



Research article

Conservation laws and travelling wave solutions of a generalized NLS-Boussinesq system using symmetries

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Abstract: This work investigates a coupled generalized Schrödinger-Boussinesq system with a p -power nonlinearity. These equations appear in several physical applications. To begin, we present a complete classification of infinitesimal point symmetries for the system. Then, low-order local conservation laws are derived using the multiplier method, providing conserved quantities such as momentum, mass, and charge. By introducing a potential variable, we derive a Lagrangian formulation for the potential system and find nonlocal conservation laws via Noether's theorem. Finally, we explore travelling wave solutions by reducing the governing equations to a coupled autonomous system of ordinary differential equations. Several families of solutions are constructed, and their physical relevance is discussed.

Keywords: Schrödinger-Boussinesq; symmetries; conservation laws; Noether's theorem; travelling wave solutions

1. Introduction

Nonlinear dispersive equations provide a fundamental framework for modelling a wide class of physical wave propagation phenomena [1, 2]. These systems arise naturally in many areas of applied mathematics and physics, serving as essential models for wave dynamics in fluids, plasmas, and elastic media.

In particular, nonlinear generalizations of integrable wave equations, such as the Boussinesq equation with p -power nonlinearity, are of great analytical interest due to their critical behavior and applications [3]. The ordinary Boussinesq equation $w_t = w_{xx} + \gamma(w^2)_{xx} + \beta w_{xxx}$ is an integrable system obtained as a dimensional reduction of the Kadomtsev-Petviashvili (KP) equation [4] and serves as a physically relevant model for bidirectional shallow water waves.

In more complex physical systems, nonlinear dispersive wave motion couples with high-frequency waves. This is typically modeled by the nonlinear Schrödinger (NLS) equation $iE_t + E_{xx} + \delta E + 2Ew = 0$, where $w(t, x)$ acts as a potential for the complex field $E(t, x)$. Physical applications include electron propagation coupled to nonlinear ion-acoustic waves in collisionless plasmas [5, 6], resonant interactions of capillary-gravity and internal water waves [7], electron motion along deformable molecular chains [8–10], and electron energy transport in anharmonic lattices [11, 12]. In the context of plasma physics, such interactions are traditionally described by the Zakharov equations, where the low-frequency motion satisfies a standard wave equation [13].

This motivates studying the coupled generalized Schrödinger-Boussinesq equations

$$\begin{aligned} iE_t + E_{xx} + \delta E + 2Ew &= 0, \\ w_{tt} - \alpha w_{xx} - \beta w_{xxxx} - \gamma(w^p)_{xx} - (|E|^2)_{xx} &= 0, \end{aligned} \quad (1.1)$$

where $p > 1$ is the power of the nonlinearity, γ its coefficient, α and β are dispersion coefficients, and δ is a constant coefficient. All parameters are nonzero real constants. Here, the complex function $E(t, x)$ denotes the envelope of high-frequency waves (or the electron wave function), and the real function $w(t, x)$ represents the amplitude of low-frequency waves.

Some papers have recently been published applying direct methods to solve different versions of system (1.1). For example, the extended simplest equation method was used in Ref. [14] to construct travelling wave solutions involving hyperbolic and rational functions. Similarly, the tanh method has been successfully applied to obtain periodic and soliton solutions for various coupled nonlinear evolution equations in the complex domain [15]. Other studies have focused on more complex structures, such as higher-order rogue waves. In this regard, the bilinear reduction method of the KP hierarchy has been used to derive rogue wave patterns that include bright, dark, and four-petal states [16, 17]. Furthermore, the principle of undetermined coefficients and Bernoulli sub-ODE methods have been used to explore topological and algebraic solutions [18]. All these works deal with the quadratic case $p = 2$.

Complementary to the construction of explicit solutions, the asymptotic behavior of solutions of integrable nonlinear wave equations has been extensively studied through the inverse scattering transform and the Deift-Zhou nonlinear steepest descent method for the associated Riemann-Hilbert problems, both for decaying initial data [19, 20] and for step-like initial data [21]. In this direction, $\bar{\partial}$ -techniques have been used to establish the soliton resolution conjecture and the asymptotic stability of N -soliton solutions on a nonzero background [22], while suitable decompositions in the complex plane of unified-transform representation formulae have revealed new long-range wave phenomena in initial-boundary value problems [23].

Despite these advances, the p -power nonlinearity generalization of the Boussinesq equation coupled to the complex NLS equation has not previously been studied in this general form. In the present work, we focus on determining symmetries, conservation laws, and travelling wave solutions for system (1.1).

To study these aspects of the generalized Schrödinger-Boussinesq system (1.1), we write the complex envelope as

$$E(t, x) = u(t, x) + iv(t, x), \quad (1.2)$$

where $u(t, x)$ and $v(t, x)$ are real functions. Substituting this expression of the complex envelope (1.2)

into system (1.1) and separating real and imaginary parts yields

$$\begin{aligned}u_t + v_{xx} + \delta v + 2vw &= 0, \\v_t - u_{xx} - \delta u - 2uw &= 0, \\w_{tt} - \alpha w_{xx} - \beta w_{xxxx} - \gamma(w^p)_{xx} - (u^2 + v^2)_{xx} &= 0.\end{aligned}\tag{1.3}$$

The outline of the paper is as follows. In Section 2, we present a complete classification of all point symmetries, providing their physical meaning and their commutator structure. In Section 3, a complete classification of local low-order conservation laws is derived using the multiplier method. The physical meaning of the conserved quantities is discussed. In Section 4, we seek nonlocal conservation laws by introducing a potential variable. To begin, it is shown that the potential system admits a Lagrangian formulation; consequently, we study its conservation laws through Noether's theorem. Furthermore, a Hamiltonian structure is determined by introducing an additional potential variable into the potential system. Then, we obtain the symmetries of the potential system and identify which of them are variational, leading to the conservation laws of the potential system. Finally, in Section 5, several families of travelling wave solutions are constructed and their physical properties are described. Conclusions are summarized in Section 6.

The mathematical setting used in the paper is calculus on jet spaces [24]. A detailed mathematical formulation of symmetries, conservation laws, and their applications to partial differential equations (PDEs) can be found in Refs. [24, 25].

2. Point symmetries

A point symmetry is a continuous transformation group on the space of independent and dependent variables (t, x, u, v, w) such that the coupled generalized Schrödinger-Boussinesq equations (1.3) are invariant. The infinitesimal form of this transformation group consists of infinitesimal transformations on (t, x, u, v, w) given by

$$\begin{aligned}t &\rightarrow t + \epsilon \tau(t, x, u, v, w) + O(\epsilon^2), \\x &\rightarrow x + \epsilon \xi(t, x, u, v, w) + O(\epsilon^2), \\u &\rightarrow u + \epsilon \eta^u(t, x, u, v, w) + O(\epsilon^2), \\v &\rightarrow v + \epsilon \eta^v(t, x, u, v, w) + O(\epsilon^2), \\w &\rightarrow w + \epsilon \eta^w(t, x, u, v, w) + O(\epsilon^2)\end{aligned}\tag{2.1}$$

where $\epsilon \in \mathbb{R}$ is the group parameter. These transformations can be written in operator form by the infinitesimal generator

$$\mathbf{X} = \tau \partial_t + \xi \partial_x + \eta^u \partial_u + \eta^v \partial_v + \eta^w \partial_w.\tag{2.2}$$

The resulting transformation group will be a point symmetry if and only if (invariance condition)

$$\begin{aligned}\text{pr}^{(2)}\mathbf{X}(u_t + v_{xx} + \delta v + 2vw)|_{\mathcal{E}} &= 0, \\ \text{pr}^{(2)}\mathbf{X}(v_t - u_{xx} - \delta u - 2uw)|_{\mathcal{E}} &= 0, \\ \text{pr}^{(4)}\mathbf{X}(w_{tt} - \alpha w_{xx} - \beta w_{xxxx} - \gamma(w^p)_{xx} - (u^2 + v^2)_{xx})|_{\mathcal{E}} &= 0,\end{aligned}\tag{2.3}$$

where $\text{pr}^{(i)}$ is the i th-order prolongation of $\mathbf{X}$ and $\mathcal{E}$ is the solution space of system (1.3). These symmetry determining Eqs (2.3) split with respect to derivatives of u, v, w , thereby yielding an overdetermined

linear system of equations for the infinitesimals $\tau, \xi, \eta^u, \eta^v, \eta^w$, along with the real parameters $\alpha, \beta, \gamma, \delta$, and $p > 1$. The following solution is obtained using MAPLE.

Theorem 1. *The infinitesimal point symmetries of the coupled generalized Schrödinger-Boussinesq equations (1.3) with $p > 1$ are comprised by the linear span of a time-translation, a space-translation, and a phase rotation:*

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = v\partial_u - u\partial_v. \quad (2.4)$$

An additional point symmetry is admitted by Eqs (1.3) only for $p = 2$:

$$\mathbf{X}_4 = 2t\partial_t + x\partial_x - 2((\delta - \frac{\alpha}{\gamma})tv + u)\partial_u + 2((\delta - \frac{\alpha}{\gamma})tu - v)\partial_v - (2w + \frac{\alpha}{\gamma})\partial_w. \quad (2.5)$$

This symmetry (2.5) represents a dilation in space-time combined with a scaling, a time-dependent phase rotation in (u, v) , and a shift in w .

The only nonzero commutators for these symmetry generators are given by

$$[\mathbf{X}_4, \mathbf{X}_1] = -2\mathbf{X}_1 + 2(\delta - \frac{\alpha}{\gamma})\mathbf{X}_3, \quad [\mathbf{X}_4, \mathbf{X}_2] = -\mathbf{X}_2. \quad (2.6)$$

For arbitrary $p > 1$, the span of the symmetry generators (2.4) comprises a three-dimensional abelian group. For $p = 2$, the span of the symmetry generators (2.4) and (2.5) represents a four-dimensional nonabelian group.

Each symmetry generator $\mathbf{X}$ yields a one-parameter (ϵ) transformation group given in operator form by $\exp(\epsilon\mathbf{X}) = I + \sum_{n=1}^{\infty} \frac{1}{n!} \epsilon^n \mathbf{X}^n$, where I denotes the identity operator. The resulting finite transformations on the variables (t, x, u, v, w) consist of the point transformations

$$(t, x, u, v, w) \rightarrow \exp(\epsilon\mathbf{X})(t, x, u, v, w). \quad (2.7)$$

From Theorem 1, the point symmetries of the coupled generalized Schrödinger-Boussinesq equations (1.3) with $p > 1$ produce the following symmetry transformation groups. For simplicity, the variables in the transformations will be arranged as t, x, u, v, w .

Proposition 1. (i) *The two translations and the phase rotation (2.4) yield the point transformation symmetry groups*

$$\begin{aligned} \mathbf{X}_1 : t &\rightarrow t + \epsilon, \\ \mathbf{X}_2 : x &\rightarrow x + \epsilon, \\ \mathbf{X}_3 : (u, v) &\rightarrow (v \sin(\epsilon) + u \cos(\epsilon), v \cos(\epsilon) - u \sin(\epsilon)), \end{aligned} \quad (2.8)$$

which together comprise a 3-dimensional abelian group.

(ii) *If $p = 2$, the symmetry generator (2.5) yields the point transformation symmetry group*

$$\begin{aligned} \mathbf{X}_4 : t &\rightarrow te^{2\epsilon}, \\ x &\rightarrow xe^{\epsilon}, \\ u &\rightarrow e^{-2\epsilon}(u \cos((\delta - \frac{\alpha}{\gamma})t(e^{2\epsilon} - 1)) - v \sin((\delta - \frac{\alpha}{\gamma})t(e^{2\epsilon} - 1))), \\ v &\rightarrow e^{-2\epsilon}(u \sin((\delta - \frac{\alpha}{\gamma})t(e^{2\epsilon} - 1)) + v \cos((\delta - \frac{\alpha}{\gamma})t(e^{2\epsilon} - 1))), \\ w &\rightarrow e^{-2\epsilon}(w + \frac{\alpha}{2\gamma}) - \frac{\alpha}{2\gamma}, \end{aligned} \quad (2.9)$$

which together with these transformations (2.8) comprises a 4-dimensional nonabelian group.

