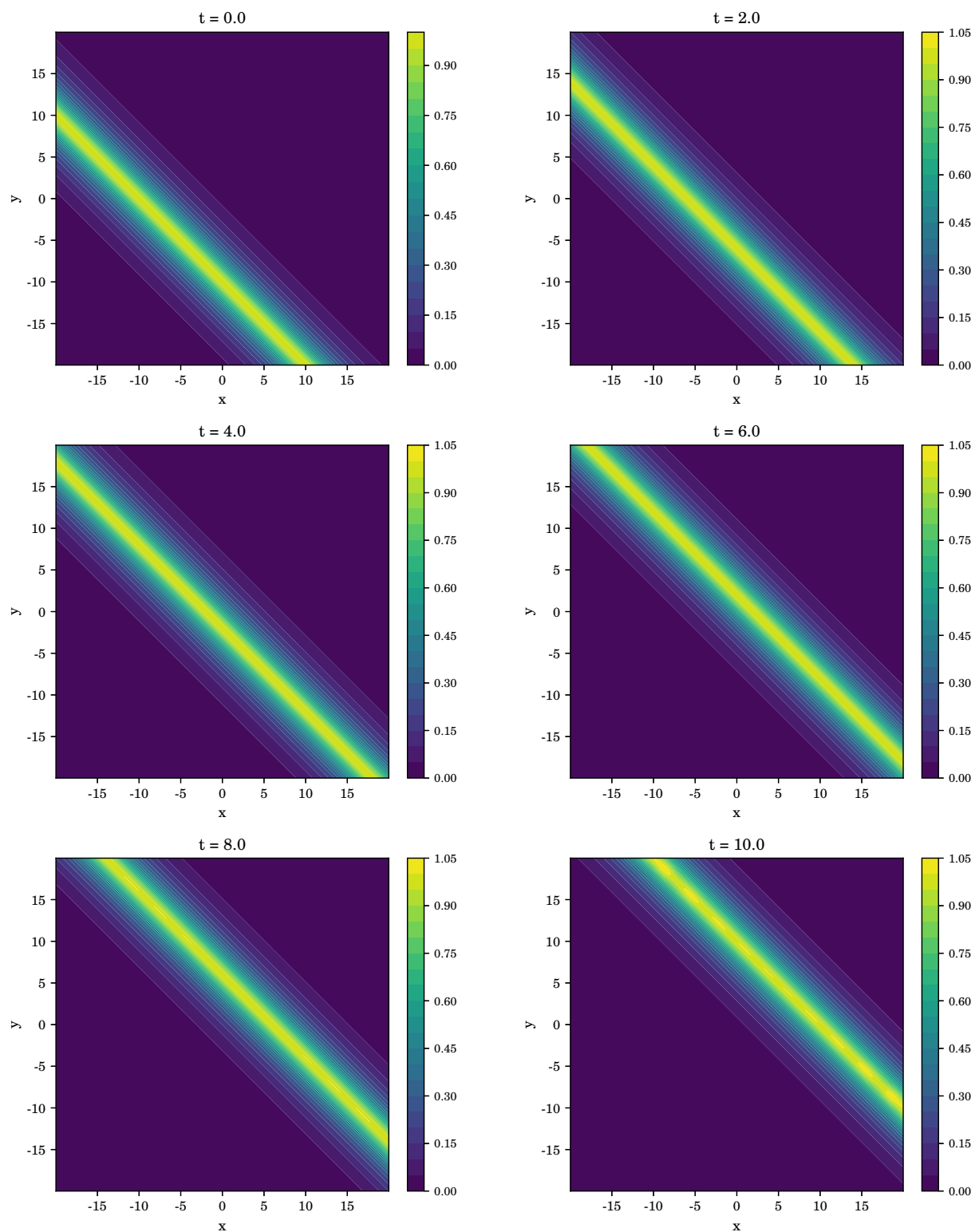


Figure 1. Wave propagation for Example 1 (Septic NLSE) with $b_1 = 0$, $b_2 = 0$, $b_3 = 1$. The contour plots show the amplitude $|u|$ at times $t = 0, 2, 4, 6, 8, 10$.