There is an alternative way to formulate infinitesimal symmetries, which will be useful for Section 4. A symmetry generator (2.2) has an equivalent evolutionary form which is given by

$$\hat{\mathbf{X}} = P^u \partial_u + P^v \partial_v + P^w \partial_w, \quad (2.10)$$

where

$$\begin{aligned} P^u &= \eta^u - \tau u_t - \xi u_x, \\ P^v &= \eta^v - \tau v_t - \xi v_x, \\ P^w &= \eta^w - \tau w_t - \xi w_x, \end{aligned} \quad (2.11)$$

are called the characteristics of the generator. The characteristic functions of the symmetry generators (2.4) and (2.5) are given by

$$\begin{aligned} P_1^u &= -u_t, & P_1^v &= -v_t, & P_1^w &= -w_t; \\ P_2^u &= -u_x, & P_2^v &= -v_x, & P_2^w &= -w_x; \\ P_3^u &= v, & P_3^v &= -u, & P_3^w &= 0; \\ P_4^u &= -2((\delta - \frac{\alpha}{\gamma})tv + u) - 2tu_t - xu_x, & P_4^v &= 2((\delta - \frac{\alpha}{\gamma})tu - v) - 2tv_t - xv_x, \\ P_4^w &= -(2w + \frac{\alpha}{\gamma}) - 2tw_t - xw_x. \end{aligned} \quad (2.12)$$

3. Local conservation laws

A local conservation law of the coupled generalized Schrödinger-Boussinesq equations (1.3) is given by the continuity equation

$$(D_t T + D_x \Phi)|_{\mathcal{E}} = 0 \quad (3.1)$$

holding on the solution space $\mathcal{E}$ of Eqs (1.3). Here D_t and D_x denote total derivatives and T and Φ , depending on t, x, u, v, w , and derivatives of u, v, w , are the conserved density and spatial flux, respectively.

Integrating the continuity Eq (3.1) over a spatial domain $\Omega \subseteq \mathbb{R}$ yields a physical balance (conservation) equation

$$\frac{dC}{dt} = -\Phi|_{\partial\Omega}|_{\mathcal{E}} \quad (3.2)$$

where

$$C = \int_{\Omega} T dx|_{\mathcal{E}}. \quad (3.3)$$

This states that the rate of change of the integral quantity (3.3) equals the net spatial flux leaving the domain Ω through its boundary $\partial\Omega$. For solutions of the coupled generalized Schrödinger-Boussinesq equations (1.3) with vanishing net flux, the conserved integral (3.3) is a conserved quantity.

A conservation law (3.1) is locally trivial if, for all solutions, $T = D_x \Theta$ and $\Phi = -D_t \Theta$ for some function Θ of t, x, u, v, w and derivatives of u, v, w . In this case, the balance equation (3.2) gives no useful information about the solutions $u(t, x), v(t, x), w(t, x)$. Likewise, two local conservation laws that differ by a locally trivial one are locally equivalent, since they contain the same information about $u(t, x), v(t, x), w(t, x)$ on Ω .

For PDEs with a Lagrangian formulation, Noether's theorem gives a correspondence between local conservation laws and symmetries. However, the coupled generalized Schrödinger-Boussinesq equations (1.3) admit no local Lagrangian in terms of $u(t, x)$, $v(t, x)$, $w(t, x)$, and their derivatives. Therefore, their symmetries do not generate conservation laws.

For non-variational PDEs, there is a modern generalization of Noether's theorem, which uses multipliers instead of symmetries. See Ref. [26] for a review and Refs. [27, 28] for further background. We now apply this generalization to determine local conservation laws.

A multiplier for the coupled generalized Schrödinger-Boussinesq equations (1.3) is a function Q of t, x, u, v, w and derivatives of u, v, w such that it is non-singular on solutions $u(t, x)$, $v(t, x)$, $w(t, x)$ of Eqs (1.3) and whose product with the governing equations is identically equal to a space-time total divergence

$$(u_t + v_{xx} + \delta v + 2vw)Q^u + (v_t - u_{xx} - \delta u - 2uw)Q^v + (w_{tt} - \alpha w_{xx} - \beta w_{xxx} - \gamma(w^p)_{xx} - (u^2 + v^2)_{xx})Q^w = D_t T + D_x \Phi, \quad (3.4)$$

where $Q = (Q^u, Q^v, Q^w)$. Consequently, on the solution space, this divergence expression gives a local conservation law (3.1). There is a one-to-one correspondence between conserved quantities (up to equivalence) and multipliers. Next, it is useful to introduce the Euler operator from variational calculus [24, 25]. This operator annihilates a function of t, x, u, v, w and derivatives of u, v, w if and only if the function is a space-time total divergence. Thus, all multipliers are determined by the condition

$$\begin{aligned} E_u((u_t + v_{xx} + \delta v + 2vw)Q^u + (v_t - u_{xx} - \delta u - 2uw)Q^v \\ + (w_{tt} - \alpha w_{xx} - \beta w_{xxx} - \gamma(w^p)_{xx} - (u^2 + v^2)_{xx})Q^w) &= 0, \\ E_v((u_t + v_{xx} + \delta v + 2vw)Q^u + (v_t - u_{xx} - \delta u - 2uw)Q^v \\ + (w_{tt} - \alpha w_{xx} - \beta w_{xxx} - \gamma(w^p)_{xx} - (u^2 + v^2)_{xx})Q^w) &= 0, \\ E_w((u_t + v_{xx} + \delta v + 2vw)Q^u + (v_t - u_{xx} - \delta u - 2uw)Q^v \\ + (w_{tt} - \alpha w_{xx} - \beta w_{xxx} - \gamma(w^p)_{xx} - (u^2 + v^2)_{xx})Q^w) &= 0, \end{aligned} \quad (3.5)$$

holding identically (off of $\mathcal{E}$), which follows from Eq (3.4). This condition yields an overdetermined linear system for Q . Typically, basic conservation laws, such as energy, momentum, and angular momentum, arise from multipliers whose differential order is lower than that of the original equations themselves. For the coupled generalized Schrödinger-Boussinesq equations (1.3), such low-order multipliers have the form

$$Q = (Q^u(t, x, u, v, w, v_x), Q^v(t, x, u, v, w, u_x), Q^w(t, x, u, v, w, w_t, u_x, v_x, w_x, w_{xx}, w_{xxx})). \quad (3.6)$$

We classify these multipliers by solving the determining equations arising from system (3.5) for Q with $p > 1$. Using MAPLE, we obtain the following result.

Proposition 2. *The low-order multipliers for the coupled generalized Schrödinger-Boussinesq equations (1.3) with $p > 1$ are comprised by the linear span of*

$$Q_1 = (0, 0, 1), \quad Q_2 = (0, 0, t), \quad Q_3 = (0, 0, x), \quad Q_4 = (0, 0, tx), \quad Q_5 = (u, v, 0). \quad (3.7)$$

The resulting conservation laws follow directly from integration by parts applied to the multiplier equation (3.4), yielding all low-order conservation laws explicitly. They can also be obtained by other methods (see Ref. [26] for details).

Theorem 2. *The local low-order conservation laws for the coupled generalized Schrödinger-Boussinesq equations (1.3) with $p > 1$ consist of*

$$T_1 = w_t, \quad \Phi_1 = -((\alpha + \gamma p w^{p-1})w_x + \beta w_{xxx} + 2(uu_x + vv_x)), \quad (3.8)$$

$$T_2 = tw_t - w, \quad \Phi_2 = -t((\alpha + \gamma p w^{p-1})w_x + \beta w_{xxx} + 2(uu_x + vv_x)), \quad (3.9)$$

$$T_3 = xw_t, \quad \Phi_3 = -x((\alpha + \gamma p w^{p-1})w_x + \beta w_{xxx} + 2(uu_x + vv_x)) \\ + (\alpha + \gamma w^{p-1})w + \beta w_{xx} + u^2 + v^2, \quad (3.10)$$

$$T_4 = x(tw_t - w), \quad \Phi_4 = -t(x((\alpha + \gamma p w^{p-1})w_x + \beta w_{xxx} + 2(uu_x + vv_x)) \\ - ((\alpha + \gamma w^{p-1})w + \beta w_{xx} + u^2 + v^2)), \quad (3.11)$$

$$T_5 = \frac{1}{2}(u^2 + v^2), \quad \Phi_5 = uv_x - vu_x. \quad (3.12)$$

3.1. Conserved quantities

The physical meaning of the previous conservation laws is now discussed.

The conserved integral associated with the first conservation law (3.8) is

$$\mathcal{P} = \int_{\Omega} w_t dx, \quad (3.13)$$

which describes conservation of momentum. The conserved quantity arising from the second conservation law (3.9) can be written as

$$C_2 = \int_{\Omega} tw_t - w dx = t\mathcal{P} - \mathcal{M}, \quad (3.14)$$

relating the conservation of momentum to the conservation of mass $\mathcal{M}$, where $\mathcal{M} = \int_{\Omega} w dx$. Next, the conserved quantity corresponding to the third conservation law (3.10)

$$C_3 = \int_{\Omega} xw_t dx \quad (3.15)$$

is related to the center of the momentum $C_3/\mathcal{P}$. The fourth conservation law (3.11) yields a conserved quantity that can be expressed as

$$C_4 = \int_{\Omega} x(tw_t - w) dx = tC_3 - \mathcal{X}, \quad (3.16)$$

which relates the center of the momentum to the conservation of the center of mass $\mathcal{X}$, where $\mathcal{X} = \int_{\Omega} xw dx$. Finally, the conserved integral associated with the last conservation law (3.12)

$$\mathcal{J} = \int_{\Omega} \frac{1}{2}(u^2 + v^2) dx \quad (3.17)$$

represents charge.

4. Nonlocal conservation laws

Wave equations often admit nonlocal conservation laws that provide conserved quantities beyond those derived from local conservation laws [25, 28]. A common approach to identifying these nonlocal conservation laws involves the introduction of a potential variable typically by reformulating the density and flux of a local conservation law into a trivial (nonlocal) form. Although the multiplier method is frequently used to identify these conservation laws expressed in terms of the potential variable, our approach for these coupled generalized Schrödinger-Boussinesq equations (1.3) focuses on identifying a potential that yields a potential system allowing a Lagrangian formulation. This enables the direct derivation of nonlocal conservation laws from variational symmetries through Noether's theorem.

4.1. Variational structure

A potential $\phi(t, x)$ via $w = \phi_x$ can be introduced in system (1.3) yielding the potential system

$$u_t + v_{xx} + \delta v + 2v\phi_x = 0, \quad (4.1a)$$

$$v_t - u_{xx} - \delta u - 2u\phi_x = 0, \quad (4.1b)$$

$$\phi_{tt} - \alpha\phi_{xx} - \beta\phi_{xxx} - \gamma((\phi_x)^p)_x - (u^2 + v^2)_x = 0, \quad (4.1c)$$

which possesses a Lagrangian given by

$$L = \frac{1}{2}(uv_t - vu_t) + \frac{1}{2}(u_x^2 + v_x^2) - (\phi_x + \frac{\delta}{2})(u^2 + v^2) + \frac{1}{2}\phi_t^2 - \frac{\alpha}{2}\phi_x^2 + \frac{\beta}{2}\phi_{xx}^2 - \frac{\gamma}{p+1}(\phi_x)^{p+1}. \quad (4.2)$$

Note that these Euler-Lagrange equations are twisted in the sense that the variational derivative of L with respect to u and v yields Eqs (4.1b) and (4.1a), respectively. This Lagrangian formulation motivates studying the symmetries and conservation laws of the potential system (4.1) using Noether's theorem.

Furthermore, by introducing an additional potential variable $\psi = \phi_t$, the potential system (4.1) can be expressed in the evolution form

$$\begin{aligned} u_t &= -v_{xx} - \delta v - 2v\phi_x, \\ v_t &= u_{xx} + \delta u + 2u\phi_x, \\ \psi_t &= \alpha\phi_{xx} + \beta\phi_{xxx} + \gamma((\phi_x)^p)_x + (u^2 + v^2)_x, \\ \phi_t &= \psi. \end{aligned} \quad (4.3)$$

Consequently, the Hamiltonian structure is given by

$$\begin{pmatrix} u \\ v \\ \psi \\ \phi \end{pmatrix}_t = J \begin{pmatrix} \delta H / \delta u \\ \delta H / \delta v \\ \delta H / \delta \psi \\ \delta H / \delta \phi \end{pmatrix}, \quad (4.4)$$

with the Hamiltonian operator [24]

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (4.5)$$

and the Hamiltonian given by

$$H = \int \left(\frac{1}{2}(u_x^2 + v_x^2) - (\phi_x + \frac{\delta}{2})(u^2 + v^2) - \frac{1}{2}\psi^2 - \frac{\alpha}{2}\phi_x^2 + \frac{\beta}{2}\phi_{xx}^2 - \frac{\gamma}{p+1}(\phi_x)^{p+1} \right) dx. \quad (4.6)$$

4.2. Point symmetries of the potential system

The formulation of the point symmetries of the potential system (4.1) proceeds in the same way as in Section 2.