Figure 1 illustrates the two-dimensional wave propagation for the septic NLSE with $|u|^6$ nonlinearity.

The septic nonlinearity produces a compact soliton profile with sharp gradients near the peak. The contour plots at times $t = 0, 2, 4, 6, 8, 10$ show that the numerical scheme captures the propagation of this higher-order soliton while maintaining its localized structure. The stronger nonlinearity results in a slightly different propagation characteristic, yet the scheme preserves the essential soliton properties without numerical instability. Figure 2 displays the corresponding diagonal cross-sections for the septic NLSE.

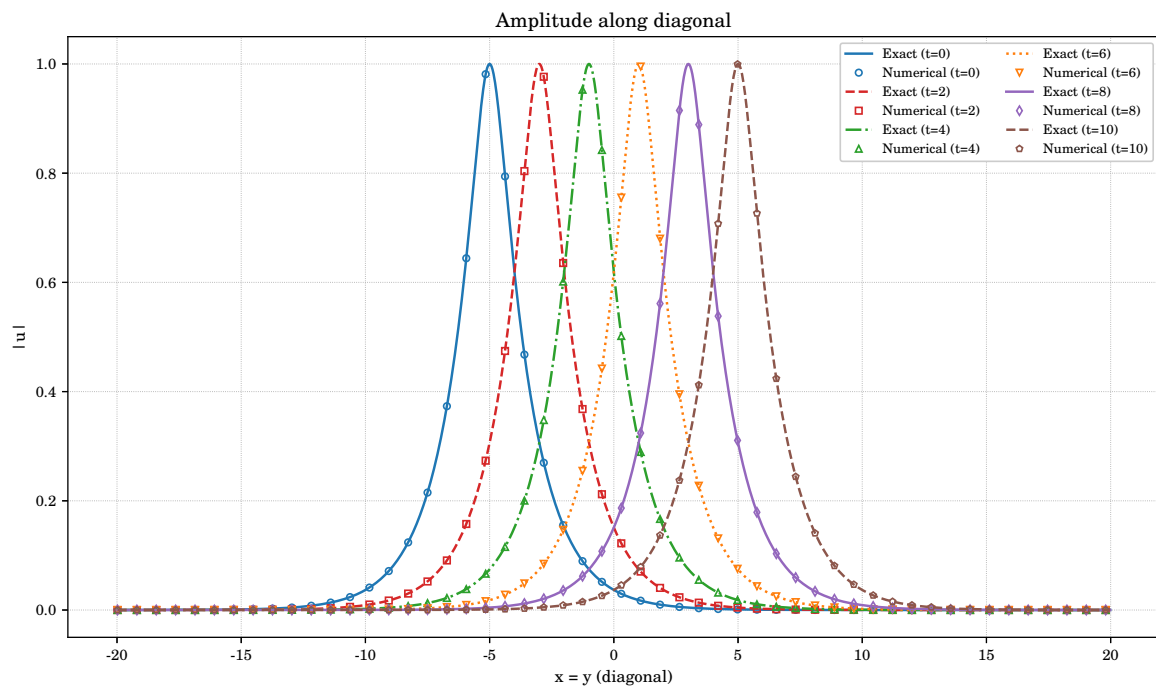


Figure 2. Cross-sectional view along the diagonal $x = y$ for Example 1 (Septic NLSE). The amplitude profiles $|u|$ at times $t = 0, 2, 4, 6, 8, 10$.