The infinitesimal point symmetries for the potential system (4.1) are given by the generator

$$\mathbf{X}^\phi = \tau \partial_t + \xi \partial_x + \eta^u \partial_u + \eta^v \partial_v + \eta^\phi \partial_\phi \quad (4.7)$$

where $\tau, \xi, \eta^u, \eta^v, \eta^\phi$ are functions of (t, x, u, v, ϕ) . The characteristic form is

$$\hat{\mathbf{X}}^\phi = P^u \partial_u + P^v \partial_v + P^\phi \partial_\phi \quad (4.8)$$

where

$$\begin{aligned} P^u &= \eta^u - \tau u_t - \xi u_x, \\ P^v &= \eta^v - \tau v_t - \xi v_x, \\ P^\phi &= \eta^\phi - \tau \phi_t - \xi \phi_x. \end{aligned} \quad (4.9)$$

The following determining equations are solved using MAPLE:

$$\begin{aligned} \text{pr}^{(2)} \mathbf{X}^\phi (u_t + v_{xx} + \delta v + 2v\phi_x)|_{\mathcal{E}^\phi} &= 0, \\ \text{pr}^{(2)} \mathbf{X}^\phi (v_t - u_{xx} - \delta u - 2u\phi_x)|_{\mathcal{E}^\phi} &= 0, \\ \text{pr}^{(4)} \mathbf{X}^\phi (\phi_{tt} - \alpha \phi_{xx} - \beta \phi_{xxx} - \gamma((\phi_x)^p)_x - (u^2 + v^2)_x)|_{\mathcal{E}^\phi} &= 0, \end{aligned} \quad (4.10)$$

where $\text{pr}^{(i)}$ is the i th-order prolongation of $\mathbf{X}^\phi$ and $\mathcal{E}^\phi$ is the solution space of system (4.1).

Theorem 3. *The infinitesimal point symmetries of the potential system (4.1) with $p > 1$ are comprised by the linear span of a time-translation, a space-translation, a phase rotation, a translation in ϕ , and a time-dependent shift in ϕ :*

$$\mathbf{X}_1^\phi = \partial_t, \quad \mathbf{X}_2^\phi = \partial_x, \quad \mathbf{X}_3^\phi = v\partial_u - u\partial_v, \quad \mathbf{X}_4^\phi = \partial_\phi, \quad \mathbf{X}_5^\phi = t\partial_\phi. \quad (4.11)$$

An additional point symmetry is admitted by system (4.1) only for $p = 2$:

$$\mathbf{X}_6^\phi = 2t\partial_t + x\partial_x - 2((\delta - \frac{\alpha}{\gamma})tv + u)\partial_u + 2((\delta - \frac{\alpha}{\gamma})tu - v)\partial_v - (\phi + \frac{\alpha}{\gamma}x)\partial_\phi. \quad (4.12)$$

This symmetry (4.12) represents a dilation in space-time combined with a scaling, a time-dependent phase rotation in (u, v) , and a space-dependent shift in ϕ .

These point symmetries have the following characteristic functions:

$$P_1^u = -u_t, \quad P_1^v = -v_t, \quad P_1^\phi = -\phi_t; \quad (4.13)$$

$$P_2^u = -u_x, \quad P_2^v = -v_x, \quad P_2^\phi = -\phi_x; \quad (4.14)$$

$$P_3^u = v, \quad P_3^v = -u, \quad P_3^\phi = 0; \quad (4.15)$$

$$P_4^u = 0, \quad P_4^v = 0, \quad P_4^\phi = 1; \quad (4.16)$$

$$P_5^u = 0, \quad P_5^v = 0, \quad P_5^\phi = t; \quad (4.17)$$

$$P_6^u = -2((\delta - \frac{\alpha}{\gamma})tv + u) - 2tu_t - xu_x, \quad P_6^v = 2((\delta - \frac{\alpha}{\gamma})tu - v) - 2tv_t - xv_x, \\ P_6^\phi = -(\phi + \frac{\alpha}{\gamma}x) - 2t\phi_t - x\phi_x. \quad (4.18)$$

Symmetries $\mathbf{X}_1^\phi$, $\mathbf{X}_2^\phi$, $\mathbf{X}_3^\phi$, and $\mathbf{X}_6^\phi$ are inherited from the coupled generalized Schrödinger-Boussinesq equations (1.3) via the projection

$$P^w = D_x P^\phi, \quad (4.19)$$

which arises directly from $w = \phi_x$. The translation $\mathbf{X}_4^\phi$ and the time-dependent shift $\mathbf{X}_5^\phi$ are “hidden” symmetries which exist only for the potential system (4.1).

The specific correspondence between the inherited point symmetries of the potential system (4.1) and the point symmetries of system (1.3) is given by

$$\mathbf{X}_1^\phi \leftrightarrow \mathbf{X}_1; \quad \mathbf{X}_2^\phi \leftrightarrow \mathbf{X}_2; \quad \mathbf{X}_3^\phi \leftrightarrow \mathbf{X}_3; \quad \mathbf{X}_6^\phi \leftrightarrow \mathbf{X}_4. \quad (4.20)$$

4.3. Variational symmetries and Noether's theorem

Next, we determine which of the symmetries (4.13)–(4.18) admitted by the potential system (4.1) are variational. Since the potential system (4.1) has a Lagrangian formulation (4.2), Noether's theorem is used to obtain conservation laws arising from variational symmetries.

The condition for a symmetry to be a variational symmetry is given by

$$E_w(E_w(L)P) = 0 \quad (4.21)$$

where E_w denotes the Euler operator with respect to a variable w and L is the Lagrangian (4.2). Substituting each of the point symmetries (4.13)–(4.18) into the variational symmetry condition (4.21) yields the variational symmetries that will lead to conservation laws. A straightforward computation gives the following result.

Proposition 3. *The infinitesimal point symmetries (4.11) of the potential system (4.1) are variational symmetries. The infinitesimal point symmetry (4.12) is not variational.*

Then, we apply integration by parts to obtain the resulting conservation laws.

Theorem 4. *The conservation laws produced by the variational point symmetries (4.11) of the potential system (4.1) with $p > 1$ are given, respectively, by*

$$T_1^\phi = \frac{\gamma}{p+1}\phi_x^{p+1} + \frac{1}{2}(\alpha\phi_x^2 + 2(u^2 + v^2)\phi_x + \delta(u^2 + v^2) - \beta\phi_{xx}^2 + \phi_t^2 - v_x^2 - u_x^2), \quad (4.22)$$

$$\Phi_1^\phi = -\gamma\phi_t\phi_x^p - (\alpha\phi_x + \beta\phi_{xxx} + u^2 + v^2)\phi_t + \beta\phi_{tx}\phi_{xx} + v_tv_x + u_tu_x;$$

$$T_2^\phi = \phi_t\phi_x - vu_x, \quad (4.23)$$

$$\Phi_2^\phi = -\frac{\gamma}{p+1}\phi_x^{p+1} - \frac{1}{2}(\alpha\phi_x^2 + 2\beta\phi_{xxx}\phi_x - \delta(u^2 + v^2) - \beta\phi_{xx}^2 + \phi_t^2 - v_x^2 - u_x^2 - 2vu_t);$$

$$T_3^\phi = \frac{1}{2}(u^2 + v^2), \quad (4.24)$$

$$\Phi_3^\phi = uv_x - vu_x;$$

$$\begin{aligned} T_4^\phi &= \phi_t, \\ \Phi_4^\phi &= -(\alpha + \gamma\phi_x^{p-1})\phi_x - \beta\phi_{xxx} - u^2 - v^2; \end{aligned} \quad (4.25)$$

$$\begin{aligned} T_5^\phi &= t\phi_t - \phi, \\ \Phi_5^\phi &= -t((\alpha + \gamma\phi_x^{p-1})\phi_x + \beta\phi_{xxx} + u^2 + v^2). \end{aligned} \quad (4.26)$$

Note that $\phi_t = \partial_x^{-1}(w_t)$ arises from $w = \phi_x$. Therefore, the conservation laws involving ϕ_t , such as (T_1^ϕ, Φ_1^ϕ) , (T_2^ϕ, Φ_2^ϕ) , (T_4^ϕ, Φ_4^ϕ) , and (T_5^ϕ, Φ_5^ϕ) , are nonlocal conservation laws for system (1.3). It can also be seen that (T_3^ϕ, Φ_3^ϕ) is locally equivalent to (T_5, Φ_5) (3.12) from the original system (1.3).

5. Travelling wave solutions

A travelling wave solution of the coupled generalized Schrödinger-Boussinesq equations (1.3) is of the form

$$u = U(\zeta), \quad v = V(\zeta), \quad w = W(\zeta), \quad (5.1)$$

where $\zeta = x - ct$ is the invariant of the symmetry $\mathbf{X}_1 + c\mathbf{X}_2$ and c is the wave speed.

Substitution of expressions (5.1) into Eqs (1.3) yields the nonlinear ordinary differential equations (ODEs)

$$V'' - cU' + \delta V + 2VW = 0, \quad (5.2a)$$

$$U'' + cV' + \delta U + 2UW = 0, \quad (5.2b)$$

$$\beta W'''' + (\alpha - c^2 + \gamma p W^{p-1})W'' + \gamma(p-1)pW^{p-2}W'^2 + 2(VV')' + 2(UU')' = 0. \quad (5.2c)$$

By integrating the third equation (5.2c) twice with respect to ζ , we obtain

$$\beta W'' + (\alpha - c^2 + \gamma W^{p-1})W + V^2 + U^2 + k_1\zeta + k_0 = 0, \quad (5.3)$$

where k_1 and k_0 are two integration constants.

The difference between multiplying the first equation (5.2a) by U and the second equation (5.2b) by V yields

$$UV'' - VU'' - c(VV' + UU') = 0. \quad (5.4)$$

Integrating this Eq (5.4) leads to

$$UV' - VU' - \frac{1}{2}c(V^2 + U^2) + k_2 = 0, \quad (5.5)$$

where k_2 is an integration constant and which can be solved in terms of V setting $k_2 = 0$,

$$V(\zeta) = \tan(\frac{1}{2}c\zeta + \zeta_0)U(\zeta) \quad (5.6)$$

with ζ_0 an integration constant. This choice of k_2 is strictly required for localized solitary waves in order to satisfy the asymptotic boundary conditions, where U and V , along with their derivatives, vanish at infinity.

Substituting this expression of V (5.6) into the first and second equations (5.2a) and (5.2b), we observe that these two equations are proportional, and thus we only focus on the simplest one

$$U'' + c \tan(\tfrac{1}{2}c\zeta + \zeta_0)U' + \tfrac{1}{2}c^2 \tan(\tfrac{1}{2}c\zeta + \zeta_0)^2 U + (\tfrac{1}{2}c^2 + \delta + 2W)U = 0. \quad (5.7)$$

We also substitute the expression of V (5.6) in the integrated third equation (5.3) and, to preserve the same localized asymptotic behavior, we set $k_1 = 0$ and $k_0 = 0$, obtaining

$$\beta W'' + (\alpha - c^2 + \gamma W^{p-1})W + (1 + \tan(\tfrac{1}{2}c\zeta + \zeta_0)^2)U^2 = 0. \quad (5.8)$$

Furthermore, for periodic wave solutions, this choice of k_1 and k_0 uniquely isolates the fundamental family of symmetric Jacobi elliptic solutions without introducing non-physical background shifts, ensuring a natural and direct reduction to the solitary wave limit. The structure of Eq (5.8) naturally leads to the following transformation of the dependent variable U ,

$$U(\zeta) = \cos(\tfrac{1}{2}c\zeta + \zeta_0)\hat{U}(\zeta). \quad (5.9)$$

Then, the non-autonomous system formed by (5.7) and (5.8) is reduced to the coupled autonomous system given by

$$\hat{U}'' + (\lambda + 2W)\hat{U} = 0, \quad (5.10a)$$

$$\beta W'' + (\mu + \gamma W^{p-1})W + \hat{U}^2 = 0, \quad (5.10b)$$

with

$$\lambda = \tfrac{c^2}{4} + \delta, \quad \mu = \alpha - c^2. \quad (5.11)$$

Multiplying Eqs (5.10a) and (5.10b) by $\hat{U}'$ and W' , respectively, and adding the resulting equations leads to the following relation:

$$\hat{U}'\hat{U}'' + \beta W'W'' + \lambda\hat{U}\hat{U}' + \mu WW' + \gamma W^p W' + 2\hat{U}\hat{U}'W + \hat{U}^2 W' = 0, \quad (5.12)$$

which can be directly integrated with respect to ζ . This yields a first integral of system (5.10) that represents its Hamiltonian,

$$\mathcal{H} := \tfrac{1}{2}(\hat{U}')^2 + \tfrac{\beta}{2}(W')^2 + \tfrac{\lambda}{2}\hat{U}^2 + \tfrac{\mu}{2}W^2 + \tfrac{\gamma}{p+1}W^{p+1} + \hat{U}^2 W = E_0, \quad (5.13)$$

where E_0 represents a constant energy. It is important to note that this Hamiltonian is valid for an arbitrary power p . The energy of the system can also be expressed in terms of a potential

$$E_0 = \tfrac{1}{2}(\hat{U}')^2 + \tfrac{\beta}{2}(W')^2 + P(\hat{U}, W) \quad (5.14)$$

where

$$P(\hat{U}, W) = \tfrac{\lambda}{2}\hat{U}^2 + \tfrac{\mu}{2}W^2 + \tfrac{\gamma}{p+1}W^{p+1} + \hat{U}^2 W. \quad (5.15)$$

There are now different ways to find solutions $\hat{U}(\zeta)$ and $W(\zeta)$ of system (5.10).

5.1. Periodic elliptic and soliton solutions

A first way is to consider a form for $W(\zeta)$ depending on the form of $\hat{U}(\zeta)$. We consider a generalized algebraic ansatz of the form

$$W(\zeta) = C_1 \hat{U}^r(\zeta) + C_2, \quad (5.16)$$

where $C_1 \neq 0$, $r \neq 0$, and C_2 are arbitrary constants to be determined. Substituting W (5.16) into Eq (5.10a), we obtain

$$\hat{U}'' = -(\lambda + 2C_2)\hat{U} - 2C_1\hat{U}^{r+1}. \quad (5.17)$$

Without loss of generality, we assume $r \neq -2$, since the following algebraic procedure yields only constant solutions when $r = -2$.

Multiplying Eq (5.17) by $\hat{U}'$ and integrating with respect to ζ , yields

$$\frac{1}{2}(\hat{U}')^2 = -\frac{1}{2}(\lambda + 2C_2)\hat{U}^2 - \frac{2C_1}{r+2}\hat{U}^{r+2} - k, \quad (5.18)$$

where k is an integration constant.

For the proposed ansatz to yield an exact solution, Eq (5.18) must identically satisfy the Hamiltonian defined in Eq (5.13). Substituting the ansatz (5.16) into Eq (5.14), we get

$$\frac{1}{2}(\hat{U}')^2(1 + \beta C_1^2 r^2 \hat{U}^{2r-2}) + \hat{P}(\hat{U}) = E_0 \quad (5.19)$$

where

$$\hat{P}(\hat{U}) = \frac{\mu}{2}(C_1 \hat{U}^r + C_2)^2 + \frac{\gamma}{p+1}(C_1 \hat{U}^r + C_2)^{p+1} + \frac{1}{2}(\lambda + 2C_2 + 2C_1 \hat{U}^r)\hat{U}^2 \quad (5.20)$$

is obtained by substituting W into the potential P (5.15).

Eliminating the common term $\frac{1}{2}(\hat{U}')^2$ between Eqs (5.18) and (5.19) yields

$$-(k + \frac{1}{2}(\lambda + 2C_2)\hat{U}^2 + \frac{2C_1}{r+2}\hat{U}^{r+2})(1 + \beta C_1^2 r^2 \hat{U}^{2r-2}) + \hat{P}(\hat{U}) = E_0. \quad (5.21)$$

Now, we balance the highest power of $\hat{U}$ arising from the product of the first two terms in Eq (5.21), that is, $\hat{U}^{3r}$, and the highest power emerging from $\hat{P}(\hat{U})$, $\hat{U}^{r(p+1)}$. Equating these dominant exponents gives $3r = r(p+1)$. This equation yields a specific power of $p = 2$ for the nonlinearity required for system (5.10) to admit exact solutions of the form (5.16).

Having established $p = 2$, we proceed to analyze the remaining powers in Eq (5.21) which takes the form

$$\begin{aligned} & \frac{1}{3} \frac{C_1^3}{r+2} ((r+2)\gamma - 6r^2\beta)\hat{U}^{3r} + \frac{C_1^2}{2}(2C_2(\gamma - \beta r^2) - \beta r^2\lambda + \mu)\hat{U}^{2r} \\ & + \frac{C_1 r}{r+2}\hat{U}^{r+2} - k\beta C_1^2 r^2 \hat{U}^{2r-2} + C_1 C_2 (C_2\gamma + \mu)\hat{U}^r + \frac{C_2^2}{6}(3\mu + 2C_2\gamma) - k = E_0. \end{aligned} \quad (5.22)$$

To ensure the existence of periodic wave solutions, the term $k\beta C_1^2 r^2 \hat{U}^{2r-2}$ cannot be isolated. Therefore, it requires a balancing term among the powers in Eq (5.22), namely $\hat{U}^{3r}$, $\hat{U}^{2r}$, $\hat{U}^{r+2}$, $\hat{U}^r$, and $\hat{U}^0$. This naturally leads to the case $r = 2$. Setting $r = 2$ prevents the system from becoming overdetermined and thus restricting the generality of the solution. From a mathematical perspective, however, it is also of significant interest to examine the admissible cases $r = 1$ and $r = 4$. Unlike $r = 2$, where the wave speed c is a free parameter, the cases $r = 1$ and $r = 4$ yield a residual term in the balancing process that uniquely determines the wave speed.

5.1.1. Solutions for $r = 2$

Setting $p = 2$ and $r = 2$ in Eq (5.21), the resulting polynomial in $\hat{U}$ involves the powers $\hat{U}^6$, $\hat{U}^4$, $\hat{U}^2$, and $\hat{U}^0$. The coefficient of the highest power $\hat{U}^6$ is proportional to $\gamma - 6\beta$ and cannot be balanced by any other term, so its vanishing is forced by the ansatz itself, yielding

$$\gamma = 6\beta. \quad (5.23)$$

Beyond its algebraic origin, condition (5.23) has an independent meaning: $\gamma = 6\beta$ is precisely the value at which system (1.3) with $p = 2$ reduces, after a suitable scaling, to the classical Schrödinger-Boussinesq system, which is integrable, possessing the Painlevé property and admitting Bäcklund transformations [29]. For generic β and γ the system is therefore not integrable.

Now, we equate the coefficients of the remaining powers in Eq (5.21), namely $\hat{U}^4$, $\hat{U}^2$, and $\hat{U}^0$, which lead to the following system:

$$1 + C_1(\mu + 4\beta(C_2 - \lambda)) = 0, \quad (5.24a)$$

$$C_2\mu + 2\beta(3C_2^2 - 2C_1k) = 0, \quad (5.24b)$$

$$2(E_0 + k) - C_2^2(4C_2\beta + \mu) = 0. \quad (5.24c)$$

From Eq (5.24a) we obtain

$$C_1 = \frac{1}{4\beta(\lambda - C_2) - \mu}. \quad (5.25)$$

Equation (5.24b) dictates the fundamental nature of the resulting dynamics, establishing the conditions for periodic elliptic solutions ($k \neq 0$) and solitons ($k = 0$).

Soliton solutions:

We examine the case of localized solitary waves. A soliton requires the integration constant to be identically zero (i.e., $k = 0$) to ensure appropriate decay at infinity. Setting $k = 0$ in Eq (5.24b) leads to

$$C_2(\mu + 6\beta C_2) = 0. \quad (5.26)$$

This constraint yields two different subcases.

- If $C_2 = 0$, then $C_1 = \frac{1}{4\beta\lambda - \mu}$, and Eq (5.18) takes the form

$$(\hat{U}')^2 = -\lambda\hat{U}^2 - \frac{1}{4\beta\lambda - \mu}\hat{U}^4, \quad (5.27)$$

whose solution is given by

$$\hat{U}(\zeta) = \pm \sqrt{-\lambda(4\beta\lambda - \mu)} \operatorname{sech}(\sqrt{-\lambda}(\zeta - \zeta_1)), \quad (5.28)$$

with ζ_1 an arbitrary constant. Using ansatz (5.16), and transformations (5.9) and (5.6), we obtain the following solution of system (5.2),

$$U(\zeta) = \pm \sqrt{-\lambda(4\beta\lambda - \mu)} \cos(\tfrac{1}{2}c\zeta + \zeta_0) \operatorname{sech}(\sqrt{-\lambda}(\zeta - \zeta_1)), \quad (5.29)$$

$$V(\zeta) = \pm \sqrt{-\lambda(4\beta\lambda - \mu)} \sin(\tfrac{1}{2}c\zeta + \zeta_0) \operatorname{sech}(\sqrt{-\lambda}(\zeta - \zeta_1)), \quad (5.30)$$

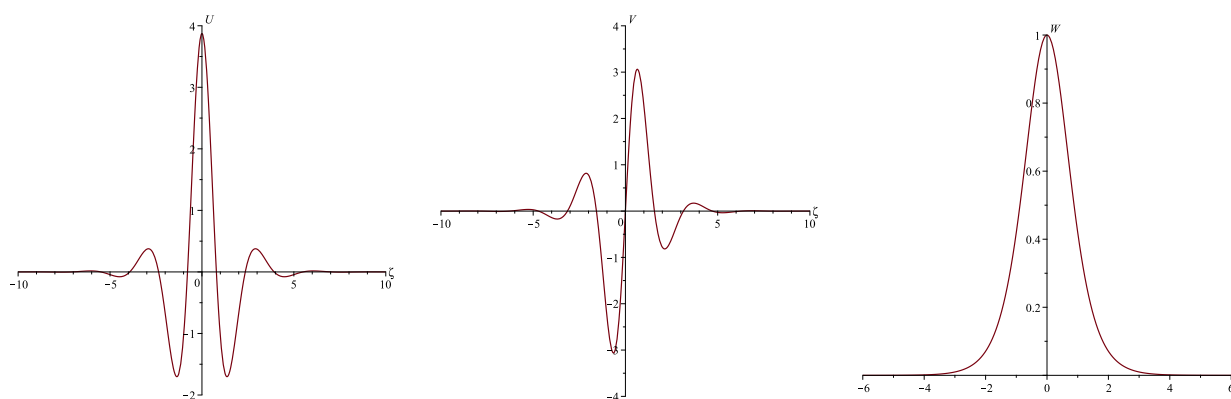


Figure 1. Solutions from Eqs (5.29) (left), (5.30) (middle), and (5.31) (right) where $\alpha = -3$, $\delta = -5$, $c = 4$, $\beta = 1$, $\zeta_0 = 0$, and $\zeta_1 = 0$, with positive signs in U , V and λ , μ defined by Eq (5.11).

$$W(\zeta) = -\lambda \operatorname{sech}(\sqrt{-\lambda}(\zeta - \zeta_1))^2. \quad (5.31)$$

Square-root arguments are always taken as positive, which sets the parameter constraints for each solution in this section.

Since the trigonometric factors cancel in $U^2 + V^2$, the modulus of the complex envelope (1.2), $|E| = \sqrt{U^2 + V^2}$, and the long-wave component W are both bright (elevation) solitons on a zero background (see Figure 1). Physically, the solutions for U and V represent wave packets where high-frequency oscillations from the cos and sin terms are confined within a localized envelope. Their spatial localization is set by the hyperbolic secant function, which concentrates the solution's energy in a finite region and yields exponential decay as $\zeta \rightarrow \pm\infty$. The sech term controls the width and peak amplitude of the soliton and thus the global dynamics, while the trigonometric terms only fix the internal phase structure.