5.3. Example 2: Cubic-quintic-septic NLSE

We now consider the general case where cubic, quintic, and septic nonlinearities are present simultaneously. This test verifies that the scheme preserves its convergence properties when multiple nonlinear mechanisms interact. We consider the full cubic-quintic-septic NLSE with $b_1 = 1$, $b_2 = 0.1$, and $b_3 = 0.01$, where an exact solution is not available. The computation is initialized with a Gaussian pulse with momentum:

$$u(x, y, 0) = \exp\left(-\frac{(x+5)^2 + (y+5)^2}{4}\right) \exp\left(i\frac{x+y+10}{2}\right). \quad (115)$$

Since no exact solution exists for this case, we employ Cauchy error analysis to verify the convergence properties of the scheme. The computation is performed on the domain $\Omega = [-50, 50] \times [-50, 50]$. The results are summarized in Tables 4 and 5.

As shown in Table 4, the spatial convergence rates $r_{h,2}$ and $r_{h,\infty}$ approach 4.0, confirming fourth-order accuracy in space. Similarly, Table 5 demonstrates second-order temporal accuracy, with convergence

Table 4. Cauchy spatial errors and convergence rates for Example 2 (Cubic-quintic-septic NLSE) when $h_x = h_y = h$, $\tau = h^2$, $T = 1$.

Coarse h	Fine h	$\ e^n\ _2$	$r_{h,2}$	$\ e^n\ _\infty$	$r_{h,\infty}$
1/4	1/8	9.7424×10^{-3}	\	3.6132×10^{-3}	\
1/8	1/16	6.1828×10^{-4}	3.9779	2.2219×10^{-4}	4.0234
1/16	1/32	3.8674×10^{-5}	3.9988	1.3881×10^{-5}	4.0006
1/32	1/64	2.4172×10^{-6}	4.0000	8.6775×10^{-7}	3.9997

Table 5. Cauchy temporal errors and convergence rates for Example 2 (Cubic-quintic-septic NLSE) when $h_x = h_y = 1/8$, $T = 1$.

Coarse τ	Fine τ	$\ e^n\ _2$	$r_{\tau,2}$	$\ e^n\ _\infty$	$r_{\tau,\infty}$
1/8	1/16	2.8270×10^{-2}	\	1.1490×10^{-2}	\
1/16	1/32	7.3550×10^{-3}	1.9425	2.7074×10^{-3}	2.0854
1/32	1/64	1.8641×10^{-3}	1.9802	6.6728×10^{-4}	2.0205
1/64	1/128	4.6777×10^{-4}	1.9946	1.6624×10^{-4}	2.0050

rates approaching 2.0 under refinement. These results validate that the ADI compact scheme with Strang splitting maintains its high-order accuracy for the full polynomial nonlinearity when the three nonlinear terms interact.

To demonstrate the stability and accuracy of the scheme for extended integration times, we compute the numerical solution on $\Omega = [-50, 50] \times [-50, 50]$ with $h = 0.1$ and $\tau = h^2$ up to $T = 10$. The evolution of the wave amplitude is visualized in Figures 3 and 4.

Figure 5 presents the time evolution of the relative errors in mass $M(t)$ and energy $E(t)$ conservation over the interval $[0, 10]$. The mass is preserved to machine precision (relative error on the order of 10^{-15}), confirming the exact mass conservation property established in Theorem 4.1. The energy exhibits relative oscillations on the order of 10^{-4} , which remain bounded throughout the integration period. These oscillations are characteristic of symplectic integrators and reflect the Hamiltonian structure preservation of the Strang splitting method. Figure 6 shows the relative errors in the momentum components $P_x(t)$ and $P_y(t)$ over the same time interval. Due to the symmetric initial condition and diagonal propagation direction, both momentum components exhibit similar error patterns. The relative errors remain on the order of 10^{-4} throughout the simulation, confirming that the scheme maintains the momentum-conservation properties of the continuous system.

6. Conclusions

We have presented an efficient numerical framework for the two-dimensional cubic-quintic-septic nonlinear Schrödinger equation based on the synergistic coupling of Strang splitting, compact Padé discretization, and ADI factorization. The resulting scheme achieves fourth-order spatial accuracy on a three-point stencil without auxiliary boundary points, while the splitting strategy eliminates the need for iterative nonlinear solvers by treating the septic nonlinearity through exact phase rotation.

Our theoretical contributions encompass three main aspects. First, we derived the continuous conservation laws governing mass, energy, and momentum in the two-dimensional setting, providing

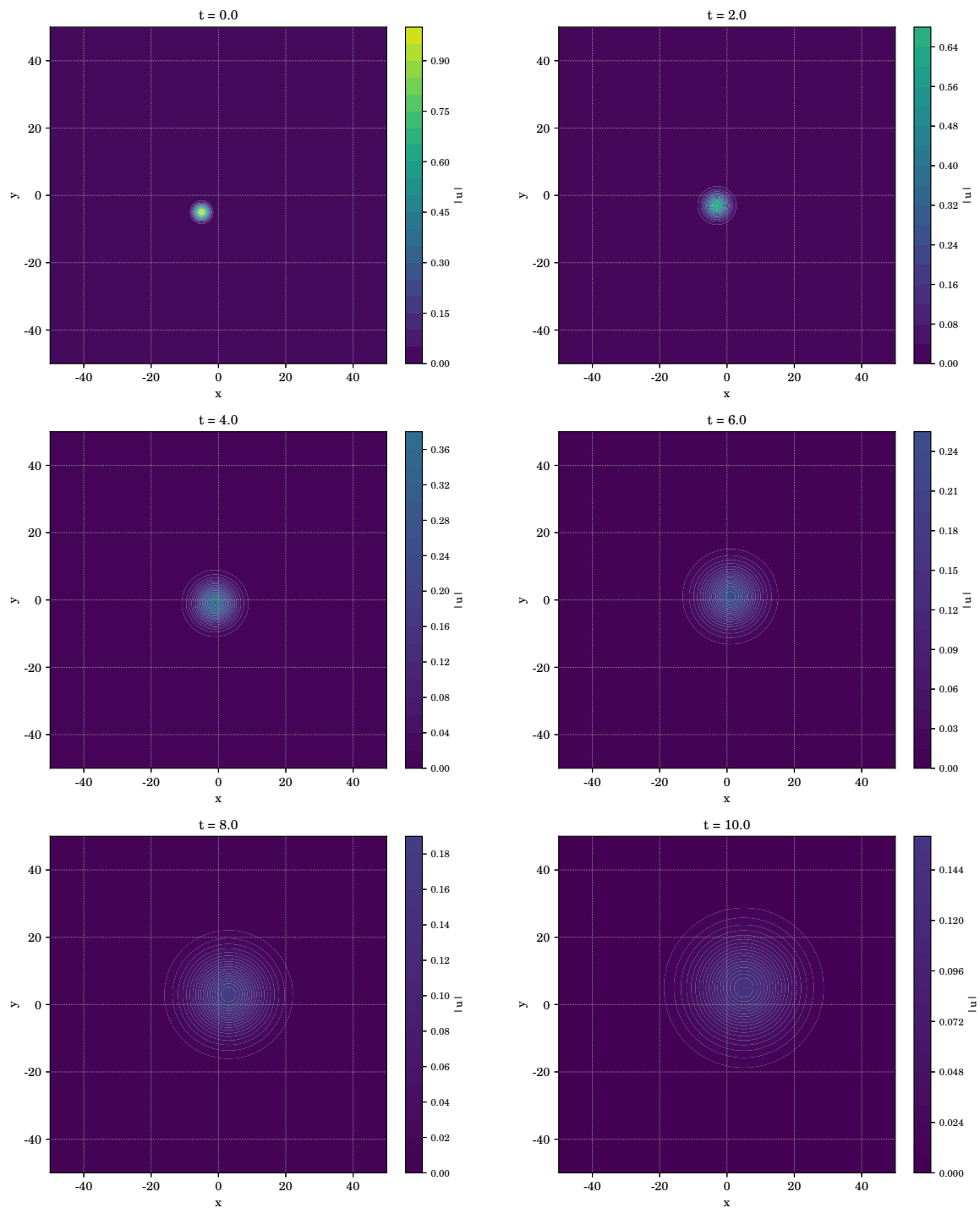


Figure 3. Wave propagation for Example 2 (Cubic-quintic-septic NLSE) with $b_1 = 1$, $b_2 = 0.1$, $b_3 = 0.01$. The contour plots show the amplitude $|u|$ at times $t = 0, 2, 4, 6, 8, 10$.