Here the envelope $|E|$ has amplitude $\sqrt{-\lambda(4\beta\lambda - \mu)}$, the long-wave component W has amplitude $-\lambda$, and both share the common width $1/\sqrt{-\lambda}$, so that λ alone controls the localization, taller humps of W being necessarily narrower, while μ only rescales the amplitude of the envelope. This illustrates a general feature of the reduction: since system (5.10) depends on the original parameters only through λ and μ , together with β and γ , all the solutions constructed below are governed by these combinations rather than by c , δ , and α , separately; the argument of the hyperbolic or elliptic function fixes the spatial scale of each profile and the prefactor fixes its amplitude.

- If $C_2 \neq 0$, then

$$C_1 = \frac{3}{12\beta\lambda - \mu}, \quad C_2 = -\frac{\mu}{6\beta}, \quad E_0 = \frac{\mu^3}{216\beta^2}, \quad (5.32)$$

and Eq (5.18) takes the form

$$(\hat{U}')^2 = \left(\frac{\mu}{3\beta} - \lambda\right)\hat{U}^2 + \frac{3}{\mu - 12\beta\lambda}\hat{U}^4, \quad (5.33)$$

whose solution is given by

$$\hat{U}(\zeta) = \pm \frac{1}{3} \sqrt{15\lambda\mu - 36\beta\lambda^2 - \mu^2/\beta} \operatorname{sech}(\sqrt{\mu/(3\beta) - \lambda}(\zeta - \zeta_1)), \quad (5.34)$$

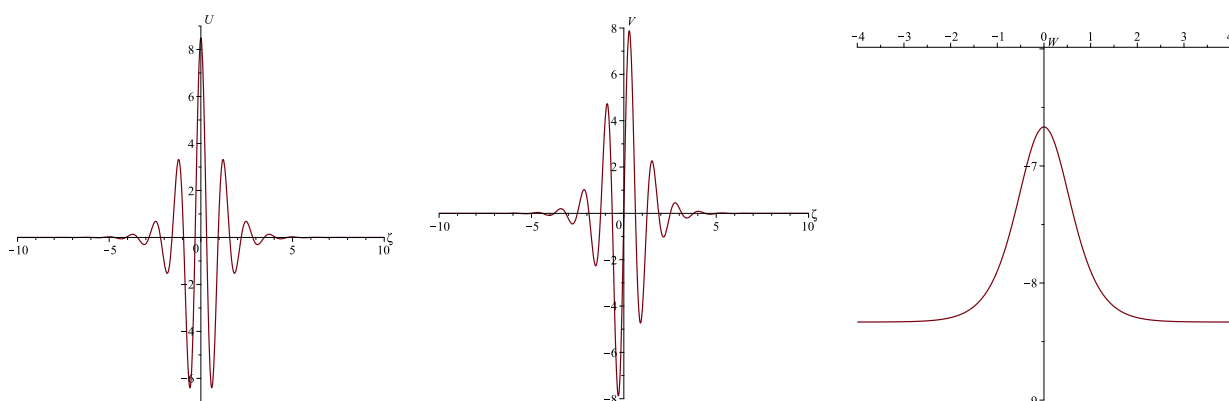


Figure 2. Solutions from Eqs (5.35) (left), (5.36) (middle), and (5.37) (right) where $\alpha = 150$, $\delta = -10$, $c = 10$, $\beta = 1$, $\zeta_0 = 0$, and $\zeta_1 = 0$, with positive signs in U , V and λ, μ defined by Eq (5.11).

with ζ_1 an arbitrary constant. Using ansatz (5.16), and transformations (5.9) and (5.6), we obtain the following solution of system (5.2),

$$U(\zeta) = \pm \frac{1}{3} \sqrt{15\lambda\mu - 36\beta\lambda^2 - \mu^2/\beta} \cos(\frac{1}{2}c\zeta + \zeta_0) \operatorname{sech}(\sqrt{\mu/(3\beta) - \lambda}(\zeta - \zeta_1)), \quad (5.35)$$

$$V(\zeta) = \pm \frac{1}{3} \sqrt{15\lambda\mu - 36\beta\lambda^2 - \mu^2/\beta} \sin(\frac{1}{2}c\zeta + \zeta_0) \operatorname{sech}(\sqrt{\mu/(3\beta) - \lambda}(\zeta - \zeta_1)), \quad (5.36)$$

$$W(\zeta) = (\frac{\mu}{3\beta} - \lambda) \operatorname{sech}(\sqrt{\mu/(3\beta) - \lambda}(\zeta - \zeta_1))^2 - \frac{\mu}{6\beta}. \quad (5.37)$$

These solutions share the same localized oscillatory structure as the previous case (see Figure 2): the trigonometric factors cancel in $U^2 + V^2$, so the envelope modulus is again a bright soliton decaying to zero and the oscillations of U and V are pure phase modulations. The difference lies in W (5.37), which now rides on the nonzero background $-\frac{\mu}{6\beta}$; reality of the argument of the hyperbolic secant forces $\frac{\mu}{3\beta} - \lambda > 0$, so W still forms an elevation above its asymptotic level. This pedestal shifts the effective potential felt by $\hat{U}$ in (5.10a) from λ to $\lambda + 2C_2$, so that the width of the pair is now governed by $1/\sqrt{\frac{\mu}{3\beta} - \lambda}$ instead of $1/\sqrt{-\lambda}$.

Periodic elliptic solutions:

Consider $k \neq 0$. From system (5.24) we obtain that C_1 is given by Eq (5.25), and k and E_0 are given by

$$k = \frac{C_2}{4\beta}(6C_2\beta + \mu)(4\beta(\lambda - C_2) - \mu), \quad E_0 = \frac{C_2^2}{2}(4C_2\beta + \mu) - k. \quad (5.38)$$

Therefore, Eq (5.18) takes the form

$$(\hat{U}')^2 = -2k + \sigma \hat{U}^2 - C_1 \hat{U}^4, \quad (5.39)$$

with

$$\sigma = -\frac{c^2}{4} - 2C_2 - \delta, \quad (5.40)$$

which admits the solution given by

$$\hat{U}(\zeta) = \pm \sqrt{\frac{\sigma + \sqrt{\sigma^2 - 8kC_1}}{2C_1}} \operatorname{cn}((\sigma^2 - 8kC_1)^{1/4}(\zeta - \zeta_1), \frac{1}{2}(1 + \frac{\sigma}{\sqrt{\sigma^2 - 8kC_1}})), \quad (5.41)$$

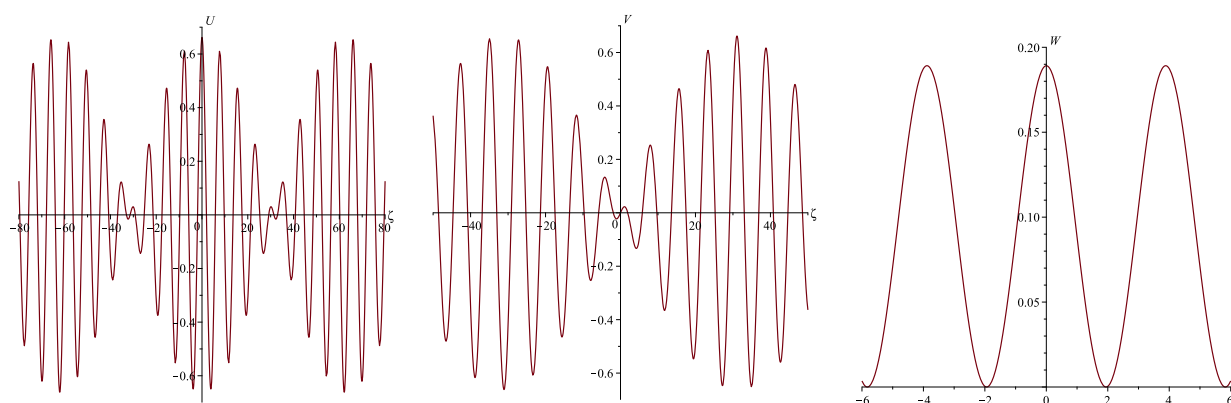


Figure 3. Solutions from Eqs (5.42) (left), (5.43) (middle), and (5.44) (right) where $\alpha = -2$, $\delta = 0.2$, $c = 0.1$, $\beta = 0.5$, $C_2 = 0.05$, $\zeta_0 = 0$, and $\zeta_1 = 0$, with positive signs in U , V and σ , k , C_1 defined respectively by Eqs (5.40), (5.38), (5.25).

where cn denotes the Jacobi elliptic cosine function with ζ_1 an arbitrary constant. Using ansatz (5.16), and transformations (5.9) and (5.6), we obtain the following solution of system (5.2),

$$U(\zeta) = \pm \sqrt{\frac{\sigma + \sqrt{\sigma^2 - 8kC_1}}{2C_1}} \cos\left(\frac{1}{2}c\zeta + \zeta_0\right) \text{cn}\left((\sigma^2 - 8kC_1)^{1/4}(\zeta - \zeta_1), \frac{1}{2}\left(1 + \frac{\sigma}{\sqrt{\sigma^2 - 8kC_1}}\right)\right), \quad (5.42)$$

$$V(\zeta) = \pm \sqrt{\frac{\sigma + \sqrt{\sigma^2 - 8kC_1}}{2C_1}} \sin\left(\frac{1}{2}c\zeta + \zeta_0\right) \text{cn}\left((\sigma^2 - 8kC_1)^{1/4}(\zeta - \zeta_1), \frac{1}{2}\left(1 + \frac{\sigma}{\sqrt{\sigma^2 - 8kC_1}}\right)\right), \quad (5.43)$$

$$W(\zeta) = \frac{1}{2}(\sigma + \sqrt{\sigma^2 - 8kC_1}) \text{cn}\left((\sigma^2 - 8kC_1)^{1/4}(\zeta - \zeta_1), \frac{1}{2}\left(1 + \frac{\sigma}{\sqrt{\sigma^2 - 8kC_1}}\right)\right)^2 + C_2. \quad (5.44)$$

The periodic profiles in Figure 3 are cnoidal wave solutions expressed with the Jacobi elliptic function $\text{cn}(\zeta, m)$ [30], representing a train of nonlinear waves. A key property of these elliptic solutions is their connection to the solitary waves discussed earlier: localized solitons (on zero and nonzero backgrounds) arise as a limiting case of these periodic patterns. As the elliptic modulus $m \rightarrow 1$, the cnoidal wave period diverges, the pulse train breaks, and one obtains a sech-type soliton.

5.1.2. Solutions for $r = 1$

Next, we seek solutions for Eq (5.18) for $r = 1$. To determine U , V , and W of system (5.2), ansatz (5.16) and transformations (5.9) and (5.6) can be employed.

Analogously to the previous case, we equate the coefficients of the different powers in Eq (5.21), namely $\hat{U}^3$, $\hat{U}^2$, $\hat{U}^1$, and $\hat{U}^0$, and we get the following system:

$$1 + C_1^2(\gamma - 2\beta) = 0, \quad (5.45a)$$

$$C_2(C_2\gamma + \mu) = 0, \quad (5.45b)$$

$$2C_2(\beta - \gamma) + \beta\lambda - \mu = 0, \quad (5.45c)$$

$$6(E_0 + k) + 6C_1^2k\beta - C_2^2(2C_2\gamma + 3\mu) = 0. \quad (5.45d)$$

From Eq (5.45a), we obtain

$$C_1 = \mp \frac{1}{\sqrt{2\beta - \gamma}}, \quad (5.46)$$

with $2\beta > \gamma$. Equation (5.45b) leads to two different subcases: $C_2 = 0$ and $C_2 = -\frac{\mu}{\gamma}$.