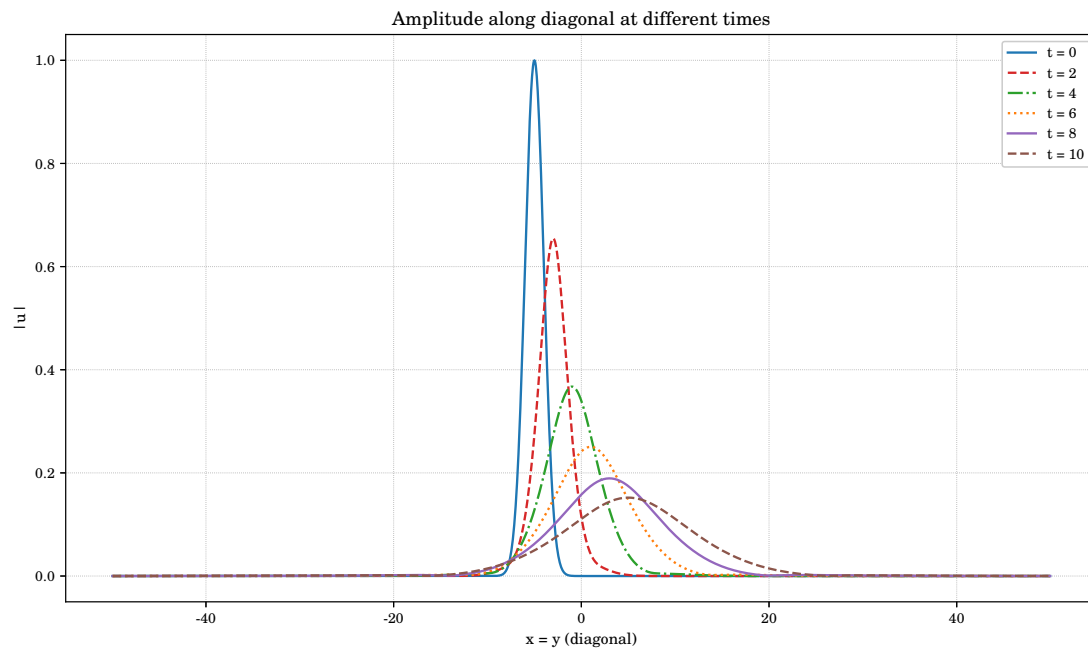


Figure 4. Cross-sectional view along the diagonal $x = y$ for Example 2 (Cubic-quintic-septic NLSE). The amplitude profiles $|u|$ at times $t = 0, 2, 4, 6, 8, 10$ demonstrate the pulse propagation and deformation along the direction of motion.

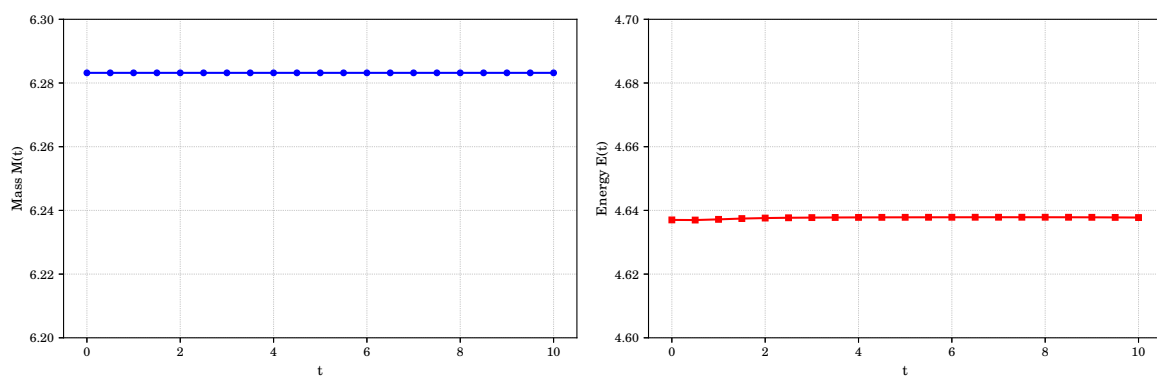


Figure 5. Mass and energy conservation for Example 2 (Cubic-quintic-septic NLSE). The relative errors of mass $M(t)$ and energy $E(t)$ are shown over the time interval $[0, 10]$.

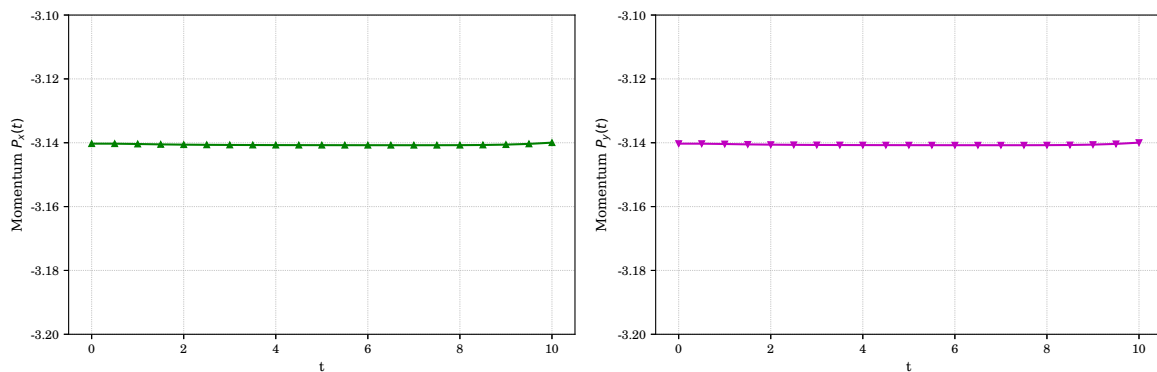


Figure 6. Momentum conservation for Example 2 (Cubic-quintic-septic NLSE). The relative errors of momentum components $P_x(t)$ and $P_y(t)$ are shown over the time interval $[0, 10]$.

the physical invariants that structure-preserving discretizations must respect. Second, we proved that the proposed ADI compact scheme is unconditionally stable in the discrete L^2 norm and conserves mass exactly at every time step, a property inherited from the unitary character of both the linear ADI sweeps and the nonlinear sub-flow. Third, we established an unconditional convergence rate of $O(\tau^2 + h_x^4 + h_y^4)$ in the discrete L^2 norm, together with a maximum-norm bound obtained through discrete Sobolev embedding, thereby rigorously accounting for the combined effects of Strang splitting, ADI factorization, and compact spatial truncation.

The numerical experiments confirm the predicted convergence orders and reveal that the scheme maintains machine-precision mass conservation over long integration times, while the energy and momentum exhibit the bounded oscillations typical of symplectic splitting methods. These features make the proposed algorithm particularly attractive for simulating high-intensity optical beams and Bose–Einstein condensates, where the septic nonlinearity induces sharp transverse gradients and long-time accuracy is essential.

Several extensions of the present work are worth pursuing. In particular, the ADI framework can be naturally extended to three-dimensional problems, as demonstrated for related models in recent works [30–32]. The incorporation of external trapping potentials or spatially varying dispersion would broaden the applicability to ultracold atom systems and inhomogeneous optical media. Moreover, adaptive time-stepping strategies tailored to the operator-splitting structure could further enhance computational efficiency near transient events such as collapse or vortex nucleation. Finally, the rigorous treatment of soliton radiation and transient dynamics in the two-dimensional septic regime remains an open analytical challenge that we hope to address in subsequent investigations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Weizhong Dai is a guest editor for the special issue of the ERA and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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