- If $C_2 = 0$, from Eqs (5.45c) and (5.45d) we obtain

$$\mu = \beta\lambda, \quad (5.47)$$

$$E_0 = \frac{k(\gamma-3\beta)}{2\beta-\gamma}, \quad (5.48)$$

and Eq (5.18) takes the form

$$(\hat{U}')^2 = -2k - \frac{\alpha+4\delta}{4+\beta}\hat{U}^2 \pm \frac{4}{\sqrt[3]{2\beta-\gamma}}\hat{U}^3. \quad (5.49)$$

Condition (5.47) implies that the wave speed c cannot be arbitrary; rather, it is uniquely determined by a consistency condition that involves the parameters of the original PDE system, given by

$$c^2 = \frac{4(\alpha-\beta\delta)}{4+\beta}. \quad (5.50)$$

Equation (5.49) can be mapped into the canonical form of the Weierstrass elliptic function $\wp(\zeta)$ defined by the ODE

$$(\wp'(\zeta))^2 = 4\wp(\zeta)^3 - g_2\wp(\zeta) - g_3 \quad (5.51)$$

through the transformation

$$\hat{U}(\zeta) = \pm \sqrt{2\beta-\gamma}(3\wp(\zeta) + \frac{Q}{4}), \quad (5.52)$$

where

$$g_2 = \frac{Q^2}{12}, \quad g_3 = \frac{Q^3}{216} + \frac{2k}{9(2\beta-\gamma)}, \quad (5.53)$$

with

$$Q = \frac{\alpha+4\delta}{4+\beta}. \quad (5.54)$$

Thus, we find that the solution to Eq (5.49) is given by

$$\hat{U}(\zeta) = \pm \sqrt{2\beta-\gamma}(3\wp(\zeta - \zeta_1; g_2, g_3) + \frac{Q}{4}), \quad (5.55)$$

with ζ_1 an arbitrary constant, and g_2 , g_3 , and Q respectively given by Eqs (5.53) and (5.54).

In order to obtain real bounded solutions, we assume that the associated cubic polynomial

$$\pm \frac{4}{\sqrt[3]{2\beta-\gamma}}\hat{U}^3 - Q\hat{U}^2 - 2k = 0 \quad (5.56)$$

has three real roots.

Let us first assume that we have three distinct real roots u_1 , u_2 , and u_3 (therefore $k \neq 0$). We define an auxiliary angle

$$\theta = \arccos\left(1 + \frac{48k}{Q^3(2\beta-\gamma)}\right) \quad (5.57)$$

for $-2 \leq \frac{48k}{Q^3(2\beta-\gamma)} \leq 0$, in order that θ is a real number. By using Vieta's trigonometric method, we obtain that the roots are given by

$$\begin{aligned} u_1 &= \pm \frac{\sqrt{2\beta-\gamma}Q}{4}(1 + 2\cos(\frac{\theta}{3})), \\ u_2 &= \pm \frac{\sqrt{2\beta-\gamma}Q}{4}(1 + 2\cos(\frac{\theta-2\pi}{3})), \\ u_3 &= \pm \frac{\sqrt{2\beta-\gamma}Q}{4}(1 + 2\cos(\frac{\theta-4\pi}{3})). \end{aligned} \quad (5.58)$$

Now, we distinguish different subcases depending on the coefficients of $\hat{U}^3$ and $\hat{U}^2$ in the original ODE (5.49).

- Original equation sign (+) and $Q > 0$, then $u_1 > u_2 > u_3$ and the wave is confined to the lower bound of the potential well, oscillating between the smallest roots (i.e., $u_3 \leq \hat{U} \leq u_2$) which leads to the sn-type periodic solution given by

$$\hat{U}(\zeta) = u_3 + (u_2 - u_3) \operatorname{sn} \left(\sqrt{\frac{u_1 - u_3}{3\sqrt{2\beta - \gamma}}} (\zeta - \zeta_1), \frac{u_2 - u_3}{u_1 - u_3} \right)^2. \quad (5.59)$$

- Original equation sign (+) and $Q < 0$, then $u_3 > u_2 > u_1$ and the wave remains confined at the bottom of the potential well, bounded by the smallest roots (i.e., $u_1 \leq \hat{U} \leq u_2$) which leads to the sn-type periodic solution given by

$$\hat{U}(\zeta) = u_1 + (u_2 - u_1) \operatorname{sn} \left(\sqrt{\frac{u_3 - u_1}{3\sqrt{2\beta - \gamma}}} (\zeta - \zeta_1), \frac{u_2 - u_1}{u_3 - u_1} \right)^2. \quad (5.60)$$

- Original equation sign (-) and $Q > 0$, then $u_3 > u_2 > u_1$ and the wave is confined to the peak of the inverted potential, oscillating between the two largest roots (i.e., $u_2 \leq \hat{U} \leq u_3$) which leads to the cn-type periodic solution given by

$$\hat{U}(\zeta) = u_2 + (u_3 - u_2) \operatorname{cn} \left(\sqrt{\frac{u_3 - u_1}{3\sqrt{2\beta - \gamma}}} (\zeta - \zeta_1), \frac{u_3 - u_2}{u_3 - u_1} \right)^2. \quad (5.61)$$

- Original equation sign (-) and $Q < 0$, then $u_1 > u_2 > u_3$ and the wave remains confined to the potential crest, bounded by the largest roots (i.e., $u_2 \leq \hat{U} \leq u_1$) which leads to the cn-type periodic solution given by

$$\hat{U}(\zeta) = u_2 + (u_1 - u_2) \operatorname{cn} \left(\sqrt{\frac{u_1 - u_3}{3\sqrt{2\beta - \gamma}}} (\zeta - \zeta_1), \frac{u_1 - u_2}{u_1 - u_3} \right)^2. \quad (5.62)$$

In solutions (5.59)–(5.62), u_1 , u_2 , and u_3 are given by Eq (5.58) and ζ_1 represents an arbitrary constant.

These solutions illustrated in Figure 4 clearly represent periodic wave solutions, distinct from the localized solitary waves. These profiles describe a continuous train of nonlinear waves that maintain a constant amplitude and wavelength across the spatial domain.

Assuming $k = 0$, the lack of a linear term in the original governing equation (5.49) implies that the double root must occur precisely at the origin and the asymptotic background of the wave is strictly forced to vanish. Consequently, Jacobi elliptic functions reduce to hyperbolic secant functions, excluding the existence of standard dark solitons. The existence of a physical solitary wave requires $Q < 0$ to maintain a real-valued phase velocity. Under this necessary constraint, Eq (5.49) with $k = 0$ admits the localized soliton solution given by

$$\hat{U}(\zeta) = \pm \frac{3\sqrt{2\beta - \gamma}Q}{4} \operatorname{sech} \left(\frac{1}{2} \sqrt{-Q} (\zeta - \zeta_1) \right)^2, \quad (5.63)$$

with ζ_1 an arbitrary constant and Q given by Eq (5.54). Depending on the sign ($\pm$) in the original equation, solution (5.63) is a localized pulse of positive or negative amplitude, both decaying exponentially to the vanishing background at a rate governed by Q . In either case the envelope modulus $|E|$ is a bright soliton, since the sign of $\hat{U}$ only affects the internal phase of the wave packet.

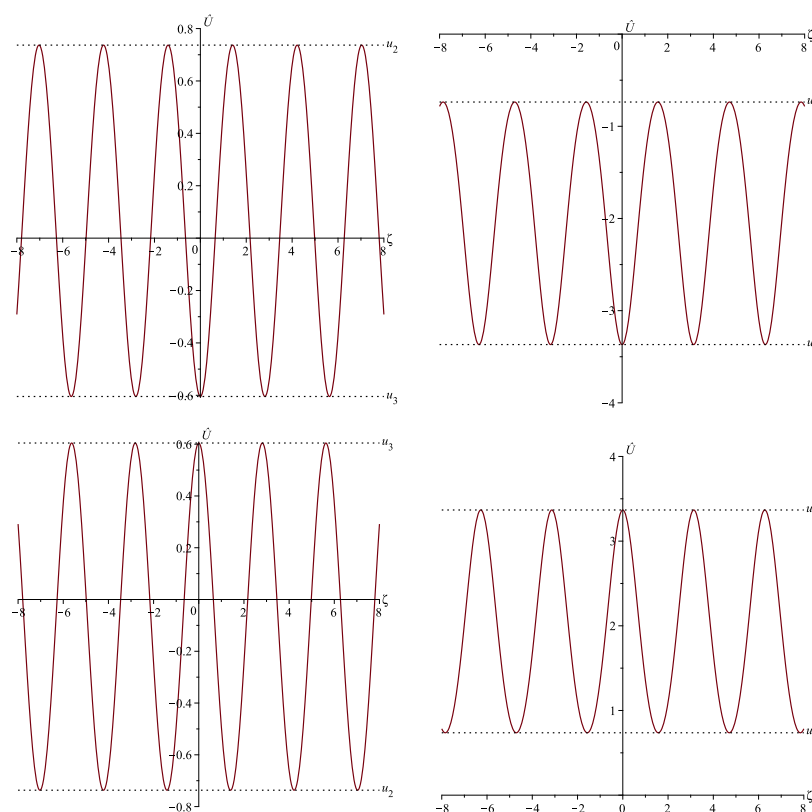


Figure 4. Left panel: Solutions from Eqs (5.59) (top) and (5.61) (bottom) where $\alpha = 10$, $\delta = 1$, $\beta = 1$, $\gamma = 1$, $k = -1$, and $\zeta_1 = 0$, with positive (top) and negative (bottom) signs in u_1 , u_2 , and u_3 . Right panel: Solutions from Eqs (5.60) (top) and (5.62) (bottom) where $\alpha = 10$, $\delta = 1$, $\beta = -5$, $\gamma = -11$, $k = 1$, and $\zeta_1 = 0$, with positive (top) and negative (bottom) signs in u_1 , u_2 , and u_3 . Here, u_1 , u_2 , and u_3 are given by Eq (5.58). Parameters Q and θ are defined by Eqs (5.57) and (5.54).

- If $C_2 = -\frac{\mu}{\gamma}$, from Eqs (5.45c) and (5.45d) we obtain

$$\mu = \frac{\beta\gamma\lambda}{2\beta-\gamma}, \quad (5.64)$$

$$E_0 = \frac{k(\gamma-3\beta)}{2\beta-\gamma} + \frac{\beta^3\gamma\lambda^3}{6(2\beta-\gamma)^3}, \quad (5.65)$$

and Eq (5.18) takes the form

$$(\hat{U}')^2 = -2k - \left(\frac{(2\beta-\gamma)(\alpha+4\delta)}{\beta(8+\gamma)-4\gamma} - \frac{2\beta\lambda}{2\beta-\gamma} \right) \hat{U}^2 \pm \frac{4}{3\sqrt{2\beta-\gamma}} \hat{U}^3. \quad (5.66)$$

Analogously, condition (5.64) implies that the wave speed c cannot be arbitrary; rather, it is uniquely determined by a consistency condition that involves the parameters of the original PDE system, given by

$$c^2 = \frac{4(2\alpha\beta-\alpha\gamma-\beta\gamma\delta)}{\beta(8+\gamma)-4\gamma}. \quad (5.67)$$

Equation (5.66) can be mapped into the canonical form of the Weierstrass elliptic (5.51) through the transformation

$$\hat{U}(\zeta) = \pm \sqrt{2\beta-\gamma} \left(3\wp(\zeta) + \frac{\hat{Q}}{4} \right), \quad (5.68)$$

where

$$g_2 = \frac{\hat{Q}^2}{12}, \quad g_3 = \frac{\hat{Q}^3}{216} + \frac{2k}{9(2\beta-\gamma)}, \quad (5.69)$$

with

$$\hat{Q} = \frac{(2\beta-\gamma)(\alpha+4\delta)}{\beta(8+\gamma)-4\gamma} - \frac{2\beta\lambda}{2\beta-\gamma}. \quad (5.70)$$

Thus, we find that the solution to Eq (5.49) is given by

$$\hat{U}(\zeta) = \pm \sqrt{2\beta-\gamma} (3\wp(\zeta - \zeta_1; g_2, g_3) + \frac{\hat{Q}}{4}), \quad (5.71)$$

with ζ_1 an arbitrary constant, and g_2 , g_3 , and $\hat{Q}$, respectively, given by Eqs (5.69) and (5.70).

Real bounded solutions can be obtained for this case following the same procedure as previously for $C_2 = 0$, substituting Q by $\hat{Q}$.

5.1.3. Solutions for $r = 4$

Finally, we seek solutions for Eq (5.18) for $r = 4$. Similarly to the previous cases, one can determine U , V , and W of system (5.2), using ansatz (5.16) along with transformations (5.9) and (5.6).

First, the coefficient of the highest power $\hat{U}^{12}$ is proportional to $\gamma - 16\beta$ and cannot be balanced by any other term, so its vanishing is forced by the ansatz itself, yielding

$$\gamma = 16\beta. \quad (5.72)$$

Then, equating the coefficients of the remaining powers in Eq (5.21), namely $\hat{U}^8$, $\hat{U}^6$, $\hat{U}^4$, and $\hat{U}^0$, we obtain the following system

$$1 - 24C_1k\beta = 0, \quad (5.73a)$$

$$C_2(C_2\gamma + \mu) = 0, \quad (5.73b)$$

$$\mu - 16\beta\lambda = 0, \quad (5.73c)$$

$$6(E_0 + k) - C_2^2(32C_2\beta + 3\mu) = 0. \quad (5.73d)$$

Equation (5.73a) yields

$$C_1 = \frac{1}{24k\beta}. \quad (5.74)$$

This condition ensures the non-existence of soliton solutions. Moreover, condition (5.73c) implies that the wave speed c cannot be arbitrary; and it is determined by a consistency condition that involves the parameters of the original PDE system, given by

$$c^2 = \frac{\alpha - 16\beta\delta}{1 + 4\beta}. \quad (5.75)$$

Equation (5.73b) leads to two different subcases: $C_2 = 0$ and $C_2 = -\frac{\mu}{\gamma}$.

- If $C_2 = 0$, from Eq (5.73d) we obtain

$$E_0 = -k, \quad (5.76)$$

and Eq (5.18) takes the form

$$(\hat{U}')^2 = -2k - \frac{\alpha+4\delta}{4+16\beta} \hat{U}^2 - \frac{1}{36k\beta} \hat{U}^6. \quad (5.77)$$

This Eq (5.77) can be mapped into the canonical form of the Weierstrass elliptic (5.51) through the transformation

$$\hat{U}(\zeta) = \pm \sqrt{-\frac{2k}{\wp(\zeta) + \hat{Q}/12}}, \quad (5.78)$$

where

$$g_2 = \frac{\hat{Q}^2}{12}, \quad g_3 = \frac{\hat{Q}^3}{216} + \frac{4k}{9\beta}, \quad (5.79)$$

with

$$\hat{Q} = \frac{\alpha + 4\delta}{1 + 4\beta}. \quad (5.80)$$

Thus, we find that the solution to Eq (5.77) is given by

$$\hat{U}(\zeta) = \pm \sqrt{-\frac{2k}{\wp(\zeta - \zeta_1; g_2, g_3) + \hat{Q}/12}}, \quad (5.81)$$

with ζ_1 an arbitrary constant, and g_2 , g_3 , and $\hat{Q}$ respectively given by Eqs (5.79) and (5.80).

- If $C_2 = -\frac{\mu}{\gamma}$, from Eq (5.73d) we obtain

$$E_0 = -k + \frac{\beta \hat{Q}^3}{24}, \quad (5.82)$$

where $\hat{Q}$ is given by Eqs (5.80) and (5.18) takes the form

$$(\hat{U}')^2 = -2k + \frac{\alpha + 4\delta}{4 + 16\beta} \hat{U}^2 - \frac{1}{36k\beta} \hat{U}^6. \quad (5.83)$$

This Eq (5.83) can be mapped into the canonical form of the Weierstrass elliptic (5.51) through the transformation

$$\hat{U}(\zeta) = \pm \sqrt{-\frac{2k}{\wp(\zeta) - \hat{Q}/12}}, \quad (5.84)$$

where

$$g_2 = \frac{\hat{Q}^2}{12}, \quad g_3 = -\frac{\hat{Q}^3}{216} + \frac{4k}{9\beta}, \quad (5.85)$$

with $\hat{Q}$ given by Eq (5.80). Thus, we find that the solution to Eq (5.83) is given by

$$\hat{U}(\zeta) = \pm \sqrt{-\frac{2k}{\wp(\zeta - \zeta_1; g_2, g_3) - \hat{Q}/12}}, \quad (5.86)$$

with ζ_1 an arbitrary constant, and g_2 , g_3 , and $\hat{Q}$, respectively, given by Eqs (5.85) and (5.80).

Bounded real solutions for these cases can be obtained in the same way as in the previous case.

5.2. Explicit sech-type solutions

A second way to find solutions for system (5.10) is by starting with a specified form for $\hat{U}(\zeta)$ and $W(\zeta)$. Let us consider $p = 2$. We now seek sech-type solutions for $\hat{U}(\zeta)$ and $W(\zeta)$ by applying the extended tanh method [31]. Consider two polynomials

$$\hat{U}(\zeta) = a_0 + \sum_{j=1}^n a_j \tanh(m\zeta)^j + \sum_{j=1}^n b_j \tanh(m\zeta)^{-j}, \quad (5.87)$$

$$W(\zeta) = s_0 + \sum_{l=1}^q s_l \tanh(m\zeta)^l + \sum_{l=1}^q h_l \tanh(m\zeta)^{-l}, \quad (5.88)$$

where m, n, q , and a_j, b_j, s_l, h_l ($j = 1, \dots, n, l = 1, \dots, q$) are unknown constants. The degrees n and q are determined by balancing the respective powers of $\tanh(m\zeta)$ that occur in the highest nonlinear terms and in the highest derivative terms in system (5.10). This yields

$$n = 2, \quad q = 2. \quad (5.89)$$

Therefore, $\hat{U}(\zeta)$ and $W(\zeta)$ have the following form:

$$\hat{U}(\zeta) = a_0 + a_1 \tanh(m\zeta) + a_2 \tanh(m\zeta)^2 + \frac{b_1}{\tanh(m\zeta)} + \frac{b_2}{\tanh(m\zeta)^2}, \quad (5.90)$$

$$W(\zeta) = s_0 + s_1 \tanh(m\zeta) + s_2 \tanh(m\zeta)^2 + \frac{h_1}{\tanh(m\zeta)} + \frac{h_2}{\tanh(m\zeta)^2}. \quad (5.91)$$

We next substitute this ansatz (5.90) and (5.91) into system (5.10), rewrite the resulting expression in terms of $\tanh(m\zeta)$, and separately put to zero the coefficient of each power of $\tanh(m\zeta)$. This yields a system of algebraic equations for $m, a_0, a_1, a_2, b_1, b_2, s_0, s_1, s_2, h_1, h_2$. Solving this system with the aid of MAPLE, we obtain the following solutions.

5.2.1. Solution family (i)

This solution family has

$$\begin{aligned} a_0 = 0, \quad a_1 = \pm \frac{1}{2} \sqrt{4\beta\lambda^2 - \lambda\mu}, \quad a_2 = 0, \quad b_1 = \mp \frac{1}{2} \sqrt{4\beta\lambda^2 - \lambda\mu}, \quad b_2 = 0, \\ s_0 = -\frac{\lambda}{2}, \quad s_1 = 0, \quad s_2 = \frac{\lambda}{4}, \quad h_1 = 0, \quad h_2 = \frac{\lambda}{4}, \quad \gamma = 6\beta, \quad m = \pm \frac{\sqrt{-\lambda}}{2}. \end{aligned} \quad (5.92)$$

Substitution of this algebraic solution into the ansatz (5.90) and (5.91) and then the expressions obtained into Eqs (5.9) and (5.6) yield the solution

$$U(\zeta) = \pm \frac{1}{2} \sqrt{4\beta\lambda^2 - \lambda\mu} \cos(\tfrac{1}{2}c\zeta + \zeta_0)(\tanh(m\zeta) - \coth(m\zeta)), \quad (5.93)$$

$$V(\zeta) = \pm \frac{1}{2} \sqrt{4\beta\lambda^2 - \lambda\mu} \sin(\tfrac{1}{2}c\zeta + \zeta_0)(\tanh(m\zeta) - \coth(m\zeta)), \quad (5.94)$$

$$W(\zeta) = -\frac{\lambda}{2} + \frac{\lambda}{4}(\tanh(m\zeta)^2 + \coth(m\zeta)^2). \quad (5.95)$$

Arguments within the square roots are always taken as positive. This sets parametric constraints for each solution in this section.

The solution families of this section have $\lambda = \frac{c^2}{4} + \delta$ and $\mu = \alpha - c^2$ as previously determined in Eq (5.11).

A remarkable feature of this family is the different roles played by the trigonometric functions; see Figure 5. The coth terms make U and W blow up at the origin, whereas the sin factor compensates the simple pole of the hyperbolic cotangent and keeps V bounded, provided $\zeta_0 = 0$ as in Figure 5. These solutions are therefore singular: they solve the reduced system classically only on the half-lines $\zeta < 0$ and $\zeta > 0$ separately, but they do not represent physical solutions on the whole line, unlike the bounded solutions obtained above.

5.2.2. Solution family (ii)

This solution family has

$$a_0 = 0, \quad a_1 = \pm \frac{1}{6} \sqrt{\frac{(3\beta\lambda - \mu)(12\beta\lambda - \mu)}{\beta}}, \quad a_2 = 0, \quad b_1 = \mp \frac{1}{6} \sqrt{\frac{(3\beta\lambda - \mu)(12\beta\lambda - \mu)}{\beta}}, \quad b_2 = 0, \quad (5.96)$$

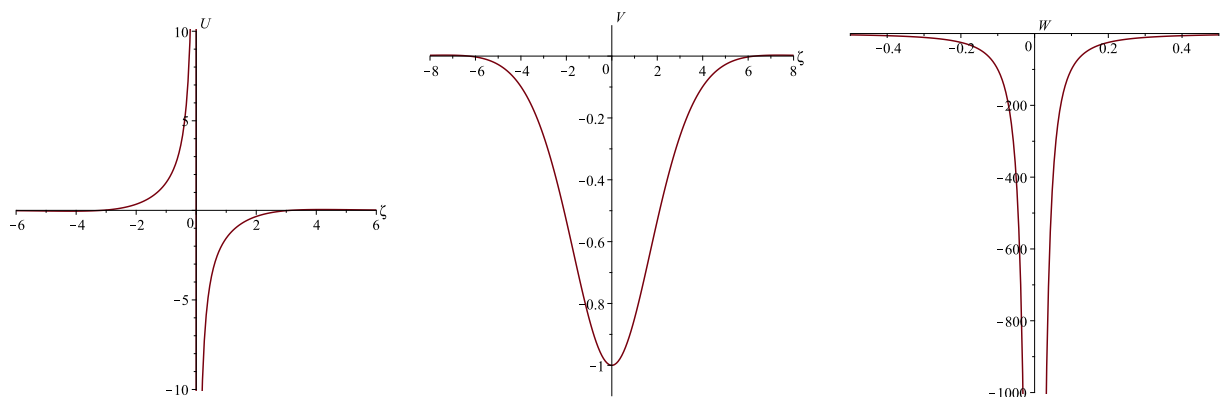


Figure 5. Solutions from Eqs (5.93) (left), (5.94) (middle), and (5.95) (right) where $\alpha = 2$, $\delta = -1$, $c = 1$, $\beta = 1$, $\zeta_0 = 0$, and $m = \sqrt{-\lambda}/2$, with positive signs in U , V and λ, μ defined by Eq (5.11).

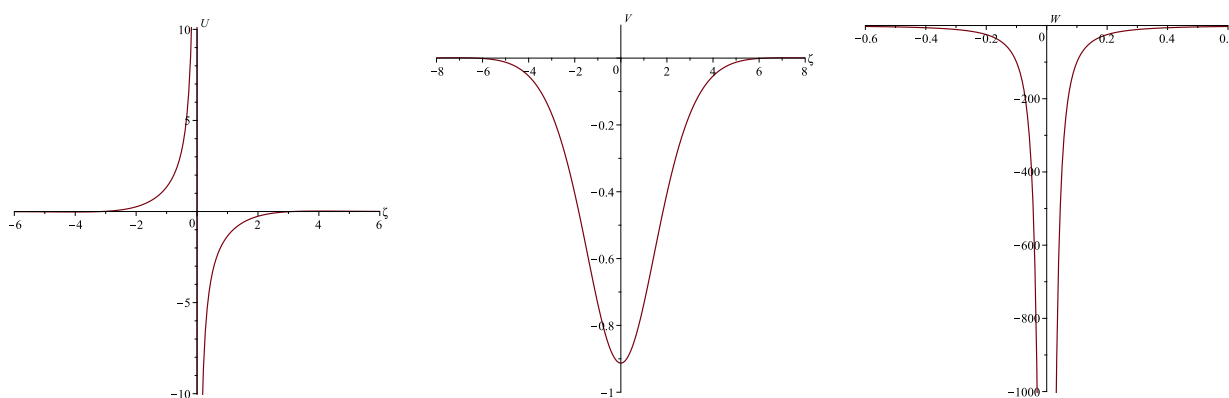


Figure 6. Solutions from Eqs (5.97) (left), (5.98) (middle), and (5.99) (right) where $\alpha = 2$, $\delta = -1$, $c = 1$, $\beta = 1$, $\zeta_0 = 0$, and $m = \sqrt{(\mu - 3\beta\lambda)/(12\beta)}$, with positive signs in U , V and λ, μ defined by Eq (5.11).

$$s_0 = -\frac{\lambda}{2}, \quad s_1 = 0, \quad s_2 = \frac{3\beta\lambda - \mu}{12\beta}, \quad h_1 = 0, \quad h_2 = \frac{3\beta\lambda - \mu}{12\beta}, \quad \gamma = 6\beta \quad m = \pm \sqrt{\frac{\mu - 3\beta\lambda}{12\beta}}.$$

Substitution of this algebraic solution into the ansatz (5.90) and (5.91) and then the expressions obtained into Eqs (5.9) and (5.6) yield the solution

$$U(\zeta) = \pm \frac{1}{6} \sqrt{\frac{(3\beta\lambda - \mu)(12\beta\lambda - \mu)}{\beta}} \cos\left(\frac{1}{2}c\zeta + \zeta_0\right)(\tanh(m\zeta) - \coth(m\zeta)), \quad (5.97)$$

$$V(\zeta) = \pm \frac{1}{6} \sqrt{\frac{(3\beta\lambda - \mu)(12\beta\lambda - \mu)}{\beta}} \sin\left(\frac{1}{2}c\zeta + \zeta_0\right)(\tanh(m\zeta) - \coth(m\zeta)), \quad (5.98)$$

$$W(\zeta) = -\frac{\lambda}{2} + \frac{3\beta\lambda - \mu}{12\beta}(\tanh(m\zeta)^2 + \coth(m\zeta)^2). \quad (5.99)$$

This second solution family in Figure 6 behaves qualitatively like the previous one, differing mainly in its specific parametric constraints on the amplitudes.

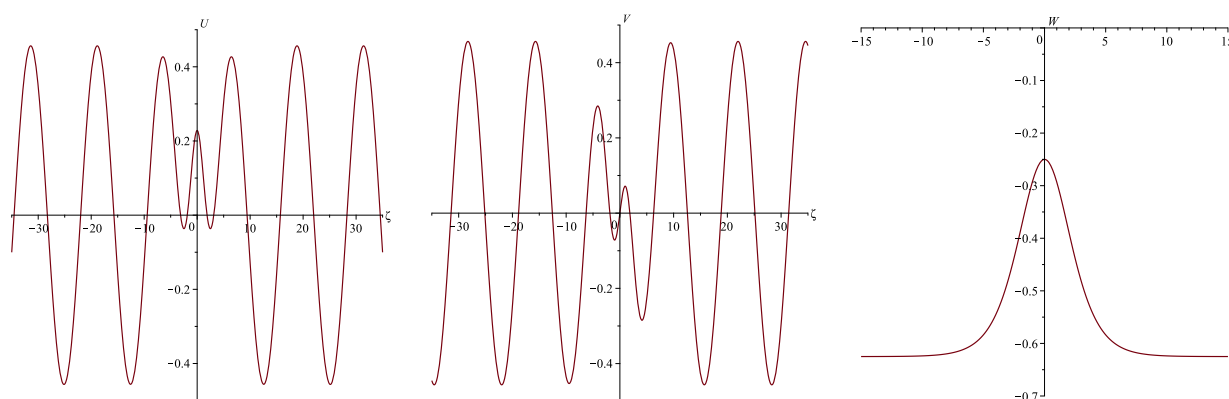


Figure 7. Solutions from Eqs (5.101) (left), (5.102) (middle), and (5.103) (right) where $\alpha = 1/2$, $\delta = 1$, $c = 1$, $\beta = 1$, $\zeta_0 = 0$, and $m = \frac{1}{2} \sqrt{-\mu/\beta}$, with positive signs in U , V and λ , μ defined by Eq (5.11).

5.2.3. Solution family (iii)

This solution family has

$$a_0 = \mp \mu \sqrt{\frac{\lambda}{8(\beta\lambda + \mu)}}, \quad a_1 = 0, \quad a_2 = \pm 3\mu \sqrt{\frac{\lambda}{8(\beta\lambda + \mu)}}, \quad b_1 = 0, \quad b_2 = 0, \quad (5.100)$$

$$s_0 = -\frac{2\beta\lambda + 3\mu}{4\beta}, \quad s_1 = 0, \quad s_2 = \frac{3\mu}{4\beta}, \quad h_1 = 0, \quad h_2 = 0, \quad \gamma = \frac{2\mu\beta}{\beta\lambda + \mu}, \quad m = \pm \frac{1}{2} \sqrt{-\frac{\mu}{\beta}}.$$

Substitution of this algebraic solution into the ansatz (5.90) and (5.91) and then the expressions obtained into Eqs (5.9) and (5.6) yield the solution

$$U(\zeta) = \mp \mu \sqrt{\frac{\lambda}{8(\beta\lambda + \mu)}} \cos(\tfrac{1}{2}c\zeta + \zeta_0)(1 - 3 \tanh(m\zeta)^2), \quad (5.101)$$

$$V(\zeta) = \mp \mu \sqrt{\frac{\lambda}{8(\beta\lambda + \mu)}} \sin(\tfrac{1}{2}c\zeta + \zeta_0)(1 - 3 \tanh(m\zeta)^2), \quad (5.102)$$

$$W(\zeta) = -\frac{2\beta\lambda + 3\mu}{4\beta} + \frac{3\mu}{4\beta} \tanh(m\zeta)^2. \quad (5.103)$$

The third solution family in Figure 7 differs markedly from the previous two, remaining finite and smooth for all $\zeta \in (-\infty, \infty)$. Since the factor $(1 - 3 \tanh(m\zeta)^2)$ tends to -2 as $|\zeta| \rightarrow \infty$ and equals 1 at the origin, the envelope modulus $|E|$ does not decay, it approaches a nonzero oscillatory background and forms a localized depression around $\zeta = 0$, where its value is half the asymptotic one, vanishing at the two points where $\tanh(m\zeta)^2 = 1/3$. The long-wave component W is a localized elevation on the nonzero background $-\lambda/2$, the departure from that level being $-3\mu/(4\beta)$, which is always positive by the reality condition $\mu/\beta < 0$, so that $|E|$ is of dark type on a nonzero background, while W is of bright type.

5.2.4. Solution family (iv)

This solution family has

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = \pm \frac{1}{4} \sqrt{3\lambda(4\beta\lambda - \mu)}, \quad b_1 = 0, \quad b_2 = \mp \frac{1}{4} \sqrt{3\lambda(4\beta\lambda - \mu)}, \quad (5.104)$$

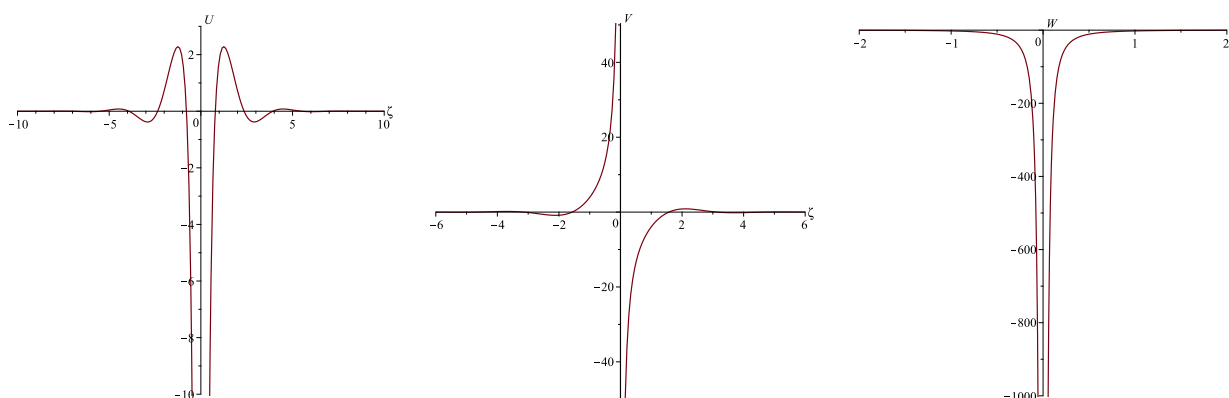


Figure 8. Solutions from Eqs (5.105) (left), (5.106) (middle), and (5.107) (right) where $\alpha = 17$, $\delta = -5$, $c = 4$, $\beta = 1$, $\zeta_0 = 0$, and $m = \sqrt{-\lambda}/2$, with positive signs in U , V and λ , μ defined by Eq (5.11).

$$s_0 = -\frac{3\lambda}{2}, \quad s_1 = 0, \quad s_2 = \frac{3\lambda}{4}, \quad h_1 = 0, \quad h_2 = \frac{3\lambda}{4}, \quad \gamma = \frac{2\beta\lambda + \mu}{3\lambda}, \quad m = \pm \frac{\sqrt{-\lambda}}{2}.$$

Substitution of this algebraic solution into the ansatz (5.90) and (5.91) and then the expressions obtained into Eqs (5.9) and (5.6) yield the solution

$$U(\zeta) = \pm \frac{1}{4} \sqrt{3\lambda(4\beta\lambda - \mu)} \cos(\frac{1}{2}c\zeta + \zeta_0)(\tanh(m\zeta)^2 - \coth(m\zeta)^2), \quad (5.105)$$

$$V(\zeta) = \pm \frac{1}{4} \sqrt{3\lambda(4\beta\lambda - \mu)} \sin(\frac{1}{2}c\zeta + \zeta_0)(\tanh(m\zeta)^2 - \coth(m\zeta)^2), \quad (5.106)$$

$$W(\zeta) = -\frac{3\lambda}{2} + \frac{3\lambda}{4}(\tanh(m\zeta)^2 + \coth(m\zeta)^2). \quad (5.107)$$

Unlike the first singular families, where the sin factor kept V bounded, the $\coth^2$ term here introduces a second-order singularity. As shown in Figure 8, all functions blow up at the origin. The trigonometric functions sin and cos cannot cancel this quadratic divergence, producing vertical asymptotes with amplitudes tending to $\pm\infty$, depending on the coefficient signs. As $|\zeta| \rightarrow \infty$, the term $(\tanh^2(m\zeta) - \coth^2(m\zeta))$ vanishes because both factors approach 1, so U and V decay to zero. Likewise, W vanishes in the far field, as the constant term $-\frac{3\lambda}{2}$ is exactly cancelled by the asymptotic limit of the hyperbolic terms. Thus, the entire solution family approaches a vacuum state at infinity. As in the previous cases, these solutions are singular (non-physical) and are defined classically only on each of the half-lines $\zeta < 0$ and $\zeta > 0$.

6. Conclusions

In this work, we have investigated several fundamental aspects of a generalized Schrödinger-Boussinesq system with a p -power nonlinearity, focusing on its symmetries, conservation laws, and travelling wave solutions.

First, a complete classification of infinitesimal point symmetries has been presented, along with a description of their physical significance. For a general power $p > 1$, the system admits a three-dimensional Lie algebra. In particular, for the specific case $p = 2$, the symmetry group extends to a four-dimensional algebra due to the emergence of an additional scaling symmetry. Using the multiplier method, we have derived low-order local conservation laws, including conservation of mass, momentum, and charge.

Furthermore, the introduction of a potential variable has enabled a Lagrangian formulation for the potential system, allowing Noether's theorem to be applied. We have demonstrated that all point symmetries of the potential system are variational, with the exception of the scaling symmetry for $p = 2$. Consequently, these variational symmetries have yielded all conservation laws for the potential system. As a result, nonlocal conservation laws have been found for the original system.

Finally, the system of PDEs has been reduced to a system of ODEs through a travelling wave reduction. This reduction has allowed the construction of several families of travelling wave solutions. We have also identified the parametric constraints to ensure the real value of these solutions. Graphical representations are provided, and their meaning is discussed.

While the Riemann-Hilbert and $\bar{\partial}$ -techniques used to prove soliton resolution and asymptotic stability [22] require an underlying Lax pair, the methods employed here need no integrability assumption, and our solutions provide explicit candidates for such an analysis when $p = 2$ and $\gamma = 6\beta$, which is a case known to be integrable [29]. The results presented here contribute to the understanding of nonlinear wave interactions in dispersive media and provide a rigorous mathematical framework for further studies on the stability and long-term dynamics of coupled Schrödinger-Boussinesq equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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